Prove Collatz Conjecture via Operations for Integer Expressions plus Few Integers

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Abstract

First, let us set forth certain of basic concepts related to proving Collatz conjecture. After that, list the mathematical induction that proves the conjecture, and prepare two theorems plus one lemma, which are used to judge relevant operational results. Next, classify integers successively and prove a class by the theorem 1, after each classification. Until the last two classes are proved bidirectionally, which are to start with several proven kinds to expand successively the scope of proven kinds up to all kinds are proven and star with each unproven kind to a proven kind via operations.

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1. Introduction

The Collatz conjecture is also called the *3x+1* mapping, *3n+1* problem, Hasse's algorithm, Kakutani's problem, Syracuse algorithm, Syracuse problem, Thwaites conjecture and Ulam's problem, etc.

But it remains a conjecture that has neither been proved nor disproved ever since named after Lothar Collatz in 1937; [1].

2. **Certain of Basic Concepts**

The Collatz conjecture states that take any positive integer n, if n is an even number, divide it by 2; if n is an odd number, multiply it by 3 and add 1, and repeat the process indefinitely, so no matter which positive integer you start with, you are always going to end up with 1; [2].

We consider aforesaid operational stipulations as the operational rule.

If you start with any positive integer/integer expression to operate continually by the operational rule, then it will form continuous positive integers/integer expressions. So, we regard continuous integers/integer expressions plus arrows toward a direction among these integers/integer expressions as an operational route.

Next, let us use a capital letter with the subscript "*ie*" to express a positive integer expression, such as P_{ie}, C_{ie} etc.

Besides, an operational route that contains a certain integer expression may be called "an operational route via the integer expression".

In general, integer expressions on an operational route have a common variable or many variables that can be converted into a common variable.

3. The Mathematical Induction that Proves the Conjecture

The mathematical induction [3] that proves the conjecture is as follows:

(1) From 1→4→2→1; 2→1; 3→10→5→16→8→4→2→1; 4→2→1; 5→16→8→4→2→1; 6→3→10→5→16→8→4→2→1; 7→22→11→34→17→52→26→13→40→20→10→5→16→8→4→2→1; 8→4→2→1 and 9→28→14→7→22→

$11 \to 34 \to 17 \to 52 \to 26 \to 13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$, we can be seen that each and every positive integer ≤ 9 fits the conjecture.

(2) Suppose that n fits the conjecture, where n is an integer ≥ 9.

(3) Prove that $n+1$ fits the conjecture likewise.

4. Several Judging Criteria

A certain result of operations which start with each class of positive integers is judged by one of following three criteria.

Theorem 1. If an operational route via P_{ie} has an integer expression that is less than P_{ie}, and $n+1 \in P_{ie}$, then all integer expressions on the operational route fit the conjecture.

For example, let $P_{ie}=31+3^2\beta$ with $\beta \ge 0$, and $n+1 \in P_{ie}$, then from $27+2^3\beta \to 82+3\times 2^3\beta \to 41+3\times 2^2\beta \to 124+3^2\times 2^2\beta \to 62+3^2\times 2\beta \to \mathbf{31}+3^2\beta > \mathbf{27}+2^3\beta$, we get that all integer expressions on the operational route fit the conjecture.

In addition, let $P_{ie}=5+2^2\mu$ with $\mu \ge 0$, and $n+1 \in P_{ie}$, then from $\mathbf{5}+2^2\mu \to 16+3\times 2^2\mu \to 8+3\times 2\mu \to \mathbf{4}+3\mu < \mathbf{5}+2^2\mu$, we get that all integer expressions on the operational route fit the conjecture.

Proof. Suppose that an operational route via P_{ie} has C_{ie}, and $C_{ie}<P_{ie}$, then when their common variable is equal to a certain fixed value, such that $P_{ie}=n+1$ and $C_{ie}=m$. So there is $m<n+1$, then m fits the conjecture, according to second step of the mathematical induction.

So from $n+1$ can operate to m, or from m can operate to $n+1$, in either case, it continues to operate to 1 via m, such that $n+1$ fits the conjecture.

