The Operative Set Theory with Application to Goldbach Conjecture

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Abstract

We defined the arithmetic operations on the integer sets. These new set operators bring the new properties and theorems to the integer sets. Therefore, we called the integer sets with these new arithmetic operators as the "operative set" and proved its properties and theorems, which are the building blocks of the "operative set theory". Then we used the operative set theory to prove the GOLDBACH CONJECTURE.

1 Introduction

The operative set is defined as the integer set with the arithmetic operators, i.e., set addition, set subtraction, set multiplication, and set division. An operative set possesses all properties of a set, and additionally it allows its elements to participate in the arithmetic operations, leading to the new operative sets. Because these arithmetic operators are new to the set concept, there are a series of interesting properties and theorems on the integer sets. For an intuitive expression of the integer set with new arithmetic operators, we called it as the "operative set" and denoted an operative set as a capital letter with bracket, e.g., $\big[\mathbb{A}\big]$, which can distinguish the operative set from the conventional set $\mathbb{A}$.

Let $[\mathbb{A}]$ and $[\mathbb{B}]$ denote two operative sets, and let $a_i \in [\mathbb{A}]$ and $b_j \in [\mathbb{B}]$, we define the operation $\odot$, such that there exist an operative set $[\mathbb{C}]$ where all elements in $[\mathbb{C}]$ result from the operation $a_i \odot b_j$ for every i and j. For example, let's set the operation $\odot$ as the summation $+$, then we have

$$\begin{aligned}
[1,2,3] + [7,9,10] &= [1+7, 1+9, 1+10, 2+7, 2+9, 2+10, 3+7, 3+9, 3+10] \\
&= [8, 10, 11, 9, 11, 12, 10, 12, 13] \\
&= [8, 9, 10, 11, 12, 13]
\end{aligned}$$

The elements of the sets used in this paper are integers. For ease of reading, we define the following symbols for these integer sets, where the sets are conventionally denoted with capital letters and the elements of the set are denoted with small letters. Without specific explanation, all numbers in this article are positive.

Symbol	Definition
$[\mathbb{O}]$	The set that includes all positive odd numbers; $o_i \in [\mathbb{O}]$ represents an odd number.
$[\mathbb{O}^c]$	The set that includes all odd composite numbers; $o_i^c \in [\mathbb{O}^c]$ represents an odd composite number.
$[\mathbb{P}]$	The set that includes all odd prime numbers; $p_i \in [\mathbb{P}]$ represents an odd prime number.
$[\mathbb{E}]$	The set that includes all even numbers; $e_i \in [\mathbb{E}]$ represents an even number.
$\left[\mathbb{X} \geqslant x_0\right]$	A subset of the set $[\mathbb{X}]$ that includes all numbers of $[\mathbb{X}]$ greater than x_0. It can be expressed as $\left[\mathbb{X} \geqslant x_0\right] = [x : x \in [\mathbb{X}] \text{ and } x \geqslant x_0]$. For example, $\left[\mathbb{E} \geqslant 6\right]$ represents the set that includes all even numbers greater than or equal to 6.
$\varnothing$	The empty set.

The definitions of the equality and inequality of the operative set are the same as those of the set. For example,

$$\begin{aligned}
[3, 5, 17, 9, 11] &= [3, 5, 9, 11, 17] \\
[3, 4, 5] &\neq [3, 7, 8] \\
[\mathbb{P}] \cup [\mathbb{O}^c] &= [\mathbb{O} \geqslant 3]
\end{aligned}$$

2 Addition of the operative sets

Given two operative sets $[\mathbb{A}] = [a_1, a_2, \cdots]$ and $[\mathbb{B}] = [b_1, b_2, \cdots]$, we define their addition operation as building a new set $[\mathbb{A}] + [\mathbb{B}]$, whose elements are the sum of every member of $[\mathbb{A}]$ and every member of $[\mathbb{B}]$. This can be expressed as

$$[\mathbb{A}] + [\mathbb{B}] = \Big[a_i + b_j : a_i \in [\mathbb{A}] \text{ and } b_j \in [\mathbb{B}]\Big]$$

Note that since $[\mathbb{A}] + [\mathbb{B}]$ is a set, those duplicated numbers in the set need to be uniquified so that the set $[\mathbb{A}] + [\mathbb{B}]$ complies with the fundamental requirement that each element in the set is unique. For example,

$$\begin{aligned}
[1, 2, 3] + [7, 9, 10] &= [1+7, 1+9, 1+10, 2+7, 2+9, 2+10, 3+7, 3+9, 3+10] \\
&= [8, 10, 11, 9, 11, 12, 10, 12, 13] \\
&= [8, 9, 10, 11, 12, 13]
\end{aligned}$$

We define the following properties for the set addition operation:

(1) The operands can be empty sets. We define $[\mathbb{A}] + \varnothing = [\mathbb{A}]$.

