Poincaré and Geometrization Conjectures: mathematical connections between String Theory, Ricci Flow and Number Theory

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Riassunto

La "congettura di Poincaré" afferma che: "Ogni 3-varietà semplicemente connessa chiusa (ossia compatta e senza bordi) ed orientabile è omeomorfa a una sfera tridimensionale". (In topologia, un omeomorfismo è una corrispondenza biunivoca fra due spazi topologici che ne preserva la topologia. In altre parole, è una funzione tra due spazi topologici con la proprietà di essere continua, invertibile e di avere l'inversa continua. Due spazi omeomorfi godono delle stesse proprietà topologiche (separabilità, connessione, compattezza…). Informalmente, due spazi sono omeomorfi se possono essere deformati l'uno nell'altro senza "strappi", "sovrapposizioni" o "incollature").

Detto con termini diversi, la congettura dice che la sfera è l'unica varietà tridimensionale "senza buchi", cioè dove qualsiasi cammino chiuso può essere contratto fino a diventare un punto. Il padre dell'enunciato è il matematico francese Henri Poincaré. La Congettura che prende il suo nome nacque nel 1904, mentre lo studioso stava lavorando ai fondamenti di quella che poi sarà chiamata topologia algebrica. L'enunciato di Poincaré tenta di dimostrare che la sfera è il più semplice campo in cui un qualsiasi cammino chiuso possa essere contratto fino a diventare un punto (ogni varietà chiusa n dimensionale omotopicamente equivalente alla n-sfera è omeomorfa alla n-sfera. In topologia, due funzioni continue da uno spazio topologico ad un altro sono dette "omotope", se una delle due può essere "deformata con continuità" nell'altra e tale trasformazione è detta "omotopia" fra le due funzioni).

Dato che la sfera è la più semplice delle superfici bidimensionali, Poincaré, nel 1904, ipotizzò che così fosse anche in dimensione superiore, congetturando che la tri-sfera è l'unica superficie tridimensionale chiusa (e orientabile, per essere precisi) sulla quale tutte le curve chiuse sono deformabili l'una nell'altra. Una formulazione della congettura di Poincaré a n dimensioni è la seguente: Ogni varietà chiusa n dimensionale omotopicamente equivalente alla n-sfera è omeomorfa alla n-sfera. Questa definizione è equivalente alla congettura di Poincaré nel caso n=3. Le difficoltà maggiori sorgono per le dimensioni n = 3 e n = 4. I matematici cinesi Zhu Xiping e Cao Huaidong, seguendo la strada indicata dal matematico russo G. Perelman, hanno trovato la soluzione di questo grande enigma delle scienze esatte. La congettura di Poincaré avrebbe ripercussioni sulle possibili topologie della teoria delle stringhe e delle varie altre teorie della gravitazione quantistica. Alcune teorie della fisica moderna si formalizzano, infatti, mediante strutture geometriche aventi un numero di dimensioni maggiore di tre: la relatività einsteiniana prevede uno spazio-tempo quadridimensionale, mentre la più recente teoria delle stringhe ipotizza l'esistenza di dieci dimensioni fisiche, sei delle quali «compattificate» in iperspazi minuscoli con una geometria e una topologia molto intricate.

Nella presente tesi vengono evidenziate le connessioni matematiche ottenute tra la Congettura di Poincarè, la Congettura di Geometrizzazione di Thurston, la Teoria di Stringa ed alcuni settori della Teoria dei Numeri.
Vengono prima descritti alcuni fondamentali risultati matematici inerenti la Congettura di Poincarè, ottenuti dal matematico russo G. Perelman e sviluppati dai matematici cinesi Huai-Dong Cao e Xi-Ping Zhu, le cui dimostrazioni possono essere considerate come le conseguenze finali della teoria del flusso di Ricci di Hamilton-Perelman. A questi concetti seguiranno quelli riguardanti la Teoria di Stringa, precisamente, la 3D stringy-gravity per comprendere la congettura di Thurston, i buchi neri tri-dimensionali in teoria di stringa, ed infine l'azione effettiva di una D2-brana frazionaria e quella al contorno di una D3-brana frazionaria
Concluderemo, inoltre, evidenziando le correlazioni ottenute tra alcune equazioni inerenti le Congetture di Poincarè e di Thurston e (i) i settori prima menzionati inerenti la Teoria di Stringa, (ii) alcune formule che riguardano la Teoria dei Numeri, precisamente, π e la Costante di Legendre "c", ed, infine, (iii) il modello di Palumbo applicato alla Teoria di Stringa.

1. Poincarè and Geometrization conjectures. [1]

A Riemannian metric g_{ij} is called **Einstein** if $R_{ij} = \lambda g_{ij}$ for some constant λ. A smooth manifold M with an Einstein metric is called an **Einstein manifold**. If the initial metric is Ricci flat, so that $R_{ij} = 0$, than any Ricci flat metric is a stationary solution of the Ricci flow. This happens, for example, on a flat torus or on any K3-surface with a Calabi-Yau metric.
The equation $R_{ij} + \nabla_i \nabla_j f = 0$, or $Ric + \nabla^2 f = 0$, (1.1) is the **steady gradient Ricci soliton** equation. The equation for a homothetic Ricci soliton (**expanding Ricci soliton**) is

$$2R_{ij} + g_{ik}\nabla_j V^k + g_{jk}\nabla_i V^k - 2\lambda g_{ij} = 0, \quad (1.2)$$

or for a homothetic gradient Ricci soliton,

$$R_{ij} + \nabla_i \nabla_j f - \lambda g_{ij} = 0, \quad (1.3)$$

where λ is the homothetic constant. For $\lambda > 0$ the soliton is **shrinking**, for $\lambda < 0$ it is expanding. The case $\lambda = 0$ is a steady Ricci soliton, the case V = 0 (or f being a constant function) is an Einstein metric. The Ricci flow on a Kahler manifold is called the **Kahler-Ricci flow**. A Ricci soliton to the Kahler-Ricci flow is called a **Kahler-Ricci soliton**.

PROPOSITION A.

On a compact n-dimensional manifold M, a gradient steady or expanding Ricci soliton is necessarily an Einstein metric.
Let g_{ij} be a complete steady gradient Ricci soliton on a manifold M so that

$$R_{ij} + \nabla_i \nabla_j f = 0.$$

Taking the trace, we get $R + \Delta f = 0$. (1.4)

Now, for the equation $\nabla_i R_{jk} - \nabla_j R_{ik} + R_{ijkl}\nabla_l f = 0$,
taking the trace on j and k, and using the contracted second Bianchi identity

$$\nabla_j R_{ij} = \frac{1}{2}\nabla_i R, \quad (1.5)$$

we get $\nabla_i R - 2R_{ij}\nabla_j f = 0$. Then $\nabla_i\left(|\nabla f|^2 + R\right) = 2\nabla_j f\left(\nabla_i\nabla_j f + R_{ij}\right) = 0$.
Therefore

$$R + |\nabla f|^2 = C, \quad (1.6)$$

for some constant C. Taking the difference of (1.4) and (1.6), we get

$$\Delta f - |\nabla f|^2 = -C. \quad (1.7)$$

Then, by integrating (1.7) we obtain

$$\int_M |\nabla f|^2 dV = 0.$$

Therefore f is a constant and g_{ij} is Ricci flat.
We assume M is a compact n-dimensional manifold and consider the following functional

$$F(g_{ij}, f) = \int_M \left(R + |\nabla f|^2\right)e^{-f} dV \quad (1.8)$$

of Perelman defined on the space of Riemannian metrics, and smooth functions on M. Here R is the scalar curvature of g_{ij}.

LEMMA 1. (Perelman).

If $\delta g_{ij} = v_{ij}$ and $\delta f = h$ are variations of g_{ij} and f respectively, then the first variation of F is given by

$$\delta F(v_{ij}, h) = \int_M \left[-v_{ij}\left(R_{ij} + \nabla_i\nabla_j f\right) + \left(\frac{v}{2} - h\right)\left(2\Delta f - |\nabla f|^2 + R\right)\right]e^{-f} dV$$

where $v = g^{ij}v_{ij}$.

PROPOSITION 1. (Perelman).

Let $g_{ij}(t)$ and $f(t)$ evolve according to the coupled flow

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij},$$

$$\frac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - R. \text{ Then}$$

$$\frac{d}{dt}F\left(g_{ij}(t), f(t)\right) = 2\int_M \left|R_{ij} + \nabla_i\nabla_j f\right|^2 e^{-f} dV$$

and $\int_M e^{-f} dV$ *is constant. In particular* $F(g_{ij}(t), f(t))$ *is non-decreasing in time and the monotonicity is strict unless we are on a steady gradient soliton.*

PROPOSITION 2.

On a compact manifold, a steady or expanding breather is necessarily an Einstein metric.

Now, we introduce the following important functional, also due to Perelman,

$$W(g_{ij}, f, \tau) = \int_M \left[\tau\left(R + |\nabla f|^2\right) + f - n\right](4\pi\tau)^{-\frac{n}{2}} e^{-f} dV \quad (1.9)$$

where g_{ij} is a Riemannian metric, f is a smooth function on M, and τ is a positive scale parameter. The functional W is invariant under simultaneous scaling of τ and g_{ij}, and invariant under diffeomorphism. Namely, for any positive number "a" and any diffeomorphism φ

$$W(a\varphi^* g_{ij}, \varphi^* f, a\tau) = W(g_{ij}, f, \tau). \quad (1.10)$$

Similar to Lemma 1, we have the following first variation formula for W.

LEMMA 2. (Perelman).

If $v_{ij} = \delta g_{ij}$, $h = \delta f$, *and* $\eta = \delta\tau$, *then*

$$\delta W(v_{ij}, h, \eta) = \int_M -\tau v_{ij}\left(R_{ij} + \nabla_i \nabla_j f - \frac{1}{2\tau} g_{ij}\right)(4\pi\tau)^{-\frac{n}{2}} e^{-f} dV +$$
$$+ \int_M \left(\frac{v}{2} - h - \frac{n}{2\tau}\eta\right)\left[\tau\left(R + 2\Delta f - |\nabla f|^2\right) + f - n - 1\right](4\pi\tau)^{-\frac{n}{2}} e^{-f} dV + \int_M \eta\left(R + |\nabla f|^2 - \frac{n}{2\tau}\right)(4\pi\tau)^{-\frac{n}{2}} e^{-f} dV .$$

Here $v = g^{ij} v_{ij}$ as before.
The following result is analogous to Proposition 1.

PROPOSITION 3.

If $g_{ij}(t)$, $f(t)$ *and* $\tau(t)$ *evolve according to the system*

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij},$$
$$\frac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau},$$
$$\frac{\partial \tau}{\partial t} = -1,$$

then we have the identity

$$\frac{d}{dt}W\left(g_{ij}(t), f(t), \tau(t)\right) = \int_M 2\tau\left|R_{ij} + \nabla_i\nabla_j f - \frac{1}{2\tau}g_{ij}\right|^2 (4\pi\tau)^{-\frac{n}{2}} e^{-f} dV$$

and $\int_M (4\pi\tau)^{-\frac{n}{2}} e^{-f} dV$ *is constant. In particular* $W\left(g_{ij}(t), f(t), \tau(t)\right)$ *is non-decreasing in time and the monotonicity is strict unless we are on a shrinking gradient soliton.*
Now we set

$$\mu(g_{ij}, \tau) = \inf\left\{W(g_{ij}, f, \tau)\left| f \in C^\infty(M), \frac{1}{(4\pi\tau)^{n/2}}\int_M e^{-f} dV = 1\right.\right\} \quad (1.11)$$

and
$$\nu(g_{ij}) = \inf\left\{W(g, f, \tau)\left| f \in C^\infty(M), \tau > 0, \frac{1}{(4\pi\tau)^{n/2}}\int e^{-f} dV = 1\right.\right\}. \quad (1.12)$$

Note that if we let $u = e^{-f/2}$, then the functional W can be expressed as

$$W(g_{ij}, f, \tau) = \int_M \left[\tau\left(Ru^2 + 4|\nabla u|^2\right) - u^2 \log u^2 - nu^2\right](4\pi\tau)^{-\frac{n}{2}} dV$$

and the constraint $\int_M (4\pi\tau)^{-\frac{n}{2}} e^{-f} dV = 1$ becomes $\int_M u^2 (4\pi\tau)^{-\frac{n}{2}} dV = 1$. Thus $\mu(g_{ij}, \tau)$ corresponds to the best constant of a logarithmic Sobolev inequality. Since the non-quadratic term is sub-critical, it is rather straightforward to show that

$$\inf\left\{\int_M \left[\tau\left(4|\nabla u|^2 + Ru^2\right) - u^2 \log u^2 - nu^2\right](4\pi\tau)^{-\frac{n}{2}} dV \left| \int_M u^2 (4\pi\tau)^{-\frac{n}{2}} dV = 1\right.\right\}$$

is achieved by some nonnegative function $u \in H^1(M)$ which satisfies the Euler-Lagrange equation

$$\tau(-4\Delta u + Ru) - 2u\log u - nu = \mu(g_{ij}, \tau)u.$$

Then, for the (1.11), we have

$$\tau(-4\Delta u + Ru) - 2u\log u - nu = \inf\left\{W(g_{ij}, f, \tau)\left| f \in C^\infty(M), \frac{1}{(4\pi\tau)^{n/2}}\int_M e^{-f} dV = 1\right.\right\}u. \quad (1.13)$$

Now we relating the quantity H (or v) and the W-functional of Perelman defined in (1.9). Observe that v happens to be the integrand of the W-functional,

$$W(g_{ij}(t), f, \tau) = \int_M v dV.$$

Hence, when M is compact,

$$\frac{d}{d\tau}W = \int_M \left(\frac{\partial}{\partial\tau}v + Rv\right)dV = -2\tau\int_M \left|Ric + \nabla^2 f - \frac{1}{2\tau}g\right|^2 u dV \le 0,$$

or equivalently,

$$\frac{d}{dt}W\left(g_{ij}(t), f(t), \tau(t)\right) = \int_M 2\tau \left| R_{ij} + \nabla_i \nabla_j f - \frac{1}{2\tau} g_{ij} \right|^2 \frac{1}{(4\pi\tau)^{n/2}} e^{-f} dV ,$$

which is the same as stated in Proposition 3.

