Links between string theory and the Riemann's zeta function

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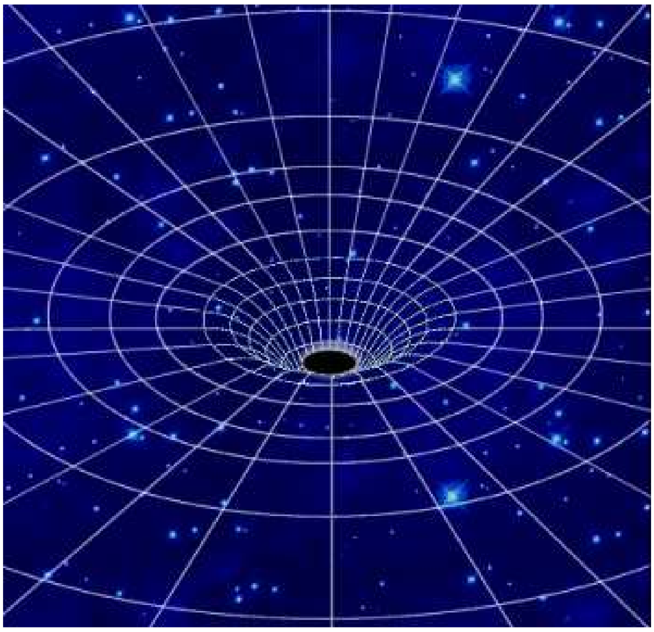
There is a connection between string theory and the Riemann's zeta function: this is an interesting way, because the zeta is related to prime numbers and we have seen on many occasions how nature likes to express himself through perfect laws or mathematical models.



Not least the situation that certain stable energy levels of atoms could be associated with non-trivial zeros of the Riemann's zeta. In [6] for example has been shown the binding of the Riemann zeta and its non-trivial zeros with quantum physics through the Law of *Montgomery-Odlyzko.*

The law of Montgomery-Odlyzko says that "the distribution of the spacing between successive non-trivial zeros of the Riemann zeta function (normalized) is identical in terms of statistical distribution of spacing of eigenvalues in an GUE operator", which also represent dynamical systems of subatomic particles!

In [6] the authors showed all the mathematical and theoretical aspects related to the Riemann's zeta, while in [9] showed the links of certain formulas of number theory with the golden section and other areas such as string theory. The authors have proposed a solution of the Riemann hypothesis (RH) and the conjecture on the multiplicity of nontrivial zeros, showing that they are simple zeros [7][8].

In [10] [11] have proposed hypotheses equivalent RH, in [12] [13] the authors have presented informative articles on the physics of extra dimensions, string theory and M-theory, in [15] the conjecture Yang and Mills, in [16] the conjecture of Birch and Swinnerton-Dyer.

A mathematical aspect of string theory

During the experiments in particle accelerators, physicists had observed that the spin of a hadron is never greater than a certain multiple of the root of its energy, but no simple hadronic model was able to explain these relationships and the behaviour of hadrons (see [12]).

Gabriele Veneziano in 1968 found that a function in complex variables created by the Swiss mathematician Leonhard Euler, could be the right answer: the beta function, was perfectly suited to data obtained *strong nuclear interaction*.

Veneziano applied the Beta function to the strong interaction:

$$\beta\left(p,q\right)=\int_0^1 x^{p-1}\left(1-x\right)^{q-1}dx=\frac{\Gamma\left(p\right)\Gamma\left(q\right)}{\Gamma\left(p+q\right)} \qquad (1)$$

with $\mathrm{Re}\left(p\right)>0,\mathrm{Re}\left(q\right)>0$

but no one could explain why it works well. In 1970, *Yoichiro Nambu*, *Holger Bech Nielsen*, and *Leonard Susskind* presented a physical explanation of the extraordinary precision of the theoretical formula of Euler: representing the nuclear force by vibrating strings to a single dimension, showed that the function of Euler was a good description.

The *Beta function* is also called the Euler's integral of the first kind [6]; it is given by the integral defined:

$$\beta(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt \qquad (2)$$

where both x and y have positive real part and not null (if they were, the integral does not converge to a finite number).

This historical function was studied first by Euler, then by Legendre, and Jacques Binet. It is a *symmetric function*, i.e. its value does not change by exchanging the arguments:

$$\beta\left(x,y\right)=\beta\left(y,x\right)$$

Furthermore, we have also the following identities:

- $\beta(1,1)=1$
- $\beta(1/2,1/2)=\pi$

We can write the Beta function in the following various modes:

$$\beta(x,y)=\frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)} \qquad (3)$$

$$\beta(x,y) = 2\int_0^{\pi/2} \sin^{2x-1}\theta\cos^{2y-1}\theta\, d\theta, \qquad \Re(x) > 0,\ \Re(y) > 0 \tag{4}$$

$$\beta(x,y) = \int_0^{\infty} \frac{t^{x-1}}{(1+t)^{x+y}}\, dt, \qquad \Re(x) > 0,\ \Re(y) > 0 \tag{5}$$

$$\beta(x,y) = \frac{1}{y}\sum_{n=0}^{\infty}(-1)^n \frac{(y)_{n+1}}{n!(x+n)} \tag{6}$$

where Γ(x) s the *Gamma function*, due to Euler, is a *meromorphic function*, continuous on the positive real numbers, which extends the concept of factorial to complex numbers, meaning that for every non-negative integer n we have:

$$\Gamma(n+1) = n!, \tag{7}$$

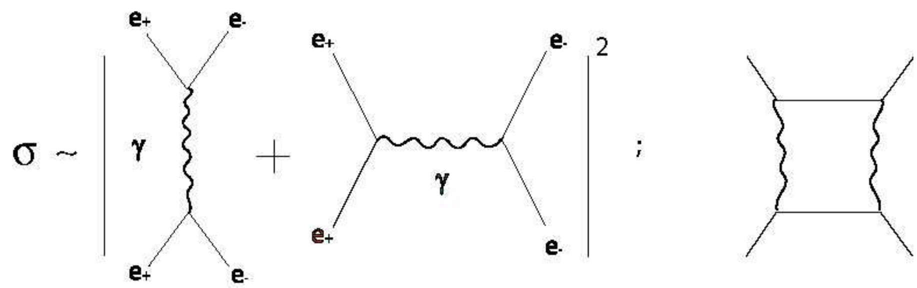
where *n*! is the *factorial*.

While the Gamma function describes the factorial of integers, the Beta function can describe the Newton binomial coefficients:

$$\binom{n}{k} = \frac{1}{(n+1)\mathrm{B}(n-k+1, k+1)} \tag{8}$$

Physical meaning of the result of G. Veneziano

The collision processes (see figure) have a key role, both in terms of experimental and theoretical physics of elementary particles, and are the primary tool for the study of their interactions. Since the beginning of the atomic theory of the nature of the atom was studied with "techniques impact": particles shot into the atom (see Rutherford experiment for example).



