Bayesian Updating Quaternion Probability☆

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Abstract

The quaternion is an effective tool to evaluate uncertainty, and it has been studied widely. However, what is the quaternion probability still an open question. This paper has proposed the quaternion probability, which is the extent of classical probability and plural probability with the aid of quaternion. The quaternion probability can apply classical probability theory to the four-dimensional space. Based on the quaternion probability, the quaternion probability multiplication has been proposed, which is a method of multiplication conforming to the law of quaternion multiplication. Under the bayesian environment, the quaternion full joint probability and the quaternion conditional probability is proposed, which can apply the quaternion probability to address the issues of quantum decision making. Numerical examples are applied to prove the efficiency of the proposed model. The experimental results show that the proposed model can apply the quaternion theory to the bayesian updating effectively and successfully.

Keywords: Quaternion probability, Bayesian updating, Quantum decision making

1. Introduction

There are a lot of uncertainties in the real world [1, 2, 3]. In order to deal with the uncertainties, many mathematical theories are proposed [4, 5, 6], such as quantum theory [7, 8, 9], intuitionistic fuzzy sets [10, 11], orthopair fuzzy sets [12, 13], complex networks [14, 15], Dempster-Shafer evidence theory [16, 17, 18], entropy [19, 20, 21], belief entropy [22, 23, 24], game theory [25, 26] and von neumann entropy [27, 28]. Among these theories and models, the bayesian updating model have high efficiency in dealing with the uncertainty [29, 30, 31]. The so-called bayesian updating model based on bayesian law, which means that when the analysis sample is large enough to be close to the population number, the probability of the occurrence of events in the sample will

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be close to the probability of the occurrence of events in the population [32, 33]. The bayesian updating model is one of the most effective theoretical models in uncertain knowledge expression and reasoning [34]. It has become a hot research topic in recent years [35, 36]. Relying on the association between nodes, the bayesian updating model can accurately grasp the overall information of the logic networks, which means that it can evaluate objects more comprehensively [37, 38]. Relying on the advantages on representing uncertainty, the related bayesian methods and concepts have been widely studied by scholars at home and abroad [39, 40].

Quaternions are mathematical concepts invented by Irish mathematician William Rowan Hamilton [41], which has great promise for discovery [42, 43, 44]. The multiplication of quaternions does not conform to the commutative law [45, 46]. If the set of quaternions is considered as a multidimensional real number space, a quaternion represents a four-dimensional space, which is a two-dimensional space relative to a plural numbers [47, 48]. According to the superiority of quaternions in the representation of four-dimensional spatial information, the quaternions have been applied in many fields [49, 50]. However, how to apply quaternions to bayesian updating is still an open issue.

This paper proposes the bayesian updating quaternion probability, which apply the quaternion theory to the bayesian updating. The quaternion probability is the extent of classical probability and plural probability, which can apply probability theory to the four-dimensional space. The quaternion probability multiplication is a method of multiplication conforming to the law of quaternion multiplication, which means that the quaternion probability multiplication doesn't conform to the commutative law of multiplication. However, in special case, the quaternion probability multiplication conforms to the commutative law of multiplication. With the aid of quaternion probability multiplication and quaternion probability, the quaternion full joint probability and the quaternion conditional probability can apply the quaternion probability theory to deal with the issues of quantum decision making effectively.

The remain of this paper is structured as follows. Section 2 introduces the preliminary. Section 3 presents the bayesian updating quaternion probability. Section 4 illustrates the flexibility and accuracy of the bayesian updating quaternion probability. Section 5 summarizes the whole paper.

2. Preliminaries

In this section, bayesian formula [51], quaternion [52, 53, 54] are briefly introduced.

2.1. Bayesian Formula

Definition 2.1. *(The Full Probability Bayesian Formula) [51]*

$$Pr(X_1, X_2, \ldots, X_n) = \prod_{i=1}^{n} Pr(X_i|Parents(X_i)) \tag{1}$$

Where X is the variables list and $Parents(X_i)$ represents the nodes in the bayesian network, which are pointing to X_i.

