On attractivity for ψ-Hilfer fractional differential equations systems

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Abstract

In this paper, we investigate the existence of a class of globally attractive solutions of the Cauchy fractional problem with the ψ-Hilfer fractional derivative using the measure of noncompactness. An example is given to illustrate our theory.

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1 Introduction

Consider the following Cauchy fractional problem

$$\left\{ \begin{array}{rcl} {}^{\mathsf{H}}\mathscr{D}_{0+}^{\nu,\eta;\psi}\theta(t) & = & u(t,\theta(t)), \quad t \in (0,\infty) \\ I_{0+}^{1-\gamma;\psi}\theta(0) & = & \theta_0 \end{array} \right. \tag{1.1}$$

where ${}^{\mathsf{H}}\mathscr{D}_{0+}^{\nu,\eta;\psi}\theta(\cdot)$ is the ψ-Hilfer fractional derivative of order $0 < \nu < 1$ and type $0 \leq \eta \leq 1$, $\mathcal{I}_{0+}^{1-\gamma;\psi}\theta(\cdot)$ is the fractional integral of order γ, with $0 \leq \gamma < 1$ with respect to another function, $u : [0,\infty) \times \Omega \to \Omega$ is a continuous function satisfying some conditions and θ_0 is a element of the Banach space.

The theory of fractional differential equations can be found in for example [1, 5, 12, 18, 19, 22, 24, 27, 31]. Existence, uniqueness and Ulam-Hyers stabilities of solutions of differential and integrodifferential equations was studied using the ψ-Hilfer fractional derivative in [17, 23, 18, 19, 20, 22, 24, 26]. In 2013 Hernandez et al. [14], proposed a different approach to abstract fractional differential equations if one considers the existence of non-local mild solutions. In 2018, Sousa and Oliveira [18], investigated the Ulam-Hyers stability of an fractional integrodifferential equation using the Banach fixed point theorem and in 2019, Liu et al. [15] considered the ψ-Hilfer fractional derivative, and investigated the Ulam-Hyers stability of a fractional delay differential equation. We also refer the reader to [4, 3, 11, 13, 27, 26, 25]. Attractivity of mild solutions of fractional differential and integrodifferential equations was considered in [1, 2, 10, 11, 14, 16, 28]. Chang et al. [8], investigated the asymptotic decay of some operators via fixed point theorems and they considered the existence and uniqueness for a class of mild solutions of Sobolev fractional differential equations. In 2008 Banas and O'Regan [6] investigated the existence and local attractivity of solutions of a quadratic Volterra integral equation of fractional order in Banach spaces and in 2012 Chen et al. [9] considered the global attractivity of solutions of fractional differential equations in the Riemann-Liouville fractional derivative sense, using the Krasnoselskii fixed-point theorem and the Schauder fixed point theorem. Motivated by the above we will investigate the existence of globally attractivity solutions to the ψ-Hilfer Cauchy fractional problem (1.1) (paying attention to some particular cases of the ψ-Hilfer fractional derivative) .

This paper is organized as follows. In section 2 we present the definitions of the ψ-Riemann-Liouville fractional integral and the ψ-Hilfer fractional derivative and some important results. In section 3 we investigate the globally attractivity existence of solutions of the Cauchy fractional problem. An example is given to illustrate our results.

2 Preliminaries

Let $J=[a,b]$ $(-\infty<a<b<+\infty)$ be a finite interval of $\mathbb{R}$ and Ω a Banach space.

The space $\mathcal{C}(J,\Omega)$ of continuous functions θ on J has the norm given by [18, 21]

$$\|\theta\| := \sup_{t\in J}|\theta(t)|.$$

We have n-times absolutely continuous functions given by

$$\mathcal{AC}^n(J,\Omega)=\{u: J\longmapsto\Omega, u^{(n-1)}\in\mathcal{AC}(J,\Omega)\}.$$

In particular, $\mathcal{AC}^1(J,\Omega)=\mathcal{AC}(J,\Omega)$.

The weighted space $\mathcal{C}_{\gamma,\psi}(J,\Omega)$ of functions θ on $(a,b]$ is defined by [18, 21]

$$\mathcal{C}_{\gamma,\psi}(J,\Omega)=\{\theta:(a,b]\to\Omega, (\psi(t)-\psi(a))^\gamma\theta(t)\in\mathcal{C}(J,\Omega)\}$$

with $0\leq\gamma<1$ and the norm is given by

$$\begin{aligned}\|\theta\|_{C_{\gamma,\psi}(J,\Omega)} &= \|(\psi(t)-\psi(a))^\gamma\theta(t)\|_{C_{\gamma,\psi}(J,\Omega)}\\ &= \max_{t\in J}|(\psi(t)-\psi(a))^\gamma\theta(t)|\cdot\end{aligned}$$

The weighted space $\mathcal{C}^n_{\gamma,\psi}(J,\Omega)$ of functions θ on $(a,b]$ is defined by

$$\mathcal{C}^n_{\gamma,\psi}(J,\Omega)=\{\theta:(a,b]\to\Omega, \theta(t)\in\mathcal{C}^{n-1}(J,\Omega), \theta^{(n)}\in\mathcal{C}_{\gamma,\psi}(J,\Omega)\}$$

with $0\leq\gamma<1$ and the norm is given by

$$\|\theta\|_{C^n_{\gamma,\psi}(J,\Omega)}=\sum_{k=0}^{n-1}\left\|\theta^{(k)}\right\|_{\mathcal{C}(J,\Omega)}+\left\|\theta^{(n)}\right\|_{\mathcal{C}_{\gamma,\psi}(J,\Omega)}.$$

For $n = 0$, we have $\mathcal{C}^0_{\gamma,\psi}(J,\Omega) = \mathcal{C}_{\gamma,\psi}(J,\Omega)$ and

$$\mathcal{C}^{\nu,\eta}_{\gamma,\psi}(J,\Omega) = \left\{\theta \in \mathcal{C}_{\gamma,\psi}(J,\Omega),\ {}^{\mathsf{H}}\mathscr{D}^{\nu,\eta;\psi}_{a^+}\theta \in \mathcal{C}_{\gamma,\psi}(J,\Omega)\right\}$$

with $\gamma = \nu + \eta(1-\nu)$.

