RELATIONS BETWEEN THE GAUSS-EISENSTEIN PRIME NUMBERS AND THEIR CORRELATION WITH SOPHIE GERMAIN PRIMES

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Abstract

In this paper we examine the relations between the Gauss prime numbers and the Eisenstein prime numbers and their correlation with Sophie Germain primes. Furthermore, we have described also various mathematical connections with some equations concerning the string theory.

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1. GAUSSIAN INTEGER

A **Gaussian integer** is a complex number whose real and imaginary part are both integers. The Gaussian integers, with ordinary addition and multiplication of complex numbers, form an integral domain, usually written as **Z**[*i*].

Definition

Formally, Gaussian integers are the set

$$Z[i] = \{a + bi | a, b \in Z\}.$$

Norm

The norm of a Gaussian integer is the natural number defined as

$$N(a + bi) = a^2 + b^2.$$

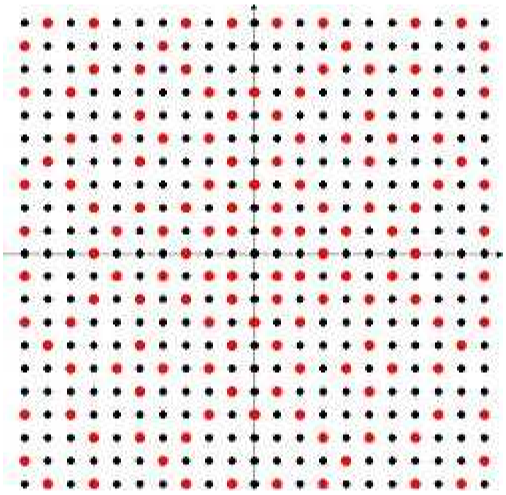
The norm is multiplicative:

$$N(zw) = N(z)N(w).$$

The units of **Z**[*i*] are therefore precisely those elements with norm 1, i.e. the elements :

$$1, -1, i, -i.$$

Gaussian primes in the complex plane are marked in red:



A Gaussian integer $a + bi$ is a Gaussian prime if and only if either:

- *a is prime number and b* is zero, so we have a prime number of the form 4k+3
- both are nonzero and $p = a^2 + b^2 = (a+bi)\,(a-bi)$ is a prime number (which will *not* be of the form 4k+3).

The prime number 2 is a special case because is the only ramified prime in **Z**[*i*]).

The integer 2 factors as $2 = (1+i)(1-i) = i(1-i)^2$ as a Gaussian integer, the second factorization (in which *i* is a unit) showing that 2 is divisible by the square of a Gaussian prime; it is the unique prime number with this property.

Instead, the prime number 5 is a unique factorization

$$5 = (2+i)(2-i).$$

Gaussian primes are infinite, because they are infinite primes.

1.1 REMARKS

The prime numbers that are congruent to 3 (mod.4) are also Gaussian primes.

Since there are infinitely many rational primes of the form 4k +3, there are infinitely many necessarily Gaussian primes.

The prime numbers congruent to 1 (mod.4) are the product of two distinct Gaussian primes. In fact, the primes of the form 4k +1 can always be written as the sum of two squares (Fermat's theorem on the sum of two squares)

so we have:

$$p = a^2 + b^2 = (a+bi)(a-bi)$$

Numerical form.

Since the form 4k + 3 corresponds to the form 6k '+1 with k' <k, for example, 19 = 4 * 4 +3 = 6 * 3 +1, with k > k ', in fact 4 > 3, the Gaussian primes are all of the form 6k '+ 1, and not of the form 6k' - 1, these being also of the form 4k +1, and then no-Gaussian primes.

1.2 On some equations regarding Lemmas and Theorems of Gaussian integers, mod-gaussian convergence of a sum over primes and Gaussian primes.

Now we see some Lemmas and Theorems concerning the ring of Gaussian integers $Z[i]$, $N(a+bi)=a^2+b^2$. We note that the letter $\boldsymbol{\rho}$ denote the Gaussian prime.

Lemma 1

$$\underset{s=1}{res}\, F_k(s)x^s/s = C_k x, \qquad \underset{s=1}{res}\, F_{k*}(s)x^s/s = C_{k*}x, \quad (1.1)$$

where

$$C_k = \frac{\pi}{4}\prod_{\boldsymbol{\rho}}\left(1+\sum_{a=2}^{\infty}\frac{\tau_k(a)-\tau_k(a-1)}{N^a(\boldsymbol{\rho})}\right), \quad (1.2) \qquad C_{k*} = \frac{\pi}{4}\prod_{\boldsymbol{\rho}}\left(1+\sum_{a=2}^{\infty}\frac{\boldsymbol{t}_k(a)-\boldsymbol{t}_k(a-1)}{N^a(\boldsymbol{\rho})}\right), \quad (1.3)$$

As a consequence of the following representation,

$$F_k(s) = Z(s)Z^{k-1}(2s)Z^{(k-k^2)/2}(5s)Z^{(-k^3+6k^2-5k)/6}(6s)\times Z^{(k^3-4k^2+3k)/2}(7s)Z^{(3k^4-26k^3+57k^2-34k)/24}(8s)f_k(s)$$

(1.4)

we have

$$\frac{F_k(s)}{Z(s)} = \prod_{\boldsymbol{\rho}}\left(1+\sum_{a=1}^{\infty}\frac{\tau_k(a)}{N^{as}(\boldsymbol{\rho})}\right)(1-\boldsymbol{\rho}^{-1}) = \prod_{\boldsymbol{\rho}}\left(1+\sum_{a=2}^{\infty}\frac{\tau_k(a)-\tau_k(a-1)}{N^{as}(\boldsymbol{\rho})}\right) \quad (1.5)$$

and so function $F_k(s)/Z(s)$ is regular in the neighbourhood of $s=1$. At the same time we have

$$\operatorname*{res}_{s=1} Z(s) = L(1,\chi_4)\operatorname*{res}_{s=1} \zeta(s) = \frac{\pi}{4}, \quad (1.6)$$

which implies (1.2). Numerical values of C_k and C_{k*} can be, for example: $C_2 \approx 1{,}156101$, $C_{2*} \approx 1{,}524172$.

We note that the value 1,156101 is very near to the value 1,15522604 and that the value 1,524172 is very near to the value 1,52313377 ($1{,}1561 \approx 1{,}1552$; $1{,}5241 \approx 1{,}5231$). Values concerning the **new universal musical system based on fractional powers of Phi and Pigreco.**

Theorem 1

$$M_k(x) = C_k x + O\left(x^{1/2}\log^{3+4(k-1)/3} x\right), \qquad (1.7) \qquad M_{k*}(x) = C_{k*} x + O\left(x^{1/2}\log^{3+2(k^2+k-2)/3} x\right), \quad (1.8)$$

where C_k *and* C_{k*} *were defined in* (1.2) *and* (1.3).