Definition 2.2. *(The Conditional Probability Bayesian Formula) [51]*

$$Pr(X|e) = \alpha[\Sigma_{y \in Y} Pr(X, e, y)] \tag{2}$$

Where $\alpha = \frac{1}{\Sigma_{x \in X} Pr(X,e)}$. e is the list of observed variables nodes and y is the remaining unobserved variables nodes in the bayesian network, the α is the normalization factor for the distribution $Pr(X|e)$.

2.2. Quaternion

Quaternions are a four dimensional hypercomplex numbers system, which are an extension of complex numbers to four-dimensional space [55, 56, 57]. The definition of quaternions as follow:

Definition 2.3. *(Quaternion) [41]*

$$q = a + bi + cj + dk \tag{3}$$

Given two quaternions, $q_1 = a + bi + cj + dk$ and $q_2 = e + fi + gj + hk$, the definition of quaternions multiplication as follow:

Definition 2.4. *(Quaternion Multiplication) [41]*

$$\begin{aligned} q_1 q_2 = &(ae - (bf + cg + dh)) + \\ &(be + af + ch - dg)i + \\ &(ce + ag + df - bh)j + \\ &(de + ah + bg - cf)k \end{aligned} \tag{4}$$

3. The proposed method

In order to extend the bayesian updating to four-dimensional space, this paper proposes bayesian updating quaternion probability, which includes several models.

Given a probability distribution P, the definition of quaternion probability distribution as follow:

Definition 3.1. *(Quaternion Probability)*

$$PQ(A) = P(A)e^{u\theta} \tag{5}$$

Where, $P(A) = \sqrt{a^2 + b^2 + c^2 + d^2}$ is based on a quaternion $q = a + bi + cj + dk = \sqrt{a^2 + b^2 + c^2 + d^2}e^{u\theta}$. $u = u_x i + u_y j + u_z k$ is unit vector. $\theta = \arccos(\frac{a}{\sqrt{a^2+b^2+c^2+d^2}})$.

In this way, we define the probability of occurrence of event A is $P(A)|e^{u\theta}|$.

Given two quaternion probabilities, $PQ(A)=\sqrt{a^2+b^2+c^2+d^2}e^{u_1\theta_1}$ and $PQ(B)=\sqrt{e^2+f^2+g^2+h^2}e^{u_2\theta_2}$ with two unit vector $u=u_{1x}i+u_{1y}j+u_{1z}k$ and $u=u_{2x}i+u_{2y}j+u_{2z}k$ respectively, the definition of quaternions multiplication as follow:

Definition 3.2. *(Quaternion Probability Multiplication)*

$$PQ(A)PQ(B)=P(A)P(B)e^{u_1\theta_1}e^{u_2\theta_2} \tag{6}$$

In this way, we define the probability of occurrence of events A and B is $\sqrt{a^2+b^2+c^2+d^2}\sqrt{e^2+f^2+g^2+h^2}|e^{u_1\theta_1}e^{u_2\theta_2}|$.

Theorem 3.1. *If the three imaginary values of the base spaces of quaternions,* $u_1=u_{1x}i+u_{1y}j+u_{1z}k$ *and* $u_2=u_{2x}i+u_{2y}j+u_{2z}k$*, are proportional, which mean that* $\frac{u_{1x}}{u_{2x}}=\frac{u_{1y}}{u_{2y}}=\frac{u_{1z}}{u_{2z}}$*, then the quaternion probability distribution satisfies commutative law of multiplication as follow:*

$$PQ(A)PQ(B)=PQ(B)PQ(A)$$

Proof 3.1. *Relying on the equation of Eq.(3.1), we have equations as follows:*

$$PQ(A)=P(A)e^{u_1\theta_1}=\sqrt{a^2+b^2+c^2+d^2}e^{u_1\theta_1}$$

$$PQ(B)=P(B)e^{u_2\theta_2}=\sqrt{e^2+f^2+g^2+h^2}e^{u_2\theta_2}$$

Where $\theta_1=\arccos(\frac{a}{\sqrt{a^2+b^2+c^2+d^2}})$ *and* $\theta_2=\arccos(\frac{e}{\sqrt{e^2+f^2+g^2+h^2}})$.