(2) The addition of an integer x and an integer set $[\mathbb{A}]$, is defined as building a new set, whose elements are the sum of x and every member of $[\mathbb{A}]$. Actually, $x + [\mathbb{A}]$ is equivalent to $[x] + [\mathbb{A}]$. Based on this definition, the addition operation of two sets $[\mathbb{A}]$ and $[\mathbb{B}]$ actually can be decomposed as the process: $\mathbb{A}$ shall be added to every element in $[\mathbb{B}]$, and then every element in $[\mathbb{A}]$ shall be added to every element

in $[\mathbb{B}]$. This process is shown as below:

$$\begin{aligned}
[\mathbb{A}] + [\mathbb{B}] &= [a_1, a_2, \cdots, a_n] + [\mathbb{B}] \\
&= (a_1 + [\mathbb{B}]) \cup (a_2 + [\mathbb{B}]) \cup \cdots (a_n + [\mathbb{B}]) \\
&= \Big(a_1 + [b_1, b_2, \cdots, b_m]\Big) \cup \Big(a_2 + [b_1, b_2, \cdots, b_m]\Big) \cup \cdots \cup \Big(a_n + [b_1, b_2, \cdots, b_m]\Big) \\
&= \Big[a_1 + b_1, a_2 + b_1, \cdots, a_n + b_1, \cdots, a_n + b_1, a_n + b_2, \cdots, a_n + b_m\Big]
\end{aligned}$$

Given two integer sets $[\mathbb{A}]$ and $[\mathbb{B}]$, we also define the **selective addition operation**, whose resulting set satisfies some condition, such as all its elements are greater than or equal to a number x. This is formally defined as below:

$$[\mathbb{A} + \mathbb{B} \geqslant x] = \Big[a_i + b_j : a_i \in [\mathbb{A}] \text{ and } b_j \in \mathbb{B} \text{ and } a_i + b_j \geqslant x\Big]$$

This is a set addition operation with the element constraint. This operation is denoted as $\big[\mathbb{A} + \mathbb{B} \geqslant x\big]$, where the square brackets $\big[\ \big]$ are used to highlight this is a set whose elements satisfy some constraint. In this article, we often use the constraints, such as $\big[6 \leqslant \big(\mathbb{P} + \mathbb{P}\big) \leqslant e\big]$, which represents a set that comes from the addition of two odd prime sets $[\mathbb{P}] + [\mathbb{P}]$ and all its elements are $\geqslant 6$ and $\leqslant e$. For simplicity, it is also denoted $\big[6 \leqslant \mathbb{P} + \mathbb{P} \leqslant e\big]$.

Example: Calculate the selective addition of $[3, 5, 7, 9]$ and $[1, 2, 3]$ such that the elements of the resulting set are $\geqslant 10$. We denote the resulting set as $\Big[[3, 5, 7, 9] + [1, 2, 3] \geqslant 10\Big]$ and calculate it as below:

$$\Big[[3, 5, 7, 9] + [1, 2, 3] \geqslant 10\Big] = [7 + 3, 9 + 1, 9 + 2, 9 + 3] = [10, 11, 12]$$

3 Theorems for the set addition

Theorem 1. Let $p_i \in \mathbb{P}$ be an odd prime number that is greater than 3, there is at least one odd composite number in $p_i + 2$ and $p_i + 4$.

Proof. $\because p_i > 3, \therefore p_i \neq 3$. Let $p_i = 3q + r$, where q is a positive integer and $r = 1$ or 2. q can be analyzed in the following two cases. In either case, the conclusion holds. Therefore, the conclusion is proved.

Case I (q is an even number): When $r = 1$, it can be proved that $p_i + 2$ is an odd composite number. Let $p_i = 3q + 1$

$\therefore g_i + 2 = 3q + 1 + 2 = 3(q + 1)$

$\therefore g_i + 2$ is an odd composite number.

Case II (q is an odd number): When $r = 2$, it can be proved that $p_i + 4$ is an odd composite number. Let $p_i = 3q + 2$

$\because g_i + 4 = 3q + 2 + 4 = 3(q + 2)$ and q is an odd number

$\therefore q + 2$ is an odd number

$\therefore p_i + 4$ is an odd composite number.

$\square$

Based on Theorem 1, we have its following corollaries, which can be simply proved. So we ignore their proofs here.

Corollary 1. In $\left[\mathbb{O} > 3\right]$, there does not exist three consecutive odd numbers, all of which are prime numbers.

Corollary 2. In the set $\left[\mathbb{O} > 3\right]$, among three consecutive odd numbers, there is at least one odd composite number.

Corollary 3. In the set $\left[\mathbb{O} \geqslant 9\right]$, if there exist odd prime numbers between two consecutive composite odd numbers, then the number of the odd prime numbers between them cannot be more than 2.

Theorem 2. $[\mathbb{P}] + [\mathbb{O}^c] = \left[\mathbb{E} \geqslant 12\right]$

Proof. Let e be any element of the set $\left[\mathbb{E} \geqslant 12\right]$, we have

$$e = 3 + (e-3) = 5 + (e-5) = 7 + (e-7)$$

$\because$ e is an even number, $\therefore$ $e-7, e-5, e-3$ are three consecutive odd numbers. Based on Corollary 2, there exist at least one odd composite number

$\therefore$ $e \in [\mathbb{P}] + [\mathbb{O}^c]$, $\therefore$ $\left[\mathbb{E} \geqslant 12\right] \subseteq [\mathbb{P}] + [\mathbb{O}^c]$

Let $p_x + o_{c_y}$ be any element of $[\mathbb{P}] + [\mathbb{O}^c]$. Obviously $p_x + o_{c_y} \in \left[\mathbb{E} \geqslant 12\right]$ $\therefore$ $[\mathbb{P}] + [\mathbb{O}^c] \subseteq \left[\mathbb{E} \geqslant 12\right]$.

$\therefore$ $[\mathbb{P}] + [\mathbb{O}^c] = \left[\mathbb{E} \geqslant 12\right]$ □

Theorem 3. $If [\mathbb{A}] = [\mathbb{B}], [\mathbb{C}] = [\mathbb{D}]$, then $[\mathbb{A}] + [\mathbb{C}] = [\mathbb{B}] + [\mathbb{D}]$

Proof. Let a_i be any member of $[\mathbb{A}]$ and c_j be any member of $[\mathbb{C}]$. Then $a_i + c_j$ is a member of $[\mathbb{A}] + [\mathbb{C}]$.

