Riemann Hypothesis

Shekhar Suman

April 2020

Subject Classification code- 11M26

Keywords- Riemann Zeta function; Analytic Continuation; Critical strip; Critical line.

1 Abstract

The Riemann Zeta function is defined as

$\zeta(s) = \sum_{n=1}^{\infty} 1/n^s, \ Re(s) > 1$

The Zeta function is holomorphic in the complex plane except for a pole at $s = 1$. *The trivial zeros of* $\zeta(s)$ *are* $-2, -4, -6, ...$ *Its non trivial zeros lie in the critical strip* $0 < Re(s) < 1$.

The Riemann Hypothesis states that all the non trivial zeros lie on the critical line $Re(s) = 1/2$.

2 Proof

Analytic continuation of $\zeta(s)$ *is defined as* [*see* 1, *p*.14 , *Eq*. 2.1.4]

$$\zeta(s) = s\int_1^\infty \frac{[x]-x+\frac{1}{2}}{x^{s+1}}dx + \frac{1}{s-1} + \frac{1}{2} \quad (1)$$

Here [.] *denotes the Greatest Integer Function.*

Let, $s = \sigma + it$.

For $0 < \sigma < 1$, [*see* 1, *p*.14],

$$\frac{s}{2}\int_1^\infty \frac{1}{x^{s+1}}dx = \frac{1}{2}.$$

So, using $\frac{1}{2} = \frac{s}{2}\int_1^\infty \frac{1}{x^{s+1}}dx$ *in* (1),

$$\zeta(s) = s\int_1^\infty \frac{[x]-x+\frac{1}{2}}{x^{s+1}}dx + \frac{1}{s-1} + \frac{s}{2}\int_1^\infty \frac{1}{x^{s+1}}dx$$

$$\zeta(s) = s\int_1^\infty \frac{[x]-x+\frac{1}{2}}{x^{s+1}}dx + \frac{s}{2}\int_1^\infty \frac{1}{x^{s+1}}dx + \frac{1}{s-1}$$

$$\zeta(s) = s\int_1^\infty \frac{[x]-x+\frac{1}{2}}{x^{s+1}}dx + s\int_1^\infty \frac{1/2}{x^{s+1}}dx + \frac{1}{s-1}$$

$$\zeta(s) = s\int_1^\infty \frac{[x]-x+1}{x^{s+1}}dx + \frac{1}{s-1}$$

Let, ρ *be a non trivial zero of the Riemann Zeta Function.*

Then, $\zeta(\rho) = 0$; $0 < Re(\rho) < 1$

$$\zeta(\rho) = \rho\int_1^\infty \frac{[x]-x+1}{x^{\rho+1}}dx + \frac{1}{\rho-1} = 0$$

$$\int_1^\infty \frac{[x]-x+1}{x^{\rho+1}}dx = \frac{1}{\rho(1-\rho)};\ 0 < Re(\rho) < 1 \quad (2)$$

The functional equation of the Riemann Zeta function is [*see* [1], *p*.22, 2.6.4],

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s).$$

So, by functional equation if ρ *is a zero of the Riemann Zeta function then* $1-\rho$ *is also a zero* [*see* [1], *p*.30].

$\zeta(\rho) = 0 \Rightarrow \zeta(1-\rho) = 0.$

$\zeta(1-\rho) = (1-\rho)\int_1^\infty \frac{[x]-x+1}{x^{2-\rho}}dx - \frac{1}{\rho} = 0;$

$$\int_1^\infty \frac{[x]-x+1}{x^{2-\rho}}dx = \frac{1}{\rho(1-\rho)};\ 0 < Re(\rho) < 1 \tag{3}$$

Equating left sides of equation (2) *and* (3),

$\int_1^\infty \frac{[x]-x+1}{x^{\rho+1}}dx = \int_1^\infty \frac{[x]-x+1}{x^{2-\rho}}dx$

$$\int_1^\infty([x]-x+1)(\frac{1}{x^{\rho+1}} - \frac{1}{x^{2-\rho}})dx = 0 \tag{4}$$

$Let, \rho = \sigma + it\ ; 0 < \sigma < 1,$

Since, $0 < \sigma < 1$ *so we discuss* 2 *cases*

$1/2 \leq \sigma < 1$ *and* $0 < \sigma \leq 1/2.$

Case $1 : 1/2 \leq \sigma < 1.$

Putting, $\rho = \sigma + it$ *in equation* (4),

$\int_1^\infty([x]-x+1)(\frac{1}{x^{\sigma+1+it}} - \frac{1}{x^{2-\sigma-it}})dx = 0.$