In the figure above there are two Feynman diagrams "tree" for the process of impact $e^+e^- \rightarrow e^+e^-$ between a *positron* (e^+) and a *electron* (e^-) and an example of diagram of higher order with a closed loop or "loop". The lines can be associated with trajectories of particles involved in the process, and the heads of their electromagnetic interactions. Lowest order contributions involving the exchange of a photon γ, channels s (vertical) and t (horizontal), respectively, and the dominant contribution to the cross section σ is related to the square of module of their sum.
We saw in [12] that one of the greatest difficulties in QED is the presence of quantum vacuum and virtual particles, which contribute in the interactions giving rise to other particles and so on.

This fact underlies the technique of “Feynman diagrams”, which allows to link the probability of reaction, known as "cross sections" for the elementary processes in which particles reagents generate reaction products through the formation of other particles in intermediate states.

The sum of diagrams related to a certain process combines the theory of probability amplitude, a complex number whose square module essentially determines the cross sections. But the proliferation of particles subject to strong interactions has long limited the application of these methods because of the extreme intensity of these nuclear forces, and then in the '60s many efforts have been devoted to the problem of characterizing the cross sections or "S matrix", a collection of the corresponding amplitudes of probability.

String theory originated precisely in this context, when the use of Quantum Field Theory and the corresponding Feynman diagrams seemed impossible for the strong interactions.

In this context, places the result of Veneziano. In (2), with variables x and y, Veneziano identified the angles of impact and the energies of particles involved in the collision. In general, the Feynman diagrams depend on these magnitudes, but does not show any symmetry under their individual trade, and therefore the peculiar function B was his obvious symmetry under exchange of two variables x and y, which in this context is defined "*planar duality*”.

B (x, y) also has infinitely many "poles" for x = 0, -1, -2, ... and similarly for y, in the neighborhood of which essentially acts as the function 1 / z near the origin for z = 0.

Singularity of this type are characteristic of the contributions of lower order (without the "loops" or "tree" as the two diagrams on the left in picture above), whose intermediate states involving many types of particles, one at a time and therefore reported their exchange.

It was therefore clear that the Veneziano amplitude had originated from a theory much more complex than any other previously known, with countless types of particles, all bosons, of mass and spin increasing.

Links between Gamma and Beta

The (3) proves the product of two factorials as:

$$\Gamma(\mathrm{x})\Gamma(\mathrm{y}) = \int_0^\infty \ e^{-u}u^{x-1}\mathrm{d}u \int_0^\infty \ e^{-v}v^{y-1}\mathrm{d}v.$$

Now let $u \equiv a^2, v \equiv b^2$ such that:

$$\begin{aligned}\Gamma(\mathrm{x})\Gamma(\mathrm{y}) &= 4\int_0^\infty \ e^{-a^2}a^{2x-1}\mathrm{d}a \int_0^\infty \ e^{-b^2}b^{2y-1}\mathrm{d}b \\ &= \int_{-\infty}^\infty \int_{-\infty}^\infty \ e^{-(a^2+b^2)}|a|^{2x-1}|b|^{2y-1}\,\mathrm{d}a\,\mathrm{d}b.\end{aligned}$$

Transform in polar coordinates with $a = r\cos\theta$, $b = r\sin\theta$:

$$
\begin{aligned}
\Gamma(\mathrm{x})\Gamma(\mathrm{y}) &= \int_0^{2\pi} \int_0^{\infty} e^{-r^2}|r\cos\theta|^{2x-1}|r\sin\theta|^{2y-1} r\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^{\infty} e^{-r^2} r^{2x+2y-2}\,\mathrm{d}r \int_0^{2\pi} |\cos^{2x-1}\theta\sin^{2y-1}\theta|\,\mathrm{d}\theta \\
&= 2\int_0^{\infty} e^{-r^2} r^{2(x+y-1)}\,\mathrm{d}(r^2) \int_0^{\pi/2} \cos^{2x-1}\theta\sin^{2y-1}\theta\,\mathrm{d}\theta \\
&= 2\Gamma(x+y) \int_0^{\pi/2} \cos^{2x-1}\theta\sin^{2y-1}\theta\,\mathrm{d}\theta \\
&= \Gamma(x+y)\beta(x,\, y).
\end{aligned}
$$

then:

$$\beta(x,\, y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}.$$

The derivative of the beta function can be written using, again, the Gamma function and *digamma function* $\psi(x)$:

$$\frac{\partial}{\partial x}\beta(x,y) = \beta(x,y)\left(\frac{\Gamma'(x)}{\Gamma(x)} - \frac{\Gamma'(x+y)}{\Gamma(x+y)}\right) = \beta(x,y)(\psi(x) - \psi(x+y))$$

Links between the Beta and the Riemann's Zeta

In [6] we have seen that if the real part of complex number z is positive, then the integral

$$\Gamma(z) = \int_0^\infty t^{z-1}\, e^{-t}\, dt$$

converges absolutely and represents the Gamma function.

Using analytic continuation, the Γ converges for z with real part not positive, if not whole. Using integration by parts, one can show that:

$$\Gamma(z+1) = z\Gamma(z)\,.$$

Since Γ (1) = 1, this relationship implies, for all natural numbers n, which

$$\Gamma(n+1) = n!$$

Other definitions are:

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\; n^z}{z\;(z+1)\cdots(z+n)}$$

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}$$

where γ is the *constant of Euler-Mascheroni*.

In [6] we saw that other important properties of the gamma function is the *reflection formula* of Euler:

$$\Gamma(1-z)\ \Gamma(z) = \frac{\pi}{\operatorname{sen}(\pi z)}$$

and:

$$\Gamma(z)\ \Gamma\left(z + \frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\ \Gamma(2z).$$

This is a particolar case of

$$\Gamma(z)\ \Gamma\left(z + \frac{1}{m}\right)\ \Gamma\left(z + \frac{2}{m}\right) \cdots \Gamma\left(z + \frac{m-1}{m}\right) = (2\pi)^{(m-1)/2}\ m^{1/2-mz}\ \Gamma(mz).$$

Derivatives of the Gamma function can be expressed in terms of itself and other functions, such as:

$$\Gamma'(z) = \Gamma(z)\, \psi_0(z).$$

were ψ_0 is the *function poligamma* of zero order. Specifically:

$$\Gamma'(1) = -\gamma.$$

We know that is:

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi},$$

for *z=1/2* in the reflection formula, or with Beta function in (1/2, 1/2).

Frequently utilized in statistics is the integral

$$\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}}\, dx = \sqrt{2\pi}$$

It is obtained by placing:

$$\frac{x^2}{2} = t$$

$$x = \sqrt{2t}$$

$$dx = \sqrt{2}\frac{1}{2\sqrt{t}}dt$$

$$\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}}dx = 2\int_0^{+\infty} e^{-\frac{x^2}{2}}dx = 2\int_0^{+\infty} \frac{\sqrt{2}}{2}t^{-\frac{1}{2}}e^{-t}dt = \sqrt{2}\Gamma\left(\frac{1}{2}\right) = \sqrt{2\pi}$$

Other proprieties are:

$$\Gamma\left(\frac{n}{2}\right) = \frac{(n-2)!!}{2^{(n-1)/2}}\sqrt{\pi} = \binom{\frac{n}{2}-1}{\frac{n-1}{2}}\frac{n-1}{2}!\sqrt{\pi}$$
$$\Gamma\left(-\frac{n}{2}\right) = \frac{\sqrt{\pi}}{\binom{-\frac{1}{2}}{\frac{n+1}{2}}\frac{n+1}{2}!}$$

where n!! is the *semifactorial*.

