Matter-waves, amplitudes, and signals

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Summary

This paper further explores intuitions we highlighted in previous papers already:

1. The concept of the matter-wave traveling through the vacuum, an atomic lattice or any medium can be equated to the concept of an electric or electromagnetic *signal* traveling through the same medium.
2. There is no need to model the matter-wave as a wave packet: a single wave – with a precise frequency and a precise wavelength – will do.
3. If we do want to model the matter-wave as a wave *packet* rather than a single wave with a precisely defined frequency and wavelength, then the uncertainty in such wave packet reflects our own *limited knowledge* about the momentum and/or the velocity of the particle that we think we are representing. The uncertainty is, therefore, not inherent to Nature, but to our limited *knowledge* about the initial conditions.
4. The fact that such wave *packets* usually dissipate very rapidly, reflects that even our limited knowledge about initial conditions tends to become equally rapidly irrelevant. Indeed, as Feynman puts it, "the tiniest irregularities" tend to get magnified very quickly at the micro-scale.

All of the above makes us agree with what Hendrik Antoon Lorentz noted a few months before his demise: there is no reason whatsoever "to elevate indeterminism to a philosophical principle."

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Matter-waves, amplitudes, and signals

Uncertainty, superluminal velocities, dispersion, and non-localization

Most interpretations, including Louis de Broglie's own reading[1], of the concept of the matter-wave raise logical as well as practical issues. Let us start with the logic.

1. The assumption of an *Ungenauigkeit* or an *Ungewissheit*[2] in the energy value to be used in the (complex-valued) wavefunction leads us to consider a wave *packet* as a potentially truthful representation of the matter-particle. At the same time, we assume a *precise* relation between the frequencies and wavelengths ($f_i = E_i/h$ and with $\lambda_i = h/p_i = h/m_i v$), so as to be able to derive a precise group velocity of the wavepacket, which corresponds to the equally precise (classical) velocity of the particle: no uncertainty here!

This is not consistent: why would we assume an uncertainty in the energy E or in the momentum **p** if there is no uncertainty in the classical velocity of the particle? If there is uncertainty – but *no* uncertainty in the velocity – then the $E = mc^2$ relation tells us that all of the uncertainty must reside in the rest mass of the particle. We see no *a priori* reason for such assumption.

This particularity also triggers a practical issue: wavepackets usually dissipate away—*un*like real-life matter-particles, which do not exhibit such behavior. In fact, the dissipation relation which is assumed to apply to the matter-wave ensures a very rapid dissipation! Let us first fully explore the logic of our first remark before we move to that.

2. That one *must* assume a precise and unique *phase* velocity for each of the component waves of the wavepacket so as to be able to derive a precise velocity *v* for the wave packet can easily be shown by re-writing the $\lambda_i = h/p_i = h/m_i v$ relation as:

$$v = \frac{hc^2}{E_i\lambda_i} = \frac{hc^2}{hf_i\lambda_i} = \frac{c^2}{f_i\lambda_i} = \frac{c^2}{v_i}$$

This shows the velocity *v* is constant if, and *only* if, $f_i\lambda_i = v_i$ is the same for each *component* wave *i*. constant. The equation also shows that the phase velocity $v_i = f_i\lambda_i$ is equal to the ratio of lightspeed and the relative velocity of the particle: $v_i = c^2/v = c/\beta \geq c$. This is a standard result in mainstream theory. However, we note this phase velocity is superluminal if, and – we should note again – *only* if, and we

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[1] We use Louis de Broglie's paper for the 5th Solvay Conference (*La Nouvelle Dynamique des Quanta*, 1927) as a truthful representation of his views.

[2] We prefer to use Heisenberg's very first references to the concept of uncertainty to avoid the metaphysical connotation it acquired when, paraphrasing H.A. Lorentz, physicists decided to discard the assumption of causality by elevating indeterminism to a philosophical principle. Heisenberg originally used the term *Ungenauigkeit* to refer to a disturbance of the phenomenon by the measurement. One example of this are Compton scattering experiments, in which the classical velocity of an electron (both its direction as well as its magnitude) is changed as result of the probing by photons. The use of photons that have lower energy (and, therefore, longer wavelengths) will limit the impact of the measurement but increase the imprecision. Hence, because the wavelength of the photon determines the order of magnitude of the relevant length scale, a compromise must be made between the precision of the measurement and the disturbance it causes. As for the term *Ungewissheit*, this is yet another term we use to avoid referring to Heisenberg's metaphysical *Unbestimmtheitsprinzip* or *Unschärferelation*. We use it to refer to our *lack of knowledge* about the initial conditions: this uncertainty, too, has nothing to do with reality if we use Wittgenstein's definition of reality: "*Die Welt ist alles, was der Fall ist.*"

choose to represent the particle by a wave *packet* rather than a single wave only.

For a single wave, we get the velocity we would get for a particle with zero rest mass: *c*. We think lightspeed is possible.[3] In contrast, superluminal velocities for *any* wave – including any component of a group of waves – cannot be consistent with special relativity theory. Indeed, it would seem to be logical to assume that the vacuum in which matter-wave travels must be the same vacuum as the one in which electromagnetic waves travel, and the properties of this vacuum emerge from Maxwell's equations which – among other things – tell us that waves should not be travelling faster than light. The assumption of superluminal velocities, therefore, appears to be incongruent.[4] However, such assumption is the corollary of the dispersion or dissipation relation we are assuming to be correct.

The latter remark leads us to the practical issue with the concept of the wavepacket as a representation of matter-particles, which we may useful illustrate by quoting Prof. H. Pleijel, then Chairman of the Nobel Committee for Physics of the Royal Swedish Academy of Sciences, as he dutifully makes the following comments on the nature of the new 'matter waves' in the ceremonial speech for the 1933 Nobel Prize, which was awarded to Heisenberg for nothing less than "*the creation of quantum mechanics*"[5]:

> "Matter is formed or represented by a great number of this kind of waves which have somewhat different velocities of propagation and such phase that they combine at the point in question. Such a system of waves forms a crest which propagates itself with quite a different velocity from that of its component waves, this velocity being the so-called group velocity. Such a wave crest represents a material point which is thus either formed by it or connected with it, and is called a wave packet. […] As a result of this theory on is forced to the conclusion to conceive of matter as not being durable, or that it can have definite extension in space. The waves, which form the matter, travel, in fact, with different velocity and must, therefore, sooner or later separate. Matter changes form and extent in space. The picture which has been created, of matter being composed of unchangeable particles, must be modified."

This sounds very familiar: it is the basics of the basics of the mainstream interpretation of de Broglie's matter-wave, in fact. The problem is this: it is, obviously, untrue. *Real-life particles – electrons or atoms traveling in space – do not dissipate*. They might change form and extent in space when traveling

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[3] Particles with zero rest mass must always travel at the speed of light because of Newton's force law: even the tiniest force on them causes infinite acceleration. This logical conclusion makes one wonder if a matter-particle can actually have zero rest mass. The ring current or *Zitterbewegung* model of an electron – which we will discuss in this paper – deals with that question by distinguish between the charge and the particle. The rest mass of the charge is zero but the rest mass of the particle is not. We get a mass without mass model of an electron, or of any matter-particle: its mass is the equivalent mass of the energy in the oscillation of the charge.

[4] Mainstream physicists sometimes justify the assumption of superluminal matter-waves by saying these waves cannot transmit any signal. We do not buy this argument because pair production and annihilation would logically amount to the creation and destruction of matter-waves. Hence, one may imagine a matter-wave switched on or off. Such on- or off-switching would amount to a signal with matter-waves which, individually, would travel faster than light.

[5] To be precise, Werner Heisenberg actually got a postponed prize from 1932: it is Erwin Schrödinger and Paul A.M. Dirac who, jointly, got the actual 1933 prize. Prof. Pleijel therefore acknowledges all three more or less equally in the introduction of his speech: "This year's Nobel Prizes for Physics are dedicated to the new atomic physics. The prizes, which the Academy of Sciences has at its disposal, have namely been awarded to those men, Heisenberg, Schrödinger, and Dirac, who have created and developed the basic ideas of modern atomic physics."

through one or more slits but, as Feynman and all mainstream physicists equally dutifully note in the context of the double-slit experiment with electrons: electrons – or matter-particles in general – always come in identical *lumps* at the backstop. All these lumps are the same size, only whole lumps arrive, and they arrive one at a time.[6]

3. We must, finally, note a very practical issue with the mainstream interpretation of de Broglie's wavepacket: the issue of non-localization. Richard Feynman states this problem as follows:

> "If an amplitude to find a particle at different places is given by $e^{i(\omega \cdot t - k \cdot x)}$, whose absolute square is a constant, that would mean that the probability of finding a particle is the same at all points. That means we do not know *where* it is—it can be anywhere—there is a great uncertainty in its location. On the other hand, if the position of a particle is more or less well known and we can predict it fairly accurately, then the probability of finding it in different places must be confined to a certain region, whose length we call Δx. Outside this region, the probability is zero. Now this probability is the absolute square of an amplitude, and if the absolute square is zero, the amplitude is also zero, so that we have a wave train whose length is Δx, and the wavelength (the distance between nodes of the waves in the train) of that wave train is what corresponds to the particle momentum."[7]

Indeed, one of the properties of the idea of a particle is that it must be *somewhere* at any point in time, and that somewhere must be defined in terms of one-, two- or three-dimensional physical space. The idea of a wave train or a wave*packet* does *not* solve that problem. A *composite* wave with a finite or infinite number of component waves with (phase) frequencies ν_i and wavelengths λ_i is still what it is: an oscillation which repeats itself in space and in time. It is, therefore, all over the place, unless we would limit its domain to some randomly or non-randomly Δx space. However, that amounts to a rather random intervention.