Let (a,b) $(-\infty \le a < b \le +\infty)$ and $\nu > 0$. Also let $\psi(t)$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $\psi'(t)$ on (a,b). The ψ-Riemann-Liouville fractional integral (left-sided) of a function θ with respect to another function ψ on $[a,b]$ is defined by [17, 23]

$$\mathcal{I}^{\nu,\psi}_{a^+}\theta(t) = \frac{1}{\Gamma(\nu)}\int_a^t \Theta^{\nu-1}_{\psi}(s,t)\theta(s)\,\mathrm{d}s. \tag{2.1}$$

where $\Theta^{\nu-1}_{\psi}(s,t) := \psi'(s)(\psi(t)-\psi(s))^{\nu-1}$. Similarly one can define the ψ-Riemann-Liouville fractional integral (right-sided).

Let $n-1 < \nu < n$, with $n \in \mathbb{N}$, $J = [a,b]$ is an interval such that $-\infty \le a < b \le +\infty$ and $\theta, \psi \in \mathcal{C}^n(J,\mathbb{R})$ are two functions such that ψ is increasing and $\psi(t) \neq 0$, for all $t \in J$. The ψ-Hilfer fractional derivative (left-sided), denoted by ${}^{\mathsf{H}}\mathscr{D}^{\nu,\eta;\psi}_{a^+}(\cdot)$ of a function θ of order ν and type $0 \le \eta \le 1$, is defined by [17, 23]

$${}^{\mathsf{H}}\mathscr{D}^{\nu,\eta;\psi}_{a^+}\theta(t) = \mathcal{I}^{\eta(n-\nu);\psi}_{a^+}\left(\frac{1}{\psi'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \mathcal{I}^{(1-\eta)(n-\nu);\psi}_{a^+}\theta(t) \tag{2.2}$$

where $I^{\nu;\psi}_{a^+}(\cdot)$ is the fractional integral given in (2.1). Similarly one can define the ψ-Hilfer fractional derivative (right-sided).

Proposition 2.1 [17] *Let $\nu > 0$ and $\delta > 0$. If $\theta(t) = (\psi(t)-\psi(0))^{\delta-1}$ then*

$$\mathcal{I}^{\nu;\psi}_{0^+}\theta(t) = \frac{\Gamma(\delta)}{\Gamma(\nu+\delta)}(\psi(t)-\psi(0))^{\nu+\delta-1}. \tag{2.3}$$

Proposition 2.2 [17] *Let $\nu > 0$, then*

$${}^{\mathsf{H}}\mathscr{D}^{\nu,\eta;\psi}_{0^+}(\psi(t)-\psi(0)) = 0 \tag{2.4}$$

with ${}^{\mathsf{H}}\mathscr{D}^{\nu,\eta;\psi}_{0^+}(\cdot)$ is the ψ-Hilfer fractional derivative.

Assumes that the operator $u : [0,\infty) \times \Omega \to \Omega$ is continuous. The Cauchy fractional problem (1.1) is equivalent to the integral Volterra equation,

$$\theta(t) = \frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}\theta_0 + \frac{1}{\Gamma(\nu)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\nu-1}u(s,\theta(s))\,\mathrm{d}s \tag{2.5}$$

with $t > 0$.

Let $\mathcal{C}^0_{\gamma,\psi}([t_0,\infty),\Omega) = \{\theta \in \mathcal{C}_{\gamma,\psi}([t_0,\infty),\Omega);\ \lim_{t\to\infty}|\theta(t)| = 0\}\cdot$ Note $\mathcal{C}^0_{\gamma,\psi}([0,\infty),\Omega)$ is a Banach space. We need also the following generalized Arzelà-Ascoli theorem [29].

Lemma 2.3 [29] *The set* $\mathcal{H} \subset \mathcal{C}^0([0,\infty),\Omega)$ *is relatively compact if and only if the following conditions hold:*

1. *for any* $T > 0$*, the function in* $\mathcal{H}$ *are equicontinuous on* $[0,T]$*;*
2. *for any* $t \in [0,\infty)$*,* $\mathcal{H}(t) = \{\theta(t) : \theta \in \mathcal{H}\}$ *is relatively compact in* Ω*;*
3. $\lim\limits_{t\to\infty} |\theta(t)| = 0$ *uniformly for* $\theta \in \mathcal{H}$*.*

Lemma 2.4 [12, 33] *The noncompact measure* $\mu(\cdot)$ *satisfies:*

1. *If for all bounded subsets* B_1, B_2 *of* Ω *implies* $\mu(B_1) \leq \mu(B_2)$*;*
2. *If* $\mu(\{x\} \cup B) = \mu(B)$ *for every* $x \in \Omega$ *and every nonempty subset* $B \subseteq \Omega$*;*
3. $\mu(B) = 0$ *if and only if* B *is relatively compact in* Ω*;*
4. $\mu(B_1 + B_2) \leq \mu(B_1) + \mu(B_2)$*, where* $B_1 + B_2 = \{x + y; x \in B_1, y \in B_2\}$*;*
5. $\mu(B_1 \cup B_2) \leq \max\{\mu(B_1), \mu(B_2)\}$*;*
6. $\mu(\lambda B) \leq |\lambda| \mu(B)$ *for any* $\lambda \in \mathbb{R}$*.*