By Perron formula and by the following expressions,

$$\tau_{k*}^{(e)}(n) << n^{\varepsilon}, \qquad t_k^{(e)}(\alpha) << N^{\varepsilon}(\alpha), \qquad t_{k*}^{(e)}(\alpha) << N^{\varepsilon}(\alpha), \quad (1.9)$$

for $c = 1 + 1/\log x$, $\log T \approx \log x$ we have

$$M_k(x) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} F_k(s)\frac{x^s}{s}ds + O\left(\frac{x^{1+\varepsilon}}{T}\right), \quad (1.10)$$

that, for the eq. (1.7), can be rewritten also as follows:

$$M_k(x) = C_k x + O\left(x^{1/2}\log^{3+4(k-1)/3} x\right) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} F_k(s)\frac{x^s}{s}ds + O\left(\frac{x^{1+\varepsilon}}{T}\right). \quad (1.10b)$$

Suppose $d = 1/2 - 1/\log x$. Let us shift the interval of integration to $[d - iT, d + iT]$. To do this consider an integral about a closed rectangle path with vertexes in $d - iT$, $d + iT$, $c + iT$ and $c - iT$. There are two poles in $s = 1$ and $s = 1/2$ inside the contour. The residue at $s = 1$ was calculated in (1.1). The residue at $s = 1/2$ is equal to $Dx^{1/2}$, $D =$ const and will be absorbed by error term (see below). Identity (1.4) implies

$$F_k(s) = Z(s) Z^{k-1}(2s) f_k(s), \quad (1.11)$$

where $f_k(s)$ is regular for $\mathcal{R}s > 1/3$, so for each $\varepsilon > 0$ it is uniformly bounded for $\mathcal{R}s > 1/3 + \varepsilon$. Let us estimate the error term. The error term absorbs values of integrals about three sides of the integration's rectangle. We take into account $Z(s) = \zeta(s) L(s, \chi_4)$. On the horizontal segments we have

$$\int_{d+iT}^{c+iT} Z(s) Z^{k-1}(2s) \frac{x^s}{s} ds << \max_{\sigma \in [d,c]} Z(\sigma + iT) Z^{k-1}(2\sigma + 2iT) x^{\sigma} T^{-1} <<$$
$$<< x^{1/2} T^{2\theta - 1} \log^{4(k-1)/3} T + x T^{-1} \log^{4/3} T. \quad (1.12)$$

It is well-known that $\zeta(s) \approx (s-1)^{-1}$ in the neighbourhood of $s = 1$. So on $[d, d+i]$ we get

$$\int_{d}^{d+i} Z(s) Z^{k-1}(2s) \frac{x^s}{s} ds << x^{1/2} \int_0^1 \zeta^{k-1}(2d + 2it) dt << x^{1/2} \int_0^1 \frac{dt}{|it - 1/\log x|^{k-1}} << x^{1/2} \log^{k-1} x, \quad (1.13)$$

and for the rest of the vertical segment we have

$$\int_{d+i}^{d+iT} Z(s) Z^{k-1}(2s) \frac{x^s}{s} ds << \left(\int_1^T |\zeta(1/2 + it)|^4 \frac{dt}{t} \int_1^T |L(1/2 + it, \chi_4)|^4 \frac{dt}{t} \right)^{1/4} \left(\int_1^T |Z(1 + 2it)|^{2(k-1)} \frac{dt}{t} \right)^{1/2} <<$$
$$<< x^{1/2} \left(\log^5 T \cdot \log^{8(k-1)/3+1} T \right)^{1/2} << x^{1/2} \log^{3+4(k-1)/3} T. \quad (1.14)$$

The choice $T = x^{1/2+\varepsilon}$ finishes the proof of (1.7).

In conclusion, we note that multiplying both the sides of (1.14) for 8, we obtain the equivalent expression:

$$8\int_{d+i}^{d+iT} Z(s)Z^{k-1}(2s)\frac{x^s}{s}ds << 8\left(\int_1^T |\zeta(1/2+it)|^4 \frac{dt}{t}\int_1^T |L(1/2+it,\chi_4)|^4 \frac{dt}{t}\right)^{1/4}\left(\int_1^T |Z(1+2it)|^{2(k-1)}\frac{dt}{t}\right)^{1/2} <<$$
$$<< 8x^{1/2}\left(\log^5 T\cdot\log^{8(k-1)/3+1}T\right)^{1/2} << 8x^{1/2}\log^{3+4(k-1)/3}T\,, \quad (1.14b)$$

that we can connect to the modes corresponding to the physical vibrations of the superstrings by the following Ramanujan modular equation:

$$8 = \frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (1.14c)$$

Thence, we obtain the following mathematical connection:

$$8\int_{d+i}^{d+iT} Z(s)Z^{k-1}(2s)\frac{x^s}{s}ds << 8\left(\int_1^T |\zeta(1/2+it)|^4 \frac{dt}{t}\int_1^T |L(1/2+it,\chi_4)|^4 \frac{dt}{t}\right)^{1/4}\left(\int_1^T |Z(1+2it)|^{2(k-1)}\frac{dt}{t}\right)^{1/2} <<$$
$$<< 8x^{1/2}\left(\log^5 T\cdot\log^{8(k-1)/3+1}T\right)^{1/2} << 8x^{1/2}\log^{3+4(k-1)/3}T \Rightarrow$$
$$\Rightarrow \frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (1.14d)$$

Theorem 2

Let $x = e^{\log T/N}$ *and* N *such that* x *and* $N/\log\log T \to \infty$ *as* $T \to \infty$. *Then*

$$e^{u^2(\log\log x+\gamma)/4}\frac{1}{T}\int_T^{2T} e^{iu\sum_{p\le x}\frac{\sin(t\log p)}{\sqrt{p}}}dt \to \Phi(u) \quad as \quad T\to\infty \quad (1.15)$$

locally uniformly for $u\in R$. *Here* γ *denotes Euler's constant and* Φ *is the analytic function given by*

$$\Phi(u)=\prod_{p\in P}\left(1-\frac{1}{p}\right)^{-u^2/4} J_0\left(\frac{u}{\sqrt{p}}\right) \quad (1.16)$$

where J_0 *denotes the zeroth Bessel function.*

Corollary 1

Assume RH. Let U_T *be random variables uniformly distributed on* $[T,2T]$. *Then the family* $(1/((\log\log T)/2))\mathrm{Im}\log\zeta(1/2+iU_T)$ *satisfies the large deviation principle with speed* $1/((\log\log T)/2)$ *and rate function* $I(h)=h^2/2$. *For instance,*

$$\frac{1}{(\log\log T)/2}\log\left(\frac{1}{T}\lambda(\{t\in[T,2T]:\mathrm{Im}\log\zeta(1/2+it)\ge h(\log\log T)/2\})\right)\to -h^2/2 \qquad as \qquad T\to\infty$$
(1.17)

where $h>0$ *and* λ *denotes the Lebesgue measure.*

Let x be a positive real number and denote by $p_1, p_2,\dots, p_n$ the prime numbers not exceeding x. We have

$$\frac{1}{T}\int_T^{2T}\left|\sum_{p\le x}\frac{1}{p^{1/2+it}}\right|^{2k}dt=\frac{1}{T}\int_T^{2T}\left|\sum_{\lambda_1+\ldots+\lambda_n=k}\binom{k}{\lambda_1\ldots\lambda_n}\left(p_1^{-\lambda_1}\ldots p_n^{-\lambda_n}\right)^{1/2+it}\right|^2 dt \qquad (1.18)$$