We now use the Ricci-flatness of the metric $\tilde{g}$ to interpret the Bishop-Gromov relative volume comparison theorem which will motivate another monotonicity formula for the Ricci flow.

Consider a metric ball in $\left(\tilde{M}, \tilde{g}\right)$ centred at some point $(p, s, 0) \in \tilde{M}$. Note that the metric of the sphere S^N at $\tau = 0$ degenerates and it shrinks to a point. Then the shortest geodesic $\gamma(\tau)$ between $(p, s, 0)$ and an arbitrary point $(q, \bar{s}, \bar{\tau}) \in \tilde{M}$ is always orthogonal to the S^N fibre. The length of $\gamma(\tau)$ can be computed as

$$\int_0^{\bar{\tau}} \sqrt{\left(\frac{N}{2\tau} + R\right) + \left|\dot{\gamma}(\tau)\right|^2_{g_{ij}(\tau)}} d\tau = \sqrt{2N\bar{\tau}} + \frac{1}{\sqrt{2N}} \int_0^{\bar{\tau}} \sqrt{\tau}\left(R + \left|\dot{\gamma}(\tau)\right|^2_{g_{ij}}\right) d\tau + O\left(N^{-\frac{3}{2}}\right).$$

Thus a shortest geodesic should minimize

$$L(\gamma) = \int_0^{\bar{\tau}} \sqrt{\tau}\left(R + \left|\dot{\gamma}(\tau)\right|^2_{g_{ij}}\right) d\tau .$$

Let $L(q, \bar{\tau})$ denote the corresponding minimum. We claim that a metric sphere $S_{\tilde{M}}\left(\sqrt{2N\bar{\tau}}\right)$ in $\tilde{M}$ of radius $\sqrt{2N\bar{\tau}}$ centred at $(p, s, 0)$ is $O(N^{-1})$-close to the hypersurface $\{\tau = \bar{\tau}\}$. Indeed, if $(x, s', \tau(x))$ lies on the metric sphere $S_{\tilde{M}}\left(\sqrt{2N\bar{\tau}}\right)$, then the distance between $(x, s', \tau(x))$ and $(p, s, 0)$ is

$$\sqrt{2N\bar{\tau}} = \sqrt{2N\tau(x)} + \frac{1}{\sqrt{2N}} L(x, \tau(x)) + O\left(N^{-\frac{3}{2}}\right)$$

which can be written as

$$\sqrt{\tau(x)} - \sqrt{\bar{\tau}} = -\frac{1}{2N} L(x, \tau(x)) + O\left(N^{-2}\right) = O\left(N^{-1}\right).$$

This shows that the metric sphere $S_{\tilde{M}}\left(\sqrt{2N\bar{\tau}}\right)$ is $O\left(N^{-1}\right)-$close to the hypersurface $\{\tau = \bar{\tau}\}$. Thus, we have

$$Vol\left(S_{\tilde{M}}\left(\sqrt{2N\bar{\tau}}\right)\right) \approx (2N)^{\frac{N}{2}} \omega_N \int_M \left(\sqrt{\bar{\tau}} - \frac{1}{2N} L(x, \bar{\tau}) + O\left(N^{-2}\right)\right)^N dV_M$$

$$\approx (2N)^{\frac{N}{2}} \omega_N \int_M \left(\sqrt{\bar{\tau}} - \frac{1}{2N} L(x, \bar{\tau}) + o\left(N^{-1}\right)\right)^N dV_M ,$$

where ω_N is the volume of the standard N-dimensional sphere. Now the volume of Euclidean sphere of radius $\sqrt{2N\bar{\tau}}$ in R^{n+N+1} is

$$Vol\left(S_{R^{n+N+1}}\left(\sqrt{2N\bar{\tau}}\right)\right) = (2N\bar{\tau})^{\frac{N+n}{2}} \omega_{n+N} .$$

Thus we have

$$\frac{Vol\left(S_{\tilde{M}}\left(\sqrt{2N\bar{\tau}}\right)\right)}{Vol\left(S_{R^{n+N+1}}\left(\sqrt{2N\bar{\tau}}\right)\right)} \approx const \cdot N^{-\frac{n}{2}} \cdot \int_M (\bar{\tau})^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\sqrt{\bar{\tau}}} L(x,\bar{\tau})\right\} dV_M .$$

Since the Ricci curvature of $\tilde{M}$ is zero (modulo N^{-1}), the Bishop-Gromov volume comparison theorem suggests that the integral

$$\tilde{V}(\bar{\tau}) \stackrel{\Delta}{=} \int_M (4\pi\bar{\tau})^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\sqrt{\bar{\tau}}} L(x,\bar{\tau})\right\} dV_M ,$$

which we will call Perelman's reduced volume, should be non-increasing in $\bar{\tau}$.
Now the **Li-Yau-Perelman distance** $l = l(q,\bar{\tau})$ is defined by $l(q,\bar{\tau}) = L(q,\bar{\tau})/2\sqrt{\bar{\tau}}$. We thus have the following

LEMMA 3.

For the Li-Yau-Perelman distance $l(q,\bar{\tau})$ defined above, we have

$$\frac{\partial l}{\partial \bar{\tau}} = -\frac{l}{\bar{\tau}} + R + \frac{1}{2\bar{\tau}^{3/2}} K, \ (1.14) \qquad |\nabla l|^2 = -R + \frac{l}{\bar{\tau}} - \frac{1}{\bar{\tau}^{3/2}} K, \ (1.15) \qquad \Delta l \le -R + \frac{n}{2\bar{\tau}} - \frac{1}{2\bar{\tau}^{3/2}} K, \ (1.16)$$

in the sense of distributions. Moreover, the equality in (1.16) holds if and only if we are on a gradient shrinking soliton.

COROLLARY 3.1

Let $g_{ij}(\tau), \tau \ge 0$, be a family of metrics evolving by the Ricci flow $\frac{\partial}{\partial \tau} g_{ij} = 2R_{ij}$ on a compact n-dimensional manifold M. Fix a point p in M and let $l(q,\tau)$ be the Li-Yau-Perelman distance from $(p,0)$. Then for all τ,

$$\min\{l(q,\tau) | q \in M\} \le \frac{n}{2}.$$

As consequence of Lemma 3, we obtain

$$\frac{\partial l}{\partial \bar{\tau}} - \Delta l + |\nabla l|^2 - R + \frac{n}{2\bar{\tau}} \ge 0, \quad \text{or equivalently} \quad \left(\frac{\partial}{\partial \bar{\tau}} - \Delta + R\right)\left[(4\pi\bar{\tau})^{-\frac{n}{2}} \exp(-l)\right] \le 0.$$

If M is compact, we define **Perelman's reduced volume** by

$$\tilde{V}(\tau) = \int_M (4\pi\tau)^{-\frac{n}{2}} \exp[-l(q,\tau)] dV_\tau(q),$$

where dV_τ denotes the volume element with respect to the metric $g_{ij}(\tau)$. Note that Perelman's reduced volume resembles the expression in Huisken's monotonicity formula for the mean curvature flow. It follows that

$$\frac{d}{d\bar{\tau}}\int_M (4\pi\bar{\tau})^{-\frac{n}{2}}\exp(-l(q,\bar{\tau}))dV_\tau(q)=\int_M\left[\frac{\partial}{\partial\bar{\tau}}\left((4\pi\bar{\tau})^{-\frac{n}{2}}\exp(-l(q,\bar{\tau}))\right)+R(4\pi\bar{\tau})^{-\frac{n}{2}}\exp(-l(q,\bar{\tau}))\right]dV_{\bar{\tau}}(q)\leq$$

$$\leq\int_M\Delta\left((4\pi\bar{\tau})^{-\frac{n}{2}}\exp(-l(q,\bar{\tau}))\right)dV_{\bar{\tau}}(q)=0\,.$$

This says that if M is compact, then Perelman's reduced volume $\tilde{V}(\tau)$ in nonincreasing in τ; moreover, the monotonicity is strict unless we are on a gradient shrinking soliton.

THEOREM 1 (Perelman's Jacobian comparison theorem).

Let $g_{ij}(\tau)$ be a family of complete solutions to the Ricci flow $\frac{\partial}{\partial\tau}g_{ij}=2R_{ij}$ on a manifold M with bounded curvature. Let $\gamma:[0,\bar{\tau}]\to M$ be a shortest L-geodesic starting from a fixed point p. Then ***Perelman's reduced volume element*** $(4\pi\tau)^{-\frac{n}{2}}\exp(-l(\tau))J(\tau)$ *is nonincreasing in τ along γ.*

THEOREM 2 (Monotonicity of Perelman's reduced volume).

Let g_{ij} be a family of complete metrics evolving by the Ricci flow $\frac{\partial}{\partial\tau}g_{ij}=2R_{ij}$ on a manifold M with bounded curvature. Fix a point p in M and let $l(q,\tau)$ be the reduced distance from $(p,0)$. Then (i) Perelman's reduced volume $\tilde{V}(\tau)=\int_M(4\pi\tau)^{-\frac{n}{2}}\exp(-l(q,\tau))dV_\tau(q)$ is finite and nonincreasing in τ; (ii) the monotonicity is strict unless we are on a gradient shrinking soliton.

Now, we have

$$\lim_{\tau\to0^+}\tau^{-\frac{n}{2}}J(\tau)=1,\ (1.17)\quad\text{and}\qquad l(0)=|v|^2.\ (1.18)$$

Combining (1.17) and (1.18) with Theorem 1, we get

$$\tilde{V}(\tau)=\int_M(4\pi\tau)^{-\frac{n}{2}}\exp(-l(q,\tau))dV_\tau(q)\leq\int_{T_pM}(4\pi\tau)^{-\frac{n}{2}}\exp(-l(\tau))J(\tau)\big|_{\tau=0}dv=$$

$$=(4\pi)^{-\frac{n}{2}}\int_{R^n}\exp\left(-|v|^2\right)dv<+\infty\ .\ (1.18a)$$

This proves that Perelman's reduced volume is always finite and hence well defined.

THEOREM 3 (No local collapsing theorem I).

Given any metric g_{ij} on an n-dimensional compact manifold M. Let $g_{ij}(t)$ be the solution to the Ricci flow on $[0,T)$, with $T<+\infty$, starting at g_{ij}. Then there exist positive constants κ and ρ_0 such that for any $t_0\in[0,T)$ and any point $x_0\in M$, the solution $g_{ij}(t)$ is κ-noncollapsed at (x_0,t_0) on all scales less than ρ_0.

We have that for k large enough, $\tilde{V}_k(\varepsilon_k r_k^2) \leq 2\varepsilon_k^{\frac{n}{2}}$. (Step 1)

We estimate the integral of $\tilde{V}_k(\varepsilon_k r_k^2)$ as follows,

$$\tilde{V}_k(\varepsilon_k r_k^2) = \int_M (4\pi\varepsilon_k r_k^2)^{-\frac{n}{2}} \exp(-l(q,\varepsilon_k r_k^2)) dV_{t_k-\varepsilon_k r_k^2}(q) = \int_{L\exp_{\{|v|\leq\frac{1}{4}\varepsilon_k^{-1/2}\}}(\varepsilon_k r_k^2)} (4\pi\varepsilon_k r_k^2)^{-\frac{n}{2}} \exp(-l(q,\varepsilon_k r_k^2)) dV_{t_k-\varepsilon_k r_k^2}(q) +$$

$$+ \int_{M\setminus L\exp_{\{|v|\leq\frac{1}{4}\varepsilon_k^{-1/2}\}}(\varepsilon_k r_k^2)} (4\pi\varepsilon_k r_k^2)^{-\frac{n}{2}} \exp(-l(q,\varepsilon_k r_k^2)) dV_{t_k-\varepsilon_k r_k^2}(q). \qquad (1.19)$$

The second term on the right hand side of (3.19) can be estimated as follows

$$\int_{M\setminus L\exp_{\{|v|\leq\frac{1}{4}\varepsilon_k^{-1/2}\}}(\varepsilon_k r_k^2)} (4\pi\varepsilon_k r_k^2)^{-\frac{n}{2}} \exp(-l(q,\varepsilon_k r_k^2)) dV_{t_k-\varepsilon_k r_k^2}(q) \leq \int_{\{|v|>\frac{1}{4}\varepsilon_k^{-1/2}\}} (4\pi\tau)^{-\frac{n}{2}} \exp(-l(\tau)) I(\tau)\big|_{\tau=0} dv$$

$$= (4\pi)^{-\frac{n}{2}} \int_{\{|v|>\frac{1}{4}\varepsilon_k^{-1/2}\}} \exp(-|v|^2) dv \leq \varepsilon_k^{\frac{n}{2}}, \quad (1.20)$$

for k sufficiently large. Combining (1.19)-(1.20), we have the relation of Step 1.
We next have that (Step 2)

$$\tilde{V}_k(t_k) = (4\pi t_k)^{-\frac{n}{2}} \int_M \exp(-l(q,t_k)) dV_0(q) > C' \quad (1.20a)$$

for all k, where C' is some positive constant independent of k. It suffices to show the Li-Yau-Perelman distance $l(\cdot,t_k)$ is uniformly bounded from above on M. Combining Step 1 with Step 2, and using the monotonicity of $\tilde{V}_k(\tau)$, we get $C' < \tilde{V}_k(t_k) \leq \tilde{V}_k(\varepsilon_k r_k^2) \leq 2\varepsilon_k^{\frac{n}{2}} \to 0$ as $k \to \infty$.