$\int_1^\infty([x]-x+1)(\frac{x^{-it}}{x^{\sigma+1}} - \frac{x^{it}}{x^{2-\sigma}})dx = 0.$

$\int_1^\infty([x]-x+1)(\frac{e^{-it(lnx)}}{x^{\sigma+1}} - \frac{e^{it(lnx)}}{x^{2-\sigma}})dx = 0.$

$\int_1^\infty([x]-x+1)(\frac{cos(tlnx)}{x^{\sigma+1}} - \frac{cos(t(lnx))}{x^{2-\sigma}})dx +$

$i\int_1^\infty([x]-x+1)(\frac{-sin(tlnx)}{x^{\sigma+1}} - \frac{sin(t(lnx))}{x^{2-\sigma}})dx = 0$

Equating Real part to zero,

$\int_1^\infty([x]-x+1)(\frac{cos(tlnx)}{x^{\sigma+1}} - \frac{cos(t(lnx))}{x^{2-\sigma}})dx = 0$

$\int_1^\infty([x]-x+1)(\frac{1}{x^{\sigma+1}} - \frac{1}{x^{2-\sigma}})cos(tlnx)dx = 0$

$$\int_1^\infty([x]-x+1)(\frac{1}{x^{2-\sigma}} - \frac{1}{x^{\sigma+1}})cos(tlnx)dx = 0 \tag{5}$$

$$Let, I = \int_1^\infty ([x]-x+1)(\frac{1}{x^{2-\sigma}} - \frac{1}{x^{\sigma+1}})cos(tlnx)dx = 0 \tag{6}$$

$Claim : For,\ 1/2 \leq \sigma < 1,\ \frac{1}{x^{2-\sigma}} - \frac{1}{x^{\sigma+1}} \geq 0.$

$\frac{1}{x^{2-\sigma}} - \frac{1}{x^{\sigma+1}}$

$= e^{(\sigma-2)lnx} - e^{-(\sigma+1)lnx}$

$= \frac{e^{(2\sigma-1)lnx}-1}{e^{(\sigma+1)lnx}}.$

$For,\ 1/2 \leq \sigma < 1,\ 2\sigma - 1 \geq 0.$

$for\ x \geq 1,\ lnx \geq 0.$

$(2\sigma - 1)lnx \geq 0.$

$e^{(2\sigma-1)lnx} \geq 1.$

$e^{(2\sigma-1)lnx} - 1 \geq 0.$

$So,\ \frac{1}{x^{2-\sigma}} - \frac{1}{x^{\sigma+1}} = \frac{e^{(2\sigma-1)lnx-1}}{e^{(\sigma+1)lnx}} \geq 0.$

$$\frac{1}{x^{2-\sigma}} - \frac{1}{x^{\sigma+1}} \geq 0. \tag{7}$$

which proves the claim.

Equation (6) *gives,*

$\int_1^\infty ([x] - x + 1)(\frac{1}{x^{\sigma+1}} - \frac{1}{x^{2-\sigma}})cos(tlnx)dx = 0.$

$0 = |\int_1^\infty ([x] - x + 1)(\frac{1}{x^{\sigma+1}} - \frac{1}{x^{2-\sigma}})cos(tlnx)dx|$

$0 \leq \int_1^\infty \mid ([x] - x + 1)(\frac{1}{x^{\sigma+1}} - \frac{1}{x^{2-\sigma}})cos(tlnx) \mid dx$

$0 \leq \int_1^\infty \mid ([x]{-}x{+}1) \mid\mid (\frac{1}{x^{\sigma+1}}{-}\frac{1}{x^{2-\sigma}}) \mid\mid cos(tlnx) \mid dx$ (8)

$0 \leq x - [x] < 1$

$0 < [x] - x + 1 \leq 1.$

$Also, \mid cos(tlnx) \mid \leq 1.$

$Using\ the\ above\ inequalities\ in$ (8),

$0 \leq \int_1^\infty \mid (\frac{1}{x^{\sigma+1}} - \frac{1}{x^{2-\sigma}}) \mid dx.$

$Since,\ by$ (7)