But certainly intriguing is the link between the Gamma function and the Riemann’s zeta and between the Gamma and Beta, and consequently the relationship between Beta and the Riemann zeta:

$$\zeta(z) = \zeta(1-z)\Gamma(1-z)2^z\pi^{z-1}\sin\frac{1}{2}\pi z$$

So it exists a link between the String Theory and the function Riemann’s zeta.

L-function and modular forms

The next question to ask is: "If the Riemann hypothesis is true, nontrivial zeros are simple and there is a connection between the Riemann zeta function and the strong interactions or ones with string theory, how can use those mathematical results with the theory of strings and branes, extra dimensions or the M-theory? In a world of Calabi-Yau to 10 or 11 dimensions, the Riemann’s zeta that can offer us?

In [15] we examined the *conjecture of Birch and Swinnerton-Dyer*. The mathematical theory that leads to this conjecture has the basic elements that are useful to the theory of open strings or closed, due to the elliptic curves, p-adic numbers, Riemann’s zeta, Dirichlet L-function and the modular forms.

For simplicity, suppose that the curve that we consider is an elliptic curve E defined over rational numbers **Q**. Denoted by **Z** the set of integers, suppose that E is defined by an equation of the form

$$y^2 = x^3 + ax + b, \text{ con } a,b \in \mathbf{Z},\ \Delta_E = 4a^3 + 27b^2 \neq 0.$$

We define the function L as:

$$L(E, s) = \prod_{p \neq \Delta E} \frac{1}{1 - a_p p^{-s} + p^{1-2s}}$$

where

$$a_p = 1 + p - \#E(Fp)$$

In this definition of a_p we see E that is an elliptic curve on the field Fp, where the coefficients a and b of E are the classes of module n. The corresponding factor p is the inverse of the numerator of the zeta of E on Fp.

The analogy between ζ(s) and L(E,s) can be introduced geometrically, if we call P a geometric point is ζ(s) = L(P,s).

In [15] we saw that *Hasse's theorem* (the Riemann hypothesis for E / Fp) implies that the infinite product defining L (E, s) converges to a differentiable function (in a complex sense) in the half plane R(s) > 3/2. While for the *Taylor-Wiles Theorem*: L (E, s) can be extended to a differentiable function over all the complex plane.

Theorem of Weil (demonstrated in this case by Hasse in 1931) follows from equality $a_p = \alpha p + \beta p$, where αp e βp are complex numbers with absolute value $p^{1/2}$. From theorem Wiles-Taylor follows the demonstration of *Fermat's last theorem*.

The *conjecture of Birch and Swinnerton-Dyer* said: The equation $y^2 = x^3 + ax + b$ has infinitely many solutions in rational numbers if and only if L (E, 1) = 0. It follows that if $y^2 = x^3 + ax + b$ has infinitely many solutions in rational numbers, then L (E, 1) = 0.

The *Riemann hypothesis* for L (E, s): The nontrivial zeros of L (E, s) are concentrated on the vertical line R (s) = 1.

L(E,s) satisfies a functional equation with respect to the transformation s →2 − s, while ζ(s) satisfies a functional equation respect to s →1 − s. In other words, the zeros must be on one line of symmetry for the functional equation.

The final way to deal with the L-functions is the " Laglands philosophy " and should incorporate the theorem of Wiles. We write L(E,S) as an infinite series:

$$L(E,s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

Wiles has considered the inverse Mellin transform of L (E, s):

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$$

where z is a variable in the complex upper half plane: $H = \{ z \in \mathbb{C} ; \Im(z) > 0 \}$

The theorem of Wiles continues with algebraic techniques saying that f(z) is a *modular form*, for example as:

- f (z) is a differentiable function on H, which satisfies a suitable condition of growth for $\Im(z) \to \infty$;
- $f(\frac{az+b}{cz+d}) = (cz+d)^2 f(z)$ for all matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that a,b,c,d ∈ Z, ad-bc=1 and N|c for a positive integer N

The properties of L (E, s) result (with the *theorem of Hecke*) from analytical properties of f (z).

To summarize, the properties of modular functions provide an access to the analytic properties of the function L (E, s) (geometrically defined) and then, by means of the *conjecture of Birch and Swinnerton-Dyer*, to the rational solutions of the equation of E. *Modular forms* are very important in *string theory*.

Moving in the opposite direction to what we said earlier, it is noted that sometimes it is the geometry that allows access to the properties of modular forms.

We consider the function Δ:

$$\Delta(z) = e^{2\pi iz}\prod_{n=1}^{\infty}\left(1-e^{2\pi inz}\right)^{24} = \sum_{n=1}^{\infty}\tau(n)e^{2\pi inz}$$

where the coefficients of Fourier τ(n), are the *coefficients of Ramanujan* and Δ is a modular form:

$$\Delta(\frac{az+b}{cz+d}) = (cz+d)^{12}\Delta(z)$$

for all matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that a,b,c,d $\in$Z, ad-bc=1. Specifically Δ has weight 12 while the F associated with L (E, s) has weight 2.

In particular, the *Deligne's Theorem* says that: $|\tau(n)| = O(n^{11/2+\varepsilon})$ for each ε > 0.

The proof of this conjecture does not follow directly from the analytical properties of Δ(z), from which $\tau(n) = O(n^6)$; but is a consequence of the proof of the *Weil conjectures* obtained by Deligne. First, observe that $\tau(n) = O(n^6)$ follow from τ(p) for p prime number (*Ramanujan conjecture, proved by Mordell*).

The crucial point is to show that τ(p) depends on the number of points of an algebraic variety over Fp and this is not a curve in this case, but a variety of size 11.

This type of geometry offers what string theory will be treated.

The p-adic beta functions in superstring theory. [16]

In the ordinary case it is known that the four-point tree amplitude for the open superstring has the form

$$A_{4p}(k_1\cdot;k_2\cdot;k_3\cdot;k_4\cdot) = -\frac{g^2}{2}\frac{\Gamma\left(-\frac{s}{2}\right)\Gamma\left(-\frac{t}{2}\right)}{\Gamma\left(1-\frac{s}{2}-\frac{t}{2}\right)}K(k_1\cdot,k_2\cdot,k_3\cdot,k_4\cdot) \quad (1)$$

where $s=-(k_1+k_2)^2$ and $t=-(k_2+k_3)^2$ are the Mandelstam variables. Since $4P=4B$, $4F$ or $2B2F$, the amplitude (1) accordingly depends on the polarization vectors of the massless vector particles and on the ten-dimensional Majorana-Weyl spinor wave functions.
The simplest manner to obtain a p-adic analogue of the Veneziano amplitude is to replace the ordinary gamma functions in the Veneziano amplitude

$$A(a,b,c)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \quad (2)$$

by their p-adic analogues, i.e.

