Square Convergence Algorithm to Approach Pi

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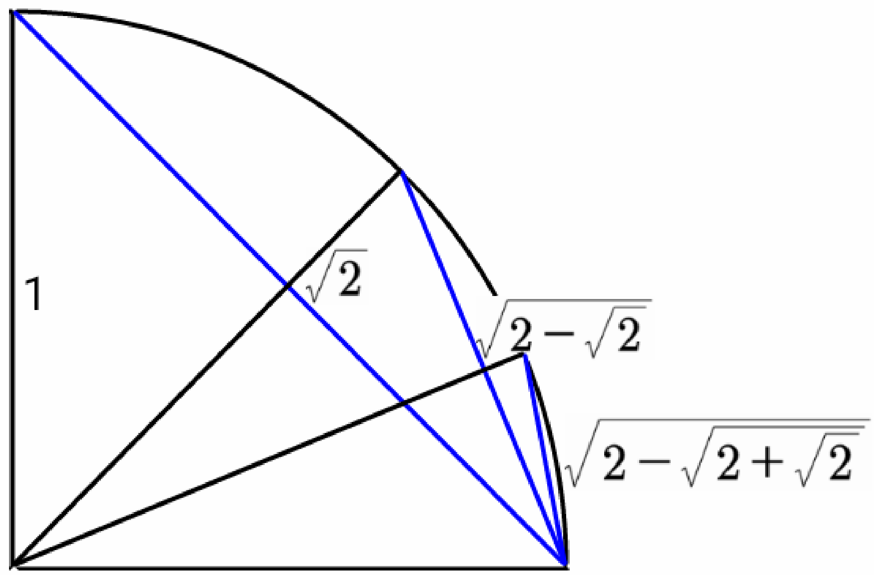
Abstract

One of the objectives of this article is to contribute to the further development and improvement of similar algorithms. I will first outline the steps that gradually lead to this algorithm and give some instructions on how to use it. There are several experimentation possibilities that can lead to improved performance. This also depends on the technical characteristics of the computer on which the program will run.

Geometric approach

Our story begins with a geometric approach to π, which is shown in the figure below.



The rays are perpendicular to the corresponding chords. The figure shows the first three steps. It is a very easy method, which leads us to the following known relationship.

$$\pi = \lim_{n\to\infty} 2^n \underbrace{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+...+\sqrt{2}}}}}}_{n\ roots}$$

The chord is multiplied by 2^n in order to approach the π from the sum of the lengths of the sides of the polygon.

From now on we will use the following symbolism for the various approaches of π.

$$\pi_n = 2^n \underbrace{\sqrt{2-\sqrt{2+\sqrt{2+...+\sqrt{2}}}}}_{n\ \text{roots}} = 2^{n+1}\sin\left(\frac{90^\circ}{2^n}\right)$$

It is always true that $\pi_{n-1}<\pi_n<\pi$. A repetitive method that would approach the π based on this relationship would require the extraction of time-consuming square roots. To get rid of the roots, we first transfer all the terms to the left member of equality. So we will have

$$(...(((\pi_n/2^n)^2-2)^2-2)^2-...-2)^2-2=0$$

If we now replace πn with a positive number p, where $p<\pi_n$, then the last representation will be equal to a number r, where $r\neq0$. Then it will be true that

$$p \lesssim p+r \leq \pi_n$$

So, by assigning the value of $p+r$ to p and repeating this process, we will end up marginally to equality $p=\pi_n$.

However, the convergence in π accelerates if after each execution of the process $p=p+r$ we multiply n by an integer such as 2, eg $n=2n$.

We also need an upper barrier for π. To do this, when exiting the program, we need to calculate the length of the side of the outer polygon that surrounds the circle of the above figure. In this way we trap the π between two polygons. This is easily done and leads to the following relationships.

$$\Pi_n = \pi_n^2/\pi_{n-1}$$

$$\pi_n \lesssim \pi \lesssim \Pi_n$$

You may think that the average of the values πn and Π_n would give a better approach. This is correct, but there is a better ratio. After studying the data, I came up with the following improved approaches.