Since, we have that $u_1=u_{1x}i+u_{1y}j+u_{1z}k$ *and* $u_2=u_{2x}i+u_{2y}j+u_{2z}k$. *Then, we can obtain the equations as follow:*

$$PQ(A)=\sqrt{a^2+b^2+c^2+d^2}(cos(\theta_1)+sin(\theta_1)(u_{1x}i+u_{1y}j+u_{1z}k))$$

$$PQ(B)=\sqrt{e^2+f^2+g^2+h^2}(cos(\theta_2)+sin(\theta_2)(u_{2x}i+u_{2y}j+u_{2z}k))$$

Then, we can get the law of multiplication of $PQ(A)PQ(B)$ *as follow:*

$$\begin{aligned}
&PQ(A)PQ(B)=\sqrt{a^2+b^2+c^2+d^2}\sqrt{e^2+f^2+g^2+h^2}(\\
&(cos(\theta_1)cos(\theta_2)-(sin(\theta_1)u_{1x}sin(\theta_2)u_{2x}+sin(\theta_1)u_{1y}sin(\theta_2)u_{2y}+sin(\theta_1)u_{1z}sin(\theta_2)u_{2z}))+\\
&(sin(\theta_1)u_{1x}cos(\theta_2)+cos(\theta_1)sin(\theta_2)u_{2x}+sin(\theta_1)u_{1y}sin(\theta_2)u_{2z}-sin(\theta_1)u_{1z}sin(\theta_2)u_{2y})i+\\
&(sin(\theta_1)u_{1y}cos(\theta_2)+cos(\theta_1)sin(\theta_2)u_{2y}+sin(\theta_1)u_{1z}sin(\theta_2)u_{2x}-sin(\theta_1)u_{1x}sin(\theta_2)u_{2z})j+\\
&(sin(\theta_1)u_{1z}cos(\theta_2)+cos(\theta_1)sin(\theta_2)u_{2z}+sin(\theta_1)u_{1x}sin(\theta_2)u_{2y}-sin(\theta_1)u_{1y}sin(\theta_2)u_{2x})k)
\end{aligned}$$

Then, we can get the law of multiplication of $PQ(B)PQ(A)$ *as follow:*

$PQ(B)PQ(A) = \sqrt{e^2+f^2+g^2+h^2}\sqrt{a^2+b^2+c^2+d^2}($
$(cos(\theta_2)cos(\theta_1) - (sin(\theta_2)u_{2x}sin(\theta_1)u_{1x} + sin(\theta_2)u_{2y}sin(\theta_1)u_{1y} + sin(\theta_2)u_{2z}sin(\theta_1)u_{1z}))+$
$(sin(\theta_2)u_{2x}cos(\theta_1) + cos(\theta_2)sin(\theta_1)u_{1x} + sin(\theta_2)u_{2y}sin(\theta_1)u_{1z} - sin(\theta_2)u_{2z}sin(\theta_1)u_{1y})i+$
$(sin(\theta_2)u_{2y}cos(\theta_1) + cos(\theta_2)sin(\theta_1)u_{1y} + sin(\theta_2)u_{2z}sin(\theta_1)u_{1x} - sin(\theta_2)u_{2x}sin(\theta_1)u_{1z})j+$
$(sin(\theta_2)u_{2z}cos(\theta_1) + cos(\theta_2)sin(\theta_1)u_{1z} + sin(\theta_2)u_{2x}sin(\theta_1)u_{1y} - sin(\theta_2)u_{2y}sin(\theta_1)u_{1x})k)$

If we want to get $PQ(A)PQ(B) = PQ(B)PQ(A)$, then the condition such that $\frac{u_{1x}}{u_{2x}} = \frac{u_{1y}}{u_{2y}} = \frac{u_{1z}}{u_{2z}}$ should be satisfied.

Then, we get the conclusion that if the three imaginary values of the base spaces of quaternions are proportional, then the quaternion probability distribution satisfies commutative law of multiplication.