DEFINITION 1

A solution $g_{ij}(x,t)$ to the Ricci flow on the manifold M, where either M is compact or at each time t the metric $g_{ij}(\cdot,t)$ is complete and has bounded curvature, is called a **singularity model** if it is not flat and of one of the following three types:
Type I: The solution exists for $t \in (-\infty, \Omega)$ for some constant Ω with $0 < \Omega < +\infty$ and $|Rm| \leq \Omega/(\Omega - t)$ everywhere with equality somewhere at t = 0;
Type II: The solution exists for $t \in (-\infty, +\infty)$ and $|Rm| \leq 1$ everywhere with equality somewhere at t = 0;
Type III: The solution exists for $t \in (-A, +\infty)$ for some constant A with $0 < A < +\infty$ and $|Rm| \leq A/(A + t)$ everywhere with equality somewhere at t = 0.

We state a result of Sesum on compact Type I singularity models. Recall that Perelman's functional W is given by

$$W(g,f,\tau)=\int_M (4\pi\tau)^{-\frac{n}{2}}\left[\tau\left(|\nabla f|^2+R\right)+f-n\right]e^{-f}dV_g$$

with the function f satisfying the constraint $\int_M (4\pi\tau)^{-\frac{n}{2}}e^{-f}dV_g=1$.
Furthermore, we have also that

$$\mu(g(t))=\inf\left\{W(g(t),f,T-t)\middle|\int_M (4\pi(T-t))^{-\frac{n}{2}}e^{-f}dV_{g(t)}=1\right\}.$$

As shown by Natasa Sesum, Type I assumption guarantees the boundedness of $\mu(g(t))$, while the compactness assumption of the rescaling limit guarantees the existence of the limit for the minimizing functions $f(\cdot,t)$. Therefore we have

THEOREM 4 (Sesum).

Let $(M,g_{ij}(t))$ be a Type I singularity model obtained as a rescaling limit of a Type I maximal solution. Suppose M is compact: then $(M,g_{ij}(t))$ must be a gradient shrinking Ricci soliton.

THEOREM 5 (Long-time existence theorem proposed by Perelman).

For any fixed constant $\varepsilon>0$, there exist nonincreasing (continuous) positive functions $\tilde{\delta}(t)$ and $\tilde{r}(t)$, defined on $[0,+\infty)$, such that for an arbitrarily given (continuous) positive function $\delta(t)$ with $\delta(t)\leq\tilde{\delta}(t)$ on $[0,+\infty)$, and arbitrarily given a compact orientable normalized three-manifold as initial data, the Ricci flow with surgery has a solution with the following properties: either

- *(i)* *it is defined on a finite interval $[0,T)$ and obtained by evolving the Ricci flow and by performing a finite number of cutoff surgeries, with each δ-cutoff at a time $t\in(0,T)$ having $\delta=\delta(t)$, so that the solution becomes extinct at the finite time T, and the initial manifold is diffeomorphic to a connected sum of a finite copies of $S^2\times S^1$ and S^3/Γ (the metric quotients of round three-sphere); or*
- (ii) *it is defined on $[0,+\infty)$ and obtained by evolving the Ricci flow and by performing at most countably many cutoff surgeries, with each δ-cutoff at a time $t\in[0,+\infty)$ having $\delta=\delta(t)$, so that the pinching assumption and the canonical neighbourhood assumption (with accuracy ε) with $r=\tilde{r}(t)$ are satisfied, and there exist at most a finite number of surgeries on every finite time interval.*

The famous **Poincarè conjecture** states that every compact three-manifold with trivial fundamental group is diffeomorphic to S^3. Let M be a compact three-manifold with trivial fundamental group. In particular, the three-manifold M is orientable. Arbitrarily given a Riemannian metric on M, by scaling we may assume the metric is normalized. With this normalized metric as initial data, we consider the solution to the Ricci flow with surgery. If one can show the solution becomes extinct in finite time, it will follow from Theorem 5 (i) that the three-manifold M is diffeomorphic to the three-sphere S^3. Such finite extinction time result was first proposed by Perelman, and, recently, Colding-Minicozzi has published a proof to it. **The combination of Theorem 5 (i) and Colding-Minicozzi's finite extinction result, gives a complete proof of the Poincarè conjecture**.
Now, we have:

THEOREM 6 (Perelman).

For any $\varepsilon > 0$ *and* $1 \le A < +\infty$, *one can find* $\kappa = \kappa(A,\varepsilon) > 0$, $K_1(A,\varepsilon) < +\infty$, $K_2 = K_2(A,\varepsilon) < +\infty$ *and* $\bar{r} = \bar{r}(A,\varepsilon) > 0$ *such that for any* $t_0 < +\infty$ *there exists* $\bar{\delta}_A = \bar{\delta}_A(t_0) > 0$ *(depending also on* ε*), nonincreasing in* t_0*, with the following property. Suppose we have a solution, constructed by Theorem 5 with the nonincreasing (continuous) positive functions* $\tilde{\delta}(t)$ *and* $\tilde{r}(t)$*, to the Ricci flow with* δ*-cutoff surgeries on time interval* $[0,T]$ *and with a compact orientable normalized three-manifold as initial data, where each* δ*-cutoff at a time t satisfies* $\delta = \delta(t) \le \tilde{\delta}(t)$ *on* $[0,T]$ *and* $\delta = \delta(t) \le \bar{\delta}_A$ *on* $\left[\frac{t_0}{2}, t_0\right]$*; assume that the solution is defined on the whole parabolic neighbourhood* $P(x_0,t_0,r_0,-r_0^2) \stackrel{\Delta}{=} \{(x,t) | x \in B_t(x_0,r_0), t \in [t_0 - r_0^2, t_0]\}$, $2r_0^2 < t_0$, *and satisfies* $|Rm| \le r_0^{-2}$ *on* $P(x_0,t_0,r_0,-r_0^2)$*, and* $Vol_{t_0}(B_{t_0}(x_0,r_0)) \ge A^{-1} r_0^3$*. Then*

(i) *the solution is* κ*-noncollapsed on all scales less than* r_0 *in the ball* $B_{t_0}(x_0, Ar_0)$*;*

(ii) *every point* $x \in B_{t_0}(x_0, Ar_0)$ *with* $R(x,t_0) \ge K_1 r_0^{-2}$ *has a canonical neighbourhood B, with* $B_{t_0}(x,\sigma) \subset B \subset B_{t_0}(x,2\sigma)$ *for some* $0 < \sigma < C_1(\varepsilon) R^{-\frac{1}{2}}(x,t_0)$*, which is either a strong* ε*-neck or an* ε*-cap;*

(iii) *if* $r_0 \le \bar{r}\sqrt{t_0}$ *then* $R \le K_2 r_0^{-2}$ *in* $B_{t_0}(x_0, Ar_0)$*.*

Here $C_1(\varepsilon)$ *is the positive constant in the canonical neighbourhood assumption.*

Now, we have that every shortest L-geodesic from (x,t_0) to the ball $B_{t_0-r_0^2}(x_0,r_0)$ is necessarily admissible. By combining with the assumption that $Vol_{t_0}(B_{t_0}(x_0,r_0)) \ge A^{-1} r_0^3$, we conclude that Perelman's reduced volume of the ball $B_{t_0-r_0^2}(x_0,r_0)$ satisfies the estimate

$$\tilde{V}_{r_0^2}\left(B_{t_0-r_0^2}(x_0,r_0)\right) = \int_{B_{t_0-r_0^2}(x_0,r_0)} (4\pi r_0^2)^{-\frac{3}{2}} \exp(-l(q,r_0^2))\, dV_{t_0-r_0^2}(q) \ge c(A) \quad (1.21)$$

for some positive constant $c(A)$ depending only on A. The union of all shortest L-geodesics from (x,t_0) to the ball $B_{t_0-r_0^2}(x_0,r_0)$, defined by $CB_{t_0-r_0^2}(x_0,r_0) = \{(y,t) | (y,t)$ lies in a shortest L-geodesic from (x,t_0) to a point in $B_{t_0-r_0^2}(x_0,r_0)\}$, forms a cone-like subset in space-time with vertex (x,t_0). Denote by $B(t)$ the intersection of the cone-like subset $CB_{t_0-r_0^2}(x_0,r_0)$ with the time-slice at t. Perelman's reduced volume of the subset $B(t)$ is given by

$$\tilde{V}_{t_0-t}(B(t)) = \int_{B(t)} (4\pi(t_0-t))^{-\frac{3}{2}} \exp(-l(q,t_0-t))\, dV_t(q).$$

Since the cone-like subset $CB_{t_0-r_0^2}(x_0,r_0)$ lies entirely in the region unaffected by surgery, we can apply Perelman's Jacobian comparison Theorem 1 and the estimate (1.21) to conclude that

$$\tilde{V}_{t_0-t}(B(t)) \geq \tilde{V}_{r_0^2}\left(B_{t_0-r_0^2}(x_0,r_0)\right) \geq c(A) \quad (1.22)$$

for all $t \in [t_0 - r_0^2, t_0]$. Consider $\tilde{B}(t_0-\xi\rho^2)$, the subset at the time-slice $\{t = t_0 - \xi\rho^2\}$ where every point can be connected to (x,t_0) by an admissible shortest L-geodesic. Perelman's reduced volume of $\tilde{B}(t_0-\xi\rho^2)$ is given by

$$\begin{aligned}\tilde{V}_{\xi\rho^2}\left(\tilde{B}(t_0-\xi\rho^2)\right) &= \int_{\tilde{B}(t_0-\xi\rho^2)} (4\pi\xi\rho^2)^{-\frac{3}{2}} \exp\left(-l(q,\xi\rho^2)\right) dV_{t_0-\xi\rho^2}(q) = \\ &= \int_{\tilde{B}(t_0-\xi\rho^2)\cap L\exp_{\left\{|v|\leq\frac{1}{4}\xi^{-\frac{1}{2}}\right\}}(\xi\rho^2)} (4\pi\xi\rho^2)^{-\frac{3}{2}} \exp\left(-l(q,\xi\rho^2)\right) dV_{t_0-\xi\rho^2}(q) + \\ &+ \int_{\tilde{B}(t_0-\xi\rho^2)\setminus L\exp_{\left\{|v|\leq\frac{1}{4}\xi^{-\frac{1}{2}}\right\}}(\xi\rho^2)} (4\pi\xi\rho^2)^{-\frac{3}{2}} \exp\left(-l(q,\xi\rho^2)\right) dV_{t_0-\xi\rho^2}(q). \quad (1.23)\end{aligned}$$

Note that the whole region $P(x,t_0,\rho,-\rho^2)$ is unaffected by surgery because $\rho \geq \frac{1}{2\eta}\tilde{r}(t_0)$ and $\bar{\delta}\left(L, t_0, \tilde{r}(t_0), \tilde{r}\left(\frac{t_0}{2}\right), \varepsilon\right) > 0$ is sufficiently small. Then, there is a universal positive constant ξ_0 such that when $0 < \xi \leq \xi_0$, there holds $L\exp_{\left\{|v|\leq\frac{1}{4}\xi^{-\frac{1}{2}}\right\}}(\xi\rho^2) \subset B_{t_0}(x,\rho)$ and the first term on right hand side of (1.23) can be estimated by

$$\int_{\tilde{B}(t_0-\xi\rho^2)\cap L\exp_{\left\{|v|\leq\frac{1}{4}\xi^{-\frac{1}{2}}\right\}}(\xi\rho^2)} (4\pi\xi\rho^2)^{-\frac{3}{2}} \exp\left(-l(q,\xi\rho^2)\right) dV_{t_0-\xi\rho^2}(q) \leq e^{C\xi}(4\pi)^{-\frac{3}{2}}\xi^{\frac{3}{2}} \quad (1.24)$$

for some universal constant C; while the second term on right hand side of (1.23) can be estimated by

$$\int_{\tilde{B}(t_0-\xi\rho^2)\setminus L\exp_{\left\{|v|\leq\frac{1}{4}\xi^{-\frac{1}{2}}\right\}}(\xi\rho^2)} (4\pi\xi\rho^2)^{-\frac{3}{2}} \exp\left(-l(q,\xi\rho^2)\right) dV_{t_0-\xi\rho^2}(q) \leq (4\pi)^{-\frac{3}{2}} \int_{\left\{|v|>\frac{1}{4}\xi^{-\frac{1}{2}}\right\}} \exp\left(-|v|^2\right) dv. \quad (1.25)$$

Since $B(t_0-\xi\rho^2) \subset \tilde{B}(t_0-\xi\rho^2)$, the combination of (1.22)-(1.25) bounds ξ from below by a positive constant depending only on A. This proves the statement (i).