4. In the next section, we will present an elegant solution to all of the issues we have highlighted here. It consists of an interpretation of the matter-wave in terms of the Schrödinger-Dirac *Zitterbewegung* motion of the electric charge. Indeed, the intuition of Einstein[8] and de Broglie in regard to the wave nature of matter is, essentially, correct. However, his modeling of it as a wave *packet* is not: **modeling matter-particles as some *linear* oscillation does not do the trick.** Of course, if that is the case, then what does?

We suggest trying something circular.

Indeed, we think the interpretation of the elementary wavefunction as representing the mentioned *Zitterbewegung* of the electric charge solves all questions: it amounts to interpreting the real and imaginary part of the elementary wavefunction as the sine and cosine components of the orbital motion of a pointlike charge. The pointlike charge and its motion combine to explain Schrödinger's discovery of the *Zitterbewegung* of the electron, which Paul Dirac describes as follows in his Nobel Prize speech:

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[6] See: Richard Feynman, Vol. III, Chapter 1, Section 4, *An experiment with electrons*.

[7] See: *Probability wave amplitudes*, in: Feynman's *Lectures on Physics*, Vol. III, Chapter 2, Section 1.

[8] Albert Einstein played a crucial role in lending credibility to Louis de Broglie's hypothesis by seconding his 1924 *PhD* thesis whose substance was poorly understood at the time.

"It is found that an electron which seems to us to be moving slowly, must actually have a very high frequency oscillatory motion of small amplitude superposed on the regular motion which appears to us. As a result of this oscillatory motion, the velocity of the electron at any time equals the velocity of light. This is a prediction which cannot be directly verified by experiment, since the frequency of the oscillatory motion is so high and its amplitude is so small. But one must believe in this consequence of the theory, since other consequences of the theory which are inseparably bound up with this one, such as the law of scattering of light by an electron, are confirmed by experiment." (Paul A.M. Dirac, *Theory of Electrons and Positrons*, Nobel Lecture, December 12, 1933)

We will come back to this. Let us, before proceeding, remind the reader of the basic concept of a wave *packet* and the dispersion relation so as to ensure easier reading of the remainder of this paper. The academic reader may not need such basics but we will also want to highlight the above-mentioned logical issues in another way. We, therefore, recommend to *not* skip the material that follows.

Wavepackets and the dispersion relation

Most physicists assume the dispersion relation that relates the f_i frequencies (or the angular frequencies $\omega_i = 2\pi f_i$) to the λ_i wavelengths (or the wave*numbers* $k = 2\pi/\lambda_i$) should be *directly* related to the relativistic energy-moment relation[9]:

$$\mathrm{E}^2 - \mathrm{p}^2c^2 = \mathrm{m}^2c^4 \Leftrightarrow \hbar^2\omega^2 - \frac{h^2}{\lambda^2}c^2 = \hbar^2\omega^2 - \frac{h^2}{(2\pi\,/\,\mathrm{k})^2}c^2 \Leftrightarrow \hbar^2\omega^2 - \hbar^2\mathrm{k}^2c^2 = \mathrm{m}^2c^4$$

$$\Leftrightarrow \frac{\hbar^2\omega^2}{c^2} - \hbar^2\mathrm{k}^2 = \mathrm{m}c^2$$

While this assumption appears to be very logical, we think these relations implicitly mix two very different energy and/or mass concepts. One should effectively remember the following:

1. The m in the $E^2 - p^2c^2 = m^2c^4$ relation is the rest mass and should, therefore, be written as m_0.[10]
2. The (linear) momentum is related to the *kinetic* energy of the particle only which only becomes a significant fraction of the total energy when the classical velocity of the particle starts approach the speed of light which – in most situations – it does *not*.

There is, of course, no reason to *not* assume the Planck-Einstein relation does *not* apply to the rest mass (on the contrary, it is generally considered to be as universal or as *fundamental* as Einstein's mass-

[9] We also have the $pc = Ev/c$ or $p = Ev/c^2$ relation, of course, but is rather significant one can*not* derive any dispersion relation from it. When doing the same substitutions ($p = \hbar k$, $E = mc^2$ and $v = p/m$), we just get the trivial $\hbar k = \hbar k$ relation:

$$\mathrm{p} = \hbar\mathrm{k} = \frac{\mathrm{E}v}{c^2} = \frac{\hbar\omega v}{c^2} = \frac{\hbar\omega\mathrm{p}}{\mathrm{m}c^2} = \frac{\hbar\omega\hbar\mathrm{k}}{\hbar\omega} = \hbar\mathrm{k} = \mathrm{p}$$

The reader may try other substitutions – such as the use of the Planck-Einstein relation ($E = \hbar\omega$), for example – but we found we always get the trivial identity. In fact, the very different result – an identity versus an inconsistency – should make us think about the (non)sense of the dispersion relation one gets out of the momentum-energy relation.

[10] The reader will probably be aware of this but, if not, we refer him to any standard textbook. For ease of reference and because of its online availability, we used Feynman's *Lectures* rather consistently over the past. The reader will find the derivation of the formula in section 5 of Chapter 16 of the first volume.

energy mass-equivalence relation E = mc^2) and we may, therefore, re-write the (dispersion) relation above as:

$$\frac{\hbar^2\omega^2}{c^2} - \hbar^2\mathrm{k}^2 = \mathrm{m}_0c^2 = \mathrm{E}_0 = \hbar\omega_0$$

This, however, leads to a contradiction when assuming phase velocities equal to $c = \lambda\cdot f = \omega/\mathrm{k}$:

$$\frac{\hbar^2\omega^2}{c^2} - \hbar^2\mathrm{k}^2 = \hbar^2\left(\frac{\omega^2}{c^2} - \mathrm{k}^2\right) = \hbar^2\left(\frac{\omega^2}{c^2} - \mathrm{k}^2\right) = \mathrm{m}_0c^2 = \mathrm{E}_0 = \hbar\omega_0 \Leftrightarrow \hbar\left(\frac{\omega^2}{c^2} - \mathrm{k}^2\right) = \omega_0$$

$$\Rightarrow \hbar(\mathrm{k}^2 - \mathrm{k}^2) = \hbar\cdot 0 = 0 \not\equiv \omega_0$$

It is a full-blown contradiction (not a mere *inconvenience*, that is) because we cannot equate ω_0 to zero: the rest mass of our particle is not zero and, therefore, ω_0 cannot be zero. The only way out is to assume phase velocities are *not* equal to *c*.[11] We effectively get these *superluminal* phase velocities when using the $v_{\text{phase}} = \omega/\mathrm{k}$ relation and the usual substitutions for the momentum (or the wavenumber):

$$v_{\text{phase}} = \frac{\omega}{\mathrm{k}} = \frac{\mathrm{E}/\hbar}{\mathrm{p}/\hbar} = \frac{\mathrm{m}c^2}{\mathrm{m}v} = \frac{c^2}{v} = c/\beta \geqq c$$

This is, in fact, what physicists since Louis de Broglie – who got a Nobel Prize for writing this *first* – have been copying consistently. However, we think of these substitutions as being misguided and – with the benefit of hindsight, of course – we are rather dumbfounded by the *ease* with which physicists – including Albert Einstein, who seconded Louis de Broglie's 1924 PhD thesis – accepted this inevitable result.

Mainstream physicists usually justify it by noting these *component* waves cannot transmit any *signal* because the group velocity – the velocity of the wave *packet* as a whole – is all that matters (pretty much *literally*, I guess), and this *group* velocity does effectively correspond to the classical velocity of the particle that the wave packet must, somehow, represent.[12] Indeed, differentiating the dispersion

[11] We have to quality this statement. To get out of the contradiction, one might also assume that each of the component waves carries an infinitesimally small part of the energy (or, what amounts to the same, the equivalent mass). It is effectively possible to work out an alternative 'mass without mass' model of a particle based on such assumption. In fact, we actually did so in our very early papers on the topic but we have abandoned such modeling in favor of the more intuitive *Zitterbewegung* or ring current model of an electron now. In any case, this is *not* how mainstream physicists proceed: they assume some uncertainty about the energy or mass but they do not think in terms of the matter-wave actually *carrying* the energy or mass of the matter-particle.

[12] They sometimes add we should not think of the matter-wave as a real wave anyway. However, this is nonsensical: why should we think of the matter-wave as *un*real while calculating *real* phase and group velocities? Also, it is pretty obvious from introductory courses that (most) physicists do think of the wave packet as actually representing the particle. If the group is *real*, then its component waves must be real too. In case the reader may not be convinced of such logic, we may, perhaps, invite him to think about one of [H.A. Lorentz's comments at the 1921 Solvay Conference] on the use of the energy-momentum tensor in relativistic wave mechanics in a reaction to Joseph Larmor on his paper on his "*Notes sur la Théorie des Electrons*": "We can (all) interpret the components of the energy-momentum tensor in whatever way we would want to but if we associate some of them with something real, then we must associate all of them with something real." Indeed, who – in the mainstream interpretation of quantum mechanics – gets to decide on what concept corresponds to something that is real and what is not?