$\because$ Given that $[\mathbb{A}] = [\mathbb{B}], [\mathbb{C}] = [\mathbb{D}]$

$\therefore$ $a_i \in [\mathbb{B}]$ and $c_j \in [\mathbb{D}]$ $a_i + c_j \in [\mathbb{B}] + [\mathbb{D}]$ □

Theorem 4. $[\mathbb{A}] + ([\mathbb{B}] \cup [\mathbb{C}]) = ([\mathbb{A}] + [\mathbb{B}]) \cup ([\mathbb{A}] + [\mathbb{C}])$

Proof. Let a_i be any element of $[\mathbb{A}]$, x_j be any element of $[\mathbb{B}] \cup [\mathbb{C}]$, i.e. $x_j \in [\mathbb{B}]$ or $x_j \in [\mathbb{C}]$,

We have $a_i + x_j \in ([\mathbb{A}] + [\mathbb{B}]) \cup ([\mathbb{A}] + [\mathbb{C}])$

$\therefore$ $[\mathbb{A}] + ([\mathbb{B}] \cup [\mathbb{C}]) \subseteq ([\mathbb{A}] + [\mathbb{B}]) \cup ([\mathbb{A}] + [\mathbb{C}])$

Reversely, $[\mathbb{A}] + ([\mathbb{B}] \cup [\mathbb{C}]) \supseteq ([\mathbb{A}] + [\mathbb{B}]) \cup ([\mathbb{A}] + [\mathbb{C}])$

$\therefore$ $[\mathbb{A}] + ([\mathbb{B}] \cup [\mathbb{C}]) = ([\mathbb{A}] + [\mathbb{B}]) \cup ([\mathbb{A}] + [\mathbb{C}])$ □

Theorem 5. If $[\mathbb{A}] \cup [\mathbb{B}] = [\mathbb{A}]$, then $[\mathbb{B}] \subseteq [\mathbb{A}]$

Proof. $\because$ $[\mathbb{A}] \cup [\mathbb{B}] = [\mathbb{A}]$, in $[\mathbb{B}]$ any member $b_i \in [\mathbb{A}]$

$\therefore$ $[\mathbb{B}] \subseteq [\mathbb{A}]$ □

Theorem 6. If $[\mathbb{B}] \subseteq [\mathbb{A}]$ or $[\mathbb{B}] \subset [\mathbb{A}]$, then $[\mathbb{A}] \cup [\mathbb{B}] = [\mathbb{A}]$

Proof. $\because$ Given that $[\mathbb{B}] \subseteq [\mathbb{A}]$ or $[\mathbb{B}] \subset [\mathbb{A}]$, for any member b_i in $[\mathbb{B}]$, $b_i \in [\mathbb{A}]$

$\therefore$ $[\mathbb{A}] \cup [\mathbb{B}] = [\mathbb{A}]$ □

Example: $\because$ $[\mathbb{P}] + [\mathbb{O}^c] \supset [\mathbb{P}] + [\mathbb{P} > 9]$, $\therefore$ $[\mathbb{P}] + [\mathbb{O}^c] = \big([\mathbb{P}] + [\mathbb{O}^c]\big) \cup \big([\mathbb{P}] + [\mathbb{P} > 9]\big)$

Theorem 7. If $[\mathbb{B}] \supset [\mathbb{A}]$ and $[\mathbb{C}] = [\mathbb{D}]$, then $[\mathbb{A}] + [\mathbb{B}] \supseteq [\mathbb{B}] + [\mathbb{D}]$

Proof. Given the assumptions that $[\mathbb{B}] \supset [\mathbb{A}]$ and $[\mathbb{C}] = [\mathbb{D}]$

$\therefore$ for any member b_i in $[\mathbb{B}]$, $b_i \in [\mathbb{A}]$, and for any member d_j in $[\mathbb{D}]$, $d_j \in [\mathbb{C}]$

$\therefore b_i + d_j \in [\mathbb{A}] + [\mathbb{C}]$

$\therefore [\mathbb{A}] + [\mathbb{B}] \supseteq [\mathbb{B}] + [\mathbb{D}]$ $\square$

Example: Prove $[\mathbb{P}] + [\mathbb{P}] + [\mathbb{O}^c] \supset [\mathbb{P}] + [\mathbb{P}] + [\mathbb{P} \geq 11]$

Proof. $\because [\mathbb{P}] + [\mathbb{O}^c] \supset [\mathbb{P}] + [\mathbb{P} \geq 11]$ and $[\mathbb{P}] = [\mathbb{P}]$

Based on the Theorem 7, we have $[\mathbb{P}] + [\mathbb{P}] + [\mathbb{O}^c] \supset [\mathbb{P}] + [\mathbb{P}] + [\mathbb{P} \geq 11]$ $\square$

According to Corollary 3 and 2 of Theorem 1: there are at most two odd prime numbers between any two consecutive odd composite numbers and there are no three consecutive odd numbers, all of which are prime numbers. The following three theorems (Theorems 9 and 10) prove a fact that adding only three consecutive even numbers with the odd composite set can make up the whole odd number set. These theorems show how the odd prime numbers are distributed throughout the odd number set by filling those "holes" between any two consecutive odd composite numbers, and this distribution is actually implemented by three consecutive even numbers. Therefore, these theorems are called the "filling-the-hole" theorems.

