On the Riemann hypothesis

Daniel Thomas Hayes

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A proposed proof of the Riemann hypothesis.

1. Introduction

The Riemann zeta function is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

for $\sigma = \mathrm{Re}(s) > 1$. For other values of s it is defined uniquely by analytic continuation, see [1]. The function $\zeta(s)$ has trivial zeros at $s = -2l$ for $l \in \mathbb{N} = \{1, 2, 3, \ldots\}$. It is known that the nontrivial zeros $s = \sigma + \mathrm{it}$ of $\zeta(s)$ satisfy the following properties.

I: If $s = \sigma + \mathrm{it}$ is a nontrivial zero of $\zeta(s)$ then $s = \sigma - \mathrm{it}$ is a nontrivial zero of $\zeta(s)$.

II: If $s = \sigma + \mathrm{it}$ is a nontrivial zero of $\zeta(s)$ then $\sigma \in (0, 1)$.

III: If $s = \sigma + \mathrm{it}$ is a nontrivial zero of $\zeta(s)$ then $s = 1 - \sigma + \mathrm{it}$ is a nontrivial zero of $\zeta(s)$.

2. Proof of the Riemann hypothesis

Theorem

All nontrivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$.

Proof

In light of [2] consider

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \log_{\mathrm{e}}(2\pi) - \frac{1}{2}\log_{\mathrm{e}}(1 - x^{-2}) \tag{2}$$

for $x \in (n+1, n+2)$ and $n \in \mathbb{N}$. Here $\psi(x)$ is a weighted prime counting function

$$\psi(x) = \sum_{p^m \leqslant x} \log_{\mathrm{e}} p \tag{3}$$

where p is prime and the sum is over all prime powers. The sum in the second term on the right of (2) is over all ρ such that $s = \rho$ is a nontrivial zero of $\zeta(s)$. The exact function $\psi(x)$ is constant on the domain between any two consecutive integers. The approximation of $\psi(x)$ with finitely many ρ values displays a Gibbs phenomenon. Differentiating (2) with respect to x yields

$$\psi'(x) = 1 - \sum_{\rho} x^{\rho-1} - \frac{1}{x^3 - x}. \tag{4}$$

Differentiating (2) twice with respect to x yields

$$\psi''(x) = -\sum_{\rho} (\rho - 1) x^{\rho-2} + \frac{3x^2 - 1}{(x^3 - x)^2}. \tag{5}$$

Rearranging (5) yields

$$\psi''(x)(x^3 - x)^2 = -\sum_{\rho} (\rho - 1) x^{\rho}(x^2 - 1)^2 + 3x^2 - 1. \tag{6}$$

Now

$$\sum_{\rho}(\rho-1)x^{\rho}=\sum_{\beta}\sum_{\gamma}(\beta+\mathrm{i}\gamma-1)\mathrm{x}^{\beta+\mathrm{i}\gamma}. \tag{7}$$

On using Euler's identity

$$\mathrm{e}^{\mathrm{i}\theta}=\cos(\theta)+\mathrm{i}\sin(\theta) \tag{8}$$

equation (7) becomes

$$\sum_{\rho}(\rho-1)x^{\rho}=\sum_{\beta}\sum_{\gamma}(\beta+\mathrm{i}\gamma-1)\mathrm{x}^{\beta}[\cos(\gamma\log_{\mathrm{e}}\mathrm{x})+\mathrm{i}\sin(\gamma\log_{\mathrm{e}}\mathrm{x})] \tag{9}$$

which expands to

$$\sum_{\rho}(\rho-1)x^{\rho}=\sum_{\beta}\sum_{\gamma}x^{\beta}[\cos(\gamma\log_{\mathrm{e}}x)(\beta-1)-\sin(\gamma\log_{\mathrm{e}}x)\gamma]$$
$$+\mathrm{i}\sum_{\beta}\sum_{\gamma}\mathrm{x}^{\beta}[\sin(\gamma\log_{\mathrm{e}}\mathrm{x})(\beta-1)+\cos(\gamma\log_{\mathrm{e}}\mathrm{x})\gamma]. \tag{10}$$

The second term on the right of (10) disappears due to I. Then (10) becomes

$$\sum_{\rho}(\rho-1)x^{\rho}=\sum_{\beta}\sum_{\gamma}x^{\beta}[\cos(\gamma\log_{\mathrm{e}}x)(\beta-1)-\sin(\gamma\log_{\mathrm{e}}x)\gamma]. \tag{11}$$

Equation (6) is then

$$\psi''(x)(x^3-x)^2=-\sum_{\beta}\sum_{\gamma}x^{\beta}[\cos(\gamma\log_{\mathrm{e}}x)(\beta-1)-\sin(\gamma\log_{\mathrm{e}}x)\gamma](x^2-1)^2+3x^2-1. \tag{12}$$

Let $x=y+c$ where $0\leqslant y\ll 1$ and c is a constant such that $x\in(n+1,n+2)$. Then (12) implies

$$\psi''(y+c)[(y+c)^3-(y+c)]^2=-\sum_{\beta}\sum_{\gamma}(y+c)^{\beta}[\cos(\gamma\log_{\mathrm{e}}(y+c))(\beta-1)$$
$$-\sin(\gamma\log_{\mathrm{e}}(y+c))\gamma][(y+c)^2-1]^2+3(y+c)^2-1. \tag{13}$$

On using a Taylor expansion (13) becomes

$$\psi''(y+c)[(y+c)^3-(y+c)]^2=-\sum_{\beta}\sum_{\gamma}(y+c)^{\beta}\{\cos(\gamma\log_{\mathrm{e}}c)(\beta-1)-\sin(\gamma\log_{\mathrm{e}}c)\gamma$$
$$+[-\sin(\gamma\log_{\mathrm{e}}c)\frac{\gamma}{c}(\beta-1)-\cos(\gamma\log_{\mathrm{e}}c)\frac{\gamma^2}{c}]y+O(y^2)\}[(y+c)^2-1]^2+3(y+c)^2-1. \tag{14}$$

Equating like coefficients of $(y+c)$ in (14), for $\beta\in(0,1)$ in accordance with II, yields a linear polynomial equation for β. Therefore only one β value is possible and it is $\beta=\frac{1}{2}$ by III. Therefore the Riemann hypothesis is true. □

References

[1] B. Riemann, Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse, *Monat. der Königl. Preuss. Akad. der Wissen. zu Berlin aus der Jahre 1859* (1860), 671680; also, *Gesammelte math. Werke und wissensch. Nachlass*, 2. Aufl. 1892, 145155.

[2] J. Vaaler, The Riemann Hypothesis Millennium Prize Problem, Lecture Video, *CLAY* (2001).