For any $W \subset C(J,\Omega)$*, we define*

$$\int_0^t W(s)\,ds = \left\{ \int_0^t u(s)\,ds : u \in W \right\}, \text{ for } t \in J. \tag{2.6}$$

Property 2.5 [12, 33] *If* $W \subset \mathcal{C}(J,\Omega)$ *is bounded and equicontinuous, them* $\overline{co}W \subset \mathcal{C}(J,\Omega)$ *is also bounded and equicontinuous.*

Property 2.6 [12, 33] *If* $W \subset \mathcal{C}(J,\Omega)$ *is bounded and equicontinuous, then* $t \to \mu(W(t))$ *is continuous on* J*, and*

$$\mu(W) = \max_{t\in J} \mu(W(t)), \ \mu\left(\int_0^t W(s)\,ds\right) \leq \int_0^t \mu(W(s))\,ds, \ \text{for } t \in J.$$

Property 2.7 [12, 33] *Let* $\{u_n\}_{n=1}^{\infty}$ *be a sequence of Bochener integrable functions from* J *in to* Ω *with* $|u_n(t)| \leq \widetilde{m}(t)$ *for almost all* $t \in J$ *and every* $n \geq 1$*, where* $\widetilde{m} \in L(J,\mathbb{R}^+)$*, then the function* $\widetilde{\Phi}(t) = \mu(\{u_n(t)\}_{n=1}^{\infty})$ *belongs to* $L(J,\mathbb{R}^+)$ *and satisfies*

$$\mu\left(\left\{ \int_0^t u_n(s)\,ds : n \geq 1 \right\}\right) \leq 2\int_0^t \widetilde{\Phi}(s)\,ds.$$

Property 2.8 [12, 33] *If* W *is bounded, then for each* $\varepsilon > 0$*, there is a sequence* $\{u_n\}_{n=1}^{\infty} \subset W$*, such that*

$$\mu(W) \leq \mu(\{u_n(t)\}_{n=1}^{\infty}) + \varepsilon.$$

3 Attractivity with ψ-Hilfer fractional derivative

In this section, we will first discuss two important results, namely, Lemma 3.1 and Lemma 3.4. Then we investigate the existence of attractive solutions of the Cauchy fractional problem.

Now we introduce the following hypothesis:

(C1) $|u(t,\theta)| \leq L(\psi(t)-\psi(0))^{-\xi_1}\,|\theta|^{\xi_2}$ for $t\in(0,\infty)$ and $\theta\in\Omega$, $L\geq 0$, $\nu<\xi_1<1$ and $\xi_2\in\mathbb{R}$;

(C2) There exists a constant $\overline{k}>0$ such that for any bounded set $E\subset\Omega$

$$\mu(u(t,E)) \leq \overline{k}\mu(E);$$

here $\mu(\cdot)$ denotes the Hausdorff measure of non compactness.

For all $\theta\in\mathcal{C}_{\gamma,\psi}([0,\infty),\Omega)$ and for a given $n\in\mathbb{N}^{+}$, we define the operator $\mathcal{T}$ by

$$\mathcal{T}\left(\theta\right)\left(t\right)=\mathcal{T}_1\left(\theta\right)\left(t\right)+\mathcal{T}_2\left(\theta\right)\left(t\right)$$

where

$$\mathcal{T}_1\left(\theta\right)\left(t\right)=\left[\left(\psi\left(t\right)-\psi\left(0\right)\right)+\frac{1}{n}\right]^{\gamma-1}\frac{\theta_0}{\Gamma\left(\gamma\right)} \tag{3.1}$$

and

$$\mathcal{T}_2\left(\theta\right)\left(t\right)=\frac{1}{\Gamma\left(\nu\right)}\int_0^t\Theta_\psi^{\nu-1}\left(s,t\right)u\left(s,\theta\left(s\right)\right)ds, \tag{3.2}$$

with $t\in[0,\infty)$.

As $0<\nu<\xi_2<1$, we can choose $\xi>0$ small enough such that $\nu+\xi-1<0$, $1-\xi_1-\xi\xi_2>0$ and $\nu+\xi-\xi_1-\xi\xi_2<0$. Note that

$$\begin{aligned}
&|(\mathcal{T}\theta)(t)|\\
=&\left|\frac{\left[(\psi(t)-\psi(0))+\frac{1}{n}\right]^{\gamma-1}}{\Gamma(\gamma)}\theta_0+\frac{1}{\Gamma(\nu)}\int_0^t\Theta_\psi^{\nu-1}(s,t)u(s,\theta(s))\,ds\right| \qquad (3.3)\\
\leq&\frac{\left[(\psi(t)-\psi(0))+\frac{1}{n}\right]^{\gamma-1}}{\Gamma(\gamma)}|\theta_0|+\frac{1}{\Gamma(\nu)}\int_0^t\Theta_\psi^{\nu-1}(s,t)\,|u(s,\theta(s))|\;ds\\
\leq&\left[(\psi(t)-\psi(0))+\frac{1}{n}\right]^{\gamma-1}\frac{|\theta_0|}{\Gamma(\gamma)}+\frac{1}{\Gamma(\nu)}\int_0^t\Theta_\psi^{\nu-1}(s,t)L(\psi(s)-\psi(0))^{-\xi_1}|\theta(s)|^{\xi_2}\,ds\\
\leq&\left[(\psi(t)-\psi(0))+\frac{1}{n}\right]^{\gamma-1}\frac{|\theta_0|}{\Gamma(\gamma)}+\frac{1}{\Gamma(\nu)}\int_0^t\Theta_\psi^{\nu-1}(s,t)L(\psi(s)-\psi(0))^{-\xi_1-\xi\xi_2}\,ds\\
\leq&\left[(\psi(t)-\psi(0))+\frac{1}{n}\right]^{\gamma+\xi-1}\frac{|\theta_0|}{\Gamma(\gamma)}+\frac{L}{\Gamma(\nu)}\int_0^t\Theta_\psi^{\nu-1}(s,t)(\psi(s)-\psi(0))^{-\xi_1-\xi\xi_2}\,ds\,.
\end{aligned}$$

