On the Riemann hypothesis

Daniel Thomas Hayes

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A proposed proof of the Riemann hypothesis.

1. Introduction

The Riemann zeta function is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

for $\sigma = \text{Re}(s) > 1$. For other values of s it is defined uniquely by analytic continuation, see [1]. The function $\zeta(s)$ has trivial zeros at $s = -2l$ for $l \in \mathbb{N} = \{1, 2, 3, \ldots\}$. It is known that the nontrivial zeros $s = \sigma + it$ of $\zeta(s)$ satisfy the following properties.

I: If $s = \sigma + it$ is a nontrivial zero of $\zeta(s)$ then $s = \sigma - it$ is a nontrivial zero of $\zeta(s)$.

II: If $s = \sigma + it$ is a nontrivial zero of $\zeta(s)$ then $\sigma \in (0, 1)$.

III: If $s = \sigma + it$ is a nontrivial zero of $\zeta(s)$ then $s = 1 - \sigma + it$ is a nontrivial zero of $\zeta(s)$.

2. Proof of the Riemann hypothesis

Theorem

All nontrivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$.

Proof

In light of [2] consider

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \log_{\mathrm{e}}(2\pi) - \frac{1}{2}\log_{\mathrm{e}}(1 - x^{-2}) \tag{2}$$

for $x \in (n, n+1)$ and $n \in \mathbb{N}$. Here $\psi(x)$ is a weighted prime counting function

$$\psi(x) = \sum_{p^m \leqslant x} \log_{\mathrm{e}} p \tag{3}$$

where p is prime and the sum is over all prime powers. The sum in the second term on the right of (2) is over all ρ such that $s = \rho$ is a nontrivial zero of $\zeta(s)$. The exact function $\psi(x)$ is constant on the domain between any two consecutive integers. The approximation of $\psi(x)$ with finitely many ρ values displays a Gibbs phenomenon. Differentiating (2) with respect to x yields

$$0 = 1 - \sum_{\rho} x^{\rho-1} - \frac{1}{x^3 - x}. \tag{4}$$

Rearranging (4) gives

$$\sum_{\rho} x^{\rho-1}\left(\frac{x^3 - x}{x^3 - x - 1}\right) = 1. \tag{5}$$

Differentiating (5) with respect to x yields

$$\sum_{\rho} x^{\rho-1}\left[(\rho - 1)\left(\frac{x^2 - 1}{x^3 - x - 1}\right) - \left(\frac{3x^2 - 1}{(x^3 - x - 1)^2}\right)\right] = 0. \tag{6}$$

Now

$$\sum_{\rho}(\rho-1)x^{\rho-1}=\sum_{\beta+i\gamma}(\beta+i\gamma-1)x^{\beta+i\gamma-1}. \tag{7}$$

On using Euler's identity

$$\mathrm{e}^{i\theta}=\cos(\theta)+i\sin(\theta) \tag{8}$$

equation (7) becomes

$$\sum_{\rho}(\rho-1)x^{\rho-1}=\sum_{\beta+i\gamma}(\beta+i\gamma-1)x^{\beta-1}[\cos(\gamma\log_{\mathrm{e}}x)+i\sin(\gamma\log_{\mathrm{e}}x)] \tag{9}$$

which expands to

$$\sum_{\rho}(\rho-1)x^{\rho-1}=\sum_{\beta+i\gamma}x^{\beta-1}[\cos(\gamma\log_{\mathrm{e}}x)(\beta-1)-\sin(\gamma\log_{\mathrm{e}}x)\gamma]$$
$$+i\sum_{\beta+i\gamma}x^{\beta-1}[\sin(\gamma\log_{\mathrm{e}}x)(\beta-1)+\cos(\gamma\log_{\mathrm{e}}x)\gamma]. \tag{10}$$

The second term on the right of (10) disappears due to I. Then (10) becomes

$$\sum_{\rho}(\rho-1)x^{\rho-1}=\sum_{\beta+i\gamma}x^{\beta-1}[\cos(\gamma\log_{\mathrm{e}}x)(\beta-1)-\sin(\gamma\log_{\mathrm{e}}x)\gamma]. \tag{11}$$

Also

$$\sum_{\rho}x^{\rho-1}=\sum_{\beta+i\gamma}x^{\beta+i\gamma-1}. \tag{12}$$

On using Euler's identity equation (12) becomes

$$\sum_{\rho}x^{\rho-1}=\sum_{\beta+i\gamma}x^{\beta-1}\cos(\gamma\log_{\mathrm{e}}x)+i\sum_{\beta+i\gamma}x^{\beta-1}\sin(\gamma\log_{\mathrm{e}}x). \tag{13}$$