and applying the mean value theorem of Montgomery and Vaughan this is equal to

$$\sum_{\lambda_1+\ldots+\lambda_n=k}\binom{k}{\lambda_1\ldots\lambda_n}^2 p_1^{-\lambda_1}\ldots p_n^{-\lambda_n}+\theta\frac{6\pi}{T}\sum_{\lambda_1+\ldots+\lambda_n=k}\binom{k}{\lambda_1\ldots\lambda_n}^2 \qquad (1.19)$$

with $|\theta|\le 1$. The absolute value of the remainder is bounded by $(6\pi/T)k!n^k$.
We note that, for the eq. (1.19) and multiplying both the sides for 4, we can rewrite the eq. (1.18) also as follows:

$$\frac{4}{T}\int_T^{2T}\left|\sum_{p\le x}\frac{1}{p^{1/2+it}}\right|^{2k}dt=\frac{4}{T}\int_T^{2T}\left|\sum_{\lambda_1+\ldots+\lambda_n=k}\binom{k}{\lambda_1\ldots\lambda_n}\left(p_1^{-\lambda_1}\ldots p_n^{-\lambda_n}\right)^{1/2+it}\right|^2 dt=$$

$$=4\sum_{\lambda_1+\ldots+\lambda_n=k}\binom{k}{\lambda_1\ldots\lambda_n}^2 p_1^{-\lambda_1}\ldots p_n^{-\lambda_n}+\theta\frac{24\pi}{T}\sum_{\lambda_1+\ldots+\lambda_n=k}\binom{k}{\lambda_1\ldots\lambda_n}^2. \qquad (1.19b)$$

The eq. (1.19b) can be related with the following Ramanujan modular equation concerning the modes corresponding to the physical vibrations of the bosonic strings:

$$24=\frac{4\left[anti\log\frac{\int_0^\infty\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2w'}dx}{e^{-\frac{\pi^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} \qquad (1.19c)$$

and we obtain:

$$\frac{4}{T}\int_T^{2T}\left|\sum_{p\le x}\frac{1}{p^{1/2+it}}\right|^{2k}dt=\frac{4}{T}\int_T^{2T}\left|\sum_{\lambda_1+\ldots+\lambda_n=k}\binom{k}{\lambda_1\ldots\lambda_n}\left(p_1^{-\lambda_1}\ldots p_n^{-\lambda_n}\right)^{1/2+it}\right|^2 dt=$$
$$=4\sum_{\lambda_1+\ldots+\lambda_n=k}\binom{k}{\lambda_1\ldots\lambda_n}^2 p_1^{-\lambda_1}\ldots p_n^{-\lambda_n}+\theta\frac{24\pi}{T}\sum_{\lambda_1+\ldots+\lambda_n=k}\binom{k}{\lambda_1\ldots\lambda_n}^2\Rightarrow$$
$$\Rightarrow\frac{4\left[anti\log\frac{\int_0^\infty\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (1.19d)$$

Applying $\sin(t\log p)=\left(p^{it}-p^{-it}\right)/2i$ and a version of the mean value theorem, we obtain:

$$\frac{1}{T}\int_T^{2T}\left(\sum_{p\le x}\frac{\sin(t\log p)}{\sqrt{p}}\right)^{2k}dt=\frac{1}{2^{2k}}\binom{2k}{k}\sum_{\lambda_1+\ldots+\lambda_n=k}\binom{k}{\lambda_1\ldots\lambda_n}^2 p_1^{-\lambda_1}\ldots p_n^{-\lambda_n}+$$
$$+\theta\frac{2D}{2^{2k}T}\sum_{j=0}^{2k}\binom{2k}{j}\sqrt{\sum_{\lambda_1+\ldots+\lambda_n=j}\binom{j}{\lambda_1\ldots\lambda_n}^2}\sqrt{\sum_{\lambda_1+\ldots+\lambda_n=2k-j}\binom{2k-j}{\lambda_1\ldots\lambda_n}^2} \quad (1.20)$$

with $|\theta|\le 1$ and D is a constant, and we have also that:

$$E\left[\left(\sum_{i=1}^n\frac{\operatorname{Im}X_i}{\sqrt{p_i}}\right)^{2k}\right]=\frac{1}{2^{2k}}\binom{2k}{k}\sum_{\lambda_1+\ldots+\lambda_n=k}\binom{k}{\lambda_1\ldots\lambda_n}^2 p_1^{-\lambda_1}\ldots p_n^{-\lambda_n}. \quad (1.21)$$

Now, for eq. (1.21) and multiplying both the sides for 8, we can rewrite the eq. (1.20) also as follows:

$$\frac{8}{T}\int_T^{2T}\left(\sum_{p\le x}\frac{\sin(t\log p)}{\sqrt{p}}\right)^{2k}dt=8E\left[\left(\sum_{i=1}^n\frac{\operatorname{Im}X_i}{\sqrt{p_i}}\right)^{2k}\right]+$$

$$+\theta\frac{16D}{2^{2k}T}\sum_{j=0}^{2k}\binom{2k}{j}\sqrt{\sum_{\lambda_1+\ldots+\lambda_n=j}\binom{j}{\lambda_1\ldots\lambda_n}^2}\sqrt{\sum_{\lambda_1+\ldots+\lambda_n=2k-j}\binom{2k-j}{\lambda_1\ldots\lambda_n}^2}. \quad (1.21b)$$

This expression, can be related with the Ramanujan modular equation that concerning the "modes" that correspond to the physical vibrations of a superstring:

$$8=\frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}, \quad (1.21c)$$

and so, we obtain the following relationship:

$$\frac{8}{T}\int_T^{2T}\left(\sum_{p\le x}\frac{\sin(t\log p)}{\sqrt{p}}\right)^{2k}dt=8E\left[\left(\sum_{i=1}^{n}\frac{\operatorname{Im}X_i}{\sqrt{p_i}}\right)^{2k}\right]+$$

$$+\theta\frac{16D}{2^{2k}T}\sum_{j=0}^{2k}\binom{2k}{j}\sqrt{\sum_{\lambda_1+\ldots+\lambda_n=j}\binom{j}{\lambda_1\ldots\lambda_n}^2}\sqrt{\sum_{\lambda_1+\ldots+\lambda_n=2k-j}\binom{2k-j}{\lambda_1\ldots\lambda_n}^2}\Rightarrow$$

$$\Rightarrow\frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (1.21d)$$

Now let $X_1, X_2,\ldots$ be an i.i.d. sequence of random variables uniformly distributed on the unit circle. We have

$$E\left[e^{iu\operatorname{Im}X_1}\right]=\frac{1}{2\pi}\int_0^{2\pi}e^{iu\sin\theta}d\theta=J_0(u). \quad (1.22)$$

Applying (1.22), Weierstrass' product formula, and Merten's formula, we obtain that $\sum_{i=1}^{\pi(x)} \operatorname{Im} X_i / \sqrt{p_i}$ converges mod-Gaussian, i.e.

$$e^{u^2(\log\log x+\gamma)/4} E\left[e^{iu \sum_{i=1}^{\pi(x)} \frac{\operatorname{Im} X_i}{\sqrt{p_i}}} \right] \to \prod_{p \in P} \left(1 - \frac{1}{p}\right)^{-u^2/4} J_0\left(\frac{u}{\sqrt{p}}\right) \quad (1.23)$$

as $x \to \infty$, locally uniformly for $u \in C$, where $\pi(x)$ denotes the number of primes not exceeding x.
By means of (1.20), (1.21), and the analogous results for odd integers, we can apply for fixed x the method of moments and obtain the following weak convergence

$$\frac{1}{T}\int_T^{2T} e^{iu \sum_{p \le x} \frac{\sin(t \log p)}{\sqrt{p}}} dt \to \prod_{p \le x} J_0\left(\frac{u}{\sqrt{p}}\right) \quad \text{as } T \to \infty. \quad (1.24)$$

The improvement of Theorem 2 follows from combining (1.23) with the following proposition.

