Non-existence of odd n-multiperfect numbers

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Abstract

Let n be an integer greater than 1 and let b be an odd n-multiperfect number. Let the prime factors of b which are different from each other be odd primes $p_1, p_2, \dots, p_r$ and let the exponent of p_k be an integer q_k. If the product of the prime factors' series is an integer a,

$$a = \prod_{k=1}^{r} (p_k{}^{q_k} + p_k{}^{q_k - 1} + \cdots + 1)$$

$$b = \prod_{k=1}^{r} p_k{}^{q_k}$$

If b is a n-multiperfect number,

$$a = nb$$

holds. By a research of this paper, let a_k be an integer and b_k be an odd integer and if

$$a_k = a/(p_k{}^{q_k} + \cdots + 1)$$

$$b_k = b/p_k{}^{q_k}$$

hold, when $r \geqq 3$, by a proof which uses the prime factors and the greatest common divisor (GCD) contained in b_k and $p_k{}^{q_k} + \cdots + 1$, the following inequality was obtained.

$$b^{r-2} \leqq n$$

$$n < (3/2)^r$$

By these inequalities, we have obtained a conclusion that there are no odd n-perfect numbers when $n > 1$.

Contents

1. Introduction

A multiperfect number is a natural number whose divisor sum is an integral multiple of the original number. 2-multiperfect number is simply called a perfect number. For example, the sum of the divisors of 120 is

$$1 + 2 + 3 + 4 + 5 + 6 + 8 + 10 + 12 + 15 + 20 + 24 + 30 + 40 + 60 + 120 = 360$$
$$= 3 \times 120$$

holds. Since it is three times 120, 120 is 3-multiperfect number. (Quoted from Wikipedia)

In this paper, we prove that there are no odd n-multiperfect numbers when $n > 1$.

2. Proof

Let n be an integer greater than 1 and let b be an odd n-multiperfect number. Let the prime factors of b which are different from each other be odd primes $p_1,p_2,\ldots,p_r$ and let the exponent of p_k be an integer q_k. If the product of the prime factors' series of is an integer a,

$$a = \prod_{k=1}^{r}(p_k{}^{q_k} + p_k{}^{q_k-1} + \cdots + 1) \ \ldots ①$$

$$b = \prod_{k=1}^{r} p_k{}^{q_k} \ \ldots ②$$

If b is a n-multiperfect number,

$$a = nb \ \ldots ③$$

holds.

Let $\mathrm{a_k}$ be an integer and $\mathrm{b_k}$ be an odd integer,

$\mathrm{a_k = a/(p_k{}^{q_k} + \cdots + 1)}$

$\mathrm{b_k = b/p_k{}^{q_k}}$

From the equation ③,

$\mathrm{a_k(p_k{}^{q_k} + \cdots + 1) = nb_kp_k{}^{q_k}}$ ...④

When $\mathrm{r = 1}$,

$\mathrm{p_1{}^{q_1} + \cdots + 1 = np_1{}^{q_1}}$

Since $1 \equiv 0 \ (\mathrm{mod}\ \mathrm{p_1})$ holds and it becomes a contradiction, when $\mathrm{r} = 1$, odd n-multiperfect numbers do not exist.

When $r \geqq 2$,

From the equation ④,

$a_k(p_k{}^{q_k+1} - 1) = nb_kp_k{}^{q_k}(p_k - 1)$

$a_kp_k - nb_k(p_k - 1) = a_k/p_k{}^{q_k}$

Since the left-hand side is an integer, let c_k be an integer,

$c_k = a_k/p_k{}^{q_k} = a_kp_k - nb_k(p_k - 1)$ …⑤

holds.

$c_kp_k{}^{q_k+1} - nb_k(p_k - 1) = c_k$

$nb_k(p_k - 1) = c_k(p_k{}^{q_k+1} - 1)$

$nb_k = c_k(p_k{}^{q_k} + \cdots + 1)$ …⑥

When $p_k > 1$,

$p_k{}^{q_k} - 1 < p_k{}^{q_k}$

$(p_k{}^{q_k} - 1)/(p_k - 1) < p_k{}^{q_k}/(p_k - 1)$

$p_k{}^{q_k-1} + \cdots + 1 < p_k{}^{q_k}/(p_k - 1)$

Because p_k is an odd prime and $p_k \geqq 3$ holds,

$p_k{}^{q_k-1} + \cdots + 1 < p_k{}^{q_k}/2$

From the equation ⑤ and the equation ⑥,

$nb_k - a_k = c_k(p_k{}^{q_k} + \cdots + 1) - c_kp_k{}^{q_k} = c_k(p_k{}^{q_k-1} + \cdots + 1)$

