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Assuming $c < rad^2(abc)$: A Proof of The abc Conjecture

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To the memory of my Father who taught me arithmetic
To my wife Wahida, my daughter Sinda and my son Mohamed Mazen

Abstract

In this paper, assuming the conjecture $c < rad^2(abc)$ true, I give, using elementary calculus, the proof of the abc conjecture proposing the constant $K(\epsilon)$. Some numerical examples are given.

1. *Introduction*

Let a positive integer $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as :

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{1.1}$$

We note:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{1.2}$$

The abc conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [**1**]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the abc conjecture is given below:

Conjecture 1. Let a, b, c positive integers relatively prime with $c = a + b$, then for each $\epsilon > 0$, there exists a constant $K(\epsilon)$ such that :

$$c < K(\epsilon).rad^{1+\epsilon}(abc), \quad K(\epsilon) \text{ depending only of } \epsilon. \tag{1.3}$$

The idea to try to write a paper about this conjecture was born after the publication of an article in Quanta magazine, in November 2018, about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [**2**]. The difficulty to find a proof of the abc conjecture is due to the incomprehensibility how the prime factors are organized in c giving a, b with $c = a + b$.

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.629912$ [**1**]. A conjecture was proposed that $c < rad^2(abc)$ [**3**].

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CONJECTURE 2. Let a, b, c positive integers relatively prime with $c = a + b$, then:

$$c < rad^2(abc) \Longrightarrow \frac{Logc}{Log(rad(abc))} < 2 \tag{1.4}$$

It is the key to resolve the *abc* conjecture. In this paper, we assume that $c < rad^2(abc)$. By elementary method we obtain a proof of the *abc* conjecture proposing also the constant $K(\epsilon)$. The paper is organized as follows: in the second section, we present the proof of the *abc* conjecture. In sections three and four, we present some numerical examples.

2. *Proof of The abc Conjecture*

Proof. Let a, b, c (respectively a, c) positive integers relatively prime with $c = a + b, a > b, b \geq 2$ (respectively $c = a + 1$, $a \geq 2$). We note $R = rad(abc)$ in the case $c = a + b$ or $R = rad(ac)$ in the case $c = a + 1$. I propose the constant $K(\epsilon)$ as:

$$\begin{cases} K_1(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)} > 1, \, \forall \, \epsilon \geq 1 \\ \\ K_2(\epsilon) = \frac{1}{\epsilon^2}.e^{\left(\frac{1}{\epsilon^2}\right)} > 1, \, \forall \epsilon \; 0 < \epsilon < 1 \end{cases} \tag{2.1}$$

2.1. *Case $c < R$:*

$c < R < R^2 < K_1(\epsilon)R^{1+\epsilon}$ for $\epsilon \geq 1$ and $c < R < R^{1+\epsilon} < K_2(\epsilon)R^{1+\epsilon}$, $\forall \epsilon \in]0, 1[$. Then the conjecture (1) is verified.

2.2. *Case $c = R$*

Case to reject as a, b, c (respectively a, c) are relatively prime.

2.3. *Case $R < c$*

2.3.1. *Case $\epsilon \geq 1$.* As $c < R^2 \Longrightarrow c < K_1(\epsilon).R^{1+\epsilon}, \forall \epsilon \geq 1$ since $K_1(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)} > 1$ and the conjecture (1) is verified.

2.3.2. *Case $0 < \epsilon < 1$.* Let for $\forall \epsilon \in]0, 1[$:

$$y(\epsilon) = -2Log\epsilon + \frac{1}{\epsilon^2} + (1 + \epsilon)LogR - Logc \tag{2.2}$$

Our main task is give the proof that $y(\epsilon) > 0 \Longrightarrow c < \frac{1}{\epsilon^2} e^{\left(\frac{1}{\epsilon^2}\right)} .R^{1+\epsilon} = K_2(\epsilon).R^{1+\epsilon}, \forall \epsilon, 0 < \epsilon < 1$.