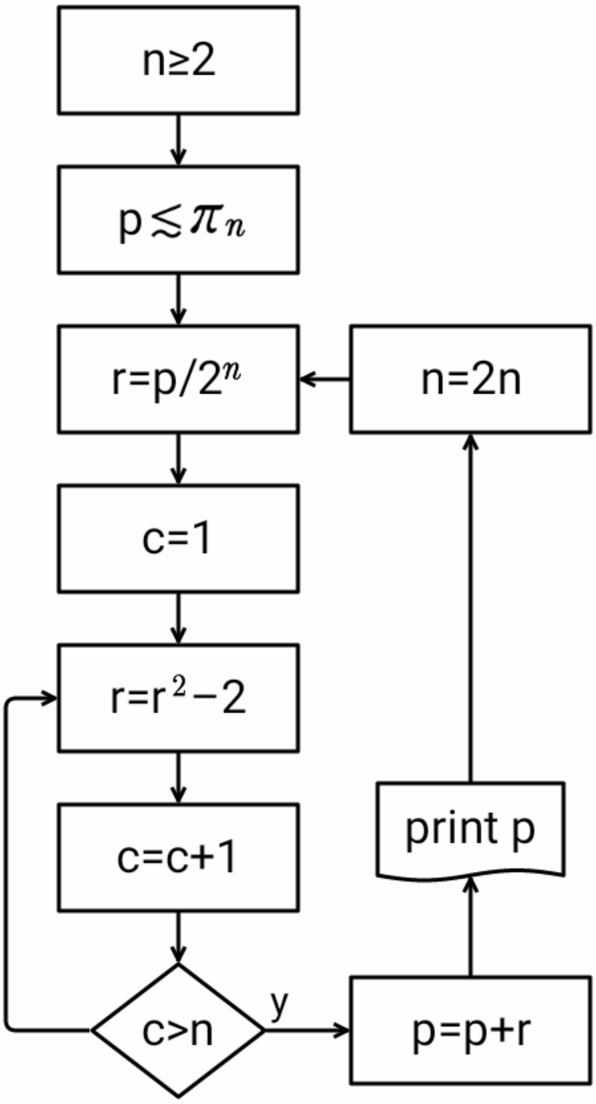
$$\pi_n \lesssim \pi \lessapprox (\pi_n/3)\,(2+\pi_n/\pi_{n-1}) \lesssim \pi_n^2/\pi_{n-1}$$

$$\pi_n\,((4^n/3)\,(2+\pi_n/\pi_{n-1})+1)\,/\,(4^n+1) \lessapprox \pi \quad (?)$$

The second of these inequalities is not very certain (hence the question mark). In the following algorithms we will not use these relationships, so as not to cause confusion.