Theorem 8 (The "Filling-the-hole" Theorem (1))**.** It requires at least three consecutive even numbers, $e, e+2, e+4$, such that $[e, e+2, e+4] + [\mathbb{O}^c]$ constitute an operative set that comprises of odd numbers greater than or equal to $e+9$. That is, $[e, e+2, e+4] + [\mathbb{O}^c] = [\mathbb{O} \geqslant e+9]$

Proof. Given the Corollary 3 of Theorem 1: there is at most two odd prime numbers between any two adjacent odd composite numbers. And given the Corollary 1 of Theorem 1, in $[\mathbb{O} > 3]$ there are no consecutive three odd prime numbers. Therefore, in $[\mathbb{O}^c]$ between any two adjacent odd composite numbers, there are at most two slots to fill.

Let o_i^c and o_{i+1}^c be any two adjacent odd composite numbers in $[\mathbb{O}^c]$.

Scenario 1. There are two holes between o_i^c and o_{i+1}^c (where $o_{i+1}^c = o_i^c + 6$): $o_i^c, \square, \square, o_{i+1}^c$. And $o_i^c + 2$ and $o_i^c + 4$ can fill these two holes. Therefore, for an even number e, we have

$$\begin{aligned}
[e, e+2, e+4] + [\mathbb{O}^c] &= [e, e+2, e+4] + [9, \square, \square, 15, \cdots, o_i^c, \square, \square, o_{i+1}^c] \\
&= [e+9, e+2+9, e+4+9, e+15, e+2+15, e+4+15, \cdots, \\
&\quad e+o_i^c, e+2+o_i^c, e+4+o_i^c, e+o_{i+1}^c, e+2+o_{i+1}^c, \\
&\quad e+4+o_{i+1}^c, \cdots] \\
&= [e+9, e+11, e+13, e+15, e+17, e+19, \cdots, e+o_i^c, \\
&\quad e+2+o_i^c, e+4+o_i^c, e+6+o_i^c, e+8+o_i^c, \\
&\quad e+10+o_i^c, \cdots]
\end{aligned}$$

The two slots $(\square, \square)$ between any two neighbor odd composite number have been filled, and the operative set contains consecutive odd numbers.

Scenario 2. When there is only one hole between the two neighbor odd composite numbers $o_i^c \square, o_{i+1}^c$. We have

$$\begin{aligned}[e, e+2, e+4] + [\mathbb{O}^c] &= [e, e+2, e+4] + [\cdots, o_i^c, \square, o_{i+1}^c, \cdots] \\ &= [\cdots, e+o_i^c, e+2+o_i^c, e+4+o_i^c, e+o_{i+1}^c, e+2+o_{i+1}^c, \\ &\quad e+4+o_{i+1}^c, \cdots] \\ &= [\cdots, e+o_i^c, e+2+o_i^c, e+4+o_i^c, e+4+o_i^c, e+6+o_i^c, \\ &\quad e+8+o_i^c, \cdots] \\ &= [\cdots, e+o_i^c, e+2+o_i^c, e+4+o_i^c, e+6+o_i^c, e+8+o_i^c, \cdots]\end{aligned}$$

So the hole between the two neighbor odd composite numbers has been filled, and the operative set contains consecutive odd numbers. Therefore, we proved that $[e, e+2, e+4] + [\mathbb{O}^c] = [\mathbb{O} \geqslant e+9]$.

$\square$

Theorem 9 (The "Filling-the-hole" Theorem (2))**.** For any subset of even numbers $[\mathbb{E}'] \subseteq [\mathbb{E}]$, let its smallest three even numbers are $e_1 < e_2 < e_3$. If e_1, e_2, and e_3 are not consecutive even numbers, then $[\mathbb{E}'] + [\mathbb{O}^c] \neq \left[\mathbb{O} \geqslant e_1 + 9\right]$.

Proof. Since e_1, e_2, and e_3 are not consecutive even numbers, we represent $e_2 = e_1 + d_1$ and $e_3 = e_1 + d_2$, where $d_1 \geqslant 2$ and $d_2 \geqslant 4$, and $d_1 = 2$ and $d_2 = 4$ cannot exist at the same time. Then we write the addition of two sets $\mathbb{E}'$ and $\left[\mathbb{O}^c\right]$ as below, where $\square$ is the hole as the placeholder to indicate the difference between $\mathbb{O}^c$ and $\mathbb{O}$:

$$\begin{aligned}[e_1, e_1+d_1, e_1+d_2] + [\mathbb{O}^c] &= [e_1, e_1+d_1, e_1+d_2] + [9, \square, \square, 15, \cdots] \\ &= [e_1+9, e_1+d_1+9, e_1+d_2+9, e_1+15, e_1+d_1+15, \\ &\quad e_1+d_2+15, \cdots]\end{aligned}$$

Let's consider the first four elements of the right side in the above equation, i.e., $e_1+9, e_1+d_1+9, e_1+d_2+9, e_1+15$. Note that all other numbers in this set are greater than e_1+15. If e_1, e_2 and e_3 are consecutive even numbers, i.e., $d_1 = 2$ and $d_2 = 4$, then they become four consecutive odd numbers: $e_1+9, e_1+11, e_1+13, e_1+15$. Unfortunately, when e_1, e_2 and e_3 are not consecutive even numbers, either "$d_1 > 2$" or "$d_1 = 2$ and $d_2 > 4$" must hold. If $d_1 > 2$, then the addition set $[\mathbb{E}'] + [\mathbb{O}^c]$ will miss $e_1 + 11$. If $d_1 = 2$ and $d_2 > 4$, then the addition set $[\mathbb{E}'] + [\mathbb{O}^c]$ will miss $e_1 + 13$. That is, either $e_1 + 11$ or $m_1 + 13$ is missing in the set $[\mathbb{E}'] + [\mathbb{O}^c]$.