□

Example 3.1. *Supposing that there is a quaternion probability distribution PQ in the space $\Omega = \{A, B\}$ as follows:*

$$PQ(A) = \sqrt{a^2+b^2+c^2+d^2}e^{u\theta_1} = \sqrt{a^2+b^2+c^2+d^2}e^{u\arccos(\frac{a}{\sqrt{a^2+b^2+c^2+d^2}})}$$

$$PQ(B) = \sqrt{e^2+f^2+g^2+h^2}e^{u\theta_2} = \sqrt{e^2+f^2+g^2+h^2}e^{u\arccos(\frac{e}{\sqrt{e^2+f^2+g^2+h^2}})}$$

Relying on the definition of quaternion probability, we can find that as follows:

If $b = c = d = f = g = h = 0$, $a = e = \frac{1}{2}$, then $PQ(A) = ae^{u\arccos(1)}$ and $PQ(B) = ee^{u\arccos(1)}$.

Since $e^{u\arccos(1)} = 1$, then we can obtain that $PQ(A) = a$ and $PQ(B) = e$. Now, we can find if the quaternion degenerates into real number, then the quaternion probability distribution will degenerate into the classical probability distribution.

Hence, we can obtain the equation as follow:

$$PQ(A)PQ(B) = P(B)P(A) = ae$$

Now, the quaternion probability distribution is satisfies the commutative law of multiplication as follow:

$$PQ(A)PQ(B) = PQ(B)PQ(A)$$

Example 3.2. *Supposing that there is a quaternion probability distribution PQ in the space $\Omega = \{A, B\}$ as follows:*

$$PQ(A)=\sqrt{a^2+b^2+c^2+d^2}e^{u\theta_1}=\sqrt{a^2+b^2+c^2+d^2}e^{u\arccos(\frac{a}{\sqrt{a^2+b^2+c^2+d^2}})}$$

$$PQ(B)=\sqrt{e^2+f^2+g^2+h^2}e^{u\theta_2}=\sqrt{e^2+f^2+g^2+h^2}e^{u\arccos(\frac{e}{\sqrt{e^2+f^2+g^2+h^2}})}$$

Where, $u=u_xi+u_yj+u_zk$ is unit vector.

If $c=d=g=h=0$, $a=b=e=-f=\frac{1}{2}$, then we can obtain

$$PQ(A)=\sqrt{a^2+b^2}e^{u\arccos(\frac{a}{\sqrt{a^2+b^2}})}$$

$$PQ(B)=\sqrt{e^2+f^2}e^{u\arccos(\frac{e}{\sqrt{e^2+f^2}})}$$

Since $c=d=g=h=0$, so the quaternion probability distribution degenerates into the plural probability distribution.

Then we can obtain as follows:

$$PQ(A)=\sqrt{a^2+b^2}(\frac{a}{\sqrt{a^2+b^2}}+\frac{bi}{\sqrt{a^2+b^2}})=a+bi$$

$$PQ(B)=\sqrt{e^2+f^2}(\frac{e}{\sqrt{e^2+f^2}}+\frac{fi}{\sqrt{e^2+f^2}})=e+fi$$

Now, we can find if the quaternion degenerates into the plural, then the quaternion probability distribution will degenerate into the plural probability distribution.

Hence, we can obtain the equation as follow:

$$PQ(A)PQ(B)=PQ(B)PQ(A)=ae-bf+(be+af)i$$

Now, the quaternion probability distribution is satisfies the commutative law of multiplication as follow:

$$PQ(A)PQ(B)=PQ(B)PQ(A)$$

Example 3.3. *Supposing that there is a quaternion probability distribution PQ in the space $\Omega=\{A,B,C\}$ as follows:*

$$PQ(A) = \sqrt{a^2+b^2+c^2+d^2}e^{u\theta_1} = \sqrt{a^2+b^2+c^2+d^2}e^{u\arccos(\frac{a}{\sqrt{a^2+b^2+c^2+d^2}})}$$

$$PQ(B) = \sqrt{e^2+f^2+g^2+h^2}e^{u\theta_2} = \sqrt{e^2+f^2+g^2+h^2}e^{u\arccos(\frac{e}{\sqrt{e^2+f^2+g^2+h^2}})}$$

$$PQ(C) = \frac{1}{2}$$

Where, $u = u_x i + u_y j + u_z k$ *is unit vector.*
If $a = b = c = d = e = \frac{1}{4}$, $f = g = h = \frac{-1}{4}$, *then we can obtain*