2. Ricci flow on compact four-manifolds with positive isotropic curvature. [2]

Let M^4 be a compact four-manifold with no essential incompressible space-form and with a metric g_{ij} of positive isotropic curvature.

THEOREM 2.1

There exists a positive constant κ_0 *with the following property. Suppose we have a four-dimensional (compact or noncompact) ancient* κ*-solution with restricted isotropic curvature pinching for some* $\kappa > 0$*. Then either the solution is* κ_0*-noncollapsed for all scales, or it is a metric quotient of the round cylinder* $R \times S^3$.

Let $g_{ij}(x,t)$, $x \in M^4$ and $t \in (-\infty, 0]$, be an ancient κ-solution with restricted isotropic curvature pinching for some $\kappa > 0$. For arbitrary point $(p, t_0) \in M^4 \times (-\infty, 0]$, we define that $\tau = t_0 - t$, for $t < t_0$,

$$l(q,\tau) = \frac{1}{2\sqrt{\tau}} \inf \left\{ \int_0^\tau \sqrt{s} \left(R(\gamma(s), t_0 - s) + |\dot{\gamma}(s)|^2_{g_{ij}(t_0 - s)} \right) ds \,\middle|\, \gamma : [0,\tau] \to M^4 \, with \, \gamma(0) = p, \gamma(\tau) = q \right\},$$

and

$$\tilde{V}(\tau) = \int_{M^4} (4\pi\tau)^{-2} \exp(-l(q,\tau)) dV_{t_0 - \tau}(q),$$

where $|\cdot|_{g_{ij}(t_0 - s)}$ is the norm with respect to the metric $g_{ij}(t_0 - s)$ and $dV_{t_0 - \tau}$ is the volume element with respect to the metric $g_{ij}(t_0 - \tau)$. Here, l is called the **reduced distance** and $\tilde{V}(\tau)$ is called the **reduced volume**. For any $A \geq 1$, one can find $B = B(A) < +\infty$ such that for every $\bar{\tau} > 1$ there holds

$$l(q,\tau) < B \text{ and } \tau R(q, t_0 - \tau) \leq B \quad (2.1)$$

whenever $\frac{1}{2}\bar{\tau} \leq \tau \leq A\bar{\tau}$ and $d^2_{t_0 - \frac{\bar{\tau}}{2}}\left(q, q\left(\frac{\bar{\tau}}{2}\right)\right) \leq A\bar{\tau}$.

Considering the reduced volume $\tilde{V}(\tau)$ of the ancient κ-solution, we have from Perelman's Jacobian comparison theorem that

$$\tilde{V}(\tau) = \int_{M^4} (4\pi\tau)^{-2} e^{-l} dV_{t_0 - \tau} \leq \int_{T_p M^4} (4\pi)^{-2} e^{-|X|^2} dX = 1.$$

Now, we denote by $\varepsilon = Vol_{t_0}(B_{t_0}(p,1))^{\frac{1}{4}}$. For any $v \in T_p M^4$, we have that one can find a L-geodesic $\gamma(\tau)$, starting at p, with $\lim_{\tau \to 0^+} \sqrt{\tau}\dot{\gamma}(\tau) = v$, which satisfies the following L-geodesic equation

$$\frac{d}{d\tau}(\sqrt{\tau}\dot{\gamma}) - \frac{1}{2}\sqrt{\tau}\nabla R + 2Ric(\sqrt{\tau}\dot{\gamma}, \cdot) = 0. \quad (2.2)$$

By integrating the L-geodesic equation we see that as $\tau \leq \varepsilon$ with the property that $\gamma(\sigma) \in B_{t_0}(p,1)$ for $\sigma \in (0, \tau]$, there holds

$$\left| \sqrt{\tau}\dot{\gamma}(\tau) - v \right| \leq C\varepsilon(|v| + 1) \quad (2.3)$$

for some universal positive constant C.

Now, for $v \in T_p M^4$ with $|v| \le \frac{1}{4}\varepsilon^{-\frac{1}{2}}$ and for $\tau \le \varepsilon$ with the property that $\gamma(\sigma) \in B_{t_0}(p,1)$ for $\sigma \in (0,\tau]$, we have

$$d_{t_0}(p,\gamma(\tau)) \le \int_0^\tau |\dot{\gamma}(\sigma)| d\sigma < \frac{1}{2}\varepsilon^{-\frac{1}{2}} \int_0^\tau \frac{d\sigma}{\sqrt{\sigma}} = 1 .$$

This shows

$$L\exp\left\{|v| \le \frac{1}{4}\varepsilon^{-\frac{1}{2}}\right\}(\varepsilon) \subset B_{t_0}(p,1) \quad (2.4)$$

where $L\exp(\cdot)(\varepsilon)$ denotes the exponential map of the L distance with parameter ε. We decompose the reduced volume $\tilde{V}(\varepsilon)$ as

$$\tilde{V}(\varepsilon) = \int_{M^4} (4\pi\varepsilon)^{-2} \exp(-l) dV_{t_0-\varepsilon} \le \int_{L\exp\left\{|v| \le \frac{1}{4}\varepsilon^{-\frac{1}{2}}\right\}(\varepsilon)} + \int_{M^4 \setminus L\exp\left\{|v| \le \frac{1}{4}\varepsilon^{-\frac{1}{2}}\right\}(\varepsilon)} (4\pi\varepsilon)^{-2} \exp(-l) dV_{t_0-\varepsilon} \quad (2.5)$$

The first term on right hand side of (2.5) can be estimated by

$$\int_{L\exp\left\{|v| \le \frac{1}{4}\varepsilon^{-\frac{1}{2}}\right\}(\varepsilon)} (4\pi\varepsilon)^{-2} \exp(-l) dV_{t_0-\varepsilon} \le e^{4\varepsilon} \int_{B_{t_0}(p,1)} (4\pi\varepsilon)^{-2} \exp(-l) dV_{t_0} \le$$

$$\le e^{4\varepsilon} (4\pi)^{-2} \varepsilon^{-2} Vol_{t_0}\left(B_{t_0}(p,1)\right) = e^{4\varepsilon} (4\pi)^{-2} \varepsilon^2 . \quad (2.6)$$

where we used (2.4) and the equivalence of the evolving metric over $B_{t_0}(p,1)$.
While the second term on the right hand side of (2.5) can be estimated as follows

$$\int_{M^4 \setminus L\exp\left\{|v| \le \frac{1}{4}\varepsilon^{-\frac{1}{2}}\right\}(\varepsilon)} (4\pi\varepsilon)^{-2} \exp(-l) dV_{t_0-\varepsilon} \le \int_{\left\{|v| > \frac{1}{4}\varepsilon^{-\frac{1}{2}}\right\}} (4\pi\tau)^{-2} \exp(-l) J(\tau)\Big|_{\tau=0} dv \quad (2.7)$$

by Perelman's Jacobian comparison theorem, where $J(\tau)$ is the Jacobian of the L-exponential map.
To evaluate $l(\cdot,\tau)$ at $\tau = 0$, we use (2.3) again to get

$$l(\cdot,\tau) = \frac{1}{2\sqrt{\tau}} \int_0^\tau \sqrt{s}\left(R + |\dot{\gamma}(s)|^2\right) ds \to |v|^2 , \quad \text{as} \quad \tau \to 0^+ , \quad \text{thus}$$

$$l(\cdot,0) = |v|^2 . \quad (2.8)$$

Hence by combining (2.7)-(2.8) we have

$$\int_{M^4 \setminus L\exp\left\{|v| \le \frac{1}{4}\varepsilon^{-\frac{1}{2}}\right\}(\varepsilon)} (4\pi\varepsilon)^{-2} \exp(-l) dV_{t_0-\varepsilon} \le (4\pi)^{-2} \int_{\left\{|v| > \frac{1}{4}\varepsilon^{-\frac{1}{2}}\right\}} \exp\left(-|v|^2\right) dv < \varepsilon^2 . \quad (2.9)$$

By summing up (2.5), (2.6) and (2.9), we obtain

$$\tilde{V}(\varepsilon) < 2\varepsilon^2 . \quad (2.10)$$

Now, by the estimates (2.1) and the κ'_0-noncollapsing of the shrinking soliton, we get

$$\tilde{V}(2\tau_k)=\int_M (4\pi(2\tau_k))^{-2}\exp(-l(q,2\tau_k))dV_{t_0-2\tau_k}(q)\geq \int_{B_{t_0-\tau_k}(q(\tau_k),\sqrt{\tau_k})}(4\pi(2\tau_k))^{-2}\exp(-l(q,2\tau_k))dV_{t_0-2\tau_k}(q)\geq\beta$$

for some universal positive constant β. By applying the monotonicity of the reduced volume and (2.10), we deduce that

$$\beta\leq\tilde{V}(2\tau_k)\leq\tilde{V}(\varepsilon)<2\varepsilon^2 .$$

This proves

$$Vol_{t_0}(B_{t_0}(p,1))\geq\kappa_0>0$$

for some universal positive constant κ_0.

LEMMA 2.1

Given $0<\varepsilon<\frac{1}{100}$, $0<\delta<\varepsilon$ *and* $0<T<+\infty$, *there exists a radius* $0<h<\delta\sigma$, *depending only on* $\delta, r(T)$ *and the pinching assumption, such that if we have a solution to the Ricci flow with surgery, with a compact four-manifold* $(M^4, g_{ij}(x))$ *with no essential incompressible space form and with positive isotropic curvature as initial data, defined on* $[0,T)$, *going singular at the time T, satisfies the a priori assumptions and has only a finite number of surgery times on* $[0,T)$, *then for each point x with* $h(x)=\overline{R}^{-\frac{1}{2}}(x)\leq h$ *in an* ε*-horn of* $(\Omega,\overline{g}_{ij})$ *with boundary in* Ω_σ, *the neighbourhood* $B_T(x,\delta^{-1}h(x))=\{y\in\Omega \mid dist_{\overline{g}_{ij}}(y,x)\leq\delta^{-1}h(x)\}$ *is a strong* δ*-neck (i.e.,* $\{(y,t)\mid y\in B_T(x,\delta^{-1}h(x)), t\in[T-h^2(x),T]\}$ *is, after scaling with factor* $h^{-2}(x)$, δ*-close (in* $C^{[\delta^{-1}]}$ *topology) to the corresponding subset of the evolving standard round cylinder* $S^3\times R$ *over the time interval* $[-1,0]$ *with scalar curvature 1 at the time zero).*

PROPOSITION 2.1

Given a compact four-manifold with positive isotropic curvature and no essential incompressible space form and given $\varepsilon>0$, *there exist decreasing sequences* $\varepsilon>\tilde{r}_j>0$, $\kappa_j>0$, $\min\{\varepsilon^2,\delta_0,\overline{\delta}\}>\tilde{\delta}_j>0$, $j=1,2,...,$ *with the following property. Define a positive function* $\tilde{\delta}(t)$ *on* $[0,+\infty)$ *by* $\tilde{\delta}(t)=\tilde{\delta}_j$ *when* $t\in[(j-1)\varepsilon^2, j\varepsilon^2)$. *Suppose we have a solution to the Ricci flow with surgery, with the given four-manifold as initial datum defined on the time interval* $[0,T)$ *and with a finite number of* δ*-cutoff surgeries such that any* δ*-cutoff surgery at a time* $t\in(0,T)$ *with* $\delta=\delta(t)$ *satisfies* $0<\delta(t)\leq\tilde{\delta}(t)$. *Then on each the time interval* $[(j-1)\varepsilon^2, j\varepsilon^2]\cap[0,T)$, *the solution satisfies the* κ_j*-noncollapsing condition on all scales less than* ε *and the canonical neighbourhood assumption (with accuracy* ε*) with* $r=\tilde{r}_j$.

LEMMA 2.2

For a given compact four-manifold with positive isotropic curvature and no essential incompressible space form and given $\varepsilon>0$, *suppose we have constructed the sequences, satisfying the above proposition for* $1\leq j\leq l$. *Then there exists* $\kappa>0$, *such that for any r,* $0<r<\varepsilon$, *one can* *find* $\tilde{\delta}$ *with* $0<\tilde{\delta}<\min\{\varepsilon^2,\delta_0,\bar{\delta}\}$, *which depends on r,* ε *and may also depend on the already constructed sequences, with the following property. Suppose we have a solution, with the given four-manifold as initial data, to the Ricci flow with surgery defined on a time interval* $[0,T]$ *with* $l\varepsilon^2\le T<(l+1)\varepsilon^2$ *such that the assumptions and conclusions of Proposition 2.1 hold on* $[0,l\varepsilon^2)$, *the canonical neighbourhood assumption (with accuracy* ε*) with r holds on* $[l\varepsilon^2,T]$, *and each* $\delta(t)$*-cutoff surgery in the time interval* $t\in[(l-1)\varepsilon^2,T]$ *has* $0<\delta(t)<\tilde{\delta}$. *Then the solution is* κ*-noncollapsed on* $[0,T]$ *for all scales less than* ε.

CLAIM 2.1

For any $L<+\infty$ *one can find* $\tilde{\delta}=\tilde{\delta}(L,r,\tilde{r}_l,\varepsilon)>0$ *with the following property. Suppose that we have a curve* γ, *parametrized by* $t\in[T_0,t_0]$, $(l-1)\varepsilon^2\le T_0<t_0$, *such that* $\gamma(t_0)=x_0$, T_0 *is a surgery time and* $\gamma(T_0)$ *lies in a 4h-collar of the middle three-sphere of a* δ*-neck with the radius h obtained in Lemma 2.1, where the* δ*-cutoff surgery was taken. Suppose also each* $\delta(t)$*-cutoff surgery in the time interval* $t\in[(l-1)\varepsilon^2,T]$ *has* $0<\delta(t)<\tilde{\delta}$. *The we have an estimate*

$$\int_0^{t_0-T_0}\sqrt{\tau}\left(R(\gamma(t_0-\tau),t_0-\tau)+|\dot{\gamma}(t_0-\tau)|^2_{g_{ij}(t_0-\tau)}\right)d\tau\ge L,\quad (2.11)$$

where $\tau=t_0-t\in[0,t_0-T_0]$.