Choosing $\delta=\xi\xi_2-\xi_1+1$ and substituting in (2.3) (Proposition 2.1), we get

$$\mathcal{I}_{0^+}^{\nu,\psi}\theta(t)=\frac{\Gamma(1-\xi_1-\xi\xi_2)}{\Gamma(1+\nu-\xi_1-\xi\xi_2)}(\psi(t)-\psi(0))^{\nu-\xi_1-\xi\xi_2}.$$

Then, choosing $T > 0$ sufficiently large, from (3.3), we have

$$
\begin{aligned}
& |\mathcal{T}(\theta)(t)| \\
\leq\ & \left[(\psi(t)-\psi(0))+\frac{1}{n}\right]^{\gamma+\xi-1}\frac{|\theta_0|}{\Gamma(\gamma)}+L\frac{\Gamma(1-\xi_1-\xi\xi_2)}{\Gamma(1+\nu-\xi_1-\xi\xi_2)}(\psi(t)-\psi(0))^{\nu-\xi_1-\xi\xi_2} \\
\leq\ & \left[(\psi(t)-\psi(0))+\frac{1}{n}\right]^{\gamma+\xi-1}\frac{|\theta_0|}{\Gamma(\gamma)}+L\frac{\Gamma(1-\xi_1-\xi\xi_2)}{\Gamma(1+\nu-\xi_1-\xi\xi_2)}(\psi(t)-\psi(0))^{\nu-\xi_1-\xi\xi_2+\xi} \\
\leq\ & 1
\end{aligned} \tag{3.4}
$$

for all $t \geq T$.

Define a set $\mathcal{Q}_{\xi;\psi}$ as follows

$$\mathcal{Q}_{\xi;\psi} = \left\{\theta(t)|\theta \in \mathcal{C}_{\gamma,\psi}([0,\infty),\Omega); \left|(\psi(t)-\psi(0))^{\xi}\theta(t)\right| \leq 1,\ t \geq T\right\}. \tag{3.5}$$

Note that by choosing $\psi(t) = t$ in (3.5), we have

$$\mathcal{Q}_{\xi} = \mathcal{Q}_{\xi,t} = \left\{\theta(t)|\theta \in \mathcal{C}_{\gamma,t}([0,\infty),\Omega); \left|t^{\xi}\theta(t)\right| \leq 1,\ t \geq T\right\}$$

and these sets are particular cases of fractional derivatives (namely Riemann-Liouville and Caputo).

Choosing $\psi(t) = t^{\rho}$, $\rho > 0$ in (3.5), we get

$$\mathcal{Q}_{\xi} = \mathcal{Q}_{\xi,t^{\rho}} = \left\{\theta(t)|\theta \in \mathcal{C}_{\gamma,t^{\rho}}([0,\infty),\Omega); \left|t^{\xi\rho}\theta(t)\right| \leq 1,\ t \geq T\right\}$$

and these sets are particular cases of fractional derivatives (namely Katugampola and Caputo-type).

Note $\mathcal{Q}_{\xi;\psi} \neq \emptyset$ and $\mathcal{Q}_{\xi;\psi}$ is a closed convex subset of $\mathcal{C}^{0}_{\gamma,\psi}([0,\infty),\Omega)$.

Lemma 3.1 *Assume* (C1) *holds. Then,* $\{\mathcal{T}\theta; \theta \in \mathcal{Q}_{\xi,\psi}\}$ *is equicontinuous and* $\lim_{t\to\infty} |\mathcal{T}\theta(t)| = 0$ *uniformly for* $\theta \in \mathcal{Q}_{\xi,\psi}$.

Proof: As $\nu - \xi_1 - \xi\xi_2 < 0$, then there exists $\widetilde{\varepsilon} > 0$ and $T_1 > 0$ large enough such that

$$\left[(\psi(t)-\psi(0))+\frac{1}{n}\right]^{\gamma-1}\frac{|\theta_0|}{\Gamma(\gamma)} < \frac{\widetilde{\varepsilon}}{4}$$

and

$$L\frac{\Gamma(1-\xi_1-\xi\xi_2)}{\Gamma(1+\nu-\xi_1-\xi\xi_2)}(\psi(t)-\psi(0))^{\nu_1-\xi_1-\xi\xi_2} < \frac{\widetilde{\varepsilon}}{4}$$

for $t \geq T_1$.

For each $\theta \in \mathcal{Q}_{\xi,\psi}$ and $t_1, t_2 \geq T_1$, we have

$$
\begin{aligned}
& |(\mathcal{T}\theta)(t_2) - (\mathcal{T}\theta)(t_1)| \\
=\ & \left|\left[(\psi(t_2)-\psi(0))+\frac{1}{n}\right]^{\gamma-1}\frac{|\theta_0|}{\Gamma(\gamma)}+\frac{1}{\Gamma(\nu)}\int_0^{t_2}\Theta_{\psi}^{\nu-1}(s,t_2)\,u(s,\theta(s))\,ds\right.
\end{aligned}
$$