We can write $y(\epsilon)$ as :

$$y(\epsilon) = \frac{\epsilon^3 LogR + \epsilon^2 LogR/c - 2\epsilon^2 Log\epsilon + 1}{\epsilon^2}$$

Let $Y(\epsilon)$:

$$Y(\epsilon) = \epsilon^3 LogR + \epsilon^2 LogR/c - 2\epsilon^2 Log\epsilon + 1 \tag{2.3}$$

For $\epsilon \in]0,1[$, the function derivative $Y'(\epsilon)$ is given by:

$$Y'(\epsilon) = 3\epsilon^2 LogR + 2\epsilon LogR/c - 4\epsilon Log\epsilon - 2\epsilon = \epsilon(3\epsilon LogR + 2LogR/c - 4Log\epsilon - 2) \tag{2.4}$$

$Y'(\epsilon) = 0$ gives, after eliminating $\epsilon = 0$, the equation:

$$e^{\epsilon.\frac{3LogR}{4}} = \epsilon.e^{\frac{1}{2}(1-LogR/c)} \tag{2.5}$$

In the interval $\epsilon \in]0,1[$, it is easy to verify that the equation (2.5) has not a solution due $\frac{3LogR}{4} > \frac{1}{2}(1 - LogR/c)$ as $c < R^2$ and R large number, then the curve $e^{\epsilon.\frac{3LogR}{4}}$ is above the line $\epsilon.e^{\frac{1}{2}(1-LogR/c)}$. We deduce that $Y'(\epsilon) > 0$. Then $Y(\epsilon)$ is an increasing function for $\epsilon \in]0,1[$ with :

$$lim_{\epsilon \longrightarrow 0} Y(\epsilon) = 1$$

$$lim_{\epsilon \longrightarrow 1} Y(\epsilon) = Log\frac{eR^2}{c} > Loge = 1$$

Hence $Y(\epsilon) > 0 \Longrightarrow \forall \epsilon \in]0,1[\ y(\epsilon) > 0$ and the proof of the *abc* conjecture for the case $\epsilon \in]0,1[$ is finished. □

We can announce the important result:

THEOREM 2.1. *Let a, b, c positive integers relatively prime with $c = a + b$. Assuming $c < rad^2(abc)$ true, then for each $\epsilon > 0$, there exists a constant $K(\epsilon)$ such that :*

$$c < K(\epsilon).rad^{1+\epsilon}(abc), \quad K(\epsilon) \text{ depending only of } \epsilon. \tag{2.6}$$

The constant $K(\epsilon)$ is defined as:

$$\begin{cases} K_1(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)} > 1, \ \forall \epsilon \geq 1 \\ \\ K_2(\epsilon) = \frac{1}{\epsilon^2}.e^{\left(\frac{1}{\epsilon^2}\right)} > 1, \ \forall \epsilon \ 0 < \epsilon < 1 \end{cases} \tag{2.7}$$

In the two following sections, we are going to verify some numerical examples.

3. *Examples : Case $c = a + 1$*

3.1. *Example 1*

The example is given by:

$$1 + 5 \times 127 \times (2 \times 3 \times 7)^3 = 19^6 \tag{3.1}$$

$a = 5 \times 127 \times (2 \times 3 \times 7)^3 = 47\,045\,880 \Rightarrow \mu_a = (2 \times 3 \times 7)^2 = 1764$ and $rad(a) = 2 \times 3 \times 5 \times 7 \times 127$, in this example, $\mu_a < rad(a)$.
$c = 19^6 = 47\,045\,881 \Rightarrow rad(c) = 19$. Then $rad(ac) = 2 \times 3 \times 5 \times 7 \times 19 \times 127 = 506\,730$.
We have $c > rad(ac)$ but $rad^2(ac) = 506\,730^2 = 256\,775\,292\,900 > c = 47\,045\,881$.

(i) - We take $\epsilon = 0.01 \Longrightarrow 1/\epsilon^2 = 1000$; $c \overset{?}{<} K_2(0.01)rad(ac)^{1.01}$. It gives:

$$c < 5.0891284815112165623923368824481e + 4352$$

and the conjecture (2) is true.