$$PQ(A) = e^{u\arccos(\frac{1}{2})}\ ,\ PQ(B) = e^{u\arccos(\frac{-1}{2})}$$

Relying on the quaternion multiplication formula, we can get that

$$\begin{aligned} PQ(A)PQ(B) &= e^{u\arccos(\frac{1}{2})}e^{u\arccos(\frac{-1}{2})} \\ &= (\frac{1}{4}+\frac{i}{4}+\frac{j}{4}+\frac{k}{4})(\frac{1}{4}-\frac{i}{4}-\frac{j}{4}-\frac{k}{4}) \\ &= \frac{-1}{4}+\frac{1}{4}(i+j+k) \end{aligned}$$

Then, we can get the probability of $PQ(A)PQ(B)$, $|PQ(A)PQ(B)| = \frac{1}{2}$.
Relying on the quaternion multiplication formula, we can get that

$$\begin{aligned} PQ(B)PQ(A) &= e^{u\arccos(\frac{-1}{2})}e^{u\arccos(\frac{1}{2})} \\ &= (\frac{1}{4}-\frac{i}{4}-\frac{j}{4}-\frac{k}{4})(\frac{1}{4}+\frac{i}{4}+\frac{j}{4}+\frac{k}{4}) \\ &= \frac{-1}{4}+\frac{1}{4}(i+j+k) \end{aligned}$$

Then, we can get the probability of $PQ(B)PQ(A)$, $|PQ(B)PQ(A)| = \frac{1}{2}$.
Hence, we can get the conclusion that $|PQ(A)PQ(B)| = |PQ(B)PQ(A)|$.

Since the quaternion multiplication indicates the rotation of the four dimensional space, so the quaternion doesn't conform to the commutative law of multiplication. However, relying on the quaternion multiplication formula, we can find the conclusion as follow. If the three imaginary values of the base spaces of quaternions, $u_1 = u_{1x}i + u_{1y}j + u_{1z}k$ *and* $u_2 = u_{2x}i + u_{2y}j + u_{2z}k$, *is proportional, which is means that* $\frac{u_{1x}}{u_{2x}} = \frac{u_{1y}}{u_{2y}} = \frac{u_{1z}}{u_{2z}}$, *then the quaternion probability satisfies commutative law of multiplication.*

Definition 3.3. *(The Quaternion Full Joint Probability)*

$$PQ(X_1, X_2, \ldots, X_n) = \prod_{i=1}^{n} PQ(X_i|Ps(X_i)) \tag{7}$$

Where X is the variables list and $Ps(X_i)$ represents the nodes, which are pointing to X_i.

Definition 3.4. *(The Quaternion Conditional Probability)*

$$PQ(X|Q) = \beta[\Sigma_{y \in Y} PQ(X, Q, y)] \tag{8}$$

Where $\beta = \frac{1}{\Sigma_{x \in X} PQ(X,Q)}$. Q is the list of observed variables nodes and y is the remaining unobserved variables nodes in the logic network, the β is the normalization factor for the distribution $PQ(X|Q)$.

4. Quantum Decision Making

An quantum application is the categorization decision-making experiment paradigm [58]. It was proposed be Townsend [59], which studies the interactions between categorization and decision making. The experiment involved two models, which are categorization decision-making(C-D) condition and decision alone(D alone). In categorization decision-making condition, participants were shown pictures of faces. Then, participants were asked to categorize the face as "bad"(B) guy or "good"(G) guy or a and make a decision to "attack"(A) or to "withdraw"(W). In decision alone condition, participants were asked to categorize the face as "bad"(B) guy or "good"(G) guy or a and make a decision to "attack"(A) or to "withdraw"(W) directly. The experiment results are shown as Tab. 1:

Table 1: The results of the C-D condition and D alone condition

$Facetype$	$P(G)$	$P(A\|G)$	$P(B)$	$P(A\|B)$	P_T	$P(A)$
$Wide$	0.84	0.35	0.16	0.52	0.37	0.39
$Narrow$	0.17	0.41	0.83	0.63	0.59	0.69

In this paper, ten experts were invited to update the probability in Tab. 1 to quaternion probability as Tab. 2.