Proposition 1

Let $c > 1$ be a constant. Define $x = e^{\log T / N}$ with $N = (c' e c^2 / 4) \log\log T$, $c' > 1$ another constant. Then for T sufficiently large independent of c and c',

$$\frac{1}{T}\int_T^{2T} e^{iu \sum_{p \le x} \frac{\sin(t \log p)}{\sqrt{p}}} dt = \prod_{p \le x} J_0\left(\frac{u}{\sqrt{p}}\right) + O\left((1/c')^N + (2c^2/\log x)^N\right) \quad (1.25)$$

uniformly for $|u| \le c$, $u \in R$. The remainder is $o(\exp(-c^2(\log\log T)/4))$, if $c' \log c' > 1/e$.

From the Taylor expansion $e^{iu} = \sum_{k \le 2N'-1} (iu)^k / k! + \theta u^{2N'} / (2N')!$, $u \in R$, with $|\theta| \le 1$ and $N' = \lfloor N \rfloor$, we obtain

$$\frac{1}{T}\int_T^{2T} e^{iu\sum_{p\le x}\frac{\sin(t\log p)}{\sqrt{p}}}\,dt = \sum_{k\le 2N'-1}\frac{(iu)^k}{k!}\frac{1}{T}\int_T^{2T}\left(\sum_{p\le x}\frac{\sin(t\log p)}{\sqrt{p}}\right)^k dt + \theta\frac{u^{2N'}}{(2N')!}\frac{1}{T}\int_T^{2T}\left(\sum_{p\le x}\frac{\sin(t\log p)}{\sqrt{p}}\right)^{2N'} dt$$

(1.26)
with $|\theta| \le 1$. The remainder is by (1.20)

$$O\left(\frac{c^{2N'}}{N'!}\frac{1}{2^{2N'}}\left(\sum_{p\le x}\frac{1}{p}\right)^{N'} + \frac{(c^2 n)^{N'}}{T}\right). \quad (1.27)$$

Using the bound $(N')! \ge (N'/e)^{N'}$, elementary results in the theory of primes, namely the formulas $\sum_{p\le x} 1/p = \log\log x + c_1 + O(1/\log x)$ and $n = \pi(x) \le 2x/\log x$, and finally $N' = \lfloor N \rfloor$, this is

$$O\left(\left(\frac{ec^2\log\log T}{4N}\right)^N + \left(\frac{2c^2}{\log x}\right)^N\right) = O\left(\left(\frac{1}{c'}\right)^N + \left(\frac{2c^2}{\log x}\right)^N\right) \quad (1.28)$$

for sufficiently large T independent of c, c' (since $c, c' > 1$).
Now let $X_1, X_2, \ldots$ be an i.d.d. sequence of random variables uniformly distributed on the unit circle. The moments in (1.26) are by (1.20) and (1.21) equal to those of the stochastic model plus a remainder which is bounded by $(2D/T)\sqrt{n^k k!}$. The resulting remainders in (1.26), $k \le 2N'-1$, add up to $O\left((c^2 n)^N / T\right) = O\left((2c^2/\log x)^N\right)$. Hence, (1.26) is equal to

$$\sum_{k\le 2N'-1}\frac{(iu)^k}{k!}E\left(\sum_{i=1}^{n}\frac{\operatorname{Im} X_i}{\sqrt{p_i}}\right)^k + O\left((1/c')^N + (2c^2/\log x)^N\right) \quad (1.29)$$

for sufficiently large T. Applying the above Taylor expansion again, we obtain

$$\prod_{p\le x} J_0\left(\frac{u}{\sqrt{p}}\right) = E\left[e^{iu\sum_{i=1}^{n}\frac{\operatorname{Im} X_i}{\sqrt{p_i}}}\right] = \sum_{k\le 2N'-1}\frac{(iu)^k}{k!}E\left(\sum_{i=1}^{n}\frac{\operatorname{Im} X_i}{\sqrt{p_i}}\right)^k + \theta\frac{u^{2N'}}{(2N')!}E\left(\sum_{i=1}^{n}\frac{\operatorname{Im} X_i}{\sqrt{p_i}}\right)^{2N'} \quad (1.30)$$

with $|\theta|\le 1$. The remainder already appeared in (1.26) and is $O\left((1/c')^N\right)$ for sufficiently large T.

Now, we note that multiplying in the eqs. (1.26) and (1.30) both the sides for 8, we obtain the following relationship:

$$\frac{8}{T}\int_T^{2T} e^{iu\sum_{p\le x}\frac{\sin(t\log p)}{\sqrt{p}}}dt = \sum_{k\le 2N'-1}\frac{(iu)^k}{k!}\frac{8}{T}\int_T^{2T}\left(\sum_{p\le x}\frac{\sin(t\log p)}{\sqrt{p}}\right)^k dt + \theta\frac{u^{2N'}}{(2N')!}\frac{8}{T}\int_T^{2T}\left(\sum_{p\le x}\frac{\sin(t\log p)}{\sqrt{p}}\right)^{2N'} dt \Rightarrow$$

$$\Rightarrow \sum_{k\le 2N'-1}\frac{8(iu)^k}{k!}E\left(\sum_{i=1}^{n}\frac{\operatorname{Im} X_i}{\sqrt{p_i}}\right)^k + \theta\frac{8u^{2N'}}{(2N')!}E\left(\sum_{i=1}^{n}\frac{\operatorname{Im} X_i}{\sqrt{p_i}}\right)^{2N'}. \quad (1.30b)$$

Also this expression, can be related with the Ramanujam modular equation that concerning the "modes" that correspond to the physical vibrations of a superstring:

$$8 = \frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos \pi t x w'}{\cosh \pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}, \quad (1.30c)$$

thence, we obtain the following mathematical connection:

$$\frac{8}{T}\int_T^{2T} e^{iu\sum_{p\le x}\frac{\sin(t\log p)}{\sqrt{p}}}dt = \sum_{k\le 2N'-1}\frac{(iu)^k}{k!}\frac{8}{T}\int_T^{2T}\left(\sum_{p\le x}\frac{\sin(t\log p)}{\sqrt{p}}\right)^k dt + \theta\frac{u^{2N'}}{(2N')!}\frac{8}{T}\int_T^{2T}\left(\sum_{p\le x}\frac{\sin(t\log p)}{\sqrt{p}}\right)^{2N'} dt \Rightarrow$$

$$\Rightarrow \sum_{k\le 2N'-1}\frac{8(iu)^k}{k!}E\left(\sum_{i=1}^{n}\frac{\operatorname{Im}X_i}{\sqrt{p_i}}\right)^k + \theta\frac{8u^{2N'}}{(2N')!}E\left(\sum_{i=1}^{n}\frac{\operatorname{Im}X_i}{\sqrt{p_i}}\right)^{2N'} \Rightarrow$$

$$\Rightarrow \frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (1.30d)$$

Now we briefly discuss Selberg's result about the rate of convergence in the central limit theorem of $\operatorname{Im}\log\zeta(1/2+it)$. From Theorem 2 we obtain