Still, we may imagine we cannot convince the reader based on epistemology only. We, therefore, also offered a non-philosophical objection in footnote 4: pair production and annihilation would logically amount to the creation and destruction of matter-waves. Hence, one may imagine a matter-wave switched on or off. Such on- or off-switching would amount to a relation and combining it with the erroneous relation for the phase velocity above yields what Louis de Broglie wanted to prove:

$$\hbar^2\omega^2 - \hbar^2\mathrm{k}^2c^2 = \mathrm{m}_0c^4 \Longrightarrow 2\hbar^2\omega d\omega = 2\hbar^2c^2\mathrm{k}d\mathrm{k}$$

$$\Leftrightarrow v_{\text{group}} = \frac{\partial\omega}{\partial\mathrm{k}} = c^2\frac{\mathrm{k}}{\omega} = \frac{c^2}{v_{\text{phase}}} = \frac{c^2\beta}{c} = v_{\text{particle}}$$

The inverse proportionality between the group and phase velocity is rather remarkable because, among other nonsensical results, it obviously implies infinite phase velocities for $v = 0$.

Another nonsensical result is this: while accepting the rather outrageous implication of superluminal phase velocities, we also accept a wave *packet* that cannot represent any real-life particle because it quickly dissipates.

We, finally, have the issue of non-localization: who gets to decide on the relevant spatial *domain* of the matter-wave? We may not know where it is – roughly or *exactly* – but the idea of it being a particle implies that it must be *some*where, right?

De Broglie's matter-wave and Schrödinger's *zbw* charge

As mentioned above, we think de Broglie's intuition in regard to the wave nature of matter is, essentially, correct but that his *modeling* of it as a wave *packet* is not: modeling matter-particles as a simple two-dimensional oscillation in space does the trick. It is, however, a rather special oscillation: we interpret the real and imaginary part of the elementary wavefunction as the sine and cosine components of the orbital motion of a pointlike charge. The pointlike charge and its motion combine to explain Schrödinger's discovery of the *Zitterbewegung* of the electron, which Dirac so aptly describes in his Nobel Prize speech (see our quote above).

The de Broglie relation (p = $h/\lambda = \hbar\mathrm{k}$) is essentially correct too, but we think it has a precise *geometric* interpretation which one can only appreciate by noting that the linear momentum of a particle is a *vector* quantity. We should, therefore, also think of Planck's quantum as a vector quantity: ***h*** has a magnitude as well as a direction. We, therefore, re-write de Broglie's relation using boldface:

$$\mathbf{p} = \frac{\boldsymbol{h}}{\lambda}$$

In a similar vein, we think of the reduced Planck constant ($\boldsymbol{\hbar} = \boldsymbol{h}/2\pi$) as a proper angular momentum, which can and should be written as $\boldsymbol{\hbar} = I\cdot\boldsymbol{\omega}$: the product of an *angular mass* (the rotational inertia I) and an orbital angular frequency ($\boldsymbol{\omega}$). This, then, also gives meaning to the concept of *spin* (which is either up or down). This oscillatory motion then also generates a classical magnetic moment which – equally classically – will precess in an external electromagnetic field.

There is no uncertainty in this model **except for the uncertainty in regard to the initial *plane* of oscillation** (which is given by the *direction* of $\boldsymbol{\hbar}$ and $\boldsymbol{\omega}$). We have explored this model *ad nauseam*[13]

signal being transmitted by matter-waves which, individually, would travel faster than light.

[13] We have documented the rather gradual development of what we refer to as a fully-fledged *realist* interpretation on Phil Gibb's alternative science site (https://vixra.org/author/jean_louis_van_belle).

elsewhere so we should limit ourselves here to the bare essentials before we proceed to the main topic of our paper, which revolves around sensible wave equations and how the matter-wave may or may not move through a medium—think of the vacuum itself here or, in a more practical context, a atomic lattice, such as semiconductor material.

Indeed, we are vaguely thinking of this paper as the very first in a series focusing on the relevance of the ring current model of matter-particles for quantum computing and semiconductor engineering. The frustration is this: while we got some good feedback from other researchers – both academic as well as non-professional or *amateur* physicists (as we are) – we think the concept of a wave equation to model the properties of free space (or the *vacuum* as we know it) is rather limited: we think Maxwell's equations model these properties quite well and we, therefore, do not see the need to add some new wave equation modeling the properties of the vacuum in the context of matter-waves: if an electromagnetic wave cannot travel any faster than light, why would the (components of the) matter-wave be able to do so?

In contrast, the phase or group velocity of a matter-wave in a crystal lattice, a conductor or semiconductor would, obviously, correspond to the velocity of the signal which may or may not reach lightspeed and, hence, our reflections in this regard may, therefore, have an impact of how we think of signals and the processing thereof.

Likewise, analyzing the uncertainty in regard to the spin of an electron (*up* or *down*) as an uncertainty in the *actual* direction of the *actual* magnetic moment of an electron may impact on how we think about the difficulties related to *decoherence* and *quantum noise* – and the related quantum error correcting (QEC) so as to improve fault-tolerance in quantum computing.

However, let us not get ahead of ourselves here and present the basics of the *Zitterbewegung* or ring current model[14] of an electron before returning to our discussion of wave equations.

The ring current model of an electron

We request the reader to think of the (elementary) wavefunction $\boldsymbol{r} = \psi = a \cdot e^{i\theta}$ as representing the *physical* position of a pointlike elementary charge – pointlike but *not dimensionless*[15] – moving at the speed of light around the center of its motion in a space that is defined by the electron's Compton radius $a = \hbar/mc$. This radius – which effectively doubles up as the *amplitude* of the wavefunction – can easily be derived from (1) Einstein's mass-energy equivalence relation, (2) the Planck-Einstein relation,

[14] We use the terms ring current model and *Zitterbewegung* model interchangeably, although we are aware of various interpretations within interpretations here: our modeling of a two-dimensional oscillation amounts to a geometric or *physical* interpretation of Einstein's mass-energy equivalence relation (we like to think it models Wheeler's 'mass without mass' intuition), which sets it apart from the model of, say, David Hestenes. We must also refer to the amazing research of the likes of Prof. Dr. Alexander Burinskii on Dirac-Kerr-Newman geometries integrating gravity and/or Prof. Dr. Giorgio Vassallo on a wide ranging of applications of the ring current model. We, therefore, think of our simple model as being essentially didactic, heuristic, or complementary. Also, for the historical record, we should mention the ring current model goes back to 1915, or even earlier, when the British chemist and physicist Alfred Lauck Parson proposed such model as part of a 'magnetic model' of the electron. We refer to Oliver Consa (2020) for a brief overview and history here.

[15] We think the *non*-zero dimension of the elementary charge explains the small anomaly in the magnetic moment which should, therefore, not be thought as being anomalous. We must thank Prof. Dr. Randolf Pohl from the Max Planck Institute for Quantum Optics for, effectively, telling us to *not* think of the anomalous magnetic moment as an anomaly. For more details, see our paper on the electron model.

and (3) the formula for a tangential velocity, as shown below:

$$\left.\begin{array}{r}\left.\begin{array}{r}\mathrm{E}=\mathrm{m}c^2\\ \mathrm{E}=\hbar\omega\end{array}\right\}\Rightarrow \mathrm{m}c^2=\hbar\omega\\ c=a\omega \Leftrightarrow a=\dfrac{c}{\omega}\Leftrightarrow \omega=\dfrac{c}{a}\end{array}\right\}\Rightarrow \mathrm{m}a^2\omega^2=\hbar\omega \Longrightarrow \mathrm{m}\frac{c^2}{\omega^2}\omega^2=\hbar\frac{c}{a}\Leftrightarrow a=\frac{\hbar}{\mathrm{m}c}$$

This exceedingly simple derivation[16] gives us a *geometric* interpretation of Prof. Dr. Patrick R. LeClair's understanding of the Compton wavelength as "the scale above which the particle can be localized in a particle-like sense." [17] It effectively amounts to a very basic ring current model[18] which allows us to interpret the elementary wavefunction $\boldsymbol{r} = a\cdot e^{i\theta} = a\cdot e^{i\theta(\mathrm{E}\cdot t - \mathbf{p}\cdot\boldsymbol{x})}$ as the position vector of Dirac's pointlike electric charge.[19]

This *Zitterbewegung* (or ring current) electron may now move linearly and we may conveniently and without any loss on the generality of the argument choose our *x*-axis in our reference frame so it coincides with the direction of travel. We are then able to write its position *along the direction of linear motion* as a simple function of time itself: $x(t) = v\cdot t$. This combination of a pointlike charge *zittering* around some center and the oscillation as a whole them moving linearly is visualized below[20]: the *radius* of the circulatory motion must effectively diminish as the electron gains speed.[21]

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[16] It is a derivation one can also use to derive a *theoretical* radius for the proton (or for any *elementary* particle, really). It works perfectly well for the muon, for example. However, for the proton, an additional assumption in regard to the proton's angular momentum and magnetic moment is needed to ensure it fits the experimentally established radius. We shared the derivation with Prof. Dr. Randolf Pohl and the PRad team but we did not receive any substantial comments so far, except for the PRad spokesman (Prof. Dr. Ashot Gasparan) confirming the Standard Model does not have any explanation for the proton radius from first principles and, therefore, encouraging us to continue our theoretical research. In contrast, Prof. Dr. Randolf Pohl suggested the concise calculations come across as numerological only. We hope this paper might help to make him change his mind!

[17] Prof. Dr. Patrick LeClair, *Introduction to Modern Physics*, Course Notes (PH253), 3 February 2019, p. 10. Also see our *physical* interpretation of Compton scattering in our previous paper(s).