$$
\begin{aligned}
&-\left[(\psi(t_1)-\psi(0))+\frac{1}{n}\right]^{\gamma-1}\frac{|\theta_0|}{\Gamma(\gamma)}-\frac{1}{\Gamma(\nu)}\int_0^{t_1}\Theta_\psi^{\nu-1}(s,t_1)\,u(s,\theta(s))\,ds\Bigg|\\
\leq\ & \left[(\psi(t_2)-\psi(0))+\frac{1}{n}\right]^{\gamma-1}\frac{|\theta_0|}{\Gamma(\gamma)}+\left[(\psi(t_1)-\psi(0))+\frac{1}{n}\right]^{\gamma-1}\frac{|\theta_0|}{\Gamma(\gamma)}+\\
&+\frac{1}{\Gamma(\nu)}\int_0^{t_2}\Theta_\psi^{\nu-1}(s,t_2)\,|u(s,\theta(s))|\;ds+\frac{1}{\Gamma(\nu)}\int_0^{t_1}\Theta_\psi^{\nu-1}(s,t_1)\,|u(s,\theta(s))|\;ds\\
\leq\ & \left[(\psi(t_2)-\psi(0))+\frac{1}{n}\right]^{\gamma-1}\frac{|\theta_0|}{\Gamma(\gamma)}+\left[(\psi(t_1)-\psi(0))+\frac{1}{n}\right]^{\gamma-1}\frac{|\theta_0|}{\Gamma(\gamma)}+\\
&+\frac{1}{\Gamma(\nu)}\int_0^{t_2}\Theta_\psi^{\nu-1}(s,t_2)\,(\psi(s)-\psi(0))^{-\xi_1-\xi\xi_2}\,ds\\
&+\frac{1}{\Gamma(\nu)}\int_0^{t_1}\Theta_\psi^{\nu-1}(s,t_1)\,(\psi(s)-\psi(0))^{-\xi_1-\xi\xi_2}ds\\
\leq\ & \left[(\psi(t_2)-\psi(0))+\frac{1}{n}\right]^{\gamma-1}\frac{|\theta_0|}{\Gamma(\gamma)}+\left[(\psi(t_1)-\psi(0))+\frac{1}{n}\right]^{\gamma-1}\frac{|\theta_0|}{\Gamma(\gamma)}+\\
&+L\frac{\Gamma(1-\xi_1-\xi\xi_2)}{\Gamma(1+\nu-\xi_1-\xi\xi_2)}(\psi(t_2)-\psi(0))^{\nu-\xi_1-\xi\xi_2}+\\
&+L\frac{\Gamma(1-\xi_1-\xi\xi_2)}{\Gamma(1+\nu-\xi_1-\xi\xi_2)}(\psi(t_1)-\psi(0))^{\nu-\xi_1-\xi\xi_2}\\
\leq\ & \left[(\psi(t_2)-\psi(0))+\frac{1}{n}\right]^{\gamma-1}\frac{|\theta_0|}{\Gamma(\gamma)}+\left[(\psi(t_1)-\psi(0))+\frac{1}{n}\right]^{\gamma-1}\frac{|\theta_0|}{\Gamma(\gamma)}+\\
&+L\frac{\Gamma(1-\xi_1-\xi\xi_2)}{\Gamma(1+\nu-\xi_1-\xi\xi_2)}\left[(\psi(t_2)-\psi(0))^{\nu-\xi_1-\xi\xi_2}+(\psi(t_1)-\psi(0))^{\nu-\xi_1-\xi\xi_2}\right]\\
<\ & \frac{\widetilde{\varepsilon}}{4}+\frac{\widetilde{\varepsilon}}{4}+\frac{\widetilde{\varepsilon}}{4}+\frac{\widetilde{\varepsilon}}{4}=\widetilde{\varepsilon},
\end{aligned}\tag{3.6}
$$

and then, we have

$$|(\mathcal{T}\theta)(t_2)-(\mathcal{T}\theta)\,(t_1)|<\widetilde{\varepsilon}.$$

For $0\leq t_1<t_2\leq T_1$ we have

$$
\begin{aligned}
&|(\mathcal{T}\theta)(t_2)-(\mathcal{T}\theta)(t_1)|\\
\leq\ & \left|\left(\left[(\psi(t_2)-\psi(0))+\frac{1}{n}\right]^{\gamma-1}-\left[(\psi(t_1)-\psi(0))+\frac{1}{n}\right]^{\gamma-1}\right)\frac{|\theta_0|}{\Gamma(\gamma)}\right|\\
&+\left|\frac{1}{\Gamma(\nu)}\left(\int_0^{t_2}\Theta_\psi^{\nu-1}(s,t_2)\,u(s,\theta(s))\,ds-\int_0^{t_1}\Theta_\psi^{\nu-1}(s,t_1)\,u(s,\theta(s))\,ds\right)\right|\\
\leq\ & \left|\left(\left[(\psi(t_2)-\psi(0))+\frac{1}{n}\right]^{\gamma-1}-\left[(\psi(t_1)-\psi(0))+\frac{1}{n}\right]^{\gamma-1}\right|\right)\frac{|\theta_0|}{\Gamma(\gamma)}\\
&+\left|\frac{1}{\Gamma(\nu)}\int_0^{t_2}\psi'(s)[(\psi(t_2)-\psi(s))^{\nu-1}-(\psi(t_1)-\psi(s))^{\nu-1}]\,|u(s,\theta(s))|\;ds\right.\\
&\left.+\frac{1}{\Gamma(\nu)}\int_{t_1}^{t_2}\Theta_\psi^{\nu-1}(s,t_2)\,|u(s,\theta(s))|\,ds\right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \left|\left(\left[(\psi(t_2)-\psi(0))+\frac{1}{n}\right]^{\gamma-1}-\left[(\psi(t_1)-\psi(0))+\frac{1}{n}\right]^{\gamma-1}\right|\right)\frac{|\theta_0|}{\Gamma(\gamma)} \\
&\quad +\frac{M}{\Gamma(\nu)}\int_0^{t_2}\psi'(s)[(\psi(t_2)-\psi(s))^{\nu-1}-(\psi(t_1)-\psi(s))^{\nu-1}]\,ds \\
&\quad +\frac{M}{\Gamma(\nu)}\int_{t_1}^{t_2}\Theta_\psi^{\nu-1}(s,t_2)\,ds \\
&\leq \left|\left(\left[(\psi(t_2)-\psi(0))+\frac{1}{n}\right]^{\gamma-1}-\left[(\psi(t_1)-\psi(0))+\frac{1}{n}\right]^{\gamma-1}\right|\right)\frac{|\theta_0|}{\Gamma(\gamma)} \\
&\quad +\frac{M}{\Gamma(\nu)}\left[\frac{(\psi(t_2)-\psi(t_1))^\nu-(\psi(t_2)-\psi(0))^\nu}{\nu}+\frac{(\psi(t_1)-\psi(0))^\nu}{\nu}\right] \\
&\quad +\frac{M}{\Gamma(\nu)}\frac{(\psi(t_2)-\psi(t_1))^\nu}{\nu}
\end{aligned}
$$