The second term on the right of (13) disappears due to I. Then (13) becomes

$$\sum_{\rho}x^{\rho-1}=\sum_{\beta+i\gamma}x^{\beta-1}\cos(\gamma\log_{\mathrm{e}}x). \tag{14}$$

Equation (6) is then

$$\sum_{\beta+i\gamma}x^{\beta-1}[\cos(\gamma\log_{\mathrm{e}}x)(\beta-1)-\sin(\gamma\log_{\mathrm{e}}x)\gamma](\frac{x^2-1}{x^3-x-1})-\sum_{\beta+i\gamma}x^{\beta-1}\cos(\gamma\log_{\mathrm{e}}x)(\frac{3x^2-1}{(x^3-x-1)^2})=0. \tag{15}$$

Let $x=y+c$ where $0\leqslant y\ll 1$ and c is a constant such that $x\in(n,n+1)$. Then (15) implies

$$\sum_{\beta+i\gamma}(y+c)^{\beta-1}[\cos(\gamma\log_{\mathrm{e}}(y+c))(\beta-1)-\sin(\gamma\log_{\mathrm{e}}(y+c))\gamma](\frac{(y+c)^2-1}{(y+c)^3-(y+c)-1})$$
$$-\sum_{\beta+i\gamma}(y+c)^{\beta-1}\cos(\gamma\log_{\mathrm{e}}(y+c))(\frac{3(y+c)^2-1}{((y+c)^3-(y+c)-1)^2})=0. \tag{16}$$

On using a Taylor expansion (16) becomes

$$\sum_{\beta+i\gamma}(y+c)^{\beta-1}\{[\cos(\gamma\log_{\mathrm{e}}c)(\beta-1)-\sin(\gamma\log_{\mathrm{e}}c)\gamma](\frac{c^2-1}{c^3-c-1})$$
$$+[[-\sin(\gamma\log_{\mathrm{e}}c)(\beta-1)\frac{\gamma}{c}-\cos(\gamma\log_{\mathrm{e}}c)\frac{\gamma^2}{c}](\frac{c^2-1}{c^3-c-1})$$
$$+[\cos(\gamma\log_{\mathrm{e}}c)(\beta-1)-\sin(\gamma\log_{\mathrm{e}}c)\gamma](\frac{-c^4+2c^2-2c-1}{(c^3-c-1)^2})]y+O(y^2)\}$$
$$-\sum_{\beta+i\gamma}(y+c)^{\beta-1}\{\cos(\gamma\log_{\mathrm{e}}c)(\frac{3c^2-1}{(c^3-c-1)^2})+[-\sin(\gamma\log_{\mathrm{e}}c)\frac{\gamma}{c}(\frac{3c^2-1}{(c^3-c-1)^2})$$
$$+\cos(\gamma\log_{\mathrm{e}}c)(\frac{-12c^7+18c^5+6c^4-8c^3+8c+2}{(c^3-c-1)^4})]y+O(y^2)\}=0. \tag{17}$$

Now (17) must be true independent of y. We then must take coefficients of $(y + c)$ in (17), for $\beta \in (0, 1)$ in accordance with II, and set them to zero. Now (17) has the form

$$\sum_{\beta\in\mathbb{R}} \sum_{\gamma\in\mathbb{R}(\beta)} (y + c)^{\beta-1} \{ \sum_{l=0}^{\infty} [f_l(\gamma, c)(\beta - 1) + g_l(\gamma, c)](y + c)^l \} = 0. \tag{18}$$

So for example, taking the the $O((y + c)^{\beta-1})$ coefficient in (18) gives

$$\sum_{\gamma\in\mathbb{R}(\beta)} [f_0(\gamma, c)(\beta - 1) + g_0(\gamma, c)] = 0 \tag{19}$$

which implies

$$\beta = -\frac{\sum_{\gamma\in\mathbb{R}(\beta)} g_0(\gamma, c)}{\sum_{\gamma\in\mathbb{R}(\beta)} f_0(\gamma, c)} + 1 = -\frac{\sum_{\gamma\in\mathbb{R}(1-\beta)} g_0(\gamma, c)}{\sum_{\gamma\in\mathbb{R}(1-\beta)} f_0(\gamma, c)} + 1 = 1 - \beta \tag{20}$$

on using III. Therefore without loss of generality $\beta = \frac{1}{2}$. □

References

[1] B. Riemann, Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse, *Monat. der Königl. Preuss. Akad. der Wissen. zu Berlin aus der Jahre 1859* (1860), 671–680; also, *Gesammelte math. Werke und wissensch. Nachlass*, 2. Aufl. 1892, 145–155.

[2] J. Vaaler, The Riemann Hypothesis Millennium Prize Problem, Lecture Video, *CLAY* (2001).