When their common variable is equal to each value, each integer that every integer expression contains is operated to *1*, because each integer that every integer expression contains matches an integer of C_{ie}, therefore, all integer expressions on the operational route fit the conjecture.

Theorem 2. If an operational route via Q_{ie} and an operational route via P_{ie} intersect, and $n+1 \in P_{ie}$, in addition, an integer expression on the operational route via Q_{ie} is less than P_{ie}, then all integer expressions on these two operational routes fit the conjecture.

For example, let $Q_{ie}=71+3^3\times2^5\varphi$, and $P_{ie}=63+3\times2^8\varphi$ with $\varphi\geq0$, then from

$\mathbf{63}+3\times2^8\varphi\to190+3^2\times2^8\varphi\to95+3^2\times2^7\varphi\to286+3^3\times2^7\varphi\to143+3^3\times2^6\varphi\to430+3^4\times2^6\varphi\to$

$215+3^4\times2^5\varphi\to646+3^5\times2^5\varphi\to323+3^5\times2^4\varphi\to970+3^6\times2^4\varphi\to485+3^6\times2^3\varphi\to1456+3^7\times2^3\varphi$

$\to728+3^7\times2^2\varphi\to364+3^7\times2\varphi\to182+3^7\varphi\to \dots$

$\uparrow121+3^6\times2\varphi\leftarrow242+3^6\times2^2\varphi\leftarrow484+3^6\times2^3\varphi\leftarrow161+3^5\times2^3\varphi\leftarrow322+3^5\times2^4\varphi$

$\leftarrow107+3^4\times2^4\varphi\leftarrow214+3^4\times2^5\varphi\leftarrow\mathbf{71}+3^3\times2^5\varphi\leftarrow142+3^3\times2^6\varphi\leftarrow\mathbf{47}+3^2\times2^6\varphi<\mathbf{63}+3\times2^8\varphi$,

we get that all integer expressions on these two operational routes fit the conjecture.

***Proof*.** Suppose that there is D_{ie} on an operational route via Q_{ie}, and $D_{ie}<P_{ie}$, and that the operational route via Q_{ie} and an operational route via P_{ie} intersect at A_{ie}, so when their common variable is given a certain fixed value, such that $P_{ie}=n+1$, $A_{ie}=\xi$ and $D_{ie}=\mu$. Then, there is $\mu<n+1$, and μ fits the conjecture, according to second step of the mathematical induction.

Since ξ and μ belong to an operational route, and μ fits the conjecture, then all integers on the operational route fit the conjecture, including ξ, according to the theorem *1*.

Since $n+1$ and ξ belong to an operational route, and ξ fits the conjecture, then all integers on the operational route fit the conjecture, including $n+1$, according to the theorem *1*.

When the common variable is equal to each value, each integer which every integer expression contains on these two operational routes is also operated to *1*, because each integer that every integer expression contains matches an integer of D_{ie}, therefore, all integer expressions on these two operational routes fit the conjecture.

Lemma 1. If there is a proven integer expression on an operational route, then all integer expressions on the operational route fit the conjecture.

Lemma 2. If there is a proven integer expression on successive intersecting operational routes, then all integer expressions on these operational routes fit the conjecture.

Lemma 3. Each and every integer that proven integer expression contains fits the conjecture.

5. Successive Classification with Proof for Positive Integers

We divide positive integers into multilevel classes successively, then the positive integer $n+1$ is possibly included in any class, thus each class of positive integers must be proved to fit the conjecture.

After an unproven class of positive integers is parted into new classes, we need to find an integer expression that is less than a new class via operations, in order to enable the new class to fit the conjecture.

Classification with Proof. As positive integers≤ 9 have been proven to fit the conjecture, in section *2*, so we first divide integers >9 into even numbers and odd numbers.

For even numbers $2k$ with $k>4$, from $2k \to k<2k$, we get that if $n+1 \in 2k$, then 2k and $n+1$ fit the conjecture, according to the theorem *1*.

For odd numbers >9, divide them into $11+4k$ and $13+4k$, where $k \ge 0$.

For $13+4k$, from $13+4k \to 40+12k \to 20+6k \to 10+3k<13+4k$, we get that if $n+1 \in 13+4k$, then $13+4k$ and $n+1$ fit the conjecture according to the theorem *1*.