$$A_p(a,b,c,)=\frac{\Gamma_p(a)\Gamma_p(b)}{\Gamma_p(a+b)}. \quad (3)$$

In eqs. (2) and (3), $a=-\alpha(s)=-1-\frac{1}{2}s$, $b=-\alpha(t)$ and $c=-\alpha(u)$, and they satisfy the mass-shell condition $s+t+u=-8$ or $a+b+c=1$. We note that $s+t+u=-8$, can be rewritten also as follow

$$s+t+u=-\frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}, \quad (4)$$

where we note that the number 8, that is a Fibonacci's number, is connected with the "modes" that correspond to the physical vibrations of a superstring by the above Ramanujan function.
Owing to the simple relation

$$\Gamma_p(y)\Gamma_p(1-y)=1 \quad (5)$$

which is a straightforward consequence of the expression

$$\Gamma_p(y)=\frac{1-p^{y-1}}{1-p^{-y}} \quad (6)$$

the p-adic amplitude (3) exhibits total crossing symmetry. $A_p(a,b,c)$ can be presented in the following form

$$A(a,b)=\int_k dx\gamma_a(x)\gamma_b(1-x), \quad (6b)$$

with $\gamma_a(x)=|x|_p^a$, where $x\in Q_p$ and $|...|_p$ denotes the p-adic norm. From an adelic point of view, the amplitudes $A_p(a,b,c)$ $(p=2,3,5,...)$ have been considered as partners of the ordinary crossing symmetry amplitude $A_\infty(a,b,c)=A(a,b,c)+A(b,c,a)+A(c,a,b)$, which can also be written in the form (6b), where $\gamma_a(x)=|x|^a$ and $x\in R$.

In an analogous way the above method can be applied to the superstring amplitude given by eq. (1) and we can look for its p-adic analogue in the form

$$A_{4P,p}(a,b,c)=-\frac{g^2}{2}\phi_p(s,t,u)K(k,\zeta) \quad (7)$$

where the kinematic factor $K(k,\zeta)$ remains unchanged. Taking $\phi_p(s,t,u)$ according to the above procedure we have

$$\phi_p(s,t,u)=\frac{\Gamma_p\left(-\frac{s}{2}\right)\Gamma_p\left(-\frac{t}{2}\right)}{\Gamma_p\left(1-\frac{s}{2}-\frac{t}{2}\right)} \quad (8)$$

where $a=-s/2$, $b=-t/2$ and $c=-u/2$ with the mass-shell condition

$$s+t+u=0. \quad (9)$$

Note that we can rewrite the eq. (7) also as follow

$$A_{4P,p}(a,b,c)=-\frac{g^2}{2}\frac{\Gamma_p\left(-\frac{s}{2}\right)\Gamma_p\left(-\frac{t}{2}\right)}{\Gamma_p\left(1-\frac{s}{2}-\frac{t}{2}\right)}K(k,\zeta). \quad (9b)$$

Using the functional equation (5), we obtain the totally crossing-symmetric amplitude

$$A_{4P,p}(a,b,c)=-\frac{g^2}{2}\Gamma_p\left(-\frac{s}{2}\right)\Gamma_p\left(-\frac{t}{2}\right)\Gamma_p\left(-\frac{u}{2}\right)K(k,\zeta). \quad (10)$$

The amplitude (10) has poles at the real points

$$s=0,\; t=0,\; u=0. \quad (11)$$

The structure of eq. (8) does not allow us to write down the amplitude (10) by means of the convolution of multiplicative characters $\pi_a(x)$, i.e.,

$$A_{4P,p}(a,b,c)\neq-\frac{g^2}{2}K(k,\zeta)\int_{Q_p}\pi_a(x)\pi_b(1-x)dx. \quad (12)$$

Furthermore, the lack of a representation of amplitude (10) in the form of the right-hand side of (12) does not permit us to extract the (st) channel amplitude from the crossing symmetric one.

Recall that the usual conformal field in a Euclidean formulation is defined by the propagator

$$\left\langle x^{\mu}(z)x^{\nu}(w)\right\rangle = -g^{\mu\nu}\ln|z-w|^{2}, \quad (13)$$

where $g^{\mu\nu}$ denotes the flat metric in D-dimensional space-time and z,w are complex variables. The tachyon vertex has the form

$$V(k,z) =: e^{ik_{\mu}x^{\mu}(z)}: \quad (14)$$

and the N-point closed string amplitude can be written as follows

$$A_{N}(k_{1},...,k_{N}) = \int\prod_{i=1}^{N}dz_{i}\left\langle\prod_{i=1}^{N}V(K_{i}Z_{i})\right\rangle = \int\prod_{i=1}^{N}dz_{i}\prod_{m<n}|z_{m}-z_{n}|^{k_{n}k_{m}/2} \qquad (15)$$

where dz is the standard measure on the complex plane. The expression (15) is divergent because of the SL(2,C) invariance and after the extraction of the group volume on mass shell $k_{i}^{2}=2$ can be presented in the Koba-Nielsen form

$$A_{N}(k_{1},...,k_{N}) = \int dV\prod_{m<n}|z_{m}-z_{n}|^{k_{m}k_{n}/2} \qquad (16)$$

where

$$dV = \pi^{3-N}\frac{|z_{a}-z_{b}|^{2}|z_{b}-z_{c}|^{2}|z_{b}-z_{c}|^{2}}{dz_{a}dz_{b}dz_{c}}\prod_{i=1}^{N}dz_{i}\,. \quad (17)$$

Thence, the eq. (16) can be rewritten also as follows

$$A_{N}(k_{1},...,k_{N}) = \int\ \pi^{3-N}\frac{|z_{a}-z_{b}|^{2}|z_{b}-z_{c}|^{2}|z_{b}-z_{c}|^{2}}{dz_{a}dz_{b}dz_{c}}\prod_{i=1}^{N}dz_{i}\prod_{m<n}|z_{m}-z_{n}|^{k_{m}k_{n}/2}\,. \quad (17b)$$

The procedure described above can be straightforwardly applied to the derivation of the open string amplitude. In such a case the integration in eq. (15) goes over the simplex on the real axis,

$$z_{1}\geq z_{2}\geq ...\geq z_{N} \qquad (18)$$

and the power of $|z_{m}-z_{n}|$ should be multiplied by 2 in accordance with the form of the propagator for an open string

$$\left\langle x^{\mu}(z)X^{\nu}(w)\right\rangle = -g^{\mu\nu}\left(\ln|z-w|^{2}+\ln|z-\overline{w}|^{2}\right). \quad (13')$$

A divergence related to the SL(2,R) invariance of the integrand can be removed by dividing it by the volume of this group. One obtains the st-channel amplitude by choosing

$$z_{1}=\infty, \quad z_{2}=1, \quad z_{N}=0\,. \quad (19)$$

Denoting the characteristic function of a simplex satisfying conditions (18) and (19) by $\theta_{[0,1,\infty]}(z_{1}...z_{N})$ we can write down

$$A_N^{open}(k_1,...,k_N)=\frac{1}{V}\int\left(\prod dz_i\right)\theta_{[0,1,\infty]}(z_3,...,z_{N-1})\left\langle\prod_{i=1}^{N}V(k_i z_i)\right\rangle=\frac{1}{V}\int\left(\prod dz_i\right)\theta_{[0,1,\infty]}(z_3,...,z_{N-1})\prod_{m<n}|z_m-z_n|^{k_m k_n} \quad (15')$$