which goes to zero as $t_2 \to t_1$; here

$$
M=\sup_{\substack{t\in[0,t_2]\\ x\in S_{\xi,\psi}}}|u(t,\theta(t))|\,.
$$

Similarly, for $t_1 < T_1 < T_2$ we have

$$
\begin{aligned}
|(\mathcal{T}\theta)(t_2)-(\mathcal{T}\theta)(t_1)| &= |\mathcal{T}\theta(t_2)-\mathcal{T}\theta(t_1)+\mathcal{T}\theta(T_1)-\mathcal{T}\theta(t_1)| \\
&\leq |\mathcal{T}\theta(t_2)-\mathcal{T}\theta(T_1)|+|\mathcal{T}\theta(T_1)-\mathcal{T}\theta(t_1)| \to 0
\end{aligned}
$$

as $t_2 \to t_1$.

Thus $\{\mathcal{T}\theta : \theta \in \mathcal{Q}_{\xi,\psi}\}$ equicontinuous.

Now, we show $\lim_{t\to\infty}|(\mathcal{T}\theta)(t)| = 0$ uniformly for $\theta \in \mathcal{Q}_{\xi,\psi}$. We have,

$$
\begin{aligned}
&|\mathcal{T}\theta(t)| \\
&= \left|\left[(\psi(t)-\psi(0))+\frac{1}{n}\right]^{\gamma-1}\frac{|\theta_0|}{\Gamma(\gamma)}+\frac{1}{\Gamma(\nu)}\int_0^t\Theta_\psi^{\nu-1}(s,t)\,u(s,\theta(s))\,ds\right| \\
&\leq \left[(\psi(t)-\psi(0))+\frac{1}{n}\right]^{\gamma-1}\frac{|\theta_0|}{\Gamma(\gamma)}+\frac{1}{\Gamma(\nu)}\int_0^t\Theta_\psi^{\nu-1}(s,t)\,(\psi(s)-\psi(0))^{-\xi_1-\xi\xi_2}\,ds \\
&\leq \left[(\psi(t)-\psi(0))+\frac{1}{n}\right]^{\gamma-1}\frac{|\theta_0|}{\Gamma(\gamma)}+\frac{L\Gamma(1-\xi_1-\xi\xi_2)}{\Gamma(1+\nu-\xi_1-\xi\xi_2)}(\psi(t)-\psi(0))^{\nu-\xi_1-\xi\xi_2}
\end{aligned}
$$

which goes to zero for $t \to \infty$ (note $0 < \gamma < 1$ and $\nu - \xi_1 - \xi\xi_2 + \xi < 0$).

Thus $\lim\limits_{t\to\infty}|\mathcal{T}\theta(t)| = 0$ uniformly for $\theta \in \mathcal{Q}_{\xi,\psi}$, which concludes the proof. □

Lemma 3.2 *Assume* (C1) *holds with* $\psi(t) = t$*. Then,* $\{\mathcal{T}\theta; \theta \in \mathcal{Q}_{\xi,t}\}$ *is equicontinuous and* $\lim_{t\to\infty}|\mathcal{T}\theta(t)| = 0$ *uniformly for* $\theta \in \mathcal{Q}_{\xi,t}$.

Proof: This follows immediately from Lemma 3.1. □

Lemma 3.3 *Assume* (C1) *holds with* $\psi(t) = t^{\rho}$. *Then* $\{\mathcal{T}\theta; \theta \in \mathcal{Q}_{\xi,t^{\rho}}\}$ *is equicontinuous and* $\lim_{t\to\infty} |\mathcal{T}\theta(t)| = 0$ *uniformly for* $\theta \in \mathcal{Q}_{\xi,t^{\rho}}$.

Proof: This follows immediately from Lemma 3.1. □

Lemma 3.4 *Suppose* (C1) *holds. Then,* $\mathcal{T}$ *takes* $\mathcal{Q}_{\xi,\psi}$ *into* $\mathcal{Q}_{\xi,\psi}$ *and is continuous on* $\mathcal{Q}_{\xi,\psi}$.

Proof: First we prove $\mathcal{T}$ takes $\mathcal{Q}_{\xi,\psi}$ into $\mathcal{Q}_{\xi,\psi}$. For $\theta \in \mathcal{Q}_{\xi,\psi}$ and from Lemma 2.3, we see that $\mathcal{T}\theta \in \mathcal{C}_{\gamma,\psi}([0,\infty),\Omega)$.