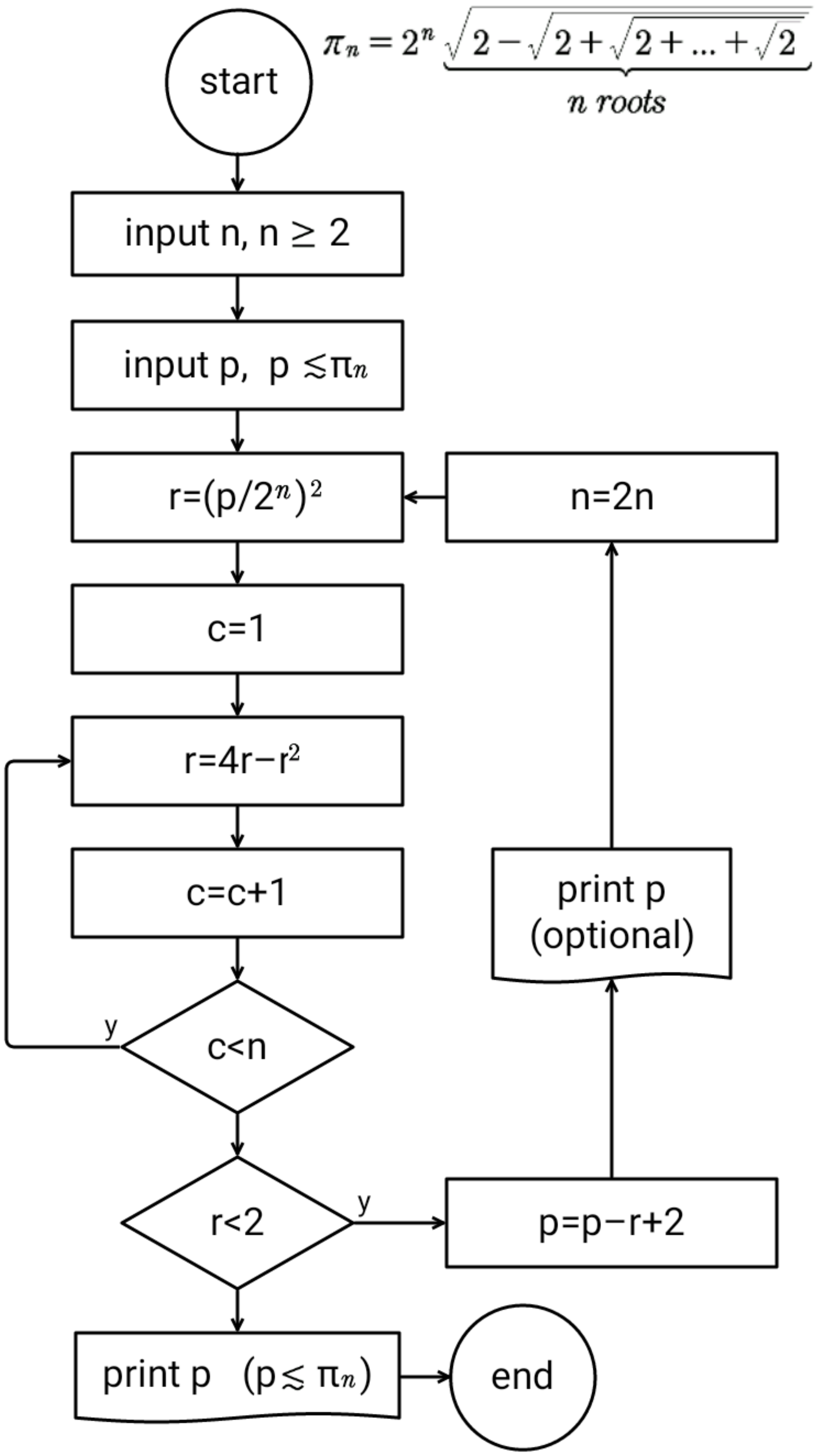
The above are implemented (in first phase) in the following algorithm.

$$\pi_n = 2^n \underbrace{\sqrt{2-\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}}_{n\ roots}$$



First of all, this algorithm inherits a defect of the root representation of π_n. Specifically, the term that follows the negative sign takes the value 1,999... before the critical digits begin to appear. This limits the accuracy to about half of the available of calculator digits. Then it will start to give wrong results. If your computer has too much memory to memorize digits, then this defect in this algorithm may not be annoying. I suggest you try giving n=p=2 at the input.

After some modifications, the above algorithm has been improved to take advantage of the ability to hold extremely small numbers that scientific pocket calculators have.



This algorithm allows π to be approached with the accuracy of all available digits of a scientific pocket calculator.

The number of relevant digits is doubled in each calculation cycle. (sentence: start with $n=p=2$)

The program is led to the output when r exceeds 2, due to the depletion of the available digits of the calculator.

The process $r=4r-r^2$ should not be written in the form $r=r(4-r)$ as this may be to the detriment of the desired accuracy.

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