The extraction of the SL(2,R) [or SL(2,C)] group volume is automatically done by introducing ghosts a and b ($a,\bar{a}$ and $b,\bar{b}$)

$$\langle b(z)c(w)\rangle=\frac{1}{|z-w|} \quad (20)$$

and replacing the vertex (14) by

$$V(z)=c(z):e^{ik_\mu x^\mu(z)}: \quad (21)$$

at the points $z_1=\infty$, $z_2=1$ and $z_N=0$. Eq. (15') can be rewritten now in the following form

$$A_N^{open}(k_1,...,k_N)=\int_{1\geq z_3\geq...\geq z_{N-1}\geq 0}\left\langle c(z_1)e^{ik_1x(z_1)}c(z_2)e^{ik_2x(z_2)}...e^{ik_{N-1}x(z_{N-1})}c(z_N)e^{ik_Nx(z_N)}\right\rangle\prod_{i=3}^{N-1}dz_i\,. \quad (22)$$

Taking into account the correlation function

$$\langle c(z_1)c(z_{N-1})c(z_N)\rangle=|z_1-z_{N-1}||z_1-z_N||z_{N-1}-z_N| \quad (23)$$

one obtain the well-known result. Note that we have introduced moduli in (23) due to the ordering of the z variables.
The p-adic generalization of the above formulae is straightforward. Let K be a quadratic extension of Q_p or an arbitrary locally compact field with norm $|...|_K$. The corresponding conformal field $x^\mu(z)$, where $z\in K$, is defined by the propagator

$$\langle x^\mu(z)x^\nu(w)\rangle=-g^{\mu\nu}\log|z-w|_K^2 \quad (24)$$

and the vertex as well as the string amplitudes acquire the form (14) and (15), respectively, where dz is now the Haar measure on K. Instead of SL(2,C) here we have SL(2,K) invariance.
For an open p-adic string amplitude we can consider formulae (13') and (14) as a definition of a conformal theory, where $z,w\in Q_p(\sqrt{\tau})$ and $|...|$ should be replaced by the p-adic norm $|...|_p$. The N-point p-adic amplitude is given by eq. (15') where $\theta_\tau(z_1...z_n)$ will denote one of the possible p-adic analogues of the characteristic function on the simplex (18), i.e.,

$$A_{N,\tau}^{open}(k_1,...,k_N)=\int_{Q_p^N}\left\langle\prod_{i=1}^{N}V(k_iz_i)\right\rangle\theta_\tau(z_1,...,z_N)\prod_{i=1}^{N}dz_i\,. \quad (25)$$

This divergent expression has an SL(2,Q_p) invariance and after extraction of the volume of the group on the mass shell $k_i^2=2$ it can be represented in the Koba-Nielsen form

$$A_{N,\tau}^{open}(k_1,...,k_N)=\int_{(Q_p)^{N-3}}\left\langle c(\infty)c(1)c(0)\prod_{i=1}^{N}V(k_i z_i)\right\rangle\theta_{\tau[0,1,\infty]}(z_3,...,z_{N-1})\prod_{i=3}^{N-1}dz_i \qquad (26)$$

where $\theta_{\tau[0,1,\infty]}(z_1...z_N)$ is now the p-adic analogue of the finite simplex defined by (18) and (19). The tree amplitude for scattering of four fermions can be given by

$$A_{4F}(k_1u_1;k_2u_2;k_3u_3;k_4u_4)=-\frac{1}{2}g^2\int dz\,\theta_{[0,1]}(z)<V_{-1/2}(\infty)V_{1/2}(1)V_{-1/2}(z)V_{1/2}(0). \qquad (27)$$

Performing the corresponding calculation we obtain

$$A_{4F}(k_1u_1;k_2u_2;k_3u_3;k_4u_4)=u^\alpha u^\beta u^\gamma u^\delta\left(-\frac{1}{2}g^2\right)\int_0^1 dz|z|^{-1+k_2+k_3}|1-z|^{-1+k_3+k_4}\left\{|1-z|\gamma^\mu_{\alpha\beta}\gamma^\mu_{\gamma\delta}-|z|\gamma^\mu_{\alpha\delta}\gamma^\mu_{\beta\gamma}\right\}=$$

$$=\frac{1}{2}g^2u^\alpha u^\beta u^\gamma u^\delta\left\{B\left(1-\frac{t}{2},-\frac{s}{2}\right)\gamma^\mu_{\alpha\beta}\gamma^\mu_{\gamma\delta}-B\left(-\frac{t}{2},1-\frac{s}{2}\right)\gamma^\mu_{\alpha\delta}\gamma^\mu_{\beta\gamma}\right\}. \qquad (28)$$

The transition amplitude from two fermions to two bosons can be written in the form

$$A_{2F\,2B}(k_1u_1;k_2u_2;k_3\zeta_3;k_4\zeta_4)=-\frac{1}{2}g^2\int dz\,\theta_{[0,1]}(z)\langle V_{-1}(\infty)V_0(1)V_{-1/2}(z)V_{-1/2}(0)\rangle. \qquad (29)$$

Performing the calculations with the corresponding correlators and imposing the kinematical conditions on the mass shell, one obtains

$$A_{2F\,2B}=-\frac{1}{2}g^2\int dz\,\theta_{[0,1]}(z)|z|^{-1+k_3k_4}|1-z|^{-1+k_2k_3}\cdot\left\{|z|\left[(\zeta_3k_2)(u_2\zeta_4 u_1)+\frac{1}{2}\zeta_4^\mu k_3^\nu\zeta_3^\lambda u_2\gamma^{[\nu}\gamma^{\lambda]}\gamma^\mu u_1\right]+\right.$$
$$\left.-|1-z|\left[(\zeta_3k_4)u_2\zeta_4 u_1+(\zeta_4k_3)u_2\zeta_3u_1-(\zeta_4\zeta_3)u_2k_3u_1\right]\right\}. \qquad (30)$$

Analogously, one can obtain the four-boson scattering amplitude. The formulae obtained above, (28) and (30), can easily be generalized to the p-adic case. As in the bosonic string case, we have to replace the standard norm by a p-adic one. In particular, the p-adic four-fermion string amplitude can be written as follows:

$$A_{4F}^{(st)}(k_1u_1;k_2u_2;k_3u_3;k_4u_4)=-\frac{1}{2}g^2u_1^\alpha u_2^\beta u_3^\gamma u_4^\delta\cdot\int_{Q_p}dz\,\theta_{\varepsilon[0,1]}(z)|z|_p^{-1-t/2}|1-z|^{-1-s/2}\left[|1-z|_p\gamma^\mu_{\alpha\beta}\gamma^\mu_{\gamma\delta}-|z|_p\gamma^\mu_{\alpha\delta}\gamma^\mu_{\beta\gamma}\right]. \qquad (31)$$