[18] In case the reader wonders what other ring current model might be on the current 'market of ideas', we are very much intrigued by electron models based on Dirac-Kerr-Newman geometries (A. Burinskii, 2018, 2019 and 2020) because they may answer the very fundamental question he asked us when we first submitted our thoughts to Prof. Dr. Alexander Burinskii: "I know many people who considered the electron as a toroidal photon and do it up to now. I also started from this model about 1969 and published an article in JETP in 1974 on it: "Microgeons with spin". [However] There was [also] this key problem: *what keeps [the pointlike charge] in its circular orbit?*" (Email from Dr. Burinskii to the author dated 22 December 2018)

[19] We must urge the reader who is not familiar with the model to clearly distinguish the idea of a charge (which we think of as being pointlike and having *zero rest mass*) and the electron (which we think of as a *charge in perpetual motion*). This combines the wave- as well as particle-like properties of a wavicle: the rest mass of the electron consists of the *relativistic* mass of the pointlike charge (which it acquires because of its lightning-like motion) and the equivalent energy of the (electromagnetic) oscillation which it generates and keeps it going.

[20] We thank Prof. Dr. Giorgio Vassallo and his publisher to let us re-use this diagram. It originally appeared in an article by Francesco Celani, Giorgio Vassallo and Antonino Di Tommaso (*Maxwell's equations and Occam's Razor*, November 2017).

[21] However, we urge the reader to imagine imagine the plane of oscillation to rotate or oscillate itself in line with our interpretation of the angular momentum vector $\boldsymbol{\hbar}$ as a vector rather than a simple scalar quantity. Also, when thinking of the electron moving linearly in an electromagnetic field, we will also have precessional motion.

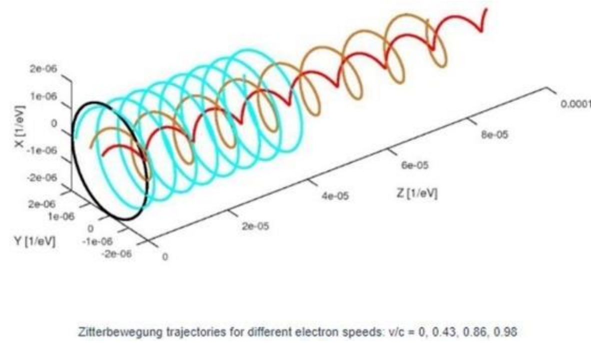

Figure 1: The Compton radius must decrease with increasing velocity

We must, of course, immediately urge the reader to imagine the plane of oscillation to rotate or oscillate itself in line with our interpretation of the angular momentum vector ***ħ*** as a vector rather than a simple scalar quantity. Also, when thinking of the electron moving linearly in an electromagnetic field, its precessional motion will yield a very different trajectory than the *Archimedes screw* or *helical motion* which is illustrated above. However, this does not much of an impact on the geometric interpretation of the *de Broglie* wavelength which we advanced in previous paper(s) but which we will not repeat here.[22]

To conclude our remarks on **Figure 1**, we just want to draw the attention of our reader to what happens when the classical (linear) velocity of the electron nears lightspeed: while its rotational motion and, therefore, its angular momentum does not vanish[23], we can see that the circumference of the oscillation – which is nothing but the (circular) Compton wavelength – turns into a *linear* wavelength in the process![24] This rather remarkable geometric property relates the ring current model electron model with our photon model, which we will not talk about either here, however.[25] All we want to do before getting into the meat of the matter of this paper – which is the subject of wave equations in atomic lattices – is to highlight the relativistic invariance of the argument of the wavefunction in this geometric interpretation of it.

We can, effectively, denote the position and time *in the reference frame of the electron itself* by x' and t'. Of course, the position of the electron particle in its own reference frame will be equal to $x'(t') = 0$ for all t', and the position and time in the two reference frames will be related by Lorentz's equations[26]:

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{vt - vt}{\sqrt{1 - \frac{v^2}{c^2}}} = 0$$

[22] See earlier references.

[23] Its radius does diminish, however, and actually goes to zero: this should be compatible with relativity theory. However, we did not yet have the time to work out the math in this regard.

[24] We may, therefore, think of the Compton wavelength as a *circular* wavelength: it is the length of a circumference rather than a linear feature!

[25] The reason is the same: we do not want to repeat text from previous papers. If interested, the reader can look up our paper on *Relativity, Light and Photons*.

[26] These are the Lorentz equations in their simplest form. We may refer the reader to any textbook here but, as usual, we like Feynman's lecture on it (chapters 15, 16 and 17 of the first volume of Feynman's *Lectures on Physics*).

$$t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Hence, if we denote the energy and the momentum of the electron in our reference frame as E_v and p = $\gamma m_0 v$, then the argument of the (elementary) wavefunction $a \cdot e^{i\theta}$ can be re-written as follows[27]:

$$\theta = \frac{1}{\hbar}(E_v t - px) = \frac{1}{\hbar}\left(\frac{E_0}{\sqrt{1-\frac{v^2}{c^2}}}t - \frac{E_0 v}{c^2\sqrt{1-\frac{v^2}{c^2}}}x\right) = \frac{1}{\hbar}E_0\left(\frac{t}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{\frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}\right) = \frac{E_0}{\hbar}t'$$

E_0 is, obviously, the rest energy and, because p' = 0 in the reference frame of the electron, the argument of the wavefunction effectively reduces to $E_0 t'/\hbar$ in the reference frame of the electron itself. The reader may also note that, besides proving that **the argument of the wavefunction is relativistically invariant**, we also demonstrated the relativistic invariance of the Planck-Einstein relation.[28] in the process! This is why we feel that the argument of the wavefunction (and the wavefunction itself) is more *real* – in a *physical* sense – than the various wave equations (Schrödinger, Dirac, or Klein-Gordon) for which it is some solution.

Having said that, wave equations is what we want to talk about it here, so let us get on with it.

Schrödinger's wave equation in free space

We quoted Dirac and his wave equation rather extensively above and, hence, the reader may wonder why we do not start with Dirac's own wave equation. The reason is this: at the occasion of the 1948 Solvay Conference, Paul Dirac himself admits his wave equation may not do the trick. Indeed, we may usefully quote the following remarks, which he makes as he tries to challenge Robert Oppenheimer's presentation of the use of perturbation theory in quantum mechanics:

> "All the infinities that are continually bothering us arise when we use a perturbation method, when we try to expand the solution of the wave equation as a power series in the electron charge. Suppose we look at the equations without using a perturbation method, then there is no reason to believe that infinities would occur. The problem, to solve the equations without using perturbation methods, is of course very difficult mathematically, but it can be done in some simple cases. For example, for a single electron by itself one can work out very easily the solutions without using perturbation methods and one gets solutions without infinities. I think it is true also for several electrons, and probably it is true generally : we would not get infinities if we solve the wave equations without using a perturbation method. However, if we look at the solutions which we obtain in this way, we meet another difficulty: namely *we have the run-away electrons appearing*. Most of the terms in our wave functions will correspond to electrons which

[27] One can use either the general E = mc^2 or – if we would want to make it look somewhat fancier – the p*c* = E*v*/*c* relation. The reader can verify they amount to the same.

[28] The relativistic invariance of the Planck-Einstein relation emerges from other problems, of course! The *added value* of the model here is the *geometric* interpretation of it: the Planck-Einstein relation models the *integrity* of the idea of a particle!

are running away[29], in the sense we have discussed yesterday. They can, therefore, not correspond to anything physical. Thus nearly all the terms in the wave functions have to be discarded, according to present ideas. *Only a small part of the wave function has a physical meaning.*"[30]

In our interpretation of matter-particles, this small part of the wavefunction is, of course, the real electron, and it is the ring current or *Zitterbewegung* electron: it is the trivial solution that Schrödinger had found, and which Dirac himself mentioned in his Nobel Prize lecture from which we quoted above. The other part of the solution(s) are, effectively, bizarre oscillations which Dirac here refers to as 'run-away electrons'.

With the benefit of hindsight, one wonders why Dirac did not see what we see now.[31] In any case, we may come back to Dirac's equation later. For the time being, we want to start where it all started[32], and so that's Schrödinger's equation. Feynman derives it – *without* the term for the electrostatic potential around a positively charged nucleus[33]:

$$\frac{\partial \psi}{\partial t} = i\frac{\hbar}{2m_{eff}}\nabla^2\psi = i\frac{\hbar}{m}\nabla^2\psi$$

What is m_{eff}? It is the concept of the *effective* mass of an electron which, in our ring current model, corresponds to the relativistic mass of the electric charge as it *zitters* around at lightspeed and so we can effectively substitute $2m_{eff}$ for the mass of the electron $m = m_e = 2m_{eff}$.[34]

So far, so good. The question now is: are we talking one wave or many waves? A wave packet or the elementary wavefunction? Let us first make the analysis for one wave only, assuming that we can

[29] This corresponds to wavefunctions dissipating away. As we noted, the problem is that the matter-particles they purport to describe obviously do *not* dissipate away.

[30] This is our translation from French: Dirac must have made his remarks in his native language (English) but we were not able to find these. See pp. 282-283 of the report of the 1948 Solvay Conference, *Discussion du rapport de Mr. Oppenheimer.*

[31] One of our correspondents wrote us this: "Remember these scientists did not have all that much to work with. Their experiments were imprecise – as measured by today's standards – and tried to guess what is at work. Even my physics professor in 1979 believed Schrödinger's equation yielded the *exact* solution (electron orbitals) for hydrogen." Hence, perhaps we should not be surprised. However, in light of the intellectual caliber of these men, we are.

[32] We probably Richard Feynman too much, but his hyperbole usually does reflect mainstream perceptions, so let us quote him once more: "Schrödinger's equation was the first quantum-mechanical equation ever known. It was written down by Schrödinger before any of the other quantum equations we have described in this book were discovered. Although we have approached the subject along a completely different route [Feynman refers to the derivation of the Schrödinger's equation wave equation in an atomic lattice here, so that is Schrödinger's equation *without* the Coulomb term], the great historical moment marking the birth of the quantum mechanical description of matter occurred when Schrödinger first wrote down his equation in 1926."

