R.H. is equivalent to the statement

$$I = \int_{-\infty}^{\infty} \frac{\log\left|\zeta\left(\frac{1}{2}+it\right)\right|}{1+4t^2}\, dt = 0$$

$$t = \frac{1}{2}\tan\theta \qquad dt = \frac{1}{2}\sec^2\theta \, d\theta$$

$$I = \frac{1}{2}\int_{-\pi/2}^{\pi/2} \log\left|\zeta\left(\frac{1}{2}+i\frac{\tan\theta}{2}\right)\right| d\theta$$

$$I = \frac{1}{2}\int_{-\pi/2}^{\pi/2} \log\left|\zeta\left(\frac{e^{i\theta}}{2\cos\theta}\right)\right| d\theta$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \log\left|\zeta\left(\frac{e^{i\theta}}{e^{i\theta}+e^{-i\theta}}\right)\right| d\theta$$

$$I = \int_{0}^{\pi/2} \log\left|\zeta\left(\frac{e^{i\theta}}{e^{i\theta}+e^{-i\theta}}\right)\right| d\theta$$

Consider $$J = \int_{0}^{\pi} \log\left|\zeta\left(\frac{e^{i\theta}}{e^{i\theta}+e^{-i\theta}}\right)\right| d\theta$$

$$J = \int_0^{\pi} \log \left| \zeta \left(\frac{e^{i\pi - i\theta}}{e^{i\pi - i\theta} + e^{-i\pi + i\theta}} \right) \right| d\theta$$

$$J = \int_0^{\pi} \log \left| \zeta \left(\frac{-e^{-i\theta}}{-e^{-i\theta} - e^{i\theta}} \right) \right| d\theta$$

$$J = \int_0^{\pi} \log \left| \zeta \left(\frac{e^{-i\theta}}{e^{i\theta} + e^{-i\theta}} \right) \right| d\theta$$

$$J = \int_0^{\pi} \log \left| \zeta \left(\frac{e^{i\theta}}{e^{i\theta} + e^{-i\theta}} \right) \right| d\theta,$$

$$\left(\because \zeta(\bar{s}) = \overline{\zeta(s)} \right)$$

$$\therefore J = 2 \int_0^{\pi/2} \log \left| \zeta \left(\frac{e^{i\theta}}{e^{i\theta} + e^{-i\theta}} \right) \right| d\theta$$

$$J = 2I$$

$$I = \frac{1}{2} \int_0^{\pi} \log \left| \zeta \left(\frac{e^{i\theta}}{e^{i\theta} + e^{-i\theta}} \right) \right| d\theta$$

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~~Boo~~ $K = \int_0^{2\pi} \log\left|\zeta\left(\frac{e^{i\theta}}{e^{i\theta}+e^{-i\theta}}\right)\right| d\theta$

$$K = \int_0^{2\pi} \log\left|\zeta\left(\frac{e^{i(2\pi-\theta)}}{e^{i(2\pi-\theta)}+e^{-i(2\pi-\theta)}}\right)\right|$$

$$K = \int_0^{2\pi} \log\left|\zeta\left(\frac{e^{-i\theta}}{e^{i\theta}+e^{-i\theta}}\right)\right|$$

$$K = \int_0^{2\pi} \log\left|\zeta\left(\frac{e^{i\theta}}{e^{i\theta}+e^{-i\theta}}\right)\right| d\theta$$

$$f(2a-x) = f(x)$$

$$K = 2\int_0^{\pi} \log\left|\zeta\left(\frac{e^{i\theta}}{e^{i\theta}+e^{-i\theta}}\right)\right| d\theta$$

$$K = 2(2I)$$

$$I = \frac{1}{4}K$$

R.H. is Equivalent to the statement

$$\int_{-\infty}^{\infty} \frac{\log\left|\zeta\left(\frac{1}{2}+it\right)\right|}{1+4t^2}\, dt = 0$$

$$I = \int_{-\infty}^{\infty} \frac{\log\left|\zeta\left(\frac{1}{2}+it\right)\right|}{1+4t^2}\, dt$$