Table 2: The quaternion probability of C-D condition and D alone condition

$Facetype$	$P(G)$	$P(A\|G)$	$P(B)$	$P(A\|B)$
$Wide$	$0.84e^{u_1\theta_1}$	$0.35e^{u_2\theta_2}$	$0.16e^{u_3\theta_3}$	$0.52e^{u_4\theta_4}$
$Narrow$	$0.17e^{u_5\theta_5}$	$0.41e^{u_6\theta_6}$	$0.83e^{u_7\theta_7}$	$0.63e^{u_8\theta_8}$

Now, relying on the equation as follow:

$$P(A) = P(G)P(A|G) + P(B)P(A|B) \tag{9}$$

In categorization decision-making condition, we can obtain the equation as follow:

$$\begin{aligned} P(A) &= 0.84e^{u_1\theta_1} * 0.35e^{u_2\theta_2} + 0.16e^{u_3\theta_3} * 0.52e^{u_4\theta_4} \\ &= 0.294e^{u_2\theta_1^{'}} + 0.0832e^{u_2\theta_2^{'}} \end{aligned} \tag{10}$$

Where, $\theta_1 + \theta_2 = \theta_1^{'}$ and $\theta_3 + \theta_4 = \theta_2^{'}$.

According to the experts, $|\theta_1^{'} - \theta_2^{'}| = 27.122807498323960042421609926546$ and $u_1 = u_2 = u_3 = u_4 = \frac{i}{3} + \frac{2j}{3} + \frac{2k}{3}$.

Then, we can get that $p(A) = 0.37$.

In decision alone condition, we can obtain the equation as follow:

$$\begin{aligned} P(A) &= 0.17e^{u_5\theta_5} * 0.41e^{u_6\theta_6} + 0.83e^{u_7\theta_7} * 0.63e^{u_8\theta_8} \\ &= 0.0697e^{u_2\theta_3^{'}} + 0.5229e^{u_2\theta_4^{'}} \end{aligned} \tag{11}$$

Where, $\theta_5 + \theta_6 = \theta_3^{'}$ and $\theta_7 + \theta_8 = \theta_4^{'}$.

According to the experts, $|\theta_3^{'} - \theta_4^{'}| = 16.700947305146110026626992621484$ and $u_5 = u_6 = u_7 = u_8 = \frac{2i}{3} + \frac{2j}{3} + \frac{k}{3}$.

Then, we can get that $p(A) = 0.59$.

Now the quaternion probability is as as Tab. 3.

Table 3: The quaternion results of the C-D condition and D alone condition

Facetype	$P(G)$	$P(A\|G)$	$P(B)$	$P(A\|B)$	P_T	$P(A)$
Wide	$0.84e^{u_1\theta_1}$	$0.35e^{u_2\theta_2}$	$0.16e^{u_3\theta_3}$	$0.52e^{u_4\theta_4}$	0.37	0.37
Narrow	$0.17e^{u_5\theta_5}$	$0.41e^{u_6\theta_6}$	$0.83e^{u_7\theta_7}$	$0.63e^{u_8\theta_8}$	0.59	0.59

In this case, the $P(A)$ is equal to P_T, either in the case of categorization decision-making condition or in the case of decision alone condition. The bayesian updating quaternion probability is under four-dimensional space, which has strong spatial and information description ability, so it can accurately get the desired result in this experiment.

5. Conclusion

This paper proposes bayesian updating quaternion probability, which is the extent of classical bayesian updating with the aid of the quaternion theory. This paper proposes quatenion probability, which is based on the quaternion theory and classical probability theory. With the definition of quaternion probability, the quaternion probability multiplication has been proposed, which satisfies the commutative law of multiplication under some special cases. The quaternion full joint probability and the quaternion are proposed, which means that the quaternion theory can be applied to the bayesian updating. Numerical examples are applied to verify the validity of the bayesian updating quaternion probability. The experimental results demonstrate that the proposed model can address the probability issues of bayesian updating with the aid of quaternion theory effectively.

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Compliance with Ethical Standards

All the authors certify that there is no conflict of interest with any individual or organization for the present work. This article does not contain any studies with human participants or animals performed by any of the authors.

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