[33] For Schrödinger's equation in free space or the same equation with the Coulomb potential see Chapters 16 and 19 of Feynman's *Lectures* on Quantum Mechanics respectively. Note that we moved the imaginary unit to the right-hand side, as a result of which the usual *minus* sign disappears: $1/i = -i$.

[34] For a derivation of the $m = 2m_{eff}$ formula, we refer the reader to our paper on the ring current model of an electron, where we write the effective mass as $m_{eff} = m_\gamma$. The *gamma* symbol (γ) refers to the photon-like character of the charge as it zips around some center at lightspeed. However, unlike a photon, a charge carries charge. Photons do not. Hence, we agree – with Burinskii – that one should not use terms such as toroidal photon to refer to the pointlike *Zitterbewegung* charge even if its rest mass – just like that of a photon – is equal to zero, which is why it (also) whizzes around at lightspeed.

effectively write ψ as the *elementary* wavefunction $\psi = a \cdot e^{i\theta} = a \cdot e^{i \cdot (k(x-\omega t)}$.

Now, two complex numbers a + *i*·b and c + *i*·d are equal if, and *only* if, their real and imaginary parts are the same, and the $\partial\psi/\partial t = i \cdot (\hbar/m) \cdot \nabla^2\psi$ equation amounts to writing something like this: a + *i*·b = *i*·(c + *i*·d). Remembering that $i^2 = -1$, you can then easily figure out that $i \cdot (c + i \cdot d) = i \cdot c + i^2 \cdot d = -d + i \cdot c$. The $\partial\psi/\partial t = i \cdot (\hbar/m) \cdot \nabla^2\psi$ wave equation therefore corresponds to the following *set* of equations[35]:

- $Re(\partial\psi/\partial t) = -(\hbar/m) \cdot Im(\nabla^2\psi) \Leftrightarrow \omega \cdot \cos(kx - \omega t) = k^2 \cdot (\hbar/m) \cdot \cos(kx - \omega t)$
- $Im(\partial\psi/\partial t) = (\hbar/m) \cdot Re(\nabla^2\psi) \Leftrightarrow \omega \cdot \sin(kx - \omega t) = k^2 \cdot (\hbar/m) \cdot \sin(kx - \omega t)$

It is, therefore, easy to see that ω and k must be related through the following relation[36]:

$$\omega = \frac{\hbar k^2}{m} = \frac{\hbar c^2 k^2}{E}$$

We can easily verify this makes perfect sense if we substitute the energy E using the Planck-Einstein relation $E = \hbar \cdot \omega$ and assuming the wave velocity is equal to *c*, which should be the case if we are talking about the same vacuum as the one through which Maxwell's electromagnetic waves are supposed to be traveling[37]:

$$\omega = \frac{\hbar k^2}{m} = \frac{\hbar c^2 k^2}{E} = \frac{\hbar c^2 k^2}{\hbar\omega} = \frac{c^2 k^2}{\omega} \Leftrightarrow \frac{\omega^2}{k^2} = \frac{(2\pi f)^2}{(2\pi/\lambda)^2} = (f\lambda)^2 = c^2 \Leftrightarrow c = f\lambda$$

We know need to think about the question we started out with: one wave or many component waves? It is fairly obvious that if we think of many component waves, each with their own frequency, then we need to think about different values m_i or E_i for the mass and/or energy of the electron as well! How can we motivate or justify this? The electron mass or energy is known, isn't it? This is where the *uncertainty* comes in: the electron may have some (classical) velocity or momentum for which we may not have a *definite* value. If so, we may assume different values for its (kinetic) energy and/or its (linear) momentum may be possible. We then effectively get various *possible* values for m, E, and p which we may denote as m_i, E_i and p_i, respectively. We can, then, effectively write our dispersion relation and, importantly, *the condition for it to make physical sense* as:

$$\omega_i = \frac{\hbar k^2}{m_i} = \frac{\hbar c^2 k_i^2}{E_i} = \frac{\hbar c^2 k_i^2}{E_i} = \frac{\hbar c^2 k_i^2}{\hbar\omega_i} = \frac{c^2 k_i^2}{\omega_i} \Leftrightarrow \frac{\omega_i^2}{k_i^2} = c^2 \Leftrightarrow c = f_i\lambda_i$$

Of course, the $c = f_i\lambda_i$ makes *a lot of* sense: we would not want the properties of the medium in which matter-particles move to be different from the medium through which electromagnetic waves are travelling: lightspeed should remain lightspeed, and waves – *matter*-waves included – should not be

[35] We invite the reader to double-check our calculations. If needed, we provide some more detail in one of our physics blog posts on the geometry of the wavefunction.

[36] If you *google* this (check out the Wikipedia article on the dispersion relation, for example), you will find this relation is referred to as a *non-relativistic* limit of a supposedly relativistically correct dispersion relation, and the various authors of such accounts will usually also add the 1/2 factor because they conveniently (but *wrongly*) forget to distinguish between the *effective* mass of the *Zitterbewegung* charge and the total energy or mass of the electron as a whole.

[37] We apologize if this sounds slightly ironic but we are actually astonished Louis de Broglie does not mind having to assume superluminal speeds for wave velocities, even if it is for *phase* rather than group velocities.

traveling faster. Let us quickly sum up our key conclusions so far:

1. If there is a matter-wave, then it must travel at the speed of light and *not*, as Louis de Broglie suggests, at some superluminal velocity.
2. If the matter-wave is a wave *packet* rather than a single wave with a precisely defined frequency and wavelength, then such wave packet will represent *our limited knowledge* about the momentum and/or the velocity of the electron. The uncertainty is, therefore, not inherent to Nature, but to our limited *knowledge* about the initial conditions.

Let us now look at Schrödinger's wave equation in the context of a crystal lattice. This should be easy enough because Feynman actually derives Schrödinger's wave equation *in the very same context*. He – and all academics who produced more recent textbooks based on his – then just substitutes the efficient mass (m_{eff}) for m_e rather than by $m_e/2$ noting, *without any explanation at all*, that the effective mass of an electron becomes the free-space mass of an electron outside of the lattice:

$$\frac{\partial\psi}{\partial t} = i\frac{\hbar}{2m_{eff}}\nabla^2\psi = i\frac{\hbar}{2m}\nabla^2\psi$$

In fact, it is a happy or not-so-happy coincidence that this 1/2 *mistake* is not noticed when adding the electrostatic potential term so as to model an atom because – as we know – the solutions to Schrödinger's equation for electron orbitals are actually solutions for orbitals with *two* electrons, so that is what then take care of the ½ factor: the only reason why Schrödinger's equation for an atom – with the electrostatic potential and the ½ factor works is because it models electron orbitals for *two* electrons. In fact, that the substitution of the *effective* mass (m_{eff}) by the *total* mass of the electron (m_e) is rather nonsensical also follows from Feynman's casual remark in another chapter which uses the same concepts to arrive at the same equation:

> "Don't forget that m_{eff} has nothing to do with the real mass of an electron. It may be quite different—although in commonly used metals and semiconductors it often happens to turn out to be the same general order of magnitude, about 0.1 to 30 times the free-space mass of the electron."[38]

In fact, we may usefully note here that, in the original 1963 print edition we bought long time ago, the general order of magnitude is actually mentioned as being equal to "about 2 to 20 times" the mass of an electron. We think we should understand all of the above as follows: the effective mass (m_{eff}) can be (almost) any ratio or multiple of the total mass (m_e) but m_{eff} being equal to m_e, *exactly*, is the exception rather than the rule! To be precise, the exception is this: the ratio between the effective mass and the total mass is actually ½ in free space – so we should actually write Schrödinger's equation in free space as $\frac{\partial\psi}{\partial t} = i\frac{\hbar}{m}\nabla^2\psi$. However, because the equation – in the context of electron orbitals – models the orbitals for *two* electrons, the ½ and 2 factor cancel each other out, and we get the Schrödinger equation we all memorized[39]:

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[38] Feynman's *Lectures*, Vol. III, Chapter 13, Section 3, *Time-dependent states*.

[39] The equation is usually written using the V(***r***) = –e²/*r*, with e the electron charge and *r* the distance between the proton and the (two) electron(s). These remarks should allow the reader to relate our formulation to the rendering he or she is used to.

$$\frac{\partial\psi}{\partial t} = i(\frac{\hbar}{2m}\nabla^2 + \frac{e^2}{\hbar r})\psi$$

There is another accident in the history of ideas which we must mention here—and one that is *definitely* unhappy: for some reason I, once again, cannot understand, Feynman follows de Broglie and earlier theorists in associating the same ½ factor in Schrödinger's equation with the ½ factor in the non-relativistic KE = mv^2/2 formula for kinetic energy and says we need a different wave equation, which is the Klein-Gordon equation.

The Klein-Gordon relation is developed in pretty much the same way as Schrödinger's equation: a *heuristic* argument is used but this time it is the relativistically correct energy-momentum relation which, combining the E = mc^2 (Planck-Einstein) and p = *h*/λ (*de Broglie*) relations, serves as the dispersion relation—as opposed to the much more intuitive $\omega = \hbar k^2/m$ relation. We already showed this wave equation just gives us dissipating or 'run-away' electrons too.[40]

In addition, our ring current model challenges the assumption that Schrödinger's equation is *not* relativistically correct: we associate the 1/2 factor with the (relativistic) effective mass of the pointlike charge: the ½ factor is, therefore, *entirely* relativistic!