Lemma

Let $x=e^{\log T/N}$ and N such that x and $N/\log\log T\to\infty$ as $T\to\infty$. Assume further than $N=O(\log\log T)$. Then

$$\sup_{a<b}\left(\frac{1}{T}\lambda\left(\left\{t\in[T,2T]:\frac{1}{\sqrt{(\log\log x+\gamma)/2}}\sum_{p\le x}\frac{\sin(t\log p)}{\sqrt{p}}\in[a,b]\right\}\right)-\int_a^b e^{-t^2/2}\frac{dt}{\sqrt{2\pi}}\right)=O\left(1/\sqrt{\log\log T}\right).$$

(1.31)

In fact, the right hand side can be replaced by $o\left(1/\sqrt{\log\log T}\right)$.

We denote by $\phi_n(u)$ the left hand side of (1.15). We can bound the right hand side by $(\log\log x/\log\log T\to 1)$

$$\frac{2}{\pi}\int_{-c\sqrt{\log\log x}}^{c\sqrt{\log\log x}} e^{-u/2}\left|\left(\phi_n\left(u/\sqrt{(\log\log x+\gamma)/2}\right)-1\right)/u\right|du + O\left(\frac{1}{c\sqrt{\log\log T}}\right). \quad (1.32)$$

We choose $c>0$ such that $J_0(u)$ has no zeros for $|u|\le c$. An inspection of the proof of Proposition 1 shows that $\phi_n(u)=\phi(u)(1+O(|u|/\log x))$, $|u|\le c$. On the other hand, we have $\phi\left(u/\sqrt{(\log\log x+\gamma)/2}\right)=1+O\left(|u|/\sqrt{\log\log T}\right)$, $|u|\le c\sqrt{\log\log x}$. Plugging in these estimates gives the lemma.
This lemma combined with the following bound

$$\left|\{t\in[T,2T]:|r_{1,x}(t)|\ge c\log\log\log T\}\right|=O\left(1/\sqrt{\log\log T}\right) \quad (1.33)$$

where $c>0$ is a constant, yields Selberg's result

$$\sup_{a<b}\left(\frac{1}{T}\lambda\left(\left\{t\in[T,2T]:\frac{\operatorname{Im}\log\zeta(1/2+it)}{\sqrt{(\log\log T)/2}}\in[a,b]\right\}\right)-\int_a^b e^{-t^2/2}\frac{dt}{\sqrt{2\pi}}\right)=O\left(\frac{\log\log\log T}{\sqrt{\log\log T}}\right). \quad (1.34)$$

Let x and y be positive real number, a_p and b_p be complex numbers with $|a_p|\le 1$ and $|b_p|\le\log p/\log x$, and k be a positive integer. By repeating the arguments in (1.18) and (1.19), we obtain

$$\frac{1}{T}\int_T^{2T}\left|\sum_{p\le x}\frac{a_p}{p^{1+2it}}\right|^{2k}dt\le k!\left(\sum_{p\le x}\frac{1}{p^2}\right)^k+6\pi k!(\pi(x))^k/T\,, \quad (1.35)$$

$$\frac{1}{T}\int_T^{2T}\left|\sum_{p\le x}\frac{b_p}{p^{1/2+it}}\right|^{2k}dt\le k!\frac{1}{(\log x)^k}\left(\sum_{p\le x}\frac{\log p}{p}\right)^k+6\pi k!(\pi(x))^k/T\,, \quad (1.36)$$

$$\frac{1}{T}\int_T^{2T}\left|\sum_{y<p\le x}\frac{a_p}{p^{1/2+it}}\right|^{2k}dt\le k!\left(\sum_{y<p\le x}\frac{1}{p}\right)^k+6\pi k!(\pi(x)-\pi(y))^k/T\,. \quad (1.37)$$

If $x\le T^{1/k}$, the first two terms are bounded by $(Ak)^k$ and the third by $(k(\log\log x-\log\log y+A))^k$, $A>0$ some constant.

Now, if we multiply for 4 both the sides of the eq. (1.36), we obtain the equivalent expression:

$$\frac{4}{T}\int_T^{2T}\left|\sum_{p\le x}\frac{b_p}{p^{1/2+it}}\right|^{2k}dt \le 4k!\frac{1}{(\log x)^k}\left(\sum_{p\le x}\frac{\log p}{p}\right)^k + 24\pi k!(\pi(x))^k/T\,, \qquad (1.37b)$$

and this expression, can be related with the following Ramanujan modular equation concerning the modes corresponding to the physical vibrations of the bosonic strings, i.e.

$$24 = \frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]},$$

and thence, we obtain the following mathematical connection:

$$\frac{4}{T}\int_T^{2T}\left|\sum_{p\le x}\frac{b_p}{p^{1/2+it}}\right|^{2k}dt \le 4k!\frac{1}{(\log x)^k}\left(\sum_{p\le x}\frac{\log p}{p}\right)^k + 24\pi k!(\pi(x))^k/T \Rightarrow$$

$$\Rightarrow \frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \qquad (1.37c)$$

Returning to the eqs. (1.35) – (1.37), for example, we obtain for a function $|g(u)|\le 1$

$$\frac{1}{T}\int_T^{2T}\left|\frac{1}{\log T^{1/V}}\sum_{n\le T^{1/V}}\frac{\Lambda(n)}{n^{1/2+it}}g\left(\frac{\log n}{\log T^{1/V}}\right)\right|^{2\lfloor V\rfloor}dt=\frac{1}{T}\int_T^{2T}\left|\sum_{p\le T^{1/V}}\frac{b_p}{p^{1/2+it}}+\sum_{p^2\le T^{1/V}}\frac{a_p}{p^{1+2it}}+O(1)\right|^{2\lfloor V\rfloor}dt\le$$
$$\le 3^{2V}\left((AV)^V+(AV)^V+O(1)^V\right).$$

(1.38)

Also here, multiplying both the sides for 8, we obtain the following similar expression:

$$\frac{8}{T}\int_T^{2T}\left|\frac{1}{\log T^{1/V}}\sum_{n\le T^{1/V}}\frac{\Lambda(n)}{n^{1/2+it}}g\left(\frac{\log n}{\log T^{1/V}}\right)\right|^{2\lfloor V\rfloor}dt=\frac{8}{T}\int_T^{2T}\left|\sum_{p\le T^{1/V}}\frac{b_p}{p^{1/2+it}}+\sum_{p^2\le T^{1/V}}\frac{a_p}{p^{1+2it}}+O(1)\right|^{2\lfloor V\rfloor}dt\le$$
$$\le 24^{2V}\left((AV)^V+(AV)^V+O(1)^V\right),$$