Now choose L = 100 in (2.11), then it follows from Claim 2.1 that there exists $\tilde{\delta}>0$, depending on r and $\tilde{r}_l$, such that as each δ-cutoff surgery at the time interval $t\in[(l-1)\varepsilon^2,T]$ has $\delta<\tilde{\delta}$, every barely admissible curve γ with endpoints (x_0,t_0) and (x,t), where $t\in[(l-1)\varepsilon^2,t_0)$, has

$$L(\gamma)=\int_0^{t_0-t}\sqrt{\tau}\left(R(\gamma(\tau),t_0-\tau)+|\dot{\gamma}(\tau)|^2_{g_{ij}(t_0-\tau)}\right)d\tau\ge 100,$$

which implies the reduced distance from (x_0,t_0) to (x,t) satisfies $l\ge 25\varepsilon^{-1}$. (2.12)
We also observe that, there exists a minimizing curve γ of $l_{\min}(t_0-(l-1)\varepsilon^2)$, defined on $\tau\in[0,t_0-(l-1)\varepsilon^2]$ with $\gamma(0)=x_0$, such that

$$L(\gamma)\le 2\cdot\left(2\sqrt{2}\varepsilon\right)<10\varepsilon.\quad (2.13)$$

Now we want to get a lower bound for the reduced volume of a ball around $\bar{x}$ of radius about $\tilde{r}_l$ at some time-slice slightly before $\bar{t}$. Since the solution satisfies the canonical neighbourhood assumption on the time interval $[(l-1)\varepsilon^2,l\varepsilon^2)$, it follows that

$$R(x,t)\le 400\tilde{r}_l^{-2}\quad (2.14)$$

for those $(x,t)\in P\left(\bar{x},\bar{t},\frac{1}{16}\eta^{-1}\tilde{r}_l,-\frac{1}{64}\eta^{-1}\tilde{r}_l^2\right)$ for which the solution is defined. Thus by combining (2.13) and (2.14), the reduced distance from (x_0,t_0) to each point of the ball $B_{\bar{t}-\frac{1}{64}\eta^{-1}\tilde{r}_l^2}\left(\bar{x},\frac{1}{16}\eta^{-1}\tilde{r}_l\right)$ is uniformly bounded by some universal constant.

We want to get a lower bound estimate for the volume of the ball $B_{t_0}(x_0,r_0)$. The reduced distance from (x_0,t_0) to each point of the ball $B_{\bar{t}-\frac{1}{64}\eta^{-1}\tilde{r}_l^2}\left(\bar{x},\frac{1}{16}\eta^{-1}\tilde{r}_l\right)$ is uniformly bounded by some universal constant. We may assume $\varepsilon>0$ is very small. Then it follows from (2.12) that the points in the ball $B_{\bar{t}-\frac{1}{64}\eta^{-1}\tilde{r}_l^2}\left(\bar{x},\frac{1}{16}\eta^{-1}\tilde{r}_l\right)$ can be connected to (x_0,t_0) by shortest L- geodesics, and all of these L-geodesics are admissible. The union of all shortest L- geodesics from (x_0,t_0) to the ball $B_{\bar{t}-\frac{1}{64}\eta^{-1}\tilde{r}_l^2}\left(\bar{x},\frac{1}{16}\eta^{-1}\tilde{r}_l\right)$ denoted by $CB_{\bar{t}-\frac{1}{64}\eta^{-1}\tilde{r}_l^2}\left(\bar{x},\frac{1}{16}\eta^{-1}\tilde{r}_l\right)$, forms a cone-like subset in space-time with the vertex (x_0,t_0). Denote B(t) by the intersection of $CB_{\bar{t}-\frac{1}{64}\eta^{-1}\tilde{r}_l^2}\left(\bar{x},\frac{1}{16}\eta^{-1}\tilde{r}_l\right)$ with the time-slice at t. The reduced volume of the subset B(t) is defined by

$$\tilde{V}_{t_0-t}(B(t))=\int_{B(t)}(4\pi(t_0-t))^{-2}\exp(-l(q,t_0-t))dV_t(q).$$

Since the cone-like subset $CB_{\bar{t}-\frac{1}{64}\eta^{-1}\tilde{r}_l^2}\left(\bar{x},\frac{1}{16}\eta^{-1}\tilde{r}_l\right)$ lies entirely in the region unaffected by surgery, we can apply Perelman's Jacobian comparison to conclude that

$$\tilde{V}_{t_0-t}(B(t))\geq\tilde{V}_{t_0-\bar{t}+\frac{1}{64}\eta^{-1}\tilde{r}_l^2}\left(B_{\bar{t}-\frac{1}{64}\eta^{-1}\tilde{r}_l^2}\left(\bar{x},\frac{1}{16}\eta^{-1}\tilde{r}_l\right)\right)\geq c(\kappa_l,\tilde{r}_l)\quad(2.15)$$

for all $t\in\left[\bar{t}-\frac{1}{64}\eta^{-1}\tilde{r}_l^2,t_0\right]$, where $c(\kappa_l,\tilde{r}_l)$ is some positive constant depending only on κ_l and $\tilde{r}_l$.

Denote by $\xi=r_0^{-1}V_0l_{t_0}\left(B_{t_0}(x_0,r_0)\right)^{\frac{1}{4}}$. Our purpose is to give a positive lower bound for ξ. We may assume $\xi<\frac{1}{4}$, thus $0<\xi r_0^2<t_0-\bar{t}+\frac{1}{64}\eta^{-1}\tilde{r}_l^2$. Furthermore, we denote by $\tilde{B}(t_0-\xi r_0^2)$ the subset of the points at the time-slice $\{t=t_0-\xi r_0^2\}$ where every point can be connected to (x_0,t_0) by an admissible shortest L- geodesic. Clearly $B(t_0-\xi r_0^2)\subset\tilde{B}(t_0-\xi r_0^2)$. Since $r_0\geq\frac{1}{2\eta}r$ and $\tilde{\delta}=\tilde{\delta}(r,\tilde{r}_l,\varepsilon)$ sufficiently small, the region $P(x_0,t_0,r_0,-r_0^2)$ is unaffected by surgery. Then by the exactly same argument as deriving (2.4), we see that there exists a universal positive constant ξ_0 such that as $0<\xi\leq\xi_0$, there holds

$$L\exp_{\left\{|v|\leq\frac{1}{4}\xi^{-\frac{1}{2}}\right\}}(\xi r_0^2)\subset B_{t_0}(x_0,r_0).\quad(2.16)$$

The reduced volume $\tilde{B}(t_0 - \xi r_0^2)$ is given by

$$\tilde{V}_{\xi r_0^2}\left(\tilde{B}(t_0 - \xi r_0^2)\right) = \int_{\tilde{B}(t_0 - \xi r_0^2)} (4\pi \xi r_0^2)^{-2} \exp(-l(q, \xi r_0^2)) dV_{t_0 - \xi r_0^2}(q) =$$
$$= \int_{\tilde{B}(t_0 - \xi r_0^2) \cap L\exp_{\left\{|v| \le \frac{1}{4}\xi^{-\frac{1}{2}}\right\}}(\xi r_0^2)} (4\pi \xi r_0^2)^{-2} \exp(-l(q, \xi r_0^2)) dV_{t_0 - \xi r_0^2}(q) +$$
$$+ \int_{\tilde{B}(t_0 - \xi r_0^2) \setminus L\exp_{\left\{|v| \le \frac{1}{4}\xi^{-\frac{1}{2}}\right\}}(\xi r_0^2)} (4\pi \xi r_0^2)^{-2} \exp(-l(q, \xi r_0^2)) dV_{t_0 - \xi r_0^2}(q). \quad (2.17)$$

By (2.16), the first term on the right hand side of (2.17) can be estimated by

$$\int_{\tilde{B}(t_0 - \xi r_0^2) \cap L\exp_{\left\{|v| \le \frac{1}{4}\xi^{-\frac{1}{2}}\right\}}(\xi r_0^2)} (4\pi \xi r_0^2)^{-2} \exp(-l(q, \xi r_0^2)) dV_{t_0 - \xi r_0^2}(q) \le e^{4\xi} \int_{B_{t_0}(x_0, r_0)} (4\pi \xi r_0^2)^{-2} \exp(-l) dV_{t_0}(q) \le e^{4\xi} (4\pi)^{-2} \xi^2 \quad (2.18)$$

And the second term on the right hand side of (2.17) can be estimated by

$$\int_{\tilde{B}(t_0 - \xi r_0^2) \setminus L\exp_{\left\{|v| \le \frac{1}{4}\xi^{-\frac{1}{2}}\right\}}(\xi r_0^2)} (4\pi \xi r_0^2)^{-2} \exp(-l(q, \xi r_0^2)) dV_{t_0 - \xi r_0^2}(q) \le \int_{\left\{|v| > \frac{1}{4}\xi^{-\frac{1}{2}}\right\}} (4\pi\tau)^{-2} \exp(-l) J(\tau)\big|_{\tau=0} dv =$$
$$= (4\pi)^{-2} \int_{\left\{|v| > \frac{1}{4}\xi^{-\frac{1}{2}}\right\}} \exp(-|v|^2) dv, \quad (2.19)$$

by using Perelman's Jacobian comparison theorem. Hence the combination of (2.15), (2.17), (2.18) and (2.19) bounds ξ from blow by a positive constant depending only on κ_l and $\tilde{r}_l$.

3. <u>A string inspired 3D Euclidean field theory as the starting point for a modified Ricci flow analysis of the Thurston Conjecture</u>. [3]

The potential importance of a 3D uniformization theorem is evident, particularly in the context of (super)membrane physics or three-dimensional quantum gravity where one should be able to perform path-integral quantization via a similar procedure to that in two dimensions.
In three dimensions there is only a conjecture due to W.P. Thurston. This conjecture states that a three-manifold with a given topology has a canonical decomposition into "simple three-manifold", each of which admits one, and only one, of eight homogeneous geometries: $H^3, S^3, E^3, S^2 \times S^1, H^2 \times S^1$.
Of the Thurston spaces, only E^3, S^3 and H^3 are solutions of Einstein gravity with an appropriate cosmological constant term. In search of a single theory from which all eight of the Thurston geometries arise, we turn to the low-energy limit of three-dimensional string theory, which has a metric $g_{\mu\nu}$, dilaton ϕ, Abelian 2-form potential $B_{(2)}$ with field strength $H_{(3)} = dB_{(2)}$ and a "constant" term in the level of the original sigma model. This theory has many more solutions than the constant curvature geometries. In fact, it has propagating modes. If the dilaton is set to a constant value, then for a given sign for the coupling of the H^2 term, the only solutions have either constant, non-negative or non-positive metrics. The value of the cosmological constant is given by a

constant of integration. We therefore modify the above 3D stringy theory by appending to it a U(1) gauge field with potential 1-form A and field strength F which couple as a "Maxwell-Chern-Simons theory". The corresponding action is given by:

$$S=\int d^3x\sqrt{g}e^{-2\phi}\left(-\chi+R+4|\nabla\phi|^2-\frac{\varepsilon_H}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}-\frac{\varepsilon_F}{2}F_{\mu\nu}F^{\mu\nu}\right)+\frac{e}{2}\varepsilon^{\mu\nu\rho}A_\mu F_{\nu\rho}\,, \quad (3.1)$$

where the last term is the Abelian Chern-Simons term for the one-form potential $A_{(1)}$, and $F_{(2)}=dA_{(1)}$. The Wess-Zumino field $B_{\mu\nu}$ is a 2-form potential whose field strength $H_{\mu\nu\rho}=\partial_{[\mu}B_{\nu\rho]}$. Hence, in 3D, the field strength is proportional to the LeviCivita *tensor*:

$$H_{\mu\nu\rho}=H(x)\eta_{\mu\nu\rho}\,, \quad (3.2)$$

where H(x) is a scalar field. The equations of motion for the "Wess-Zumino field" $B_{\mu\nu}$ are

$$H^{\nu\rho}:=\nabla_\mu\left(e^{-2\phi}H(x)\eta^{\mu\nu\rho}\right)=0\,. \quad (3.3)$$

It easy to see that the latter implies that H(x) = c = constant. Without loss of generality, this result can be substituted into the remaining equations of motion. The result is:

$$E_{\mu\nu}:=R_{\mu\nu}+2\nabla_\mu\nabla_\nu\phi-\frac{\varepsilon_H}{2}c^2e^{4\phi}g_{\mu\nu}-\varepsilon_F F_\mu^\rho F_{\nu\rho}=0\,; \quad (3.4)$$

$$J^\mu:=\varepsilon_F\nabla_\nu\left(e^{-2\phi}F^{\mu\nu}\right)-\frac{e}{2}\eta^{\mu\nu\rho}F_{\nu\rho}=0\,; \quad (3.5)$$

$$D:=-\chi+R(g)+4\Delta\phi-4|\nabla\phi|^2-\frac{\varepsilon_H}{2}c^2e^{4\phi}-\frac{\varepsilon_F}{2}F^{\mu\nu}F_{\mu\nu}\,. \quad (3.6)$$