Schrödinger's wave equation in a crystal lattice

Feynman writes the following about his quantum-mechanical derivation of Schrödinger's equation in crystal lattice:

> "We do not intend to have you think we have derived the Schrödinger equation but only wish to show you one way of thinking about it. When Schrödinger first wrote it down, he gave a kind of derivation based on some heuristic arguments and some brilliant intuitive guesses. Some of the arguments he used were even false, but that does not matter; the only important thing is that the ultimate equation gives a correct description of nature."[41]

Unfortunately, after playing with it for a while, he then discards it for the wrong reason:

> "In principle, Schrödinger's equation is capable of explaining all atomic phenomena except those involving magnetism and relativity. […] The Schrödinger equation as we have written it does not take into account any magnetic effects. It is possible to take such effects into account in an approximate way by adding some more terms to the equation. However, as we have seen in Volume II, magnetism is essentially a relativistic effect, and so a correct description of the motion of an electron in an arbitrary electromagnetic field can only be discussed in a proper relativistic equation. The correct relativistic equation for the motion of an electron was discovered by Dirac a year after Schrödinger brought forth his equation, and takes on quite a different form. We will not be able to discuss it at all here."[42]

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[40] We showed how one gets a dispersion relation out of the energy-momentum relation. We will not talk about the Klein-Gordon relation here as we (briefly) talked about it in our previous paper.

[41] *Lectures*, Vol. III, Chapter 16, p. 16-4.

[42] *Lectures*, Vol. III, Chapter 16, p. 16-13. We have re-read Feynman's *Lectures* many times now and, in discussions with fellow amateur physicists, we sometimes joke that Feynman must have had a secret copy of the truth. He clearly doesn't bother to develop Dirac's equation because – having worked with Robert Oppenheimer on the Manhattan project – he knew Dirac's

However, as we note above, Dirac's equation is actually *not* correct (its solutions are dissipating wave packets: 'run-away electrons'), and the ½ factor in Schrödinger's equation may be relativistically correct because it is related to the ½ factor in the $m_{eff} = m_e/2$ which relates the effective mass to the total mass in the ring current model of an electron! In fact, we may quickly insert the Klein-Gordon wave equation here which, according to Feynman, corrects Schrödinger's supposedly non-relativistically correct wave equation[43]:

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \frac{m^2c^2}{h^2}\psi = \nabla^2\psi$$

As explained above, we have no use for it because its solutions – any quantum-mechanical wave *packet* that comes out of it as a solution – dissipate away. Actual matter-particles do not. We do not adhere to mainstream wisdom here, according to which this equation incorporates a relativistically correct dispersion relation: superluminal wave velocities – even *phase* velocities – are effectively *not* compatible with relativity theory! Let us, therefore, explore Feynman's derivation of Schrödinger's equation – the very first wave equation in the history of quantum mechanics! – in all of its detail.

In case you wonder, we do not do this to show off Feynman's brilliance (we assume no one needs any convincing in this regard) but to get a better understanding of what might be real and what isn't and – even more importantly – to better understand the difference between a matter-wave, a signal, an amplitude and the electron itself, as per the title of this paper.

You will, of course, understand our analysis will be as *heuristic* as Feynman's because we have a clear objective in mind: we want to show the moving electron is the moving electron, and the amplitude is the signal. Hence, we request the reader to be very critical but constructive: when everything is said and done, a probability must represent something, and if it isn't the electron, then it should be the signal. What other common-sense candidates for *reality* are there? Let us, so as to focus our mind on what is what, copy Feynman's illustration of – paraphrasing Wittgenstein here[44] – whatever it is that might be

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equation only produces non-sensical 'run-away electrons'. In contrast, while noting Schrödinger's equation is non-relativistic, it is the only one he bothers to explore extensively. Indeed, while claiming the Klein-Gordon equation is the 'right one', he hardly devotes any space to it.

[43] See: Richard Feynman, *Waves in three dimensions*, *Lectures*, Vol. I, Chapter 48. With the usual hyperbole, Feynman refers to it as the 'grand equation, which corresponds to the dispersion equation for quantum-mechanical waves.' In fact, because his students are – at that point – not yet familiar with differential calculus for *vector* fields (and, therefore, not with the *Laplacian* operator ∇^2), Feynman writes it there like this:

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \frac{m^2c^2}{\hbar^2}\phi$$

We are not ashamed to admit Feynman's early introduction of this equation in his three volumes of lectures on physics which, as he clearly states in his preface, were written "to maintain the interest [in physics] of the very enthusiastic and rather smart students coming out of the high schools" did not miss their effect on us: I wrote this equation on a piece of paper on the backside of my toilet of my student room when getting my first degree (in economics) and vowed that, one day, I would understand this equation "the way I would like to understand it." We now understand it to *not* represent anything real: I now understand it to model *our* uncertainty (our lack of knowledge about the past or initial conditions) rather than any fundamental uncertainty in Nature!

[44] The first proposition of Wittgenstein's *Tractatus Logico-Philosophicus* (originally published in his native German language as the *Logisch-philosophische Abhandlung*) is this: *"Die Welt ist alles, was der Fall ist"* ("The world is all that is the case") and it is usually interpreted as Wittgenstein's definition of reality. One has to start somewhere, isn't it?

the case (**Figure 2**).

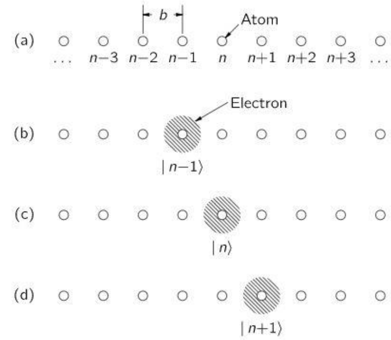

Figure 2: Feynman's idea of a moving electron in an atomic lattice

We should note, first, that this is a very classical picture: Feynman talks quantum mechanics but, to focus the mind, he pictures an electron as always being somewhere, and this *some*where is near an atom. In Feynman's words[45]:

> "We want to see what happens if we put a single electron on this line of atoms. Of course, in a real crystal there are already millions of electrons. But most of them (nearly all for an insulating crystal) take up positions in some pattern of motion each around its own atom—and everything is quite stationary. However, we now want to think about what happens if we put an extra electron in. We will not consider what the other ones are doing because we suppose that to change their motion involves a lot of excitation energy. We are going to add an electron as if to produce one slightly bound negative ion. In watching what the one extra electron does we are making an approximation which disregards the mechanics of the inside workings of the atoms. Of course the electron could then move to another atom, transferring the negative ion to another place. We will suppose that just as in the case of an electron jumping between two protons, the electron can jump from one atom to the neighbor on either side with a certain amplitude."

At this point, we may already want to distinguish between the motion of *the* electron, and the *possible* motion of a whole *series* of electrons, as illustrated in **Figure 3**. This illustration is probably not our best drawing ever but the reader will understand the idea: think of the musical chairs game but with the chairs on a line and all players agreeing to kindly move to the next chair for the new arrival and – importantly – the last person on the last chair agreeing to leave the game to get a beer.

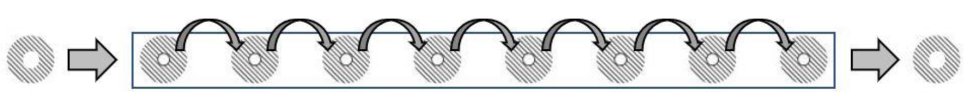

Figure 3: The idea of a moving electric signal in an atomic lattice

What happens in such game of musical chairs is the transmission of a *signal*: while each person needs some time to get up – say Δ*t* – and move to the next chair, an orchestra leader could, perhaps, make

[45] We will (mostly) quote from Chapter 13 (*Propagation in a Crystal Lattice*) as well as Chapter 16 (*Dependence of Amplitudes on Position*) in Feynman's *Lectures on Quantum Mechanics* here. Chapter 13 focuses on the wavefunction, whereas Chapter 16 focuses more on the wave equation that might generate them as solutions.

them all move at the same time and, hence, while each person would move only one chair, the transmission of the *signal* could be lightning fast.[46]

The difference between a traveling electric charge and a traveling electric signal

What we explained above is, of course, the idea of an electron *current*: we know the typical *drift* speed of a free electron is actually quite low – it collides with many other charged particles (electron shells or other free electrons) – but the *electric signal* travels much faster. Why? When charged particles are forced into the wire – or any conductor really – an equal number are forced to leave because of the repulsion between like charges. That makes it actually difficult to increase the number of charges in a volume. Even if one charge enters, another leaves almost immediately, carrying the signal rapidly forward.[47]

The question is *how* fast, *exactly*? We checked but find that textbooks will agree that the *order of magnitude* is the same as that of light but seem to shy away from actually calculating signal speeds. One course compares signal and charge *drift* velocity as follows: "Electrical signals are known to move very rapidly. Telephone conversations carried by currents in wires cover large distances without noticeable delays. Lights come on as soon as a switch is flicked. Most electrical signals carried by currents travel at speeds on the order of 10^8 m/s, a significant *fraction* of the speed of light. Interestingly, the individual charges that make up the current move much more slowly on average, typically drifting at speeds on the order of 10^{-4} m/s."[48]

We may rephrase the question: if the velocity of an electromagnetic field in free space is equal to *c*, then at what *fraction* of *c* can electric *signals* travel in a conductor, a semiconductor or whatever lattice-like structure? Lightspeed itself? Slower? Faster? Let us get back to Feynman's analysis of quantum-mechanical amplitudes and the associated wave equation(s) to see whether or not it may help to answer that question. The modeling is this[49]:

1. The assumption is that the *amplitude* for an electron to go from one atom to the next is given by $iA/\hbar$ *per unit time*. Such assumption must, of course, assume that the electron that goes from one atom to the next starts out at the first atom which must, therefore, have a different energy. We, therefore, know – we do so because all of us are (or should be) familiar with the analysis of two-state systems in

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[46] We leave the interpretation of the signal to the reader. To most, it will be this: a new person has arrived so you should move up. To the last person, however, the message is this: go and get a beer. 😊

[47] It is quite astonishing but Feynman actually makes the case that a current-carrying wire is actually neutral: it only *appears* charged when the charges inside are set in motion. See Feynman's lecture on *the relativity of magnetic and electric fields*.