$\therefore$ We have $[\mathbb{E}'] + [\mathbb{O}^c] \neq \left[\mathbb{O} \geqslant e_1 + 9\right]$. $\square$

Theorem 10 (The "Filling-the-hole" Theorem (3))**.** Given a set that comprises of even numbers, denoted as $[e_1, e_2, e_3, \cdots, e_i]$, where $e_1 < e_2 < e_3, \cdots < e_i$, if $[e_1, e_2, e_3, \cdots, e_i] + [\mathbb{O}^c] = \left[\mathbb{O} \geqslant e_1 + 9\right]$, then e_1, e_2, e_3 must be three consecutive even numbers.

Proof. Assume that e_1, e_2, e_3 are not consecutive, based on the Theorem 9, $[e_1, e_2, e_3, \cdots, e_i] + [\mathbb{O}^c] \neq [\mathbb{O} \geqslant e_1 + 9]$, which is contradictory to the given condition, therefore the assumption is not true.

$\therefore e_1, e_2, e_3$ must be three consecutive even numbers. $\square$

4 Subtraction of the operative sets

Given the operative set $[\mathbb{A}]$ and $[\mathbb{B}]$, a complete subtraction of $[\mathbb{A}] - [\mathbb{B}]$ leads to the subtraction of every member in $[\mathbb{B}]$ from every member in $[\mathbb{A}]$. For example

$[7, 8, 9, 10, 11, 12, 13] -_{\text{complete}} [6, 7, 8] = [-1, 0, 1, 2, 3, 4, 5, 6, 7]$

However, $[-1, 0, 1, 2, 3, 4, 5, 6, 7] + [6, 7, 8] \neq [7, 8, 9, 10, 11, 12, 13]$

Therefore, when we compute $[\mathbb{A}] - [\mathbb{B}]$, we shall select members from the complete subtraction $[\mathbb{C}]$ complete to constitute the operative set $[\mathbb{C}]$, such that $[\mathbb{B}] + [\mathbb{C}] = [\mathbb{A}]$. We define $[\mathbb{C}]$ to be the result of the subtraction of $[\mathbb{B}]$ from $[\mathbb{A}]$.

From the operative set $[-1, 0, 1, 2, 3, 4, 5, 6, 7]$ we select the subset $[1, 2, 3, 4, 5]$

$\because [1, 2, 3, 4, 5] + [6, 7, 8] = [7, 8, 9, 10, 11, 12, 13]$

$\therefore [7, 8, 9, 10, 11, 12, 13] - [6, 7, 8] = [1, 2, 3, 4, 5]$

We define the subtraction of operative sets as follows:

Definition 1. Let $[\mathbb{A}] -_{\text{complete}} [\mathbb{B}] = [\mathbb{C}]_{\text{complete}}$, and if we can select $[\mathbb{C}]$ from $[\mathbb{C}]_{\text{complete}}$, such that $[\mathbb{B}] + [\mathbb{C}] = [\mathbb{A}]$, then $[\mathbb{A}] - [\mathbb{B}]$ has a solution $[\mathbb{C}]$; if we fail to select $[\mathbb{C}]$ from $[\mathbb{C}]_{\text{complete}}$, such that $[\mathbb{B}] + [\mathbb{C}] = [\mathbb{A}]$, then $[\mathbb{A}] - [\mathbb{B}]$ has no solution.

Therefore, the subtraction of operative sets is not always executable.

Example: We have two sets of cards, and there is a number on each card. The first set has 5 cards with the numbers 3, 4, 6, 8, 11, respectively. The second set has 3 cards with the numbers 2, 3, 7, respectively.

Question: Does there exist the third set of cards with a number on each card, such that drawing a card of the second and third sets, respectively, and the sum of the two numbers constitute an operative set that equals to the operative set comprising of numbers in the first set of cards?

Answer: Let the numbers on the third set of the cards $\mathbb{X}$ be $[x_1, x_2, \cdots]$, we want $[x_1, x_2, \cdots] + [2, 3, 7] = [3, 4, 6, 8, 11]$.

$\therefore [3, 4, 6, 8, 11]_{\text{complete}}[2, 3, 7] = [1, 0, -2, -4, 2, -1, -3, 4, 6, 3, 8, 9, 5]$, and $\mathbb{X}$ shall only contain positive. So we need to select a subset from $[1, 2, 3, 4, 5, 6, 8, 9]$ to be $\mathbb{X}$ such that $\mathbb{X} + [2, 3, 7] = [3, 4, 6, 8, 11]$ integers. The result is that no subset would satisfy the above condition, so there is solution of the problem.

However, if we change the numbers on the 1st and 2nd card sets as follows:

- The first set has 7 cards with the numbers $7, 8, 9, 10, 11, 12, 13$ respectively.
- The second set has 3 cards with the numbers $6, 7, 8$ respectively.
- There exist the 3rd set of cards that satisfy $[x_1, x_2, \cdots] + [6, 7, 8] = [7, 8, 9, 10, 11, 12, 13]$.

$[7, 8, 9, 10, 11, 12, 13]_{\text{complete}}[6, 7, 8] = [-1, 0, 1, 2, 3, 4, 5, 6, 7]$

From the complete subtraction we further select $[1, 2, 3, 4, 5]$

$\because [1, 2, 3, 4, 5] + [6, 7, 8] = [7, 8, 9, 10, 11, 12, 13]$

$\therefore$ We have a solution, and the 3rd set card has 5 cards with the numbers $1, 2, 3, 4, 5$.