(ii) - We take $\epsilon = 0.23 \Longrightarrow 1/\epsilon^2 = 18.903591682419659735349716446 1$; $c \overset{?}{<} K_2(0.23) rad(ac)^{1.23}$. It gives:

$$c < 31\,852\,617\,193\,677\,022.347552284134714$$

and the conjecture (2) is true.

3.2. *Example 2*

We give here the example 2 from *https : //nitaj.users.lmno.cnrs.fr*:

$$3^7 \times 7^5 \times 13^5 \times 17 \times 1831 + 1 = 2^{30} \times 5^2 \times 127 \times 353 \tag{3.2}$$

$a = 3^7 \times 7^5 \times 13^5 \times 17 \times 1831 = 424\,808\,316\,456\,140\,799 \Rightarrow rad(a) = 3 \times 7 \times 13 \times 17 \times 1831 = 8497671$
$c = 1\,203\,422\,992\,793\,600 \Longrightarrow rad(c) = 2 \times 5 \times 127 \times 353 = 448310 \Longrightarrow rad(ac) = 849767 \times 448310 = 3\,809\,590\,886\,010 < c$.

We take $\epsilon = 0.85 \Longrightarrow 1/ep^2 = 1.3840830449826989619377 1$; $c \overset{?}{<} K_2(0.85) rad(ac)^{1.85}$. It gives:

$$c = 1\,203\,422\,992\,793\,600 < 1\,039\,648\,915\,260\,096\,510\,370\,849.62777$$

and the conjecture (2) is true.

4. *Examples : Case $c = a + b$*

4.1. *Example 1*

We give here the example of Eric Reyssat [**1**], it is given by:

$$3^{10} \times 109 + 2 = 23^5 = 6436343 \tag{4.1}$$

$a = 3^{10}.109 \Rightarrow \mu_a = 3^9 = 19683$ and $rad(a) = 3 \times 109$,
$b = 2 \Rightarrow \mu_b = 1$ and $rad(b) = 2$,
$c = 23^5 = 6436343 \Rightarrow rad(c) = 23$. Then $rad(abc) = 2 \times 3 \times 109 \times 23 = 15042$.
$rad^2(abc) = 226\,261\,764 > c$.

(i) - We take $\epsilon = 0.1 \Longrightarrow 1/\epsilon^2 = 100$; $c \overset{?}{<} K_2(0.1) rad(abc)^{1.1}$. It gives:

$$c = 6\,436\,343 < 1.331958473162302914862655605692 2e + 51$$

and the conjecture (2) is true.

(ii) - We take $\epsilon = 0.95 \Longrightarrow 1/\epsilon^2 = 1.1080332409972299168975 0$; $c \overset{?}{<} K_2(0.95) rad(abc)^{1.95}$. It gives:

$$c = 6\,436\,343 < 469\,365\,756.075695$$

and the conjecture (2) is true.

(iii) - We take $\epsilon = 1.20 \Longrightarrow 1/\epsilon^2 = 0.6944444$; $c \overset{?}{<} K_1(1.20) rad(abc)^{2.20}$. It gives:

$$c = 6\,436\,343 < 3\,102\,170\,586.75263511$$

and the conjecture (2) is true.

4.2. *Example 2*

The example of Nitaj about the abc conjecture [**1**] is:

$$a = 11^{16}.13^2.79 = 613\,474\,843\,408\,551\,921\,511 \Rightarrow rad(a) = 11.13.79 \tag{4.2}$$
$$b = 7^2.41^2.311^3 = 2\,477\,678\,547\,239 \Rightarrow rad(b) = 7.41.311 \tag{4.3}$$
$$c = 2.3^3.5^{23}.953 = 613\,474\,845\,886\,230\,468\,750 \Rightarrow rad(c) = 2.3.5.953 \tag{4.4}$$
$$rad(abc) = 2.3.5.7.11.13.41.79.311.953 = 28\,828\,335\,646\,110$$
$$rad^2(abc) = 831\,072\,936\,124\,776\,471\,158\,132\,100 >$$
$$c = 613\,474\,845\,886\,230\,468\,750$$

(i) - We take $\epsilon = 0.35 \Rightarrow 1/\epsilon^2 = 8.16326530612244$; $c \overset{?}{<} K_2(0.35)rad(abc)^{1.35}$. It gives:

$$c = 613\,474\,845\,886\,230\,468\,750 < 42\,450\,362\,909\,291\,733\,374\,870.441768129$$

and the conjecture (2) is true.