[48] We simply *googled* and this course (from a private course provider which we must reference and/or credit here) was just one of the many textbooks to pop up.

[49] We stick closely to Feynman's presentation here but, at the same time, we want to shorten and we do have our own nuances in the interpretation, so the reader should check with the original. We will try to limit our additions and critical remarks to footnotes.

quantum mechanics[50] – that A must be related to some energy *difference*.[51]

Feynman does not give any explanation here but we may want to think, for example, of the extra electron at atom n as occupying some orbital and its angular momentum will, therefore, be equal to some multiple of $\hbar$. Such *visualization* or modeling has the advantage that it allows us to effectively interpret $A/\hbar$ as some energy per unit time.[52]

The reader should also note that the amplitude is constant: it, therefore, increases linearly with time. Likewise, the associated *probability* (the squared absolute value) will also *not* increase exponentially but linearly.[53] So far so good.

One obvious question might be this: do Feynman allow the electron to go back? Feynman does not elaborate here but the fact of the matter is that he does. If $iA/\hbar$ is the amplitude to go from atom n to atom n+1, then its (complex) conjugate, $-iA/\hbar$ is the amplitude to go from atom n to atom n–1. Feynman also admits the wave that is the solution to this problem – which we will get in a moment – "can travel toward positive or negative x depending on the sign we have picked for k." In fact, the question runs much deeper than this but we do not want to confuse the discussion here by getting into the nitty-gritty of it.[54] One should just note that the *probabilities* that are associated with $iA/\hbar$ and $-iA/\hbar$ (and with the *real* numbers $A/\hbar$ and $-A/\hbar$) are all the same: $|iA/\hbar|^2 = |-iA/\hbar|^2 = |A/\hbar|^2 = |-A/\hbar|^2 =$

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[50] We note modern textbooks do often *not* start out with that. This is yet another reason why we prefer Feynman's lectures on quantum mechanics over more recent material: we like the logical progression from analyzing two-state system to then go to n-state systems (using matrix mechanics) to then, in the final chapters, go from discrete to continuous states. This provides for an easy flow from matrix to wave mechanics and shows the connections between both. We speak from experience here as our son was studying one of such recent textbooks and so we got a taste of how quantum mechanics is being taught nowadays. We also looked at the MIT's edX course on quantum mechanics, which is wonderful but also jumps rather straight into mathematical analyses that make it difficult to appreciate how ideas and principles were arrived at historically.

[51] However, the reader should this system is not quite the same as a simple two-state system, in which we the two energy levels can be written as $E_0 + A$ and $E_0 - A$ states. The *difference* between the two energy levels in a two-state system is, therefore, $2A$, while the difference between these two states and the *zero* state E_0 is equal to A.

[5252] See Chapter VII (*The wavefunction and the atom*) of our book on a realist interpretation of quantum mechanics, in which we assume the n^{th} orbital (this n has, obviously, nothing to do with the *atom* number n in our lattice) to pack an angular momentum that is equal to $n\cdot\hbar$. It is then easy to show – at least for a simple hydrogen atom – that this orbital will be associated with an energy that is equal to $E_n = -\frac{1}{n^2}E_R$ and an angular momentum that will be equal to $L_n = I \cdot \omega_n = n\hbar$. The Planck-Einstein relation – written as $E_n = n\cdot\hbar\cdot\omega_n$ when modeling Bohr orbitals – then tells us the (angular) *period* or *cycle time* will be equal to $\frac{T_n}{2\pi} = \frac{1}{2\pi f_n} = \frac{1}{\omega_n} = \frac{n\hbar}{E_n}$. We can, therefore, see that we may now think of equating $A/\hbar$ to $A/\hbar = E_n/n\cdot\hbar \Leftrightarrow A = E_n/n$ and that $\frac{A}{\hbar} = \frac{E_n}{n\hbar} = \frac{2\pi}{T_n} = \omega_n$ can effectively be interpreted as an *energy per unit time*. This unit time corresponds logically to an *angular* period. An angular period ($T_n/2\pi$) is the cycle time that goes with a *reduced* quantity which we get, in this case, from going from an analysis in terms of h (the physical *action* that is associated with an elementary cycle in Nature) to an analysis in terms of $\hbar$ (which is the *angular momentum* that corresponds to h).

[53] This amounts to saying the past has no influence on the future here: it is *not* because an electron has been sitting on top of an atom for a while now that there is a *greater* likelihood it will make the hop in the next time unit. That likelihood always stays the time *per time unit*. The reader can easily link this Bayesian *ex ante* and *ex post* statistics.

[54] In our paper on classical interpretations of quantum-mechanical concepts we argue that the *convention* that is used in regard to the plus or minus sign for the imaginary unit in the $ae^{-i\theta} = ae^{-i(E/\hbar)}$ also *fixes* the direction of time. Such convention is necessary because time goes in one direction only. Hence, when taking the complex conjugate of an wavefunction or an amplitude you perform a *mathematical* operation that amounts to time reversal but this does not correspond to a *real* possibility in physics: time must go in one direction only. In the mentioned paper we show this is essentially rooted in the idea of motion itself: if we would allow for time reversal, we would not be able to define well-defined equations of motion.

$A^2/\hbar^2$. The $A/\hbar$ number is, therefore, the essential quantity here and its *direction* (the imaginary unit in front of it) comes with the other conventions of quantum-mechanical modeling.

In short, all of this looks very reasonable and we should, therefore, move on and present the next assumption in Feynman's *narrative*.

2. The electron must be somewhere when it goes from one atom to another. Feynman writes the amplitude to on be on the n^{th} atom as C_n and suggests to the following general functional form for it:

$$C_n = a_n e^{-i\frac{E}{\hbar}t} = a(x_n)e^{-i\frac{E}{\hbar}t} = a(n \cdot b)e^{-i\frac{E}{\hbar}t}$$

This looks pretty good – the amplitude varies with time so as to capture our uncertainty – but why the subscript in the a_n coefficient? Should the *probability* (the absolute square of the amplitude) not be the same for every atom? No. If the electron is not anywhere near atom *n*, it will probably not get there in the next instant. Hence, these amplitudes depend one on another![55]

This looks fine too. Next assumption.

3. Feynman obviously wants to get some wavefunction for this moving electron. Hence, remembering a wavefunction usually looks like $e^{-i(\omega t - kx)}$, Feynman suggests the following *trial* solution for $a_n = a(x_n)$:

$$a(x_n) = e^{ikx_n}$$

This, then, gives us the wavefunction we had expected to see all along:

$$C_n = e^{ikx_n}e^{-i\frac{E}{\hbar}t} = e^{-i(\omega t - kx_n)}$$

We may actually wonder why he did not write it down immediately! In fact, Richard Feynman's argument is far more complicated than our summary of it above: he shows off by showing how one can associate a set of Hamiltonian equations with this system and then *buttresses* his argument by showing how the assumptions above fit in with that. However, we conveniently look at this as unnecessary luggage for the time being.

The question is this: we have some idea of what the ω frequency actually is, but what is k? It is an important question because it obviously answers one of the basic questions we started out with: what is the propagation speed of this wave? Needless to say, we will, perhaps, also want to think of it in terms of *de Broglie* relation and associate it with some idea of the *momentum* of our electron: p should, somehow, be equal to $p = \hbar \cdot k$ in quantum mechanics, right?

Maybe. Maybe not. Let us see what we can say and what we cannot say. We have the model now. Let us try to tackle all interpretational issues in the next section—if only to keep this rather long paper somewhat organized. Before we do so, we would like to note one thing here: Feynman gets a *single* wave. No wave *packet*. No uncertainty. Not in *Nature*, at least: once again, the uncertainty is only in our mind. 😊

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[55] As for the substitution of a_n, one can easily appreciate a_n is a function of *n* or, what amounts to the same, the position $x_n = n \cdot b$. The latter substitution assumes we choose the origin of our line at atom zero and then uses the *spacing* between atoms *b* to get the rather logical $x_n = n \cdot b$ equation.

The energy of an electron in a crystal lattice

Feynman starts by noting k must be a real number: if it were an imaginary number, say i·k', we'd get a *real* exponential function for the coefficient ($a_n = e^{-ik'}$) and depending on your convention for the ± sign for the imaginary unit for your wavefunction (or the ± sign for k'), the amplitudes would blow up or, else, die off. In addition, we have no coefficient in front of our $C_n = e^{-i(\omega t - kx)}$ wavefunction and its real and imaginary part, therefore, oscillate between –1 and +1. What more can we say about k? We should be able to relate it to the energy E, isn't it?

Correct, and that is where the model becomes more complicated. We joked about Feynman showing off his mathematical skills by developing a set of Hamiltonian equations but we actually need them. The amplitude to be in this or that state depends on the amplitude to be in that or this state. For a simple two-state system, we have two energy levels that are separated from the *zero* state E_0 by A. These energy levels can, therefore, be written as $E_0 + A$ and $E_0 - A$ and the energy *difference* between them is, therefore, equal to $2A$. The Hamiltonian equations for such two-state system are these[56]:

$$i\hbar\frac{dC_1}{dt} = E_0C_1 - AC_2$$
$$i\hbar\frac{dC_2}{dt} = E_0C_2 - AC_1$$

We know this modeling – with the imaginary unit (i) in these differential equations – leads to what we want to see: probabilities "sloshing back and forth" between the two states—an *oscillatory* solution rather than, say, what happens "when water leaks between two tanks", which will simply result in the water levels approaching each other (the comparison is Feynman's, of course). The reader will probably be familiar with the comparison with paired pendulums.