We will look for solutions with $\phi=0$. In this case, by taking appropriate linear combinations of the trace of (3.4) and (3.6) one obtains constraints on the Ricci scalar and electromagnetic field strengths. In particular, the Ricci scalar and the square of the field strength must both be constant:

$$R=-\frac{\varepsilon_H}{2}c^2+2\chi \quad (3.7) \qquad\qquad F^{\mu\nu}F_{\mu\nu}=2\varepsilon_F\left(\chi-\varepsilon_H c^2\right) \quad (3.8)$$

Now we define a vector field dual to the Maxwell field strength:

$$v^\mu:=\frac{1}{2}\eta^{\mu\nu\rho}F_{\nu\rho}\,, \quad (3.9)$$

where $\eta^{\mu\nu\rho}:=\varepsilon^{\mu\nu\rho}/\sqrt{g}$ is the completely skewsymmetric Levi-Civita tensor. Then the Maxwell-Chern-Simons equation (with the modified Ricci flow) $\dot g_{\mu\nu}=-2R_{\mu\nu}+\frac{\chi(M_2)}{V(M_2)}g_{\mu\nu}$, where $V(M_2):=\int_{M_2}d^2x\sqrt{g}$ is the volume of 2D manifold M_2, becomes

$$\varepsilon_F\eta^{\mu\nu\rho}\nabla_\nu v_\rho=ev^\mu\,. \quad (3.10)$$

If we multiply by $\eta_{\mu\alpha\beta}$ (contracting on μ) and use the property

$$\eta^{\mu\nu\rho}\eta_{\mu\alpha\beta} = \delta^{\nu}_{\alpha}\delta^{\rho}_{\beta} - \delta^{\nu}_{\beta}\delta^{\rho}_{\alpha}\,, \quad (3.11)$$

we get

$$2\varepsilon_F \nabla_{[\alpha}v_{\beta]} = e\eta_{\mu\alpha\beta}v^{\mu} = eF_{\alpha\beta}\,. \quad (3.12)$$

Now since $F_{\alpha\beta} = 2\partial_{[\alpha}A_{\beta]}$, it follows that *locally* there exists a smooth function σ such that

$$v_{\mu} = \varepsilon_F eA_{\mu} + \nabla_{\mu}\sigma\,. \quad (3.13)$$

From the eq. (3.8) it follows immediately that:

$$v^2 = \frac{1}{2}F^{\mu\nu}F_{\mu\nu} = \varepsilon_F\left(\chi - \varepsilon_H c^2\right). \quad (3.14)$$

Then, the gravitational equations now take the simple form:

$$E_{\mu\nu} = R_{\mu\nu} - \frac{\varepsilon_H}{2}c^2 g_{\mu\nu} - \varepsilon_F\left(v^2 g_{\mu\nu} - v_{\mu}v_{\nu}\right) = 0\,. \quad (3.15)$$

We will use the vector field v^{μ} to specify a local coordinate system in which the metric takes a particularly simple form. Choose the coordinate system $\{x^1, x^2, y\}$ so that

$$\left(\frac{\partial}{\partial y}\right)^{\mu} = v^{\mu}\,. \quad (3.16)$$

We will denote the dependence of a function f on the x^j by f(x). Then from the constancy of v_{μ} it follows that

$$g_{33} = v^2\,, \quad (3.17)$$

where v^2 is the constant given by (3.14).
Without loss of generality we can write the metric as

$$ds^2 = h_{ij}(x, y)dx^i dx^j + v^2\left(dy + a_i(x)dx^i\right)^2, \quad (3.18)$$

where the "2D metric" h_{ij} depends on *all* the coordinates x^1, x^2, y. However, A_i depend only on the x^j. This follows from the requirement that v^{μ} is tangent to a family of geodesics. The form of the metric (3.18) suggests that v^{μ} is a Killing vector for the full metric. Indeed a straightforward calculation shows that the i, j components of the Killing equation on v^{μ}

$$\nabla_i v_j + \nabla_j v_i = \dot{h}_{ij} \quad (3.19)$$

where we have defined the quantity $\dot{h}_{ij} := h_{ij,y}$. We have yet to impose the condition $\nabla_{\mu}v^{\mu} = 0$ which is equivalent to

$$h^{ij}\dot{h}_{ij} = 0\,, \quad (3.20)$$

where i, j,… indices are lowered and raised by h_{ij} and its inverse matrix h^{ij}. This means of course that $h = \det(h_{ij})$ is independent of y. Thence, we have effectively solved the Maxwell-Chern-Simons equations. The only remaining field equations are the Einstein equations (3.15), which in terms of the h "metric" reduce to

$$E_{yy} := -\frac{1}{4}\dot{h}^{ij}\dot{h}_{ij} + \frac{v^2}{2}\left(-\varepsilon_H c^2 + e^2 v^4\right) = 0; \quad (3.21)$$

$$E_{iy} := \frac{1}{2}\left[\nabla_j \dot{h}_i^j - a_j \partial_y \dot{h}_i^j + \frac{1}{2} a_i \dot{h}^{jk} \dot{h}_{jk}\right] + \frac{v^2}{2}\left(-\varepsilon_H c^2 + e^2 v^4\right) a_i = 0; \quad (3.22)$$

$$h^{ij} E_{ij} := R(h) + \nabla_k\left(\dot{h}^{ki} a_i\right) + a_k \nabla_i \dot{h}^{ki} - a_k a_i \ddot{h}^{ki} - 2\varepsilon_F v^2 + \left(1 - \frac{v^2}{2} h^{ij} a_i a_j\right)\left(-\varepsilon_H c^2 + e^2 v^4\right) = 0. \quad (3.23)$$

Now we will describe the flow suggested by our three dimensional gravity theory. The idea is that the right side of the flow has as its zeroes the solutions of the equations of motion eqs. (3.4) to eqs. (3.6):

$$\dot{g}_{\mu\nu} = -2\left[R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \phi + \left(-\frac{\varepsilon_H}{4} H_{\mu\rho\tau} H_\nu^{\rho\tau} - \varepsilon_F F_\mu^\rho F_{\nu\rho}\right)\right]; \quad (3.24)$$

$$\dot{B}_{\mu\nu} = \nabla_\rho\left(e^{-2\phi} H_{\mu\nu}^\rho\right); \quad (3.25)$$

$$\dot{A}_\mu = \nabla_\nu\left(e^{-2\phi} F_\mu^\nu\right) + \frac{e\varepsilon_F}{2} \eta_\mu^{\nu\rho} F_{\nu\rho}; \quad (3.26)$$

$$\dot{\phi} = -\chi + R(g) + 4\Delta\phi - 4|\nabla\phi|^2 - \frac{\varepsilon_H}{12} H^2 - \frac{\varepsilon_F}{2} F^2. \quad (3.27)$$

If the manifold has the topology of a Seifert bundle η over an orbifold Y, we specify $\varepsilon_H = +1, \chi = \chi(Y), e = e(\eta)$. If it is not a Seifert bundle then $\varepsilon_H = -1$.
Once the parameters (and hence topology) are specified, one begins with an arbitrary configuration of metric, dilaton field, 2-form potential $B_{\mu\nu}$ and U(1) potential A_μ as initial conditions for the flow equations (3.24-3.27). If the flow is to be useful then in the case where the flow is non-singular, the metric should reach the appropriate Thurston geometry. Only the Ricci-Hamilton flow of locally homogeneous geometries converges to the fixed points for the case of locally homogeneous and *isotropic* geometries. We shall consider a few of the details for the flow of an initial geometry which is locally $H^2 \times E^1$. Thus the metric, U(1) gauge field and dilaton are of the form:

$$ds^2 = \frac{l^2}{x_1^2}\left(D_1(t)dx_1^2 + D_2(t)dx_2^2\right) + E(t)dy^2; \quad A_\mu = \left[0, A(t)\frac{l}{x_1}, 0\right]; \quad \phi = \phi(t). \quad (3.28)$$

From the flow of the metric, we find first that the factor E(t) must be constant, and hence can be absorbed by rescaling the y-coordinate. Second, it turns out that for any value of the flow parameter t, there must be a constant α such that $D_2(t) = \alpha D_1(t)$. The constant α can be absorbed by rescaling x_2. Third, the function A(t) in the gauge potential is frozen by its flow to be a constant A(t) = a. Finally, we calculate

$$\frac{dD_1}{d\phi} := \frac{\dot{D}_1(t)}{\dot{\phi}(t)} = -2\frac{D_1(t)\left(D_1(t) - a^2\right)}{\left(D_1^2(t) + 2D_1(t) - a^2\right)}. \quad (3.29)$$

The solution is

$$\phi(D_1) = \phi_0 - \frac{1}{2}\left\{D_1 + \log\left[D_1\left(-D_1 + a^2\right)^{1+a^2}\right]\right\}. \quad (3.30)$$

Hence we find that $D_1 \to a^2$, in the limit $\phi \to \infty$. Similar behaviour occurs for the case of the locally homogeneous flow of $S^2 \times E^1$.

The above calculation suggests that the dilaton field ϕ, "normalizes" the flow and can in some sense be considered as the physical flow parameter. If we had solved the locally homogeneous flows for D_1, D_2, ϕ in terms of t, we would have found that the first two do *not* converge to a finite value as $t \to \infty$. Instead, the fields flow to their fixed points as $t \to -\infty$. In the usual Ricci flow, the locally homogeneous and isotropic geometries do not converge to their global "round" form in the limit $t \to \infty$. To accomplish this, the flow is normalized by adding to it a term $2/3rg_{\mu\nu}$, where r is the average value of the Ricci scalar over the manifold. These considerations suggest the idea that occurrence of singularities in the flow of the metric is tracked by the flow of the dilaton field, instead of the rather arbitrary parameter t.

The stringy gravity flow described in this section, is a promising approach to proving the Thurston Geometrization Conjecture. Now we conclude with the following observations:

- It is quite closely related to the Ricci flow and its various modifications considered by Hamilton, Perelman and others. Hence the recent progress made by Perelman in resolving the analytical properties of these flows will almost certainly apply to the flow described here.

- The parameters that appear in the flow are determined by the topology of the 3-manifold. This makes it easier to "input" the 3-manifold into the flow at the beginning.

- All the Thurston geometries are fixed points of the flow. Hence we can follow non-singular flows directly to the Thurston geometries.

- The dilaton field in the flow seems to track the singularities in the flow. This should streamline the procedure of performing surgery on the manifolds in regions where these singularities occur.

Furthermore, we believe that underlying the stringy flow is a quantum field theoretic understanding of the Thurston Geometrization Conjecture. In particular, if the stringy gravity on the 3-manifolds are the bulk theory, then the sigma model whose RG flow is the stringy gravity equations of motion is its holographical dual theory.

4. <u>The three dimensional charged black string solution</u>. [4]

We take the following form of anti-de Sitter space:

$$ds^2 = \left(1 - \frac{\hat{r}^2}{l^2}\right)d\hat{t}^2 + \left(\frac{\hat{r}^2}{l^2} - 1\right)^{-1} d\hat{r}^2 + \hat{r}^2 d\hat{\varphi}^2. \quad (4.1)$$

If we identify $\hat{\varphi} = \hat{\varphi} + 2\pi$, (4.1) describes a black hole.

Now we choose two constants r_+, r_- and introduce the following new coordinates $\hat{t} = (r_+ t/l) - r_- \varphi$, $\hat{\varphi} = (r_+ \varphi/l) - (r_- t/l^2)$, $\hat{r}^2 = l^2(r^2 - r_-^2)/(r_+^2 - r_-^2)$. Then the metric (4.1) becomes

$$ds^2 = \left(M - \frac{r^2}{l^2}\right)dt^2 - Jdtd\varphi + r^2 d\varphi^2 + \left(\frac{r^2}{l^2} - M + \frac{J^2}{4r^2}\right)^{-1} dr^2, \quad (4.2)$$

where the constants M and J are related to $r_\pm$ by

$$M = \frac{r_+^2 + r_-^2}{l^2} \qquad J = \frac{2r_+ r_-}{l}. \quad (4.3)$$

Identifying φ with $\varphi + 2\pi$, yields a two parameter family of black holes.
We now turn to string theory. We consider the black holes in the context of the low energy approximation, and then consider the exact conformal field theory. In three dimensions, the low energy string action is

$$S = \int d^3x \sqrt{-g}\, e^{-2\phi}\left[\frac{4}{k} + R + 4(\nabla\phi)^2 - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho}\right]. \quad (4.4)$$

The equations of motion which follow from this action are

$$R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \phi - \frac{1}{4} H_{\mu\lambda\sigma} H_\nu^{\lambda\sigma} = 0, \quad (4.5a) \qquad \nabla^\mu \left(e^{-2\phi} H_{\mu\nu\rho}\right) = 0, \quad (4.5b)$$

$$4\nabla^2\phi - 4(\nabla\phi)^2 + \frac{4}{k} + R - \frac{1}{12} H^2 = 0. \quad (4.5c)$$