Using the inequality (3.4) we have

$$\begin{aligned}
\left|(\psi(t)-\psi(0))^{\xi}(\mathcal{T}\theta)(t)\right| &= \left|(\psi(t)-\psi(0))^{\xi}\left[\left((\psi(t)-\psi(0)+\frac{1}{n}\right)^{\gamma-1}\frac{|\theta_0|}{\Gamma(\gamma)}\right.\right.\\
&\quad \left.\left.+\frac{1}{\Gamma(\nu)}\int_0^t \Theta_{\psi}^{\nu-1}(s,t)\, u(s,\theta(s))\, ds\right]\right|\\
&\leq (\psi(t)-\psi(0))^{\xi}\left[\left((\psi(t)-\psi(0)+\frac{1}{n}\right)^{\gamma-1}\frac{|\theta_0|}{\Gamma(\gamma)}\right.\\
&\quad \left.+\frac{1}{\Gamma(\nu)}\int_0^t \Theta_{\psi}^{\nu-1}(s,t_1)\,|u(s,\theta(s))|\, ds\right]\\
&\leq (\psi(t)-\psi(0))^{\xi}\left[\left((\psi(t)-\psi(0)+\frac{1}{n}\right)^{\gamma-1}\frac{|\theta_0|}{\Gamma(\gamma)}\right.\\
&\quad \left.\frac{1}{\Gamma(\nu)}\int_0^t \Theta_{\psi}^{\nu-1}(s,t_1)\,(\psi(s)-\psi(0))^{-\xi\xi_2-\xi_1}\, ds\right]\\
&\leq \left[(\psi(t)-\psi(0)+\frac{1}{n}\right]^{\xi+\gamma-1}\frac{|\theta_0|}{\Gamma(\gamma)}\\
&\quad +\frac{L\Gamma(1-\xi_1-\xi\xi_2)}{\Gamma(1+\nu-\xi_1-\xi\xi_2)}(\psi(t)-\psi(0))^{\nu+\xi-\xi_1-\xi\xi_2}\\
&\leq 1
\end{aligned}$$

for $t \geq T$.

Thus, we have $\left|(\psi(t)-\psi(0))^{\xi}\mathcal{T}\theta(t)\right| \leq 1$, so $\mathcal{T}$ takes $\mathcal{Q}_{\xi,\psi}$ into $\mathcal{Q}_{\xi,\psi}$ ($\mathcal{T}\mathcal{Q}_{\xi,\psi} \subset \mathcal{Q}_{\xi,\psi}$).

Now we prove $\mathcal{T}$ is continuous in $\mathcal{Q}_{\xi,\psi}$.

Now, for $\theta_m, \theta \in \mathcal{Q}_{\xi,\psi}$, $m = 1,2,3,\ldots$ with $\lim\limits_{m\to\infty}\theta_m = \theta$, we prove $\mathcal{T}\theta_m \to \mathcal{T}\theta$, as $m \to \infty$. For $\forall\widetilde{\varepsilon} > 0$, there exists $T_2 > 0$ large enough such that

$$(\psi(T_2)-\psi(0))^{\gamma} < \sqrt{\frac{\widetilde{\varepsilon}}{2}}$$

with $\gamma = \nu + \eta(1-\nu)$, and

$$\frac{L\Gamma(1-\xi_1-\xi\xi_2)}{\Gamma(1+\nu-\xi_1-\xi\xi_2)}(\psi(T_2)-\psi(0))^{\nu-\xi_1-\xi\xi_2} < \sqrt{\frac{\widetilde{\varepsilon}}{2}}.$$

Then, for $t > T_2$, we have

$$
\begin{aligned}
& \left|(\psi(t)-\psi(0))^{\gamma}\left(\mathcal{T}\theta_m(t)-\mathcal{T}\theta(t)\right)\right| \\
= & \left|(\psi(t)-\psi(0))^{\gamma}\left[\frac{1}{\Gamma(\nu)}\int_0^t \Theta_{\psi}^{\nu-1}(s,t)\,u(s,\theta_m(s))\,ds-\frac{1}{\Gamma(\nu)}\int_0^t \Theta_{\psi}^{\nu-1}(s,t)\,u(s,\theta(s))\,ds\right]\right| \\
\leq & (\psi(t)-\psi(0))^{\gamma}\frac{1}{\Gamma(\nu)}\int_0^t \Theta_{\psi}^{\nu-1}(s,t_1)\left(|u(s,\theta_m(s))|+|u(s,\theta(s))|\right)ds \\
\leq & \frac{2L(\psi(t)-\psi(0))^{\gamma}}{\Gamma(\nu)}\int_0^t \Theta_{\psi}^{\nu-1}(s,t_1)\,(\psi(s)-\psi(0))^{\xi\xi_2-\xi_1}\,ds \\
\leq & 2L(\psi(T_2)-\psi(0))^{\gamma}(\psi(T_2)-\psi(0))^{\nu-\xi_1-\xi\xi_2}\frac{\Gamma(1-\xi_1-\xi\xi_2)}{\Gamma(1+\nu-\xi_1+\xi\xi_2)} \\
< & 2\sqrt{\frac{\widetilde{\varepsilon}}{2}}\sqrt{\frac{\widetilde{\varepsilon}}{2}}=\widetilde{\varepsilon}.
\end{aligned}
$$

For $0 < t \leq T_2$ we have

$$
\begin{aligned}
& \left|(\psi(t)-\psi(0))^{\gamma}(\mathcal{T}x_m(t)-\mathcal{T}\theta(t))\right| \\
= & \left|(\psi(t)-\psi(0))^{\gamma}\left[\frac{1}{\Gamma(\nu)}\int_0^t \Theta_{\psi}^{\nu-1}(s,t)\,u(s,x_m(s))\,ds-\right.\right. \\
& \left.\left.\frac{1}{\Gamma(\nu)}\int_0^t \Theta_{\psi}^{\nu-1}(s,t)\,u(s,\theta(s))\,ds\right]\right| \\
\leq & (\psi(t)-\psi(0))^{\gamma}\left[\frac{1}{\Gamma(\nu)}\int_0^t \Theta_{\psi}^{\nu-1}(s,t)\left|u(s,\theta_m(s))-u(s,\theta(s))\right|ds\right].
\end{aligned}
\tag{3.7}
$$