5 Multiplication of the operative sets

Give two operative sets $[\mathbb{A}]$ and $[\mathbb{B}]$, the multiplication operation between $[\mathbb{A}]$ and $[\mathbb{B}]$ means that each member in $[\mathbb{A}]$ multiplies each member in $[\mathbb{B}]$, denoted as

$[\mathbb{A}] \times [\mathbb{B}] = [\mathbb{C}]$ or $[\mathbb{A}][\mathbb{B}] = [\mathbb{C}]$

For example, $[1,2,3]\times[2,4,6] = [1\times 2, 1\times 4, 1\times 6, 2\times 2, 2\times 4, 2\times 6, 3\times 2, 3\times 4, 3\times 6]$
$= [2,4,6,8,12,18]$

If there is a common factor for all members in an operative set, the common factor can be extracted to the outside of the operative set, resulting in the multiplication between the common factor and each member in the operative set.

For example, $[4,12,16] = 4\times[1,3,4]$

The multiplication between a number and an operative set means that the multiplication between the number and each member in the operative set.

For example, $3\times[\mathbb{O}\geqslant 3] = 3\times[3,5,7,9,\cdots]$
$= [3\times 3, 3\times 5, 3\times 7, 3\times 9,\cdots]$

It has be easily proved that the multiplications of operative sets satisfy the commutative and the associative laws, but generally do not satisfy the distributive law of multiplication, which is different from the traditional sense of the number multiplication.

The concept of multiplication of operative sets also differentiate that of the multiplications of numbers. For example, for the number multiplication 5x3 equals to summing 5 three time, i.e. $5+5+5 = 5\times 3 = 15$.

However, in the theory of the operative set, [2,7]x3 does not equals to the sum of three $[2,7]$, i.e. $[2,7]\times 3 \neq [2,7]+[2,7]+[2,7]$

Because the theory of operative set deviates from the traditional math, we may see scenarios that would not occur within the context of traditional math. Below are several examples.

Example: Let $[\mathbb{Z}^+] = [1,2,,3,\cdots]$ be the operative set of positive natural numbers. We have the following examples:

$[\mathbb{Z}^+]\times[\mathbb{Z}^+] = [1,2,,3,\cdots]\times[1,2,,3,\cdots] = [\mathbb{Z}^+]$
$[\mathbb{Z}^+]\times[\mathbb{Z}^+]\times[\mathbb{Z}^+] = [\mathbb{Z}^+]$
$[\mathbb{Z}^+]^n = [\mathbb{Z}^+]$
Analogously, $[\mathbb{O}]^n = [\mathbb{O}]$
Also, $[\mathbb{O}]\times[\mathbb{E}\geqslant 6] = [1,3,5,\cdots]\times[\mathbb{E}\geqslant 6] = [\mathbb{E}\geqslant 6]$

In the context of the computation of the operative sets, there are interesting scenarios, and we will not enumerate them here.

6 Intersection and union of the operative sets

The intersection and union of operative sets carry the same meaning as those of the sets.

The Union of operative sets: $\big[[\mathbb{A}],[\mathbb{B}]\big] = [\mathbb{A}]\cup[\mathbb{B}]$

For example, $[3,4,5]\cup[5,6,7] = [3,4,5,6,7]$, and $\big[[3,4,5],[5,6,7]\big] = [3,4,5,6,7]$

The intersection of operative sets: $[3,4,5]\cap[5,6,7] = [5]$ and $[3,4,5]\cap[6,7] = \varnothing$

Theorem 11. If $[\mathbb{A}] = [\mathbb{B}], [\mathbb{C}] = [\mathbb{D}], \cdots, [\mathbb{F}] = [\mathbb{H}]$, then $\big[[\mathbb{A}],[\mathbb{C}],\cdots,[\mathbb{F}]\big] = \big[[\mathbb{B}],[\mathbb{C}],\cdots,[\mathbb{H}]\big]$

Proof. Select any members from the left side of the equations. We have $[a_i, c_j, \cdots, f_p] \subset \big[[\mathbb{B}],[\mathbb{D}],\cdots,[\mathbb{H}]\big]$

$\therefore \big[[\mathbb{A}],[\mathbb{C}],\cdots,[\mathbb{F}]\big] \subseteq \big[[\mathbb{B}],[\mathbb{D}],\cdots[\mathbb{H}]\big]$

Conversely, $\big[[\mathbb{A}],[\mathbb{C}],\ldots,[\mathbb{F}]\big] \supseteq \big[[\mathbb{B}],[\mathbb{D}],\cdots,[\mathbb{H}]\big]$

$\therefore \big[[\mathbb{A}],[\mathbb{C}],\cdots,[\mathbb{F}]\big] = \big[[\mathbb{B}],[\mathbb{D}],\cdots,[\mathbb{H}]\big]$ □

7 Property of operative set, related concepts, and principles

The property of operative set

The theorem of Operative Set has been established based on the Set theorem. Members of an operative set are characterized by deterministic, non-sequential, and unique.

The "unique" means that members of the same operative set are different from each other, and non-repetitive. Repetitive members can be counted only once. The requirement for the members in an operative set to be unique is important, and is the basis for the operative set theory.

Belong to, inclusion, empty operative set, and union

1. For any member x and any operative set $[\mathbb{X}]$, there can be only two types of "belonging" relationship

   $x \in [\mathbb{X}]$ or $x \bar{\in} [\mathbb{X}]$

   $[\mathbb{A}] = [\mathbb{B}]$, means that $[\mathbb{A}]$ includes $[\mathbb{B}]$, and $[\mathbb{B}]$ includes $[\mathbb{A}]$.