If we assume $\phi = 0$, then (4.5b) yields $H_{\mu\nu\rho} = (2/l)\varepsilon_{\mu\nu\rho}$ where l is a constant with dimensions of length. Substituting this form of H into (4.5a) yields

$$R_{\mu\nu} = -\frac{2}{l^2} g_{\mu\nu} \quad (4.6)$$

which is exactly Einstein's equation with a negative cosmological constant. The dilaton equation (4.5c) will also be satisfied provided $k = l^2$. Thus every solution to three dimensional general relativity with negative cosmological constant, is a solution to low energy string theory with $\phi = 0, H_{\mu\nu\rho} = (2/l)\varepsilon_{\mu\nu\rho}$ and $k = l^2$. In particular, the two parameter family of black holes (4.2) is a solution with

$$B_{\varphi t} = \frac{r^2}{l}, \qquad \phi = 0 \quad (4.7)$$

where $H = dB$. We now consider the dual of this solution. Given a solution $(g_{\mu\nu}, B_{\mu\nu}, \phi)$ that is independent of one coordinate, say x, then $(\tilde{g}_{\mu\nu}, \tilde{B}_{\mu\nu}, \tilde{\phi})$ is also a solution where

$$\tilde{g}_{xx} = 1/g_{xx}, \quad \tilde{g}_{x\alpha} = B_{x\alpha}/g_{xx}, \quad \tilde{g}_{\alpha\beta} = g_{\alpha\beta} - (g_{x\alpha}g_{x\beta} - B_{x\alpha}B_{x\beta})/g_{xx}, \quad \tilde{B}_{x\alpha} = g_{x\alpha}/g_{xx},$$

$$\tilde{B}_{\alpha\beta} = B_{\alpha\beta} - 2g_{x[\alpha}B_{\beta]x}/g_{xx}, \quad \tilde{\phi} = \phi - \frac{1}{2}\ln g_{xx} \quad (4.8)$$

and α,β run over all directions except x. Applying this transformation to the φ translational symmetry of the black hole solution (4.2) (4.7) yields

$$\tilde{d}s^2 = \left(M - \frac{J^2}{4r^2}\right)dt^2 + \frac{2}{l}dtd\varphi + \frac{1}{r^2}d\varphi^2 + \left(\frac{r^2}{l^2} - M + \frac{J^2}{4r^2}\right)^{-1}dr^2, \quad \tilde{B}_{\varphi t} = -\frac{J}{2r^2}, \quad \phi = -\ln r. \quad (4.9)$$

To better understand this solution, we diagonalize the metric. Let

$$t = \frac{l(\hat{x} - \hat{t})}{\left(r_+^2 - r_-^2\right)^{1/2}}, \qquad \varphi = \frac{r_+^2 \hat{t} - r_-^2 \hat{x}}{\left(r_+^2 - r_-^2\right)^{1/2}}, \qquad r^2 = l\hat{r}. \quad (4.10)$$

Then the solution becomes

$$\tilde{d}s^2 = -\left(1 - \frac{\mathrm{M}}{\hat{r}}\right)d\hat{t}^2 + \left(1 - \frac{Q^2}{\mathrm{M}\hat{r}}\right)d\hat{x}^2 + \left(1 - \frac{\mathrm{M}}{\hat{r}}\right)^{-1}\left(1 - \frac{Q^2}{\mathrm{M}\hat{r}}\right)^{-1}\frac{l^2 d\hat{r}^2}{4\hat{r}^2}, \quad \phi = -\frac{1}{2}\ln \hat{r}l, \quad B_{\hat{x}\hat{t}} = \frac{Q}{\hat{r}} \quad (4.11)$$

where $\mathrm{M} = r_+^2 / l$ and $Q = J/2$. This is precisely the three dimensional charged black string solution.

5. On the effective action of a probe fractional D2-brane and on the boundary action of a fractional D3-brane on C^2/Z_2 orbifold. [5] [6]

We have the following solution describing N fractional D2-branes transverse to a C^2/Z_2 orbifold:

$$ds^2 = H_2^{-5/8}\eta_{\alpha\beta}dx^\alpha dx^\beta + H_2^{3/8}\left(\delta_{ij}dx^i dx^j + \delta_{rs}dx^r dx^s\right), \ (5.1a) \qquad e^\Phi = H_2^{1/4}, \ (5.1b)$$

$$\overline{C}_3 = \left(H_2^{-1} - 1\right)dx^0 \wedge dx^1 \wedge dx^2, \ (5.1c) \qquad A_1 = -4g_s\pi^2 l_s^3 N\cos\theta d\varphi, \ (5.1d)$$

$$b = \frac{(2\pi l_s)^2}{2}\left(1 - \frac{2g_s l_s N}{r}\right). \ (5.1e)$$

We will show the world-volume theory of a probe fractional D2-brane placed in the background (5.1) at some finite distance r in the transverse space $\{x^3, x^4, x^5\}$.
Let us start from the world-volume action for a single fractional D2-brane which, in the Einstein frame, is given by:

$$S_{D2}^f = -\frac{\tau_p}{2}\int d^3x e^{-\Phi/4}\sqrt{-\det\left[G_{\alpha\beta} + e^{-\Phi/2}2\pi l_s^2 F_{\alpha\beta}\right]}\left(1 + \frac{\tilde{b}}{2\pi^2 l_s^2}\right) + \frac{\tau_p}{2}\int_{\mathrm{M}_3}\left(C_3 + 2\pi l_s^2 C_1 \wedge F\right), \ (5.2)$$

where we have chosen the static gauge, hats denote pullbacks onto the brane world-volume and the fields C_3 and C_1 are given by:

$$C_3 = \overline{c}_3\left(1 + \frac{\tilde{b}}{2\pi^2 l_s^2}\right) + \frac{A_3}{2\pi^2 l_s^2} = \frac{b}{2\pi^2 l_s^2} - 1, \ (5.3a) \quad C_1 = c_1\left(1 + \frac{\tilde{b}}{2\pi^2 l_s^2}\right) + \frac{A_1}{2\pi^2 l_s^2} = -2g_s l_s N\cos\theta d\varphi.$$

(5.3b)

Thence, from (5.2), we obtain:

$$S_{D2}^{f} = -\frac{\tau_p}{2}\int d^3 x e^{-\Phi/4}\sqrt{-\det\left[G_{\alpha\beta} + e^{-\Phi/2}2\pi l_s^2 F_{\alpha\beta}\right]}\left(1+\frac{\tilde{b}}{2\pi^2 l_s^2}\right) +$$
$$+\frac{\tau_p}{2}\int_{M_3}\left[\left(\frac{b}{2\pi^2 l_s^2}-1\right)+2\pi l_s^2\left(-2g_s l_s N\cos\theta d\varphi\right)\wedge F\right]. \quad (5.3c)$$

We regard the coordinates $\{x^3, x^4, x^5\}$ transverse to the probe brane as Higgs fields of the dual gauge theory: $x^i = 2\pi l_s^2 \Phi^i$. We also define polar coordinates (μ, θ, φ) in the moduli space of the Φ^i, so that the resulting energy / radius relation is given by $r = 2\pi l_s^2 \mu$. Expanding the world-volume action for slowly varying world-volume fields and keeping only up to quadratic terms in their derivatives we easily see that position dependent terms cancel, and we are left with the following effective action:

$$S_{D2}^{f} \cong -\frac{l_s}{4g_s}\int d^3 x \frac{b}{2\pi^2 l_s^2}\left\{\frac{1}{2}\left[(\partial\mu)^2 + \mu^2\left((\partial\theta)^2 + \sin^2\theta(\partial\varphi)^2\right)\right] + \frac{1}{4}F^2\right\} - \frac{N}{8\pi}\int d^3 x \cos\theta \varepsilon^{\alpha\beta\gamma}\partial_\alpha \varphi F_{\beta\gamma}.$$
(5.4)

When b = 0, the effective tension of the probe vanishes and this means that in this case an enhançon mechanism is taking place at the radius:

$$r_e = 2 g_s l_s N. \quad (5.5)$$

Substituting in (5.4) the expression of b in terms of μ, we obtain:

$$S_{D2}^{f} = -\frac{l_s}{4g_s}\int d^3 x\left[1-\frac{g_s N}{\pi l_s \mu}\right]\left\{\frac{1}{2}\left[(\partial\mu)^2 + \mu^2\left((\partial\theta)^2 + \sin^2\theta(\partial\varphi)^2\right)\right] + \frac{1}{4}F^2\right\} - \frac{N}{8\pi}\int d^3 x \cos\theta \varepsilon^{\alpha\beta\gamma}\partial_\alpha \varphi F_{\beta\gamma}.$$
(5.6)

The moduli space of the gauge theory can be explored by means of a probe fractional D2-brane. One notices that if the enhançon radius is $r_e = g_s l_s (2N - M)$, the resulting affective action (5.6) gets modified as follows:

$$S_{probe} = -\frac{l_s}{4g_s}\int d^3 x\left[1-\frac{g_s(2N-M)}{2\pi l_s \mu}\right]\times\left\{\frac{1}{2}\left[(\partial\mu)^2 + \mu^2\left((\partial\theta)^2 + \sin^2\theta(\partial\varphi)^2\right)\right] + \frac{1}{4}F^2\right\} +$$
$$-\frac{1}{16\pi}\int d^3 x (2N-M)\cos\theta \varepsilon^{\alpha\beta\gamma}\partial_\alpha \varphi F_{\beta\gamma}. \quad (5.7)$$

The boundary action for a fractional D3-brane of type I on the orbifold C^2/Z_2 is:

$$S_{b,I}^{D3} = -\frac{T_3}{2\kappa_{orb}}\int d^4 x\sqrt{-\det G_{\mu\nu}}\left[1+\frac{1}{2\pi^2\alpha'}\sum_{i=1}^{3}F_I^i \tilde{b}^i\right] + \frac{T_3}{2\kappa_{orb}}\int\left[C_4\left(1+\frac{1}{2\pi^2\alpha'}\sum_{i=1}^{3}F_I^i \tilde{b}^i\right) + \frac{1}{2\pi^2\alpha'}\sum_{i=1}^{3}F_I^i A_4^i\right]$$
(5.8).

We can see how the scale and chiral anomalies are realized in supergravity and we consider the Dirac-Born-Infeld action and the Wess-Zumino term for a stack of N_I fractional D3-branes given by eq. (5.8). Turning on a gauge field on the world-volume of the branes and expanding the boundary action in the supergravity background up to quadratic terms in the derivatives one gets:

$$S^{I}_{gauge} = -\frac{1}{16\pi g_s}\int d^4x\sqrt{-\det G}e^{-\phi}G^{\mu\rho}G^{\nu\sigma}\frac{1}{4}F^{a}_{\mu\nu}F^{a}_{\rho\sigma}\left[1+\frac{1}{2\pi^2\alpha'}\sum_{i=1}^{3}F^{i}_{I}\tilde{b}^{i}\right]+$$
$$+\frac{1}{64\pi g_s}\int d^4x\left[C_0\left(1+\frac{1}{2\pi^2\alpha'}\sum_{i=1}^{3}F^{i}_{I}\tilde{b}^{i}\right)+\frac{1}{2\pi^2\alpha'}\sum_{i=1}^{3}F^{i}_{I}c^{i}\right]F^{a}_{\mu\nu}\tilde{F}^{a\mu\nu}\ , \quad (5.9)$$

where for simplicity we dropped the index I of the gauge fields and the metric G is the pull-back to the brane world-volume.

6. Mathematical connections between some relations concerning the Poincaré Conjecture, String Theory and some sectors of Number Theory. [7] [8]

a. **Mathematical connections with string theory**.

Now we take the equations (1.18a), (1.20), (1.20a) and (1.25), regarding the Poincaré and Geometrization Conjectures. We have that these equations can be related with the eq. (3.1) concerning the modified 3D stringy gravity, with the eq. (4.4) concerning the low energy string action in the context of the three dimensional black holes and also with the eq. (5.7) concerning the effective action of a probe fractional D2-brane.
Furthermore, we have also that the eqs. (3.1) and (4.4) can be connected with Palumbo-Nardelli model, concerning the fundamental correlation between bosonic string action and supersymmetric string action
Hence, if we take, for example, the eqs. (1.25), (3.1), (4.4) and (5.7) we obtain the following interesting connections:

$$\int_{\tilde{B}(t_0-\xi\rho^2)\setminus L\exp_{\left\{|v|\le\frac{1}{4}\xi^{-\frac{1}{2}}\right\}}(\xi\rho^2)}\left(4\pi\xi\rho^2\right)^{-\frac{3}{2}}\exp\left(-l\left(q,\xi\rho^2\right)\right)dV_{t_0-\xi\rho^2}(q)\le(4\pi)^{-\frac{3}{2}}\int_{\left\{|v|>\frac{1}{4}\xi^{-\frac{1}{2}}\right\}}\exp\left(-|v|^2\right)dv\Rightarrow$$
$$\Rightarrow\int d^3x\sqrt{g}e^{-2\phi}\left(-\chi+R+4|\nabla\phi|^2-\frac{\varepsilon_H}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}-\frac{\varepsilon_F}{2}F_{\mu\nu}F^{\mu\nu}\right)+\frac{e}{2}\varepsilon^{\mu\nu\rho}A_\mu F_{\nu\rho}\Rightarrow$$
$$\Rightarrow\int d^3x\sqrt{-g}e^{-2\phi}\left[\frac{4}{k}+R+4(\nabla\phi)^2-\frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}\right]\Rightarrow$$
$$\Rightarrow-\frac{l_s}{4g_s}\int d^3x\left[1-\frac{g_s(2N-M)}{2\pi l_s\mu}\right]\times\left\{\frac{1}{2}\left[(\partial\mu)^2+\mu^2\left((\partial\theta)^2+\sin^2\theta(\partial\varphi)^2\right)\right]+\frac{1}{4}F^2\right\}+$$
$$-\frac{1}{16\pi}\int d^3x(2N-M)\cos\theta\varepsilon^{\alpha\beta\gamma}\partial_\alpha\varphi F_{\beta\gamma}\Rightarrow$$
$$\Rightarrow\int_0^\infty\frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(|F_2|^2\right)\right]=$$
$$=-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]. \quad (6.1)$$