The explicit form of eq. (31) depends on the chosen form of the p-adic analogue of the characteristic function $\theta_{[0,1],\tau}(z)$, where τ denotes one of the three quadratic extensions. For instance, when $\tau=\varepsilon$ and

$$\theta_{\varepsilon[0,1]}(z)=\frac{1}{2}\left(Sign_\varepsilon z-Sign_\varepsilon(-1)Sign_\varepsilon(1-z)\right), \qquad (32)$$

we can write the following p-adic amplitude

$$A_{4F,p,\varepsilon}^{(st)} = u_1^{\alpha}u_2^{\beta}u_3^{\gamma}u_4^{\delta}\left(-\frac{1}{4}g^2\right)\left\{\left[B_p\left(\tilde{\pi}_{-t/2},\pi_{-s/2+1}\right)-B_p\left(\pi_{-t/2},\tilde{\pi}_{-s/2+1}\right)\right]\gamma_{\alpha\beta}^{\mu}\gamma_{\gamma\delta}^{\mu}+\right.$$
$$\left.-\left[B_p\left(\tilde{\pi}_{-t/2+1},\pi_{-s/2}\right)-B_p\left(\pi_{-t/2+1},\tilde{\pi}_{-s/2}\right)\right]\gamma_{\alpha\delta}^{\mu}\gamma_{\beta\gamma}^{\mu}\right\} \quad (33)$$

where $u_{\alpha}(i)=u_{\alpha}(k_i)$ and $B_p(\pi_a,\pi_b)$ are p-adic beta functions. So,

$$A_{4F,p,\varepsilon}^{(st)} = u_1^{\alpha}u_2^{\beta}u_3^{\gamma}u_4^{\delta}\left(-\frac{1}{2}g^2\right)\left(1-\frac{1}{p}\right)\frac{1+p^{-\frac{u}{2}-1}}{1+p^{\frac{u}{2}}}\left\{\frac{p^{t/2}-p^{-t/2-1}+p^{s/2-1}-p^{-s/2}}{\left(1-p^{s-2}\right)\left(1-p^{-t}\right)}\gamma_{\alpha\beta}^{\mu}\gamma_{\gamma\delta}^{\mu}+\right.$$
$$\left.+\frac{p^{s/2}-p^{-s/2-1}+p^{t/2-1}-p^{-t/2}}{\left(1-p^{t-2}\right)\left(1-p^{-s}\right)}\gamma_{\alpha\delta}^{\mu}\gamma_{\beta\gamma}^{\mu}\right\}. \quad (34)$$

This amplitude evidently contains poles at the real points $s=0$, $t=0$, $s=2$ and $t=2$, as well as poles at the complex points.
Performing the integration in eq. (31) over the whole field Q_p, one obtains the expression

$$A_{4F,p} = -\frac{1}{2}g^2\left[B_p\left(1-\frac{t}{2},-\frac{s}{2}\right)\gamma_{\alpha\beta}^{\mu}\gamma_{\gamma\delta}^{\mu}-B_p\left(-\frac{t}{2},1-\frac{s}{2}\right)\gamma_{\alpha\delta}^{\mu}\gamma_{\beta\gamma}^{\mu}\right] \quad (35)$$

which contains a u-pole as well.
Thence, from the eq. (31), we have that:

$$A_{4F}^{(st)}(k_1u_1;k_2u_2;k_3u_3;k_4u_4) = -\frac{1}{2}g^2u_1^{\alpha}u_2^{\beta}u_3^{\gamma}u_4^{\delta}\cdot\int_{Q_p}dz\,\theta_{\varepsilon[0,1]}(z)|z|_p^{-1-t/2}|1-z|^{-1-s/2}\left[|1-z|_p\gamma_{\alpha\beta}^{\mu}\gamma_{\gamma\delta}^{\mu}-|z|_p\gamma_{\alpha\delta}^{\mu}\gamma_{\beta\gamma}^{\mu}\right]\Rightarrow$$
$$\Rightarrow -\frac{1}{2}g^2\left[B_p\left(1-\frac{t}{2},-\frac{s}{2}\right)\gamma_{\alpha\beta}^{\mu}\gamma_{\gamma\delta}^{\mu}-B_p\left(-\frac{t}{2},1-\frac{s}{2}\right)\gamma_{\alpha\delta}^{\mu}\gamma_{\beta\gamma}^{\mu}\right]. \quad (35b)$$

p-adic, adelic and zeta-strings [17] [18] [19] [20] [21]

Like in the ordinary string theory, the starting point of p-adic strings is a construction of the corresponding scattering amplitudes. Recall that the ordinary crossing symmetric Veneziano amplitude can be presented in the following forms:

$$A_{\infty}(a,b) = g^2\int_R |x|_{\infty}^{a-1}|1-x|_{\infty}^{b-1}dx = g^2\left[\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}+\frac{\Gamma(b)\Gamma(c)}{\Gamma(b+c)}+\frac{\Gamma(c)\Gamma(a)}{\Gamma(c+a)}\right] = g^2\frac{\zeta(1-a)}{\zeta(a)}\frac{\zeta(1-b)}{\zeta(b)}\frac{\zeta(1-c)}{\zeta(c)} =$$
$$= g^2\int DX\exp\left(-\frac{i}{2\pi}\int d^2\sigma\partial^{\alpha}X_{\mu}\partial_{\alpha}X^{\mu}\right)\prod_{j=1}^{4}\int d^2\sigma_j\exp\left(ik_{\mu}^{(j)}X^{\mu}\right), \quad (1-4)$$

where $\hbar=1$, $T=1/\pi$, and $a=-\alpha(s)=-1-\frac{s}{2}$, $b=-\alpha(t)$, $c=-\alpha(u)$ with the condition $s+t+u=-8$, i.e. $a+b+c=1$.
The p-adic generalization of the above expression

$$A_\infty(a,b) = g^2 \int_R |x|_\infty^{a-1} |1-x|_\infty^{b-1} dx ,$$

is:

$$A_p(a,b) = g_p^2 \int_{Q_p} |x|_p^{a-1} |1-x|_p^{b-1} dx , \quad (5)$$

where $|\ldots|_p$ denotes p-adic absolute value. In this case only string world-sheet parameter x is treated as p-adic variable, and all other quantities have their usual (real) valuation.
Now, we remember that the Gauss integrals satisfy adelic product formula

$$\int_R \chi_\infty(ax^2 + bx) d_\infty x \prod_{p \in P} \int_{Q_p} \chi_p(ax^2 + bx) d_p x = 1, \quad a \in Q^\times, \quad b \in Q, \quad (6)$$

what follows from

$$\int_{Q_v} \chi_v(ax^2 + bx) d_v x = \lambda_v(a) |2a|_v^{-\frac{1}{2}} \chi_v\left(-\frac{b^2}{4a}\right), \quad v = \infty, 2, \ldots, p \ldots . \quad (7)$$

These Gauss integrals apply in evaluation of the Feynman path integrals

$$K_v(x'', t''; x', t') = \int_{x', t'}^{x'', t''} \chi_v\left(-\frac{1}{h} \int_{t'}^{t''} L(\dot{q}, q, t) dt\right) D_v q , \quad (8)$$

for kernels $K_v(x'', t''; x', t')$ of the evolution operator in adelic quantum mechanics for quadratic Lagrangians. In the case of Lagrangian

$$L(\dot{q}, q) = \frac{1}{2}\left(-\frac{\dot{q}^2}{4} - \lambda q + 1\right),$$

for the de Sitter cosmological model one obtains

$$K_\infty(x'', T; x', 0) \prod_{p \in P} K_p(x'', T; x', 0) = 1, \quad x'', x', \lambda \in Q , \; T \in Q^\times, \quad (9)$$

where

$$K_v(x'', T; x', 0) = \lambda_v(-8T) |4T|_v^{-\frac{1}{2}} \chi_v\left(-\frac{\lambda^2 T^3}{24} + [\lambda(x'' + x') - 2]\frac{T}{4} + \frac{(x'' - x')^2}{8T}\right). \quad (10)$$