   $[\mathbb{A}] \subset [\mathbb{B}]$, means that $[\mathbb{B}]$ includes $[\mathbb{A}]$

2. Empty operative-set

   In the set theory, we denote an empty set as a set that contains no member.

   In the operative set theory, non-empty operative set could be turned into an empty operative set. For example, for students in a classroom, the fact that the students all get out of the classroom can be denoted as [Students in Classroom]=$\varnothing$. And $[\mathbb{O}^c] = \varnothing$ means that the operative set doesn't contain any odd composite number. Because $[\mathbb{O}^c]$ contains only odd composite numbers, and if we empty all odd composite numbers, the operative set doesn't contain any members any more. Based on the definition of "belonging to" and "include", because $\varnothing$ does not contain any members, therefore no members are allowed in $\varnothing$.

3. Union of Operative Set

   If the Operative Set $[\mathbb{C}]$ is composed of all members in the operative sets $[\mathbb{A}]$ and $[\mathbb{B}]$ (repetitive members are counted only once), $[\mathbb{C}]$ is defined to be the union of $[\mathbb{A}]$ and $[\mathbb{B}]$, denoted as $[\mathbb{C}] = [[\mathbb{A}], [\mathbb{B}]]$.

   For example, $\big[[\mathbb{A}], [\mathbb{B}], [\mathbb{C}]\big]$ is a union set with the daughter sets as $[\mathbb{A}], [\mathbb{B}], [\mathbb{C}]$. The members in $\big[[\mathbb{A}], [\mathbb{B}], [\mathbb{C}]\big]$ are the union of all members in $[\mathbb{A}], [\mathbb{B}], [\mathbb{C}]$, where repetitive members can only be counted once.

   Properties of the Union-Operative-Set:

   (a) The members in the union-operative-set are the unions of the members in the daughter-operative-sets. For example, $\big[[\mathbb{A}], [\mathbb{B}], [\mathbb{C}]\big]$ contains members belonging to $[\mathbb{A}]$, $[\mathbb{B}]$, and $[\mathbb{C}]$, and does not contain members that do not belong to $[\mathbb{A}]$, $[\mathbb{B}]$, and $[\mathbb{C}]$. Therefore, once the daughter sets are determined, the members of the union-operative-set are also determined.

   (b) In a union-operative-set, each member can be found in a least one daughter set.

   (c) There is no redundant members in a union-operative-set. Any two subsets of the union-operative-set shall share no common members.

8 Application of the operative set theory to prove Goldbach Conjecture

Using the operative set, the Goldbach Conjecture can be expressed as $\big[\mathbb{E} \geqslant 6\big] = [\mathbb{P}] + [\mathbb{P}]$. We will use the operative set theory to prove Goldbach Conjecture in the following four steps:

Step 1 Prove $[\mathbb{P}] + \big[\mathbb{P} \geqslant 11\big] \subset [\mathbb{P}] + [\mathbb{O}^c]$

Step 2 Prove $\big[\mathbb{O} + \mathbb{O} \geqslant m\big] + [\mathbb{O} \geqslant 11] \subset \big[\mathbb{P} + \mathbb{P} \geqslant e\big] + [\mathbb{O}^c]$, for any even number $e \geqslant 6$

Step 3 Prove $[e, e+2, e+4] \subset \big[\mathbb{P} + \mathbb{P} \geqslant e\big]$, for any even number $e \geqslant 6$

Step 4 Prove Goldbach Conjecture: $\big[\mathbb{E} \geqslant 6\big] = [\mathbb{P}] + [\mathbb{P}]$

Step 1: Prove $[\mathbb{P}] + \big[\mathbb{P} \geqslant 11\big] \subset [\mathbb{P}] + [\mathbb{O}^c]$ (1)

Proof. According to Theorem 2, we have $[\mathbb{P}] + [\mathbb{O}^c] = \big[\mathbb{E} \geq 12\big]$. Since any element in $\mathbb{P} + \big[\mathbb{P} \geqslant 11\big]$ is an even number $\geqslant 14$, we have $\mathbb{P} + \big[\mathbb{P} \geqslant 11\big] \subset \big[\mathbb{E} \geqslant 12\big]$. Combining it with Theorem 2, we have $\mathbb{P} + \big[\mathbb{P} \geqslant 11\big] \subset \big[\mathbb{E} \geqslant 12\big] = [\mathbb{P}] + [\mathbb{O}^c]$, therefore we prove Eq.(1). □

Step 2: Prove $\Big[\mathbb{P} + \mathbb{P} \geqslant e\Big] + \Big[\mathbb{P} \geqslant 11\Big] \subset \Big[\mathbb{P} + \mathbb{P} \geqslant e\Big] + [\mathbb{O}^c]$ (2)

Proof. Let $\underbrace{p_x + p_y}_{\geqslant e} + p_z$ be any element of the set $\Big[\mathbb{P} + \mathbb{P} \geqslant e\Big] + [\mathbb{P} \geqslant 11]$, where $p_x \geqslant 3, p_y \geqslant 3, p_z \geqslant 11$. We will prove $\underbrace{p_x + p_y}_{\geqslant e} + p_z \in \Big[\mathbb{P} + \mathbb{P} \geqslant e\Big] + \mathbb{O}^c$, which can therefore prove Eq.(2).