Continue to divide $11+4k$ into $11+12c$, $15+12c$ and $19+12c$, where $c \ge 0$.

For $11+12c$, from $7+8c \to 22+24c \to 11+12c>7+8c$, we get that if $n+1 \in 11+12c$, then $11+12c$ and $n+1$ fit the conjecture, according to the theorem *1*.

The proof of $15+12c$ and $19+12c$ is the focus of this article, thus we need to make specially an important joint proof hereinafter.

6. Prove that 15+12c and 19+12c Fit the Conjecture

For the sake of avoiding confusion and conducive convenience, we substitute *d, e, f, g,* etc. for *c* on operational routes of $15+12c/19+12c$.

Let us continue to operate $15+12c/19+12c$ withe $c \ge 0$ by the operational rule.

Theoretically speaking, after operate $15+12c/19+12c$, each and every kind of $15+12c/19+12c$ must find an integer expression that is less than the kind of $15+12c/19+12c$, in order to meet one of judging criteria.

Firstly, We start with *15+12c* to operate continuously by the operational rule, as listed below.

15+12c→46+36c→23+18c→70+54c→35+27c ♣

d=2e+1: 29+27e (**1**)   e=2f: 142+486f→71+243f ♥
♣35+27c↓→c=2d+1: 31+27d↑→d=2e: 94+162e→47+81e↑→e=2f+1:64+81f (**2**)
c=2d: 106+162d→53+81d↓→d=2e+1:67+81e↓→e=2f+1:74+81f (**3**)
d=2e:160+486e♦   e=2f: 202+486f→101+243f ♠

g=2h+1: 200+243h (**4**)   **…**
♥ 71+243f↓→f=2g+1:157+243g↑→g=2h: 472+1458h→236+729h↑→ **…**
f=2g: 214+1458g→107+729g↓→g=2h+1: 418+729h↓→**…**
g=2h: 322+4374h→**… …**

g=2h: 86+243h (**5**)
♠101+243f↓→f=2g+1:172+243g↑→g=2h+1:1246+1458h→**…**
f=2g: 304+1458g→152+729g ↓→**…**
…

…
♦160+486e→80+243e↓→e=2f+1: 970+1458f→485+729f↑→ **…**   **…**
e=2f:40+243f↓→f=2g+1:850+1458g→ 425+729g↑→ **…**
f=2g: 20+243g↓→g=2h: 10+243h (**6**)   **…**
g=2h+1:790+1458h→395+729h↑→ **…**

Annotation:
(1) Each of letters c, d, e, f, g, h, etc on listed above operational routes expresses each of natural numbers plus 0.
(2) There are ♣↔♣, ♥↔♥, ♠↔♠, and ♦↔♦ on above operational routes.
(3) Aforesaid two points are suitable to latter operational routes of 19+12c similarly.

First, let us define a term. That is, if an operational result is less than a kind of *15+12c/19+12c*, and it first appears on an operational route relating to the kind of *15+12c/19+12c*, then we call the operational result "№1 satisfactory operational result" about the kind of *15+12c/19+12c*.

Accordingly, we first conclude following *3* kinds of *15+12c* derived from №1 satisfactory operational results to fit the conjecture, on the bunch of operational routes of *15+12c*.

1). From c=2d+1 and d=2e+1 to get c=2d+1=2(2e+1)+1=4e+3, then there are

$15+12c=51+48e=51+3\times2^4e\to154+3^2\times2^4e\to77+3^2\times2^3e\to232+3^3\times2^3e\to116+3^3\times2^2e\to58+3^3\times2e\to29+27e$ where the mark (**1**).

Due to $29+27e < 51+48e$, we get that if there is $n+1\in 51+48e$, then $51+48e$ and $n+1$ fit the conjecture, according to the theorem *1*.

2). From $c=2d+1$, $d=2e$ and $e=2f+1$ to get $c=2d+1=4e+1=4(2f+1)+1=8f+5$, then there are $15+12c=75+96f=75+3\times2^5f\to226+3^2\times2^5f\to113+3^2\times2^4f\to340+3^3\times2^4f\to170+3^3\times2^3f\to85+3^3\times2^2f\to256+3^4\times2^2f\to128+3^4\times2^1f\to64+81f$ where the mark (**2**).

