Riemann Hypothesis

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1 Abstract

The Riemann Zeta function is defined as

$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \ Re(s) > 1$

The Zeta function is holomorphic in the complex plane except for a simple pole at $s = 1$. *The trivial zeros of* $\zeta(s)$ *are* $-2, -4, -6, ...$ *Its non trivial zeros lie in the critical strip* $0 < Re(s) < 1$.

The Riemann Hypothesis states that all the non trivial zeros lie on the critical line $Re(s) = 1/2$.

In this article we disprove the Riemann Hypothesis.

2 Proof

We give the proof by contradction.

Let us assume that the Riemann Hypothesis is true.

Riemann Hypothesis is equivalent to the integral

equation (see [3, p.136, Corollary 8.7])

$\int_{-\infty}^{\infty} \frac{log|\zeta(1/2+it)|}{1+4t^2}\ dt = 0$

$Let,\ I = \int_{-\infty}^{\infty} \frac{log|\zeta(1/2+it)|}{1+4t^2}\ dt$

$$I = \int_{-\infty}^{\infty} \frac{log|\zeta(1/2+it)|}{1+4t^2}\ dt = 0 \quad (1)$$

$Take\ a \in R\ with\ 1/2 \leq a < 1,\ Riemann's\ \zeta\ function\ has\ no\ zeros\ in$

$Re(s) > a\ if\ and\ only\ if\ [see\ 1, p\ .499\ , Theorem\ 7.27]$

$\frac{1}{\pi}\int_0^{\infty} \frac{log|\frac{\zeta(a+it)}{\zeta(a)}|}{t^2} = \frac{\zeta'(a)}{2\zeta(a)} - \frac{1}{1-a}$

$Since\ we\ have\ assumed\ Riemann\ Hypothesis\ is\ true$

$so\ setting\ a = 1/2$

$Riemann's\ \zeta\ function\ has\ no\ zeros\ in\ Re(s) > 1/2\ \ or\ the\ Riemann\ Hypothesis\ is\ true$

$if\ and\ only\ if$

$\frac{1}{\pi}\int_0^{\infty} \frac{log|\frac{\zeta(1/2+it)}{\zeta(1/2)}|}{t^2} = \frac{\zeta'(1/2)}{2\zeta(1/2)} - 2$

$$Let, J = \frac{1}{\pi}\int_0^{\infty} \frac{log|\frac{\zeta(1/2+it)}{\zeta(1/2)}|}{t^2} \quad (2)$$

$$J = \frac{1}{\pi}\int_0^{\infty} \frac{log|\frac{\zeta(1/2+it)}{\zeta(1/2)}|}{t^2} = \frac{\zeta'(1/2)}{2\zeta(1/2)} - 2 \quad (3)$$

Substitute , t=-u

$\Rightarrow dt = -du$

$J = \frac{1}{\pi}\int_{-\infty}^{0} \frac{log|\frac{\zeta(1/2-iu)}{\zeta(1/2)}|}{u^2} du$

$By\ Schwarz\ Reflection\ principle\ ,\ \zeta(\bar{s}) = \overline{\zeta(s)}$

$|\ \zeta(1/2-iu)\ |=|\ \zeta\ (\overline{1/2+iu})\ |=|\ \overline{\zeta(1/2+iu)}\ |=|\ \zeta(1/2+iu)\ |$

$J = \frac{1}{\pi}\int_{-\infty}^{0} \frac{log|\frac{\zeta(1/2+iu)}{\zeta(1/2)}|}{u^2}$

$$J = \frac{1}{\pi}\int_{-\infty}^{0} \frac{log|\frac{\zeta(1/2+it)}{\zeta(1/2)}|}{t^2} dt \quad (4)$$

$Adding\ equations\ (2)\ and\ (4),$

$2J = \frac{1}{\pi}\int_{0}^{\infty} \frac{log|\frac{\zeta(1/2+it)}{\zeta(1/2)}|}{t^2} + \frac{1}{\pi}\int_{-\infty}^{0} \frac{log|\frac{\zeta(1/2+it)}{\zeta(1/2)}|}{t^2} dt$

$J = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{log|\frac{\zeta(1/2+it)}{\zeta(1/2)}|}{t^2} dt$

$J = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{log|\zeta(1/2+it)| - log|\zeta(1/2)|}{t^2} dt$

$J = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{log|\zeta(1/2+it)|}{t^2} - \frac{log|\zeta(1/2)|}{2\pi}\int_{-\infty}^{\infty} \frac{1}{t^2} dt$

$J = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{log|\zeta(1/2+it)|}{t^2} dt$

$t^2 < 1 + 4t^2$

$\frac{1}{t^2} > \frac{1}{1+4t^2}$

$J = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{log|\zeta(1/2+it)|}{t^2} > \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{log|\zeta(1/2+it)|}{1+4t^2}\ dt$

$J > I$

$Putting\ the\ value\ of\ J\ from\ (3)\ and\ I\ from\ (1),$

$$\frac{\zeta'(1/2)}{2\zeta(1/2)} - 2 > 0. \quad (5)$$

$From\ (see[7, Equation\ (42)\ and\ (91)]),$

$\zeta'(1/2) = -3.92264613...\ and\ \zeta(1/2) = -1.46035450880...$

$Putting\ these\ values\ of\ \zeta'(1/2)\ and\ \zeta(1/2)\ in\ Equation\ (5),$

$\frac{-3.92264613}{-2*1.46035450880} - 2 > 0.$

$\frac{-3.92264613}{-2.9207090197} - 2 > 0$

$1.343045507 - 2 > 0$

$-0.6569541493 > 0$

$which\ is\ a\ contradiction.$

$So,\ our\ assumption\ that\ Riemann\ Hypothesis\ is\ true\ is\ wrong.$

$Thus,\ we\ have\ disproved\ the\ Riemann\ Hypothesis.$

3 References

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