(1.38b)

and this expression can be related with the Ramanujan's modular equations concerning the modes corresponding to the physical vibrations of the superstrings and the bosonic strings, i.e.

$$8=\frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]},$$

$$24=\frac{4\left[anti\log\frac{\int_0^\infty\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}.$$

Thence, we can obtain, for example, the following mathematical connection:

$$\frac{8}{T}\int_T^{2T}\left|\frac{1}{\log T^{1/V}}\sum_{n\le T^{1/V}}\frac{\Lambda(n)}{n^{1/2+it}}g\left(\frac{\log n}{\log T^{1/V}}\right)\right|^{2\lfloor V\rfloor}dt=\frac{8}{T}\int_T^{2T}\left|\sum_{p\le T^{1/V}}\frac{b_p}{p^{1/2+it}}+\sum_{p^2\le T^{1/V}}\frac{a_p}{p^{1+2it}}+O(1)\right|^{2\lfloor V\rfloor}dt\le$$
$$\le 24^{2V}\left((AV)^V+(AV)^V+O(1)^V\right)\Rightarrow$$
$$\Rightarrow\frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (1.38c)$$

Now, we consider the following function:

$$F(z)=\left(e^z+1\right)^{-1}, \quad (1.39)$$

Developing in series the (1.39), we obtain:

$$F(z)=\left(e^z+1\right)^{-1}=e^{-z}\sum_{k\ge0}(-1)^k e^{zk}=\sum_{k\ge0}(-1)^k e^{-z(1+k)}. \quad (1.40)$$

The equation (1.40) is defined in the complex half-plane $R_e z\ge0$ with the exception of points in which $e^z=-1$.
Now, we examine the following function

$$\left(e^z-1\right)^{-p}, \quad p>0 \quad (1.41)$$

performing the series expansion type (1.40) of the function $\left(e^z-1\right)^{-p}$, we find:

$$\left(e^z-1\right)^{-p}=e^{-pz}\sum_{k\ge0}\binom{-p}{k}(-1)^k e^{-zk}=\sum_{k\ge0}\binom{p+k-1}{k}e^{-z(p+k)}. \quad (1.42)$$

the series on the right hand-side of equation (1.42) is defined in the half-plane $R_e z \geq 0$ with the exception of points in which $e^z = 1$.
Integrating the equation (1.42) between the limits 0 and infinity, we find:

$$\int_0^\infty \left(e^z - 1\right)^{-p} dz = \sum_{k \geq 0} \binom{p+k-1}{k} \int_0^\infty e^{-z(p+k)} dz = \sum_{k \geq 0} \binom{p+k-1}{k} \frac{1}{p+k}; \quad (1.43)$$

$$\int_0^\infty \left(e^z - 1\right)^{-p} dz = \int_0^\infty e^{-pz} \frac{dz}{\left(1-e^{-z}\right)^p} = -\int_1^0 t^{p-1}(1-t)^{-p} dt = \Gamma(p)\Gamma(1-p) \quad \text{with} \quad e^{-z} = t. \quad (1.44)$$

Thence, we can rewrite all also as follows:

$$\int_0^\infty \left(e^z - 1\right)^{-p} dz = \sum_{k \geq 0} \binom{p+k-1}{k} \int_0^\infty e^{-z(p+k)} dz = \sum_{k \geq 0} \binom{p+k-1}{k} \frac{1}{p+k} = \int_0^\infty e^{-pz} \frac{dz}{\left(1-e^{-z}\right)^p} =$$
$$= -\int_1^0 t^{p-1}(1-t)^{-p} dt = \Gamma(p)\Gamma(1-p). \quad (1.45)$$

Therefore:

$$\sum_{k \geq 0} \binom{p+k-1}{k} \frac{1}{p+k} = \Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin(\pi p)}, \qquad (0 < p < 1) \qquad (1.46)$$

For $p = \frac{1}{2}$ we have that

$$\sum_{k \geq 0} \binom{\frac{1}{2}+k-1}{k} \frac{1}{1+2k} = \frac{\pi}{2}. \quad (1.47)$$

Remembering that: $\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right)$, we obtain:

$$\binom{\frac{1}{2}+k-1}{k}=\frac{\Gamma\left(\frac{1}{2}+k\right)\Gamma(k+1)}{k!\,\Gamma\left(\frac{1}{2}\right)\Gamma(k+1)}=\frac{\Gamma(2k+1)\sqrt{\pi}}{2^{2k}(k!)^2\,\Gamma\left(\frac{1}{2}\right)}=\frac{1}{2^{2k}}\binom{2k}{k}, \quad (1.48)$$

and thence:

$$\sum_{k\geq 0}\binom{2k}{k}\frac{2^{-2k}}{1+2k}=\frac{\pi}{2}. \quad (1.49)$$

Thence, for $p=\frac{1}{2}$ the eq. (1.45), for the eqs (1.47) – (1.49), can be rewritten also as follows:

$$\int_0^\infty (e^z-1)^{-p}dz=\sum_{k\geq 0}\binom{p+k-1}{k}\int_0^\infty e^{-z(p+k)}dz=\sum_{k\geq 0}\binom{p+k-1}{k}\frac{1}{p+k}=\sum_{k\geq 0}\binom{\frac{1}{2}+k-1}{k}\frac{1}{1+2k}=\frac{\pi}{2}, \ (1.50)$$

or also as follows:

$$\int_0^\infty (e^z-1)^{-p}dz=\sum_{k\geq 0}\binom{p+k-1}{k}\int_0^\infty e^{-z(p+k)}dz=\sum_{k\geq 0}\binom{p+k-1}{k}\frac{1}{p+k}=\sum_{k\geq 0}\binom{\frac{1}{2}+k-1}{k}\frac{1}{1+2k}=$$

$$=\sum_{k\geq 0}\binom{\frac{1}{2}+k-1}{k}\frac{1}{1+2k}=\frac{\Gamma\left(\frac{1}{2}+k\right)\Gamma(k+1)}{k!\,\Gamma\left(\frac{1}{2}\right)\Gamma(k+1)}\left(\frac{1}{1+2k}\right)=\frac{\Gamma(2k+1)\sqrt{\pi}}{2^{2k}(k!)^2\,\Gamma\left(\frac{1}{2}\right)}\left(\frac{1}{1+2k}\right)=\frac{1}{2^{2k}}\binom{2k}{k}=$$

$$=\sum_{k\geq 0}\binom{2k}{k}\frac{2^{-2k}}{1+2k}=\frac{\pi}{2}. \quad (1.50b)$$