Also here we have the number 24 that correspond to the Ramanujan function that has 24 "modes", i.e., the physical vibrations of a bosonic string. Hence, we obtain the following mathematical connection:

$$K_v(x'', T; x', 0) = \lambda_v(-8T) |4T|_v^{-\frac{1}{2}} \chi_v\left(-\frac{\lambda^2 T^3}{24} + [\lambda(x'' + x') - 2]\frac{T}{4} + \frac{(x'' - x')^2}{8T}\right) \Rightarrow$$

$$\Rightarrow \frac{4\left[anti\log \frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi^2}{4} w'} \phi_{w'}(itw')}\right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} . \quad (10b)$$

The adelic wave function for the simplest ground state has the form

$$\psi_A(x) = \psi_\infty(x) \prod_{p \in P} \Omega\left(|x|_p\right) = \begin{cases} \psi_\infty(x), x \in Z \\ 0, x \in Q \setminus Z \end{cases}, \quad (11)$$

where $\Omega\left(|x|_p\right) = 1$ if $|x|_p \le 1$ and $\Omega\left(|x|_p\right) = 0$ if $|x|_p > 1$. Since this wave function is non-zero only in integer points it can be interpreted as discreteness of the space due to p-adic effects in adelic approach. The Gel'fand-Graev-Tate gamma and beta functions are:

$$\Gamma_\infty(a) = \int_R |x|_\infty^{a-1} \chi_\infty(x) d_\infty x = \frac{\zeta(1-a)}{\zeta(a)}, \quad \Gamma_p(a) = \int_{Q_p} |x|_p^{a-1} \chi_p(x) d_p x = \frac{1-p^{a-1}}{1-p^{-a}}, \quad (12)$$

$$B_\infty(a,b) = \int_R |x|_\infty^{a-1} |1-x|_\infty^{b-1} d_\infty x = \Gamma_\infty(a) \Gamma_\infty(b) \Gamma_\infty(c), \quad (13)$$

$$B_p(a,b) = \int_{Q_p} |x|_p^{a-1} |1-x|_p^{b-1} d_p x = \Gamma_p(a) \Gamma_p(b) \Gamma_p(c), \quad (14)$$

where $a,b,c \in C$ with condition $a+b+c=1$ and $\zeta(a)$ is the Riemann zeta function. With a regularization of the product of p-adic gamma functions one has adelic products:

$$\Gamma_\infty(u) \prod_{p \in P} \Gamma_p(u) = 1, \quad B_\infty(a,b) \prod_{p \in P} B_p(a,b) = 1, \quad u \ne 0,1, \quad u = a,b,c, \quad (15)$$

where $a+b+c=1$. We note that $B_\infty(a,b)$ and $B_p(a,b)$ are the crossing symmetric standard and p-adic Veneziano amplitudes for scattering of two open tachyon strings. Introducing real, p-adic and adelic zeta functions as

$$\zeta_\infty(a) = \int_R \exp\left(-\pi x^2\right) |x|_\infty^{a-1} d_\infty x = \pi^{-\frac{a}{2}} \Gamma\left(\frac{a}{2}\right), \quad (16)$$

$$\zeta_p(a) = \frac{1}{1-p^{-1}} \int_{Q_p} \Omega\left(|x|_p\right) |x|_p^{a-1} d_p x = \frac{1}{1-p^{-a}}, \quad \operatorname{Re} a > 1, \quad (17)$$

$$\zeta_A(a) = \zeta_\infty(a) \prod_{p \in P} \zeta_p(a) = \zeta_\infty(a) \zeta(a), \quad (18)$$

one obtains

$$\zeta_A(1-a) = \zeta_A(a), \quad (19)$$

where $\zeta_A(a)$ can be called adelic zeta function. We have also that

$$\zeta_A(a)=\zeta_\infty(a)\prod_{p\in P}\zeta_p(a)=\zeta_\infty(a)\zeta(a)=\int_R \exp\left(-\pi x^2\right)|x|_\infty^{a-1}d_\infty x\cdot\frac{1}{1-p^{-1}}\int_{Q_p}\Omega\left(|x|_p\right)|x|_p^{a-1}d_p x\,. \quad (19b)$$

Let us note that $\exp\left(-\pi x^2\right)$ and $\Omega\left(|x|_p\right)$ are analogous functions in real and p-adic cases. Adelic harmonic oscillator has connection with the Riemann zeta function. The simplest vacuum state of the adelic harmonic oscillator is the following Schwartz-Bruhat function:

$$\psi_A(x)=2^{\frac{1}{4}}e^{-\pi x_\infty^2}\prod_{p\in P}\Omega\left(|x_p|_p\right), \quad (20)$$

whose the Fourier transform

$$\psi_A(k)=\int\chi_A(kx)\psi_A(x)=2^{\frac{1}{4}}e^{-\pi k_\infty^2}\prod_{p\in P}\Omega\left(|k_p|_p\right) \quad (21)$$

has the same form as $\psi_A(x)$. The Mellin transform of $\psi_A(x)$ is

$$\Phi_A(a)=\int\psi_A(x)|x|^a d_A^\times x=\int_R\psi_\infty(x)|x|^{a-1}d_\infty x\prod_{p\in P}\frac{1}{1-p^{-1}}\int_{Q_p}\Omega\left(|x|_p\right)|x|^{a-1}d_p x=\sqrt{2}\Gamma\left(\frac{a}{2}\right)\pi^{-\frac{a}{2}}\zeta(a) \quad (22)$$

and the same for $\psi_A(k)$. Then according to the Tate formula one obtains (19).
The exact tree-level Lagrangian for effective scalar field φ which describes open p-adic string tachyon is

$$\mathcal{L}_p=\frac{1}{g^2}\frac{p^2}{p-1}\left[-\frac{1}{2}\varphi p^{-\frac{\Box}{2}}\varphi+\frac{1}{p+1}\varphi^{p+1}\right], \quad (23)$$

where p is any prime number, $\Box=-\partial_t^2+\nabla^2$ is the D-dimensional d'Alambertian and we adopt metric with signature $(-+...+)$. Now, we want to show a model which incorporates the p-adic string Lagrangians in a restricted adelic way. Let us take the following Lagrangian

$$L=\sum_{n\geq 1}C_n\mathcal{L}_n=\sum_{n\geq 1}\frac{n-1}{n^2}\mathcal{L}_n=\frac{1}{g^2}\left[-\frac{1}{2}\phi\sum_{n\geq 1}n^{-\frac{\Box}{2}}\phi+\sum_{n\geq 1}\frac{1}{n+1}\phi^{n+1}\right]. \quad (24)$$

Recall that the Riemann zeta function is defined as

$$\zeta(s)=\sum_{n\geq 1}\frac{1}{n^s}=\prod_p\frac{1}{1-p^{-s}}, \quad s=\sigma+i\tau, \quad \sigma>1. \quad (25)$$

Employing usual expansion for the logarithmic function and definition (25) we can rewrite (24) in the form

$$L=-\frac{1}{g^2}\left[\frac{1}{2}\phi\zeta\left(\frac{\Box}{2}\right)\phi+\phi+\ln(1-\phi)\right], \quad (26)$$

where $|\phi| < 1$. $\zeta\left(\frac{\Box}{2}\right)$ acts as pseudodifferential operator in the following way:

$$\zeta\left(\frac{\Box}{2}\right)\phi(x) = \frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk\,, \quad -k^2 = k_0^2 - \vec{k}^2 > 2+\varepsilon\,, \quad (27)$$

where $\tilde{\phi}(k) = \int e^{(-ikx)}\phi(x)dx$ is the Fourier transform of $\phi(x)$.