(ii) - We take $\epsilon = 0.80 \Rightarrow 1/\epsilon^2 = 1.5625$; $c \overset{?}{<} K_2(0.80)rad(abc)^{1.80}$. It gives:

$$c = 613\,474\,845\,886\,230\,468\,750 < 12\,591\,584\,368\,412\,779\,579\,903\,417.92517$$

and the conjecture (2) is true.

(iii) - We take $\epsilon = 1.005 \Rightarrow 1/\epsilon^2 = 0.990074503106358753496200589094\,33$;
$c \overset{?}{<} K_1(1.005)rad(abc)^{2.005}$. It gives:

$$c = 613\,474\,845\,886\,230\,468\,750 < 951\ 204\ 706\ 707\ 494\ 904\ 071\ 611\ 134.22558$$

and the conjecture (2) is true.

4.3. *Example 3*

It is of Ralf Bonse about the abc conjecture [**3**] :

$$2543^4.182587.2802983.85813163 + 2^{15}.3^{77}.11.173 = 5^{56}.245983 \tag{4.5}$$
$$a = 2543^4.182587.2802983.85813163$$
$$b = 2^{15}.3^{77}.11.173$$
$$c = 5^{56}.245983 = 3.4136998783296235160378273576449\,8e + 44$$
$$rad(abc) = 2.3.5.11.173.2543.182587.245983.2802983.85813163$$
$$rad(abc) = 1.5683959920004546031461002610848e + 33$$
$$rad^2(abc) = 2.4598659877230900595045886864951e + 66 > c$$

(i) - We take $\epsilon = 0.03 \Longrightarrow 1/\epsilon^2 = 1111.1111111111$; $c \overset{?}{<} K_2(1.03)rad(abc)^{1.03}$. It gives:

$$c < 6.1164752541929019626495294987141e + 519$$

and the conjecture (2) is true.

(ii) - We take $\epsilon = 0.5 \Longrightarrow 1/\epsilon^2 = 4$; $c \overset{?}{<} K_2(0.5)rad(abc)^{1.5}$. It gives:

$$c < 1.3565053303252801103198028639382e + 52$$

and the conjecture (2) is true.

(iii) - We take $\epsilon = 0.75 \Longrightarrow 1/\epsilon^2 = 1.7777777777777777$; $c \overset{?}{<} K_2(0.75)rad(abc)^{1.75}$. It gives:

$$c < 1.3001817590825236724478474551477e+59$$

and the conjecture (2) is true.

(iv) - We take $\epsilon = 1.019 \Longrightarrow 1/\epsilon^2 = 0.96305620107072588$; $c \overset{?}{<} K_1(1.019)rad(abc)^{2.019}$. It gives:

$$c < 2.7534367221908150906648318318746e+67$$

and the conjecture (2) is true.

Ouf! The end of the mystery!

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References

1. M. Waldschmidt, On the abc Conjecture and some of its consequences presented at The 6th World Conference on 21st Century Mathematics, Abdus Salam School of Mathematical Sciences (ASSMS), Lahore (Pakistan), March 6-9, (2013).
2. Klaus Kremmerz for Quanta Magazine, Titans of Mathematics Clash Over Epic Proof of ABC Conjecture. The Quanta Newsletter, 20 September 2018. www.quantamagazine.org. (2018).
3. P. Mihăilescu, Around ABC. European Mathematical Society Newsletter **N° 93**, September 2014. pp 29-34. (2014).

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