Multiplying both the sides of (1.42), for e^{-qz}, and integrating with respect to z, between the limits 0 and infinity, we find:

$$\int_0^\infty (e^z-1)^{-p}e^{-qz}dz=\sum_{k\geq 0}\binom{p+k-1}{k}\int_0^\infty e^{-z(q+p+k)}dz=\sum_{k\geq 0}\binom{p+k-1}{k}\frac{1}{q+p+k}\Rightarrow$$

$$\Rightarrow \int_0^{\infty} \left(e^z - 1\right)^{-p} e^{-qz} dz = \int_0^{\infty} e^{-pz} e^{-qz} \frac{dz}{\left(1-e^{-z}\right)} = -\int_1^0 t^{q+p-1} (1-t)^{-p} dt = \frac{\Gamma(q+p)\Gamma(1-p)}{\Gamma(q+1)} \quad \text{for } e^{-z} = t$$

(1.51)

Therefore:

$$\sum_{k \ge 0} \binom{p+k-1}{k} \frac{1}{q+p+k} = \frac{\Gamma(q+p)\Gamma(1-p)}{\Gamma(q+1)}, \quad (0 < p < 1, q > 0) \qquad (1.52)$$

For $p = q = 1/2$, we find:

$$\sum_{k \ge 0} \binom{2k}{k} \frac{2^{-2k}}{1+k} = 2. \qquad (1.53)$$

Thence, in conclusion, for $p = q = 1/2$, we find:

$$\int_0^{\infty} \left(e^z - 1\right)^{-p} e^{-qz} dz = \sum_{k \ge 0} \binom{p+k-1}{k} \int_0^{\infty} e^{-z(q+p+k)} dz = \sum_{k \ge 0} \binom{p+k-1}{k} \frac{1}{q+p+k} = \frac{\Gamma(q+p)\Gamma(1-p)}{\Gamma(q+1)} \Rightarrow$$

$$\Rightarrow \sum_{k \ge 0} \binom{2k}{k} \frac{2^{-2k}}{1+k} = 2. \qquad (1.53b)$$

Now, we note that the eqs. (1.50b) and (1.53b) can be related with the eqs. (1.15) and (1.32) for obtain the following mathematical relationships:

For example, multiplying the eq. (1.53b) for $\frac{1}{\pi}$, we obtain:

$$e^{u^2(\log\log x + \gamma)/4} \frac{1}{T} \int_T^{2T} e^{iu \sum_{p \le x} \frac{\sin(t \log p)}{\sqrt{p}}} dt \Rightarrow$$

$$\frac{2}{\pi} \int_{-c\sqrt{\log\log x}}^{c\sqrt{\log\log x}} e^{-u/2} \left| \left(\phi_n \left(u / \sqrt{(\log\log x + \gamma)/2} \right) - 1 \right) / u \right| du + O\left(\frac{1}{c\sqrt{\log\log T}} \right) \Rightarrow$$

$$\Rightarrow \frac{1}{\pi}\int_0^\infty \left(e^z - 1\right)^{-p} e^{-qz} dz = \frac{1}{\pi}\sum_{k\geq 0}\binom{p+k-1}{k}\int_0^\infty e^{-z(q+p+k)}dz = \frac{1}{\pi}\sum_{k\geq 0}\binom{p+k-1}{k}\frac{1}{q+p+k} = \frac{\Gamma(q+p)\Gamma(1-p)}{\pi\Gamma(q+1)} \Rightarrow$$

$$\Rightarrow \frac{1}{\pi}\sum_{k\geq 0}\binom{2k}{k}\frac{2^{-2k}}{1+k} = \frac{2}{\pi}. \quad (1.54)$$ [1]

[1] Indeed, we have that the following relation:

$$e^{u^2(\log\log x+\gamma)/4}\frac{1}{T}\int_T^{2T} e^{iu\sum_{p\leq x}\frac{\sin(t\log p)}{\sqrt{p}}} dt \Rightarrow$$

$$\frac{2}{\pi}\int_{-c\sqrt{\log\log x}}^{c\sqrt{\log\log x}} e^{-u/2}\left|\left(\phi_n\left(u/\sqrt{(\log\log x+\gamma)/2}\right)-1\right)/u\right| du + O\left(\frac{1}{c\sqrt{\log\log T}}\right). \qquad (1)$$

Form this, we can to obtain $\frac{2}{\pi}$:

$$\frac{2}{\pi} = e^{u^2(\log\log x+\gamma)/4}\frac{1}{T}\int_T^{2T} e^{iu\sum_{p\leq x}\frac{\sin(t\log p)}{\sqrt{p}}} dt \Big/$$

$$\int_{-c\sqrt{\log\log x}}^{c\sqrt{\log\log x}} e^{-u/2}\left|\left(\phi_n\left(u/\sqrt{(\log\log x+\gamma)/2}\right)-1\right)/u\right| du + O\left(\frac{1}{c\sqrt{\log\log T}}\right). \qquad (2)$$

Also the following relation yields to $\frac{2}{\pi}$. Indeed:

$$\frac{1}{\pi}\int_0^\infty \left(e^z - 1\right)^{-p} e^{-qz} dz = \frac{1}{\pi}\sum_{k\geq 0}\binom{p+k-1}{k}\int_0^\infty e^{-z(q+p+k)}dz = \frac{1}{\pi}\sum_{k\geq 0}\binom{p+k-1}{k}\frac{1}{q+p+k} = \frac{\Gamma(q+p)\Gamma(1-p)}{\pi\Gamma(q+1)} \Rightarrow$$

$$\Rightarrow \frac{1}{\pi}\sum_{k\geq 0}\binom{2k}{k}\frac{2^{-2k}}{1+k} = \frac{2}{\pi}. \qquad (3)$$

The eqs. (2) and (3), thence, are both equal to $\frac{2}{\pi}$.

The same can be make with the eq. (1.50b). As we note, the eq. (1.54) can be related with π (and thence with $\phi = \frac{\sqrt{5}+1}{2}$) by the following Ramanujan expression regarding just π:

$$\pi = \frac{24}{\sqrt{142}} \log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]. \qquad (1.55)$$

We observe that the eq. (1.55) contain the number 24, that is related to the physical vibrations of the bosonic strings. Thence, for the (1.54) for $p = q = 1/2$, we obtain the following mathematical connection:

$$e^{u^2(\log\log x+\gamma)/4} \frac{1}{T}\int_T^{2T} e^{iu\sum_{p\le x}\frac{\sin(t\log p)}{\sqrt{p}}} dt \Rightarrow$$

$$\Rightarrow \frac{1}{\pi}\int_0^\infty \left(e^z - 1\right)^{-p} e^{-qz} dz \int_{-c\sqrt{\log\log x}}^{c\sqrt{\log\log x}} e^{-u/2} \left|\left(\phi_n\left(u/\sqrt{(\log\log x+\gamma)/2}\right)-1\right)/u\right| du + O\left(\frac{1}{c\sqrt{\log\log T}}\right) \Rightarrow$$

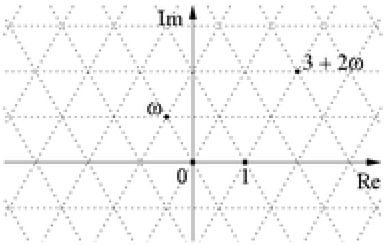
$$\Rightarrow \frac{2}{\frac{24}{\sqrt{142}}\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} \times$$

$$\int_{-c\sqrt{\log\log x}}^{c\sqrt{\log\log x}} e^{-u/2} \left|\left(\phi_n\left(u/\sqrt{(\log\log x+\gamma)/2}\right)-1\right)/u\right| du + O\left(\frac{1}{c\sqrt{\log\log T}}\right). \qquad (1.56)$$

A general relationship that link, π, prime numbers, Riemann Hypothesis and physical vibrations of bosonic strings.