Dynamics of this field ϕ is encoded in the (pseudo)differential form of the Riemann zeta function. **When the d'Alambertian is an argument of the Riemann zeta function we shall call such string a "zeta string".** Consequently, the above ϕ is an open scalar zeta string. The equation of motion for the zeta string ϕ is

$$\zeta\left(\frac{\Box}{2}\right)\phi = \frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \frac{\phi}{1-\phi} \quad (28)$$

which has an evident solution $\phi = 0$.

For the case of time dependent spatially homogeneous solutions, we have the following equation of motion

$$\zeta\left(\frac{-\partial_t^2}{2}\right)\phi(t) = \frac{1}{(2\pi)}\int_{|k_0|>\sqrt{2}+\varepsilon} e^{-ik_0t}\zeta\left(\frac{k_0^2}{2}\right)\tilde{\phi}(k_0)dk_0 = \frac{\phi(t)}{1-\phi(t)}. \quad (29)$$

With regard the open and closed scalar zeta strings, the equations of motion are

$$\zeta\left(\frac{\Box}{2}\right)\phi = \frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \sum_{n\geq 1}\theta^{\frac{n(n-1)}{2}}\phi^n\,, \quad (30)$$

$$\zeta\left(\frac{\Box}{4}\right)\theta = \frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{4}\right)\tilde{\theta}(k)dk = \sum_{n\geq 1}\left[\theta^{n^2} + \frac{n(n-1)}{2(n+1)}\theta^{\frac{n(n-1)}{2}-1}\left(\phi^{n+1}-1\right)\right], \quad (31)$$

and one can easily see trivial solution $\phi = \theta = 0$.

Mathematical connections

With regard the mathematical connections, we have the following two new interesting relationships, with the fundamental equation concerning the zeta-strings and the equation connected with the p-adic beta functions:

$$\beta\left(p,q\right)=\int_0^1 x^{p-1}\left(1-x\right)^{q-1}dx=\frac{\Gamma\left(p\right)\Gamma\left(q\right)}{\Gamma\left(p+q\right)} \quad \Rightarrow$$

$$\Rightarrow \quad dx=\sqrt{2}\frac{1}{2\sqrt{t}}dt$$

$$\int_{-\infty}^{+\infty}e^{-\frac{x^2}{2}}dx=2\int_0^{+\infty}e^{-\frac{x^2}{2}}dx=2\int_0^{+\infty}\frac{\sqrt{2}}{2}t^{-\frac{1}{2}}e^{-t}dt=\sqrt{2}\Gamma\left(\frac{1}{2}\right)=\sqrt{2\pi} \quad \Rightarrow$$

$$\Rightarrow \quad \zeta(z)=\zeta(1-z)\Gamma(1-z)2^z\pi^{z-1}\sin\frac{1}{2}\pi z \quad \Rightarrow$$

$$\Rightarrow \zeta\left(\frac{\Box}{2}\right)\phi=\frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon}e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk=\frac{\phi}{1-\phi}, \quad (32)$$

$$\beta\left(p,q\right)=\int_0^1 x^{p-1}\left(1-x\right)^{q-1}dx=\frac{\Gamma\left(p\right)\Gamma\left(q\right)}{\Gamma\left(p+q\right)} \quad \Rightarrow$$

$$\Rightarrow \quad dx=\sqrt{2}\frac{1}{2\sqrt{t}}dt$$

$$\int_{-\infty}^{+\infty}e^{-\frac{x^2}{2}}dx=2\int_0^{+\infty}e^{-\frac{x^2}{2}}dx=2\int_0^{+\infty}\frac{\sqrt{2}}{2}t^{-\frac{1}{2}}e^{-t}dt=\sqrt{2}\Gamma\left(\frac{1}{2}\right)=\sqrt{2\pi} \quad \Rightarrow$$

$$\Rightarrow \quad \zeta(z)=\zeta(1-z)\Gamma(1-z)2^z\pi^{z-1}\sin\frac{1}{2}\pi z \quad \Rightarrow$$

$$\Rightarrow A_{4F}^{(st)}\left(k_1u_1;k_2u_2;k_3u_3;k_4u_4\right)=-\frac{1}{2}g^2u_1^\alpha u_2^\beta u_3^\gamma u_4^\delta\cdot\int_{Q_p}dz\,\theta_{\varepsilon[0,1]}(z)|z|_p^{-1-t/2}|1-z|^{-1-s/2}\left[|1-z|_p\gamma_{\alpha\beta}^\mu\gamma_{\gamma\delta}^\mu-|z|_p\gamma_{\alpha\delta}^\mu\gamma_{\beta\gamma}^\mu\right]\Rightarrow$$

$$\Rightarrow -\frac{1}{2}g^2\left[B_p\left(1-\frac{t}{2},-\frac{s}{2}\right)\gamma_{\alpha\beta}^\mu\gamma_{\gamma\delta}^\mu-B_p\left(-\frac{t}{2},1-\frac{s}{2}\right)\gamma_{\alpha\delta}^\mu\gamma_{\beta\gamma}^\mu\right]. \quad (33)$$

We note also the link with π, thence with $\phi=\frac{\sqrt{5}-1}{2}$, i.e. the Aurea ratio by the simple formula

$$\arccos\phi=0,2879\pi \quad (34)$$

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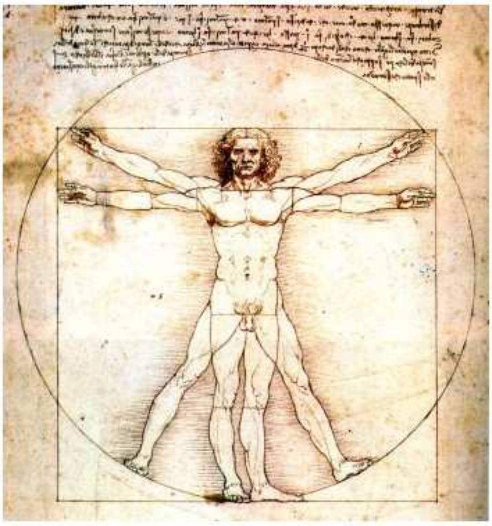

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