Substitute, $t = \frac{1}{2}\tan\theta$

$$dt = \frac{1}{2}\sec^2\theta \, d\theta$$

$$I = \frac{1}{2}\int_{-\pi/2}^{\pi/2} \log\left|\zeta\left(\frac{1+i\tan\theta}{2}\right)\right| d\theta \quad \text{— (1)}$$

$$f(\theta) = \log\left|\zeta\left(\frac{1+i\tan\theta}{2}\right)\right|$$

$$f(-\theta) = \log\left|\zeta\left(\frac{1-i\tan\theta}{2}\right)\right|$$

$$= \log\left|\overline{\zeta\left(\frac{1+i\tan\theta}{2}\right)}\right|$$

$$f(-\theta) = f(\theta) \qquad \left(\because \zeta(\bar{s}) = \overline{\zeta(s)}\right)$$

$$\therefore I = \int_{0}^{\pi/2} \log\left|\zeta\left(\frac{e^{i\theta}}{2\cos\theta}\right)\right| d\theta$$

$$I = \int_0^{\pi/2} \log \left| \zeta\left(\frac{e^{i\theta}}{e^{i\theta}+e^{-i\theta}} \right) \right| d\theta \quad - (2)$$

Let, $$J = \int_0^{\pi} \log \left| \zeta\left(\frac{e^{i\theta}}{e^{i\theta}+e^{-i\theta}} \right) \right| d\theta$$

<u>Claim</u> :- $I = \frac{1}{2} J$

Let, $$g(\theta) = \log \left| \zeta\left(\frac{e^{i\theta}}{e^{i\theta}+e^{-i\theta}} \right) \right|$$

$$g\left(\frac{2\pi}{2} - \theta\right) = \log \left| \zeta\left(\frac{-e^{-i\theta}}{-e^{-i\theta}-e^{i\theta}} \right) \right|$$

$$= \log \left| \zeta\left(\frac{e^{-i\theta}}{e^{i\theta}+e^{-i\theta}} \right) \right|$$

$$= \log \left| \overline{\zeta\left(\frac{e^{i\theta}}{e^{-i\theta}+e^{i\theta}} \right)} \right|$$

$$\left(\because \zeta(\overline{s}) = \overline{\zeta(s)} \right)$$

$$g\left(\frac{2\pi}{2} - \theta\right) = g(\theta)$$

$$g\left(2\frac{\pi}{2}-\theta\right) = g(\theta)$$

$$\therefore \int_0^{2\frac{\pi}{2}} \log\left|\xi\left(\frac{e^{i\theta}}{e^{i\theta}+e^{-i\theta}}\right)\right| d\theta$$

$$= 2\int_0^{\pi/2} \log\left|\xi\left(\frac{e^{i\theta}}{e^{i\theta}+e^{-i\theta}}\right)\right|$$

$$J = 2I$$

$$\therefore I = \frac{1}{2}\int_0^{\pi} \log\left|\xi\left(\frac{e^{i\theta}}{e^{i\theta}+e^{-i\theta}}\right)\right| d\theta$$

$$K = \int_0^{2\pi} \log\left|\xi\left(\frac{e^{i\theta}}{e^{i\theta}+e^{-i\theta}}\right)\right| d\theta$$

$$h(\theta) = \log\left|\xi\left(\frac{e^{i\theta}}{e^{i\theta}+e^{-i\theta}}\right)\right|$$

$$h(2\pi-\theta) = \log\left|\xi\left(\frac{e^{-i\theta}}{e^{i\theta}+e^{-i\theta}}\right)\right|$$

$$= \log\left|\xi\left(\frac{e^{i\theta}}{e^{i\theta}+e^{-i\theta}}\right)\right|$$

$$\left(\because \xi(\bar{s}) = \overline{\xi(s)}\right)$$

$$h(2\pi-\theta) = h(\theta)$$

$$K = \int_0^{2\pi} \log\left|\zeta\left(\frac{e^{i\theta}}{e^{i\theta}+e^{-i\theta}}\right)\right| d\theta$$

$$= 2\int_0^{\pi} \log\left|\zeta\left(\frac{e^{i\theta}}{e^{i\theta}+e^{-i\theta}}\right)\right| d\theta$$

$$K = 2(2I)$$

$$I = \frac{1}{4} K$$

$$= \frac{1}{4}\int_0^{2\pi} \log\left|\zeta\left(\frac{e^{i\theta}}{e^{i\theta}+e^{-i\theta}}\right)\right| d\theta$$

$$e^{i\theta} = z$$

$$i\, e^{i\theta}\, d\theta = dz$$

$$d\theta = \frac{dz}{iz}$$

$$I = \frac{1}{4i}\oint_{|z|=1} \frac{\log\left|\zeta\left(\frac{z^2}{1+z^2}\right)\right|}{z} dz$$