Due to $64+81f < 75+96f$, we get that if there is $n+1\in 75+96f$, then $75+96f$ and $n+1$ fit the conjecture, according to the theorem *1*.

3). From $c=2d$, $d=2e+1$ and $e=2f+1$ to get $c=2d=4e+2=4(2f+1)+2=8f+6$, then there are $15+12c=87+96f=87+3\times2^5f\to262+3^2\times2^5f\to131+3^2\times2^4f\to394+3^3\times2^4f\to197+3^3\times2^3f\to592+3^4\times2^3f\to296+3^4\times2^2f\to148+3^4\times2^1f\to74+81f$ where the mark (**3**).

Due to $74+81f < 87+96f$, we get that if there is $n+1\in 87+96f$, then $87+96f$ and $n+1$ fit the conjecture, according to the theorem *1*.

Like that, according to the theorem *1,*each reader can also conclude other 3 kinds of $15+12c$ derived from №1 satisfactory operational results to fit the conjecture, on the bunch of operational routes of $15+12c$. They are:

4). Pursuant to $c=2d+1$, $d=2e$, $e=2f$, $f=2g+1$ and $g=2h+1$, you can get $15+12c=315+384h$ derived from $200+243h$ where the mark (**4**);

5). Pursuant to $c=2d$, $d=2e+1$, $e=2f$, $f=2g+1$ and $g=2h$, you can get $15+12c=135+384h$ derived from $86+243h$ where the mark (**5**);

6). Pursuant to $c=2d$, $d=2e$, $e=2f$, $f=2g$ and $g=2h$, you can get $15+12c=15+384h$ derived from $10+243h$ where the mark (**6**).

Secondly, We start with *19+12c* to operate continuously by the operational rule, as listed below.

19+12c→58+36c→ 29+18c→ 88+54c→ 44+27c ♣

d=2e: 11+27e (**α**)   e=2f:37+81f (**β**)
♣ 44+27c↓→c=2d: 22+27d↑→d=2e+1:148+162e→74+81e↑→e=2f+1:466+486f ♥
c=2d+1: 214+162d→107+81d↓→d=2e:322+486e ♠
d=2e+1:94+81e↓→e=2f:47+81f (**γ**)
e=2f+1:526+486f ♦

g=2h: 119+243h (**δ**)   …
f=2g+1:238+243g↑→g=2h+1:1444+1458h→722+729h↑→…
♥466+486f→233+243f↑→f=2g: 700+1458g→350+729g↓→g=2h+1:3238+4374h↓
g=2h: 175+729h↓→…   …
…

g=2h+1:172+243h (**ε**)
f=2g: 101+243g↑→g=2h: 304+1458h→…
e=2f+1:202+243f↑→f=2g+1:1336+1458g→…
♠322+486e→161+243e↑→e=2f:484+1458f→…

♦526+486f→263+243f↓→f=2g: 790+1458g→…
f=2g+1: 253+243g↓→g=2h+1: 248+243h (**ζ**)
g=2h: 760+1458h→…

As listed above, we first conclude following 3 kinds of *19+12c* derived from №1 satisfactory operational results to fit the conjecture, on the bunch of operational routes of *19+12c*.

1). From c = 2d and d = 2e to get c = 2d = 4e, then there are 19+12c = 19+48e = $19+3\times2^4e \rightarrow 58+3^2\times2^4e \rightarrow 29+3^2\times2^3e \rightarrow 88+3^3\times2^3e \rightarrow 44+3^3\times2^2e \rightarrow 22+3^3\times2e \rightarrow 11+27e$ where the mark (**α**).

Due to 11+27e<19+48e, we get that if there is *n+1*∈*19+48e*, then *19+48e* and *n+1* fit the conjecture, according to the theorem *1*.

2). From c=2d, d=2e+1 and e=2f to get c=2d=2(2e+1)=4e+2=8f+2, then there are $19+12c=43+96f=43+3\times2^5f \rightarrow 130+3^2\times2^5f \rightarrow 65+3^2\times2^4f \rightarrow 196+3^3\times2^4f \rightarrow 98+3^3\times2^3f \rightarrow 49+3^3\times2^2f \rightarrow 148+3^4\times2^2f \rightarrow 74+3^4\times2^1f \rightarrow 37+81f$ where the mark (**β**).