$\frac{1}{x^{\sigma+1}} - \frac{1}{x^{2-\sigma}} \geq 0.$

$0 \leq \int_1^\infty (\frac{1}{x^{\sigma+1}} - \frac{1}{x^{2-\sigma}})dx.$

$0 \leq \frac{-1}{\sigma x^{\sigma}} - \frac{1}{(\sigma-1)x^{1-\sigma}}\Big|_1^\infty.$

$Since,\ 0 < \sigma \leq 1/2,\ so\ \frac{1}{x^{\sigma}}\ and\ \frac{1}{x^{1-\sigma}} \ \rightarrow\ 0\ as\ x\ \rightarrow\ \infty$

$0 \leq \frac{1}{\sigma} + \frac{1}{\sigma-1}$

$0 \leq \frac{1}{\sigma} - \frac{1}{1-\sigma}$

$0 \leq \frac{1-2\sigma}{\sigma(1-\sigma)}$ (9)

$Since,\ by\ Case$ (1),

$1/2 \leq \sigma < 1.$

$\Rightarrow \frac{1-2\sigma}{\sigma(1-\sigma)} \leq 0$ (10)

$Combining\ equations$ (9) and (10),

$0 \leq \frac{1-2\sigma}{\sigma(1-\sigma)} \leq 0.$

$\frac{1-2\sigma}{\sigma(1-\sigma)} = 0.$

$\sigma = 1/2.$

$Now\ we\ proceed\ to\ Case\ 2$

$Case\ 2 : 0 < \sigma \leq 1/2.$

$Let, \rho = \sigma + it, 0 < \sigma \leq 1/2.$

$Let,\ \zeta(\rho) = 0$

$The\ functional\ equation\ of\ the\ Riemann\ Zeta\ function\ is$ $[see\ [1],\ p.22, 2.6.4]$,

$$\Gamma(s/2)\pi^{-s/2}\zeta(s) = \Gamma((1-s)/2)\pi^{-(1-s)/2}\zeta(1-s).$$

$So,\ by\ functional\ equation\ if\ \rho = \sigma + it\ is\ a\ zero\ of\ the\ Riemann\ Zeta\ function\ then$
$1 - \rho = 1 - \sigma - it \quad is\ also\ a\ zero\ and\ then$
$1 - \bar{\rho} = 1 - \sigma + it \ \ is\ also\ a\ zero$ $[see\ [1], p.30]$.

$$\zeta(\rho) = 0 \Rightarrow \zeta(1-\rho) = 0.$$

$$\zeta(1-\rho) = 0 \Rightarrow \zeta(1-\bar{\rho}) = 0$$

$Since,\ \rho = \sigma + it.$

$$\zeta(1-\bar{\rho}) = 0 \Rightarrow \zeta(1-\sigma+it) = 0.$$

$$0 < \sigma \leq 1/2 \Rightarrow 1/2 \leq 1 - \sigma < 1.$$

$$\zeta(1-\sigma+it) = 0,\ 1/2 \leq 1-\sigma < 1.$$

$Let,\ \sigma' = 1 - \sigma.$

$$\zeta(\sigma'+it) = 0,\ 1/2 \leq \sigma' < 1.$$

$So,\ by\ case\ (1),$

$$\sigma' = 1/2 \Rightarrow (1-\sigma) = 1/2.$$

$\sigma = 1/2.$

So, by the above two cases we get that

$\zeta(\rho) = 0\ ; 0 < Re(\rho) < 1;\ t \in (-\infty, -1/2) \cup (1/2, \infty)$

$\Rightarrow Re(\rho) = 1/2$

, *which proves the R.H.*

3 References

1. E. C. Titchmarsh, D. R. Heath-Brown - The theory of the Riemann Zeta function [2nd ed] Clarendon Press; Oxford University Press (1986).

2. A Monotonicity of Riemann's Xi function and a reformulation of the Riemann Hypothesis, Periodica Mathematica Hungarica - May 2010.

3. Kevin Broughan - Equivalents of the Riemann Hypothesis : Arithmetic Equivalents Cambridge University Press (2017) .

4. H.M Edwards - Riemann's Zeta function- Academic Press (1974).

5.Stanislaw Saks, Antoni Zygmund Analytic Functions, 2nd Edition Hardcover(1965) .

6. Tom M. Apostol - Introduction to Analytical Number Theory (1976).

7. 4. Lars Ahlfors - Complex analysis [3 ed.] McGraw -Hill (1979).