For this system – an electron moving through this linear array of atoms – we have a very large number of equations for a very large number of atoms. These equations look like this:

$$i\hbar\frac{dC_1(t)}{dt} = E_0C_n(t) - AC_{n+1}(t) - AC_{n-1}(t)$$

We refer the reader to Feynman's rather ingenious solution to this potentially infinite set of equations.[57] One of the grand results is the formula for the energy E, which is this:

$$\mathrm{E} = \mathrm{E}_0 - 2A\cos(\mathrm{k}b)$$

Figure 4 shows a graph of E as a function of k: E – E_0 will be equal to $2A$ if cos(kb) is equal to –1 and, hence, if kb = ± π ⇔ k = ± πb. The reader can also verify the E – E_0 = –$2A$ condition, which is fulfilled if k = 0.

[56] See, for example, Feynman's development of the set of Hamiltonian equations for an ammonia maser in Chapter 8 (*The Hamiltonian matrix*) of his *Lectures*.

[57] The reader may not want to imagine an *actually* infinite array. However, the reader will understand there are a lot of atoms in a lattice which, from a practical point of view, translates into something which, from an analytical point of view, Feynman refers to as 'rather petrifying', but that is why Feynman is so famous: he solves this, and very elegantly, of course! No small-angle approximations here!

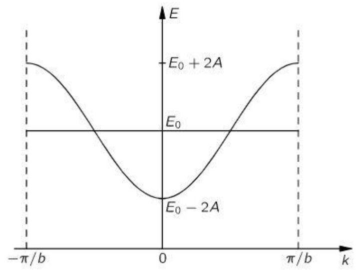

Figure 4: Energy E as a function of the parameter k

It is a very particular graph, but it will be familiar to most engineers: an electron in a crystal will always have an energy within this [$E_0 - 2A$, $E_0 + 2A$] *band* or, if we choose E_0 such that it is equal to $2A$, the [0, $2A$] band. No other energy is possible.

Let us now look at what we wanted to look at: wave velocities. Let us, to simplify things, effectively choose E_0 to be equal to $2A$, so our energy equation can be re-written as:

$$\mathrm{E} = 2A - 2A\cos \mathrm{k}b = 2A(1 - \cos \mathrm{k}b)$$

One single matter-wave or a wave *packet*?

We may now, once again, ask this question: should we think of a single wave or of a superposition of waves? Richard Feynman definitely thinks the motion of an electron should be represented by a wave *packet* and effectively calculates a *group* velocity using the familiar $v_{\text{group}} = d\omega/d\mathrm{k}$ equation. Let us go along with the argument for a while to see what does and does not make sense.

In order to proceed, Feynman needs to solve another small or not-so-small mathematical problem: how would one calculate $d\omega/d\mathrm{k}$ from the $\mathrm{E} = 2A(1 - \cos \mathrm{k}b)$ equation? Here, Feynman assumes we may use the small-angle approximation because, because of the very small space inbetween atoms, he assumes the amplitudes should only vary very little from one x_n to the next, which amounts to saying k must be very small.[58] In such case, we may, perhaps, use the small-angle approximation for the cosine ($\cos\theta = 1 - \theta^2/2$) and, therefore, write E as:

$$\mathrm{E} = 2A(1 - \cos \mathrm{k}b) = 2A\left(1 - 1 + \frac{\mathrm{k}^2 b^2}{2}\right) = A\mathrm{k}^2 b^2$$

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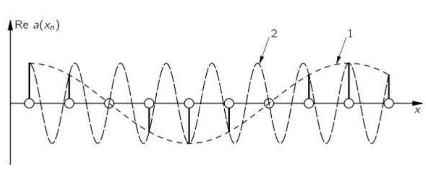
[58] That is a matter of scale, of course, and we, therefore, regret Feynman does not do a better job at motivating this assumption. In fact, it is worse than that: Feynman actually shows that, in any case, we may always replace a large k by a much smaller k by choosing a *ratio* between these two values that ensures the *physical* states that are being described are the same. The principle is illustrated below and we refer the reader to Feynman's explanation of it, which is easy enough but, in our not-so-humble view, *totally unacceptable!* Indeed, if k is related to some *real* variable (which is what Feynman actually attempts to demonstrate) such as, for example, the momentum or velocity of the charge, then one should not be allowed to do such substitutions! The reader should also note the quick transition from a precise *analytical* solution to a *make-do approximation* to ease calculations is not innocent. We think of it as very poor practice, *at best*—because the *simplification* is necessary to arrive at the desired result, which is this mystifying $p = m_{\text{eff}}v_{\text{group}} = \hbar\mathrm{k}$ relation.



Using the ω = E/ħ equation, we can now calculate wave velocities. Let us first calculate the *phase* velocity for a single wave:

$$v_{\text{phase}} = \frac{\omega}{\text{k}} = \frac{A\text{k}^2 b}{\hbar \text{k}} = \frac{A\text{k}b}{\hbar}$$

What can we do with this? Nothing much. We would need to relate *A*, somehow, to the energy of the electron or, to be precise, the *difference* in energies between an atom that has the extra electron and one that has not.[59] We would, of course, also need to find some *physical* interpretation of k.

At this point, Feynman defines the concept of the *effective* mass of an electron. However, we do not want to use this because:

1. It has no relation whatsoever with the concept of effective mass as used in the context of the ring current model, in which it is associated with the relativistic mass of the pointlike charge as it *zitters* around at lightspeed.
2. Feynman himself admits his concept of "effective mass has nothing to do with the real mass of an electron. It may be quite different—although in commonly used metals and semiconductors it often happens to turn out to be the same general order of magnitude, about 0.1 to 30 times the free-space mass of the electron."

What about the group velocity? We are not sure. We see no reason whatsoever to do what Feynman does, and that is to "make up a wave *packet* with a predominant wave number k_0, but with various other wave numbers near k_0." We can then, effectively, use the $E = Ak^2b^2$ as a dispersion relation and, therefore, calculate a group velocity which is equal to:

$$v_{\text{group}} = \frac{d\omega}{d\text{k}} = \frac{d(\frac{\text{Ak}^2\text{b}}{\hbar})}{d\text{k}} = \frac{2Ab^2}{\hbar}\text{k}$$

And, yes, we can then define a *totally artificial* concept of effective mass by *defining* it as $m_{\text{eff}} = \hbar^2/2Ab^2$ to write what Feynman apparently wants to write in some desperate effort to show the concept of a wave *packet* related to some kind of momentum concept might make sense:

$$\hbar\text{k} = \text{m}_{\text{eff}} v_{\text{group}} = \frac{\hbar^2}{2Ab^2} \cdot \frac{2Ab^2}{\hbar}\text{k} = \hbar\text{k}$$

Frankly, we do not see any use for it because of the above-mentioned reasons. In fact, we are not sure what use Feynman sees for it because he wraps up by summarizing what we have learned here as follows:

> "We have now explained a remarkable mystery—how an electron in a crystal (like an extra electron put into germanium) can ride right through the crystal and flow perfectly freely even though it has to hit all the atoms. It does so by having its amplitudes going pip-pip-pip from one atom to the next, working its way through the crystal. That is how a solid can conduct electricity."

[59] See our remarks in footnote 52.

Frankly, we knew that already, didn't we?

Moreover, we should note that even Feynman's *interpretation* of the equations he gets out of his must be plain wrong: an electron does *not* "ride right through the crystal", nor does it "flow perfectly freely even though it has to hit all the atoms." What does go "pip-pip-pip" from one atom to the next, is the *signal* and, yes, that is what these mysterious probability amplitudes actually model and/or how a solid can conduct electricity. Hence, we do *not* see any added value of Feynman's quasi-infinite set of Hamiltonian equations or of applying the *machinery* of quantum mechanics to this particular problem. Why not? We feel we just got a lot of mumbo jumbo resulting from unnecessary multiplication of concepts as opposed to a clear and unambiguous answer to the two basic questions we started out with:

1. What is the velocity of the electron itself?
2. What is the velocity of the signal?

Let us, therefore, wrap up this paper.

Conclusions

This paper basically further explored intuitions we highlighted in previous papers already:

1. The concept of the matter-wave traveling through the vacuum, an atomic lattice or any medium can be equated to the concept of an electric or electromagnetic *signal* traveling through the same medium.
2. There is no need to model the matter-wave as a wave packet: a single wave – with a precise frequency and a precise wavelength – will do.
3. If we do want to model the matter-wave as a wave *packet* rather than a single wave with a precisely defined frequency and wavelength, then the uncertainty in such wave packet reflects our own *limited knowledge* about the momentum and/or the velocity of the particle that we think we are representing. The uncertainty is, therefore, not inherent to Nature, but to our limited *knowledge* about the initial conditions.
4. The fact that such wave *packets* usually dissipate very rapidly, reflects that even our limited knowledge about initial conditions tends to become equally rapidly irrelevant. Indeed, as Feynman correctly notes, "the tiniest irregularities" tend to get magnified very quickly at the micro-scale.

All of this makes us agree with what Hendrik Antoon Lorentz stated a few months before his demise: there is no reason whatsoever "to elevate indeterminism to a philosophical principle."[60] We hope the added detail in this paper may finally convince our more skeptical readers.

Jean Louis Van Belle, 14 May 2020

[60] For more historical background and context, as well as the full translation of his remarks, we refer to our short briefing paper on the Solvay Conferences.