2. EISENSTEIN INTEGER

Eisenstein integers are the intersection points of a triangular lattice in the complex plane:



Eisenstein integers (named after Gotthold Eisenstein), are complex numbers of the form:

$$z = a + b\omega .$$

where *a* and *b* are integers and

$$\omega = \frac{1}{2}\left(-1 + i\sqrt{3}\right) = e^{2\pi i/3}$$

is a primitive (non-real) cube root of unity. The Eisenstein integers form a triangular lattice in the complex plane, in contrast with the Gaussian integers which form a square lattice in the complex plane, (see the two figures above).

2.1 EISENSTEIN PRIME

The Eisenstein primes which are equal to a natural number 3n -1 are 2, 5, 11, 17, 23, 29, 41, 47, 53, 59, 71, 83, 89 (with differences (after 5) equal to 6 (sexy numbers) or 12, for example. 41-29 = 12, 71-59 = 12, 83 = 12 -71, etc..) or even more.

For larger numbers we have in fact the biggest differences, but always of the form 6k, for example 419-401 = 18 = 6 * 3

All, less the initial prime number 2 are of the form 6k - 1 (since, for n even, we have the form 6n -1) and therefore are not Gaussian integers, which are of the form 6k +1 as already seen before.

Eisenstein primes are congruent to 2 mod 3 and Mersenne primes (except the smallest, 3) are congruent to 1 mod 3; thus no Mersenne prime is an Eisenstein prime.

3. RELATIONS BEETWEEN GAUSSIAN PRIMES AND EISENSTEIN PRIMES

We have seen that the Gaussian primes "g" are of the form:

$$g = 4k+3 \quad \text{with } k = 0, 1, 2,\ldots$$

The sequence is the following:

3, 7, 11, 19, 23, 31, 43, 47, 59, 67, 71, 79, 83, 103, 107, 127, 131, 139, 151, 163, 167, 179, 191, 199, 211, 223, 227, 239, 251, 263, 271, 283, 307, 311, 331, 347, 359, 367, 379, 383, 419, 431, 439, 443, 463, 467, 479, 487, 491, 499, 503, 523, 547, 563, 571,…

Eisenstein primes “e” are of the form:

$$e = 3k+2 \quad \text{with } k = 0, 1, 2,\ldots$$

The sequence is the following:

2, 5, 11, 17, 23, 29, 41, 47, 53, 59, 71, 83, 89, 101, 107, 113, 131, 137, 149, 167, 173, 179, 191, 197, 227, 233, 239, 251, 257, 263, 269, 281, 293, 311, 317, 347, 353, 359, 383, 389, 401, 419, 431, 443, 449, 461, 467, 479, 491, 503, 509, 521, 557, 563, 569, 587, …

From the intersection of the two sequences we obtain prime numbers of Gauss-Eisenstein.
From the two preceding formulas 3k +2 and 4k +3, since the least common multiple between 3 and 4 is 12 we have the following formula for primes common Gauss-Eisenstein ρ:

$$\rho = 12k + 11 \quad \text{with } k = 0, 1, 2,\ldots$$

The sequence is the following:

11, 23, 47, 59, 71, 83, 107, 131, 167, 179, 191, 227, 239, 251, 263, 311, 347, 359, 383, 419, 431, 443, 467, 479, 491, 503, 563, 587, 599, 647, 659, 683, 719, 743, 827, 839, 863, 887, 911, 947, 971, 983, 1019, 1031, …

It should be noted that all the prime numbers that are derived from this formula 12k +11 are all prime numbers that are simultaneously Gauss and Eisenstein primes.
These numbers are then congruent to 11 modulo 12.

We note that is 12 = 24/2, where 24 is the number connected to the modes corresponding to the physical vibrations of the bosonic strings by the following Ramanujan's modular equation:

$$24 = \frac{4\left[anti\log \frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4} w'} \phi_{w'}(itw')}\right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}.$$

3.1 CORRELATION WITH SOPHIE GERMAIN PRIMES

We observe that some of the previous sequence numbers are also Sophie Germain primes.

A Sophie Germain prime is a prime *p* such that 2*p*+1 is also prime. The number 2*p* + 1 is instead called safe prime.

The Sophie Germain prime numbers up to 1000 are:

2, 3, 5, **11**, **23**, 29, 41, 53, **83**, 89, 113, **131,** 173, **179**, **191**, 233, **239**, **251**, 281, 293, **359**, **419, 431, 443, 491,** 509, 593, 641, 653, **659**, **683**, **719**, **743**, 761, 809, **911**, 953, 1013, **1019**, **1031**, …

Let's recall the list of numbers ρ = 12k+11 with k = 0, 1, 2,…

11, **23**, 47, 59, 71, **83**, 107, **131**, 167, **179**, **191**, 227, **239**, **251**, 263, 311, 347, **359,** 383, **419**, **431**, **443**, 467, 479, **491**, 503, 563, 587, 599, 647, **659**, **683**, **719**, **743**, 827, 839, 863, 887, **911**, 947, 971, 983, **1019**, **1031**, …

Out of 40 numbers of Sophie Germain, as many as 20, are also Gaussian-Eisenstein (marked in bold in the two sequences).
So we have a new connection, with the numbers of Sophie Germain, of which ρ is a subset and is exactly the half.
We can say that 50% of the Sophie Germain primes are also Gauss-Eisenstein primes
This can be demonstrated with a simple consideration:

In the list of Sophie Germain primes, if we want to know if its any number N is a number ρ of the form 12k +11, obviously we must subtract 11 and divide by 12, if the result is integer, then N = ρ
A few examples:

3779 = Sophie German prime = prime ρ since

(3779-11)/12 = 3768/12 = 314 (integer)

Instead 4793 (Sophie German prime) it is not because (4793-11) / 12 = 4782/12 = 398,5 is not an integer, and then 4793 is not a prime number of Gauss-Eisenstein.

Now chosen any prime number N of Sophie Germain and applying the rule above, we have

(N -11) / 12 = **integer or *half-integer***

The division produces only 2 possible results, or an integer, and then N is also a prime number of Gauss-Eisenstein or is a *half-integer* of the form x+1/2 and N isn't a Gauss-Eisenstein prime.

As a result 50% of the Sophie Germain primes are also Gauss-Eisenstein primes.

Also here, we note that is present the number 12 that, as we have described above, is connected to the modes corresponding to the physical vibrations of the bosonic strings.

3.2 PROOF THAT THERE ARE INFINITELY MANY SOPHIE GERMAIN PRIMES

It is conjectured that there are infinitely many Sophie Germain primes.

Since Gauss-Eisenstein primes are infinite and there is a relationship with Sophie Germain primes, we can by induction affirm that Sophie Germain primes are infinite.

This is because the set ρ of Gauss-Eisenstein of the form 12k +11 is infinite and these numbers are an infinitely subset so that produce endless Sophie G**ermain** primes.

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4. REFERENCES

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