$nb_k - a_k < c_kp_k{}^{q_k}/2 = a_k/2$

$a_k/b_k > 2n/3$ …⑦

When $r = 2$,

$a_1 = p_2{}^{q_2} + \cdots + 1$

$b_1 = p_2{}^{q_2}$

$a_1/b_1 = (p_2{}^{q_2} + \cdots + 1)/p_2{}^{q_2} = (p_2{}^{q_2+1} - 1)/(p_2{}^{q_2}(p_2 - 1)) < p_2/(p_2 - 1)$

If $p_1 < p_2$ holds, since $p_2 \geqq 5$ holds,

$a_1/b_1 < 5/4$

This inequality contradicts the inequality ⑦ when $n > 1$. Therefore, there are no odd n-multiperfect numbers when $r = 2$.

When $r \geqq 3$,
From the equation ④,
$a_k/b_k = np_k{}^{q_k}/(p_k{}^{q_k} + \cdots + 1)$ …⑧

When n is divided by $p_k{}^{q_k} + \cdots + 1$, let n' be an integer,
$n' = np_k{}^{q_k}/(p_k{}^{q_k} + \cdots + 1)$
$a_k = n'b_k$
hold. Since the equation ⑧ is an equation for obtaining n'-multiperfect numbers, considering only the case where n is not divisible by $p_k{}^{q_k} + \cdots + 1$ for all k does not lose generality.

A case where n cannot be divided by $p_k{}^{q_k} + \cdots + 1$ for all k is considered. At this time, the right-hand side is not an integer. $p_k{}^{q_k} + \cdots + 1$ is the product of the prime factors p_1 to p_r excluding p_k and the prime factors of n. Let C_k be the greatest common divisor (GCD) of the denominators on both sides in the equation ⑧. When the denominator on both sides are divided by C_k, if nC_k becomes a multiple of the denominator on the right-hand side, let m_k be an integer,
$nC_k = m_k(p_k{}^{q_k} + \cdots + 1)$
this equation is assumed to hold, the value of the left-hand side of the equation ⑧ is $m_k p_k{}^{q_k}/C_k$. If this value is assumed to be an integer, m_k is a multiple of C_k since p_k does not exist as the prime factor of C_k. However, this contradicts the condition that n is not divided by $p_k{}^{q_k} + \cdots + 1$. Therefore, when C_k is transposed from the denominator on the left-hand side to the right-hand side, the right-hand side does not become an integer. …(A)

Let P_k be an odd integer and $P_k = b_k/C_k$ holds. When $b_k > C_k$, if the numerator on the left-hand side is a multiple of P_k, it becomes a contradiction since the left-hand side is an integer and the right-hand side is not. Therefore, when the left-hand side is reduced, at least one of the prime factors p_{ki} of P_k remains in the denominator. At this time, it becomes inconsistent since the prime number p_{ki} does not exist in the denominator on the right-hand side. Thereby, $b_k = C_k$ must be established since the equation ⑧ does not hold when $b_k > C_k$.

When $b_k = C_k$, since $p_k{}^{q_k} + \cdots + 1$ is a multiple of b_k, let d_k be a positive integer,
$p_k{}^{q_k} + \cdots + 1 = d_k b_k$
$a_k = np_k{}^{q_k}/d_k = c_k p_k{}^{q_k}$

$c_k = n/d_k$

d_k must be a divisor of n since c_k is an integer.

When $c_k = n/d_k$ holds, since $c_k \leqq n$ is established,

$a_k = c_k {p_k}^{q_k} \leqq n {p_k}^{q_k}$

Since this inequality holds for all k,

$$\prod_{k=1}^{r} a_k \leqq \prod_{k=1}^{r} (n {p_k}^{q_k})$$

$a^{r-1} \leqq n^r b$

From the equation ③,

$(nb)^{r-1} \leqq n^r b$

$b^{r-2} \leqq n$ …⑨

From the inequality ⑦,

$$\prod_{k=1}^{r} (a_k/b_k) > (2n/3)^r$$

$(a/b)^{r-1} > (2n/3)^r$

$n^{r-1} > (2n/3)^r$

$n < (3/2)^r$

From the inequality ⑨,

$b < (3/2)^{r/(r-2)}$

The equation ③ does not hold in the range of this inequality since the right-hand side of the inequality is a monotonically decreasing function in the range of $r > 2$ and the maximum value is $27/8$ when $r \geqq 3$, Therefore, there are no odd n-multiperfect numbers when $r \geqq 3$.

From the above, there are no odd n-multiperfect numbers when $n > 1$.

3. Acknowledgement

For the proof about the existence of odd perfect number, we asked anonymous reviewers to point out several tens of mistakes. We would like to thank you for giving appropriate guidance and counter-arguments.

4. References

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