Because we have proved $[\mathbb{P}] + [\mathbb{P} \geqslant 11] \subset [\mathbb{P}] + [\mathbb{O}^c]$ in Eq.(1), we have

$$p_y + p_z \in \mathbb{P} + \mathbb{O}^c \tag{3}$$

We can add p_x to the two sides of Eq.(3), and get $\underbrace{p_x + p_y}_{\geqslant e} + p_z \in p_x + [\mathbb{P}] + [\mathbb{O}^c]$. $\because \mathbb{P} + \mathbb{O}^c$ contains $p_y + p_z$,

$$\therefore \underbrace{p_x + p_y}_{\geqslant e} + p_z \in \underbrace{p_x + \mathbb{P}}_{\geqslant e} + [\mathbb{O}^c]$$

$\because \Big[p_x + \mathbb{P} \geqslant e\Big] + \mathbb{O}^c \subset \Big[\mathbb{P} + \mathbb{P} \geqslant e\Big] + \mathbb{O}^c, \therefore \underbrace{p_x + p_y}_{\geqslant e} + p_z \in \Big[\mathbb{P} + \mathbb{P} \geqslant e\Big] + [\mathbb{O}^c]$. Therefore, we proved Eq.(2), which is $\Big[\mathbb{P} + \mathbb{P} \geqslant e\Big] + [\mathbb{P} \geqslant 11] \subset \Big[\mathbb{P} + \mathbb{P} \geqslant e\Big] + [\mathbb{O}^c]$. □

Step 3: Prove $[e, e+2, e+4] \subset \big[\mathbb{P} + \mathbb{P} \geqslant e\big]$, for any even number $e \geqslant 6$ (4)

Proof. According to Conclusion of Step 2, i.e., Eq.(2), $\Big[\mathbb{P} + \mathbb{P} \geqslant e\Big] + \Big[\mathbb{P} \geqslant 11\Big]$ is a subset of $\Big[\mathbb{P} + \mathbb{P} \geqslant e\Big] + [\mathbb{O}^c]$. So we can get $\Big[\mathbb{P} + \mathbb{P} \geqslant m\Big] + [\mathbb{O}^c] = \Big(\Big[\mathbb{P} + \mathbb{P} \geqslant e\Big] + \mathbb{O}^c\Big) \bigcup \Big(\Big[\mathbb{P} + \mathbb{P} \geqslant e\Big] + \Big[\mathbb{P} \geqslant 11\Big]\Big)$, which is then transformed to the following

$$\left[\mathbb{P}+\mathbb{P} \geqslant e\right] + \mathbb{O}^c = \left(\left[\mathbb{P}+\mathbb{P} \geqslant m\right] + [\mathbb{O}^c]\right) \bigcup \left(\left[\mathbb{P}+\mathbb{P} \geqslant e\right] + \left[\mathbb{P} \geqslant 11\right]\right)$$

$$\because \text{ According to Theorem 4}$$

$$= \left[\mathbb{P}+\mathbb{P} \geqslant e\right] + \left([\mathbb{O}^c] \bigcup \left[\mathbb{P} \geqslant 11\right]\right)$$

$$\because [\mathbb{O}^c] \bigcup \left[\mathbb{P} \geqslant 11\right] = [\mathbb{O} \geqslant 9]$$

$$= \left[\mathbb{P}+\mathbb{P} \geqslant e\right] + [\mathbb{O} \geqslant 9]$$

$$\because e + [\mathbb{O} \geqslant 9] = [\mathbb{O} \geqslant m + 9]$$

$$\because \text{and it can be easily proved that } \left[\mathbb{P}+\mathbb{P} > e\right] + \left[\mathbb{O} \geqslant 9\right] \subset \left[\mathbb{O} \geqslant e+9\right]$$

$$= \left[\mathbb{O} \geqslant e + 9\right]$$

$$\therefore \left[\mathbb{P}+\mathbb{P} \geqslant e\right] + [\mathbb{O}^c] = \left[\mathbb{O} \geqslant e+9\right] \tag{5}$$

Applying the "filling-the-hole principle" (Theorem 10) to the above Eq.(5), there must exist three consecutive even numbers starting from e, i.e., $e, e+2, e+4$, satisfying the following:

$$[e, e+2, e+4] \subset \left[\mathbb{P}+\mathbb{P} \geq m\right] \tag{6}$$

Thus, we proved Eq.(4). □

Step 4: Prove GOLDBACH CONJECTURE $\left[\mathbb{E} \geqslant 6\right] = [\mathbb{P}] + [\mathbb{P}]$ (7)

Proof. We will use the mathematical induction to prove $\left[\mathbb{E} \geqslant 6\right] \subseteq [\mathbb{P}] + [\mathbb{P}]$, with two steps below:

Initial cases: Researchers have proved that any even number $6 \leqslant e_0 \leqslant 9 \times 10^8$ can be the sum of two odd prime numbers.

Inductive step Prove that for every even number e, if $e \in [\mathbb{P}] + [\mathbb{P}]$, then we have $e + 2 \in [\mathbb{P}] + [\mathbb{P}]$. According to Eq.(4), we have the conclusion $[e, e+2, e+4] \subset [\mathbb{P}+\mathbb{P} \geq e]$ for any even number $e \geqslant 6$. Therefore, it is easily to get $e + 2 \in \mathbb{P} + \mathbb{P}$ when $e \in \mathbb{P} + \mathbb{P}$.

Based on the mathematical induction principle, then we prove $\left[\mathbb{E} \geqslant 6\right] \subseteq \mathbb{P} + \mathbb{P}$.

As the sum of any two odd prime numbers is an even number $\geqslant 6$, we have $[\mathbb{P}] + [\mathbb{P}] \subseteq \left[\mathbb{E} \geqslant 6\right]$. Therefore, GOLDBACH CONJECTURE, i.e., $\left[\mathbb{E} \geqslant 6\right] = [\mathbb{P}] + [\mathbb{P}]$, is proved. □

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