Now we take the eq. (2.19) regarding the Ricci flow on compact four-manifolds with positive isotropic curvature. We note that this equation can be related with the eq. (5.9) concerning the boundary action of a fractional D3-brane on C^2/Z_2 orbifold. Also the eq. (5.9) can be connected with Palumbo-Nardelli model. Hence, we have the following connections:

$$\int_{\tilde{B}(t_0-\xi r_0^2)\setminus L\exp_{\left\{|v|\le\frac{1}{4}\xi^{-\frac{1}{2}}\right\}}(\xi r_0^2)}\left(4\pi\xi r_0^2\right)^{-2}\exp\left(-l\left(q,\xi r_0^2\right)\right)dV_{t_0-\xi r_0^2}(q)\le\int_{\left\{|v|>\frac{1}{4}\xi^{-\frac{1}{2}}\right\}}(4\pi\tau)^{-2}\exp(-l)J(\tau)\Big|_{\tau=0}dv=$$

$$=(4\pi)^{-2}\int_{\left\{|v|>\frac{1}{4}\xi^{-\frac{1}{2}}\right\}}\exp\left(-|v|^2\right)dv\Rightarrow$$

$$\Rightarrow-\frac{1}{16\pi g_s}\int d^4x\sqrt{-\det G}e^{-\phi}G^{\mu\rho}G^{\nu\sigma}\frac{1}{4}F^a_{\mu\nu}F^a_{\rho\sigma}\left[1+\frac{1}{2\pi^2\alpha'}\sum_{i=1}^{3}F_I^i\tilde{b}^i\right]+$$

$$+\frac{1}{64\pi g_s}\int d^4x\left[C_0\left(1+\frac{1}{2\pi^2\alpha'}\sum_{i=1}^{3}F_I^i\tilde{b}^i\right)+\frac{1}{2\pi^2\alpha'}\sum_{i=1}^{3}F_I^ic^i\right]F^a_{\mu\nu}\tilde{F}^{a\mu\nu}\Rightarrow$$

$$\Rightarrow\int_0^\infty\frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(|F_2|^2\right)\right]=$$

$$=-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right].\quad(6.2)$$

b. **Mathematical connection with Number Theory**.

With regard the mathematical connection with Number Theory, we have obtained some interesting relations between the equations concerning Poincarè Conjecture and Riemann zeta function, π, ϕ and Φ, hence, with the Ramanujan's modular function and Ramanujan's modular equations concerning the approximations to π.
With regard the Riemann zeta function, we take the eq. (4.11)

$$\tilde{d}s^2=-\left(1-\frac{M}{\hat{r}}\right)d\hat{t}^2+\left(1-\frac{Q^2}{M\hat{r}}\right)d\hat{x}^2+\left(1-\frac{M}{\hat{r}}\right)^{-1}\left(1-\frac{Q^2}{M\hat{r}}\right)^{-1}\frac{l^2d\hat{r}^2}{4\hat{r}^2},\quad\phi=-\frac{1}{2}\ln\hat{r}l,\quad B_{\hat{x}\hat{t}}=\frac{Q}{\hat{r}}.\quad(6.3)$$

Now, we take the Lemma 3 of Goldston-Montgomery theorem. Let $f(t)\ge 0$ a continuous function defined on $[0,+\infty)$ so that $f(t)<<\log^2(t+2)$. If $I(k)=\int_0^\infty\left(\frac{\sin ku}{u}\right)^2f(u)du=\left(\frac{\pi}{2}+\varepsilon'(k)\right)k\log\frac{1}{k}$, then

$$J(T)=\int_0^T f(t)dt=(1+\varepsilon')T\log T,\quad(6.4)$$

with $|\varepsilon'|$ small if $|\varepsilon(k)|\le\varepsilon$ uniformly for $\frac{1}{T\log T}\le k\le\frac{1}{T}\log^2T$. If now we take the expression $\phi=-\frac{1}{2}\ln\hat{r}l$, and the equation (6.4), we have the following interesting connection:

$$J(T)=\int_0^T f(t)dt=(1+\varepsilon')T\log T\Rightarrow-\frac{1}{2}\ln\hat{r}l\Rightarrow\int d^3x\sqrt{-g}e^{-2\phi}\left[\frac{4}{k}+R+4(\nabla\phi)^2-\frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}\right]\Rightarrow$$

$$\Rightarrow\int d^3x\sqrt{g}e^{-2\phi}\left(-\chi+R+4|\nabla\phi|^2-\frac{\varepsilon_H}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}-\frac{\varepsilon_F}{2}F_{\mu\nu}F^{\mu\nu}\right)+\frac{e}{2}\varepsilon^{\mu\nu\rho}A_\mu F_{\nu\rho}.\quad(6.5)$$

With regard the connections between Poincaré Conjecture, Ramanujan's modular function and Ramanujan's modular equations, we know that:

$$0{,}618033 = 1/\Phi = 1/\left(\frac{\sqrt{5}+1}{2}\right) = \frac{\sqrt{5}-1}{2} = R(q) + \frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}, \quad (6.6)$$

and

$$\pi = 2\left(\frac{\sqrt{5}+1}{2}\right) - \frac{3}{2^2\cdot 5}\left(\frac{\sqrt{5}-1}{2}\right) =$$

$$= 2\Phi - \frac{3}{2^2\cdot 5}\left[R(q) + \frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right]. \quad (6.7)$$

The equation (6.6) is the Rogers-Ramanujan identity for continued fractions related to the modular functions, thence, to the Ramanujan modular functions.
Modular functions are a subclass of the more general modular forms. An example of a modular function is the Dedekind eta function, given by the infinite product

$$e^{\pi i z/12}\prod_{m=1}^{\infty}\left(1-e^{2\pi i m z}\right). \quad (6.8)$$

Like other modular forms, this function is defined over the domain of complex numbers $z = x + iy$ where x and y are real and y > 0. For complex numbers, i is the square root of -1, i.e. $i = \sqrt{-1}$. In the function, "e" is the Euler's number (2,71828…) and π is pi (3,14159265359…).
The Dedekind eta function is defined as

$$\eta(z) = q^{1/24}\prod_{n=1}^{\infty}\left(1-q^n\right), \quad q = e^{2\pi i z}. \quad (6.9)$$

Then the modular discriminant $\Delta(z) = \eta(z)^{24}$ is a modular form of weight 12. A celebrated conjecture of Ramanujan asserted that the q^p coefficient for any prime p has absolute value $\leq 2p^{11/2}$. This was settled by Pierre Deligne as a result of his work on the Weil conjectures.
Ramanujan's function τ is defined by the expansion

$$x\prod_{1}^{\infty}\left(1-x^n\right)^{24} = \sum_{1}^{\infty}\tau(n)x^n, \quad (6.10)$$

which is valid for each complex number x such that $|x| < 1$.
The Ramanujan's function τ is related with the Rogers-Ramanujan identity (6.6). Also the Ramanujan's modular equations are related with the Ramanujan's function τ. Indeed, we have the following expressions:

$$\pi = \frac{24}{\sqrt{142}} \log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right], \quad (6.11) \text{ thence}$$

$$24 = \frac{\pi\left(\sqrt{142}\right)}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}, \quad (6.12) \text{ and}$$

$$\pi = \frac{12}{\sqrt{130}} \log\left[\frac{\left(2+\sqrt{5}\right)\cdot\left(3+\sqrt{13}\right)}{\sqrt{2}}\right] = 2\pi = \frac{24}{\sqrt{130}} \log\left[\frac{\left(2+\sqrt{5}\right)\cdot\left(3+\sqrt{13}\right)}{\sqrt{2}}\right], \quad (6.13) \text{ thence}$$

$$24 = \frac{2\pi\left(\sqrt{130}\right)}{\log\left[\frac{\left(2+\sqrt{5}\right)\cdot\left(3+\sqrt{13}\right)}{\sqrt{2}}\right]}. \quad (6.14)$$

We note easily that the pure number 24 represent the modes of Ramanujan function and, in the expressions (6.11), (6.13) is very fundamental and it is connected with π. It is interesting note also that the numbers 2, 3, 5, 7, 11 and 13 are prime numbers.
With regard the connection obtained with Poincaré Conjectures, if we take the eq. (1.25), we have the following expressions that are related with π, ϕ and Φ:

$$\int_{\tilde{B}\left(t_0-\xi\rho^2\right)\setminus L\exp_{\left\{|v|\le\frac{1}{4}\xi^{-\frac{1}{2}}\right\}}\left(\xi\rho^2\right)} \left(4\pi\xi\rho^2\right)^{-\frac{3}{2}} \exp\left(-l\left(q,\xi\rho^2\right)\right) dV_{t_0-\xi\rho^2}(q) \le (4\pi)^{-\frac{3}{2}} \int_{\left\{|v|>\frac{1}{4}\xi^{-\frac{1}{2}}\right\}} \exp\left(-|v|^2\right) dv =$$

$$= \int_{\tilde{B}\left(t_0-\xi\rho^2\right)\setminus L\exp_{\left\{|v|\le\frac{1}{4}\xi^{-\frac{1}{2}}\right\}}\left(\xi\rho^2\right)} \left\{4\left[2\Phi - \frac{3}{2^2\cdot 5}\left(R(q) + \frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f\left(-t^{1/5}\right)}\frac{dt}{t^{4/5}}\right)}\right)\right]\xi\rho^2\right\}^{-\frac{3}{2}}$$

$$\exp\left(-l\left(q,\xi\rho^2\right)\right) dV_{t_0-\xi\rho^2}(q) \le \left\{4\left[2\Phi - \frac{3}{2^2\cdot 5}\left(R(q) + \frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f\left(-t^{1/5}\right)}\frac{dt}{t^{4/5}}\right)}\right)\right]\right\}^{-\frac{3}{2}}$$

$$\int_{\left\{|v|>\frac{1}{4}\xi^{-\frac{1}{2}}\right\}} \exp\left(-|v|^2\right) dv, \quad (6.15)$$

$$\int_{\tilde{B}(t_0-\xi\rho^2)\setminus L\exp_{\left\{|v|\le\frac{1}{4}\xi^{-\frac{1}{2}}\right\}}(\xi\rho^2)}(4\pi\xi\rho^2)^{-\frac{3}{2}}\exp(-l(q,\xi\rho^2))dV_{t_0-\xi\rho^2}(q)\le(4\pi)^{-\frac{3}{2}}\int_{\left\{|v|>\frac{1}{4}\xi^{-\frac{1}{2}}\right\}}\exp\left(-|v|^2\right)dv=$$

$$=\int_{\tilde{B}(t_0-\xi\rho^2)\setminus L\exp_{\left\{|v|\le\frac{1}{4}\xi^{-\frac{1}{2}}\right\}}(\xi\rho^2)}\left\{4\left[\frac{24}{\sqrt{142}}\log\left(\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right)\right]\xi\rho^2\right\}^{-\frac{3}{2}}\exp(-l(q,\xi\rho^2))dV_{t_0-\xi\rho^2}(q)\le$$

$$\le\left\{4\left[\frac{24}{\sqrt{142}}\log\left(\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right)\right]\right\}^{-\frac{3}{2}}\int_{\left\{|v|>\frac{1}{4}\xi^{-\frac{1}{2}}\right\}}\exp\left(-|v|^2\right)dv. \quad (6.16)$$

Furthermore, these equations are related to the eq. (6.1), thence to the some equations concerning the string theory and to the Palumbo-Nardelli model.

Conclusions.

There exist an important connection between modular functions and string theory. Closed strings can be viewed as a set of loops arrayed on the manifold of space-time. The study of how simple loops behave under deformations is known as homology, and is intensely studied in K-theory. Imagine, for example, a Green's function defined on a manifold: it can be thought of defining a measure-preserving flow on the manifold. As the loops flow along the manifold, they trace out cylinders which sometimes merge and join, and sometimes split apart. Places where two cylinders join are known as pairs of pants. Riemann surfaces of negative curvature can be formed by stitching together pairs of pants. Thus, the natural setting for a string theory of closed loops is a Riemann surface.

Modular functions are used in the mathematical analysis of Riemann surfaces. Riemann surface theory is relevant to describing the behaviour of strings as they move through space-time. When strings move they maintain a kind symmetry called "conformal invariance".

Conformal invariance (also called "scale invariance") is related to the fact that points on the surface of a string's world sheet need not be evaluated in a particular order. As long as all points on the surface are taken into account in any consistent way, the physics should not change. Equations of how strings must behave when moving involve the Ramanujan function that is also related at some equations regarding the Poincaré Conjecture, as we can see easily in the present work.

When a string moves in space-time by splitting and recombining, a large number of mathematical identities must be satisfied. These are the identities of Ramanujan's modular function (and the related equations of Poincaré Conjecture). The KSV (Kikkawa-Sakita-Virasoro) loop diagrams of interacting strings can be described using modular functions.

The "Ramanujan function", an elliptic modular function that satisfies the "conformal symmetry", has 24 "modes" that correspond to the physical vibrations of a bosonic string.

When the Ramanujan function is generalized, 24 is replaced by 8 (8 + 2 = 10), hence, has 8 "modes" that correspond to the physical vibrations of a superstring.

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