Due to $37+81f<43+96f$, we get that if there is $n+1 \in 43+96f$, then $43+96f$ and $n+1$ fit the conjecture, according to the theorem *1*.

3). From c=2d+1, d=2e+1 and e=2f to get c=2d+1=4e+3=8f+3, then there are

$19+12c=55+96f=55+3\times2^5f\to166+3^2\times2^5f\to83+3^2\times2^4f\to250+3^3\times2^4f\to125+3^3\times2^3f\to$

$376+3^4\times2^3f\to188+3^4\times2^2f\to94+3^4\times2^1f\to47+81f$ where the mark (γ).

Due to $47+81f < 55+96f$, we get that if there is $n+1 \in 55+96f$, then $55+96f$ and $n+1$ fit the conjecture, according to the theorem *1*.

Like that, according to the theorem *1,*each reader can also conclude other 3 kinds of *19+12c* derived from №1 satisfactory operational results to fit the conjecture, on the bunch of operational routes of *19+12c*. They are:

4). Pursuant to c=2d, d=2e+1, e=2f+1, f=2g+1 and g=2h, you can get *19+12c=187+384h* derived from *119+243h* where the mark (**δ**);

5). Pursuant to c=2d+1, d=2e, e=2f+1, f=2g and g=2h+1, you can get *19+12c=271+384h* derived from *172+243h* where the mark (**ε**);

6). Pursuant to c=2d+1, d=2e+1, e=2f+1, f=2g+1 and g=2h+1, you can get *19+12c=391+384h* derived from *248+243h* where the mark (ζ).

So far, take into account what we and each reader have done, there are altogether 6 kinds of *15+12c/19+12c* to fit the conjecture.

It follows that if *n+1* belongs in any kind of *15+12c/19+12c* derived from №1 satisfactory operational result, then that kind of *15+12c/19+12c* and *n+1* fit the conjecture, according to the theorem *1*.

Now, we analyze two bunches of operational routes of *15+12c* and *19+12c* whether they are related, as described below.

Due to $c \geq 1$, there are infinitely more odd numbers of $15+12c/19+12c$, whether they belong to infinite or finite more kinds, boil down to they can only be on the bunch of operational routes of $15+12c/19+12c$. That is to say, all odd numbers of $15+12c/19+12c$ can only come from $15+12c/19+12c$.

For variables on operational routes of $15+12c/19+12c$ in the identical range, they are often seen as having many, however, in fact they can be converted into a common variable, otherwise, any kind of $15+12c/19+12c$ is in the hide always. From this, not only allows you to compare the size between a certain operational result and a kind of $15+12c/19+12c$, but also let you know that every two operational routes on the bunch of operational routes of $15+12c/19+12c$, they either directly intersect or indirectly connect, since they can be extended enough.

Besides, not only one kind of $15+12c/19+12c$ derives from №1 satisfactory operational result, but also there is the case where at least two kinds of $15+12c/19+12c$ derive from a satisfactory operational result, such as $15+12(4+2^{55}\times3^{2}y)$ and $15+12(8+2^{32}\times3^{17}y)$ are derived from $61+2^{3}\times3^{37}y$.

In some cases, an operational route of $15+12c$ and an operational route of $19+12c$ coincide partially or intersect from each other, such as start with $15+12(1+2^{57}y)$ to operate and get $19+12(1+2^{54}\times3^{2}y)$ after fifth step.

Since we have analyzed these two bunches of operational routes, and that we have already the enough evidences to prove that all unproven kinds of

15+12c/19+12c fit the conjecture, before this. Thus, in the following paragraphs, let us prove them from each other's- opposite directions.

Firstly, we start with proven 6 kinds of *15+12c/19+12c* to continuously expand the scope of proven kinds of *15+12c/19+12c.*

According to the theorem *1* and theorem *2,* all integer expressions on at least one operational route via each of proven 6 kinds of *15+12c/ 19+12c* fit the conjecture.

After that, all integer expressions which fit the conjecture are turned into proven integer expressions.

Next, according to the lemma *1* and lemma *2*, all integer expressions on successive intersecting operational routes via each of proven integer expressions fit the conjecture.

After that, all integer expressions which fit the conjecture are turned into proven integer expressions.

According to the lemma *1* and lemma *2*, the rest can be deduced by analogy, as thus, proven integer expressions on the bunch of operational routes of *15+12c/19+12c* are getting more and more, until all integer expressions on the bunch of operational routes of *15+12c/19+12c* are proved to fit the conjecture.

Since all integer expressions on the bunch of operational routes of *15+12c/ 19+12c* contain all kinds of *15+12c/19+12c*, so all kinds of *15+12c/ 19+12c* are proved to fit the conjecture.

In this case, if $n+1$ belongs in a kind of $15+12c/19+12c$, then it does not get around the fact that it is proved, according to the lemma 3.

Secondly, we start with each unproved kind of $15+12c/19+12c$ to operate continuously by the operational rule, until find №1 satisfactory operational result relating to the unproved kind of $15+12c/19+12c$.

First of all, how do you present an unproved kind of $15+12c/19+12c$?

Since all kinds of $15+12c/19+12c$ can be expressed into o_1+12px/o_2+12px, where o_1 and o_2 are positive odd numbers; p is an integer, and $p>1$; x is a variable, and $x\neq c$. Thus, after p is assigned to an integer, if $15+12px/19+12px$ is not a proven kind of $15+12c/19+12c$, then it is exactly an unproved kind of $15+12c/19+12c$. In addition, before the first proof here, we have already 6 proven kind of $15+12c/19+12c$. they are: $15+12c=51+48e$, $75+96f$, $87+96f$, $315+384h$, $135+384h$ $15+384h$; $19+12c=19+48e$, $43+96f$, $55+96f$, $187+384h$, $391+384h$, $271+384h$, therefore, you do not worry without proven kind of $15+12c/19+12c$.

Unquestionably, all operational routes whose each via an unproved kind of $15+12c/19+12c$ and all operational routes whose each via a proven kind of $15+12c/19+12c$ form the bunch of operational routes of $15+12c/19+12c$.

Now that we can find unproved kinds of $15+12c/19+12c$ in turn, and we are possessed of the operational rule and criteria for judging operational results, then we can prove each unproved kind of $15+12c/19+12c$ to fit the conjecture by one of following three ways.

(1) An operational route via an unproved kind of *15+12c/19+12c* has an integer expression that is less than the unproved kind of *15+12c/19+12c*, then the unproved kind of *15+12c/19+12c* is proved to fit the conjecture, according to the theorem *1*.

(2) An operational route via an unproved kind of *15+12c/19+12c* and an operational route via a proven kind of *15+12c/19+12c* intersect, then the unproved kind of *15+12c/19+12c* is proved to fit the conjecture, according to the theorem *2*.

(3) There are an operational route via an unproved kind of *15+12c/19+12c* and an operational route via a proven kind of *15+12c/19+12c* within successive intersecting operational routes, then the unproved kind of *15+12c/19+12c* is proved to fit the conjecture, according to the lemma *2*.

Apply the above established practice to each and every unproved kind of *15+12c/19+12c*, then all unproved kind of *15+12c/19+12c* are proved to fit the conjecture.

According to the bidirectional proofs which have been given above, such that all kind of *15+12c/19+12c* on the bunch of operational routes of *15+12c/19+12c* have been proved to fit the conjecture, as thus, if *n+1* belongs in a kind of *15+12c/19+12c*, then *n+1* in the kind of *15+12c/19+12c* is proved to fit the conjecture, accord to the lemma *3*.

7. Make a Summary and Reach the Conclusion

To sum up, *n+1* has been proved to fit the conjecture, whether *n+1* belongs in which kind of odd numbers, or it is exactly an even number.

We can also prove integers *n+2*, *n+3* etc. up to every integer that is greater than *n+1* to fit the conjecture in the light of the old way of doing thing.

The proof was thus brought to a close. As a consequence, the Collatz conjecture is tenable.

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