Taking the limit as $m \to \infty$ on both sides of (3.7) and using the Lebesgue dominated convergence theorem, we get

$$
\left|(\psi(t)-\psi(0))^{\gamma}\left[\mathcal{T}\theta_m(t)-\mathcal{T}\theta(t)\right]\right| \to 0\cdot
$$

Thus $\|\mathcal{T}\theta_m-\mathcal{T}\theta\|_{\mathcal{C}_{\gamma,\psi}} \to 0$ as $m \to \infty$ so $\mathcal{T}$ is continuous, which concludes the proof. □

Lemma 3.5 *Assume* (C1) *holds with* $\psi(t)=t$. *Then,* $\mathcal{T}$ *takes* $\mathcal{Q}_{\xi,t}$ *into* $\mathcal{Q}_{\xi,t}$ *and is continuous on* $\mathcal{Q}_{\xi,t}$.

Proof: This follows immediately from Lemma 3.4. □

Lemma 3.6 *Assume* (C1) *holds with* $\psi(t)=t^{\rho}$. *Then,* $\mathcal{T}$ *takes* $\mathcal{Q}_{\xi,t^{\rho}}$ *into* $\mathcal{Q}_{\xi,t^{\rho}}$ *and is continuous on* $\mathcal{Q}_{\xi,t^{\rho}}$.

Proof: This follow immediately from Lemma 3.4. □

Theorem 3.7 *Assume* (C1) *and* (C2) *hold. Then, the Cauchy fractional problem* (1.1) *admits at least one attractive solution.*

Proof: Note $\mathcal{T}:\mathcal{Q}_{\xi,\psi}\to\mathcal{Q}_{\xi,\psi}$ is bounded, continuous (see Lemma 3.4). Also $\{\mathcal{T}\theta:\theta\in\mathcal{Q}_{\xi,\psi}\}$ is equicontinuous and $\lim_{t\to\infty}|\mathcal{T}\theta(t)|=0$ uniformly for $x\in\mathcal{Q}_{\xi,\psi}$ (see Lemma 3.1), in particular $\{\mathcal{T}_2\theta:\theta\in\mathcal{Q}_{\xi,\psi}\}$.

Let's check that for any $t\in[0,\infty)$, $\{(\mathcal{T}\theta)(t):\theta\in\mathcal{Q}_{\xi,\psi}\}$ is relatively compact in Ω by using (C2). For each bounded subset $Q_0\subset\mathcal{Q}_{\xi,\psi}$, set

$$\mathcal{T}^1(Q_0)=\mathcal{T}_2(Q_0),\ \mathcal{T}^n(Q_0)=\mathcal{T}_2\left(\overline{co}\left(\mathcal{T}^{n-1}(Q_0)\right)\right),n=2,3,...$$

where $\overline{co}$ is closure convex hull [7].

Using the condition $(C2)$, Property 2.8 and Property 2.7, for any $\widetilde{\varepsilon}>0$, there is a sequence $\left\{\theta_n^{(1)}\right\}_{n=1}^{\infty}$ such that

$$\begin{aligned}
\mu\left(\mathcal{T}^1(Q_0(t))\right) &= \mu\left(\mathcal{T}_2(Q_0)\right)\\
&\leq 2\mu\left(\frac{1}{\Gamma(\nu)}\int_0^t\Theta_\psi^{\nu-1}(s,t)\,u\left(s,\left\{\theta_n^{(1)}(s)\right\}_{n=1}^{\infty}\right)ds\right)+\widetilde{\varepsilon}\\
&\leq \frac{4}{\Gamma(\nu)}\int_0^t\Theta_\psi^{\nu-1}(s,t)\,\mu\left(u\left(s,\left\{\theta_n^{(1)}(s)\right\}_{n=1}^{\infty}\right)\right)ds+\widetilde{\varepsilon}\\
&\leq \frac{4\overline{k}}{\Gamma(\nu)}\mu(Q_0)\int_0^t\psi'(s)\left(\psi(t)-\psi(s)\right)^{\nu-1}ds+\widetilde{\varepsilon}\\
&= \frac{4\overline{k}}{\Gamma(\nu+1)}\mu(Q_0)\left(\psi(t)-\psi(0)\right)^{\nu}+\widetilde{\varepsilon}.
\end{aligned}$$

Since $\widetilde{\varepsilon}>0$ is arbitrary, we get

$$\mu\left(\mathcal{T}^1(Q_0(t))\right)\leq\frac{4\overline{k}}{\Gamma(\nu+1)}\mu(Q_0)\left(\psi(t)-\psi(0)\right)^{\nu}.$$

By means of the Property 2.7 and Property 2.8, for any $\widetilde{\varepsilon}>0$, there is a sequence $\left\{\theta_n^{(2)}\right\}_{n=1}^{\infty}\subset\overline{co}\left(\mathcal{T}^1(Q_0)\right)$ such that

$$\begin{aligned}
\mu\left(\mathcal{T}^2(Q_0(t))\right) &= \mu\left(\mathcal{T}_2\left(\overline{co}\left(\mathcal{T}^1(Q_0(t))\right)\right)\right)\\
&\leq 2\mu\left(\frac{1}{\Gamma(\nu)}\int_0^t\Theta_\psi^{\nu-1}(s,t)\,u\left(s,\left\{\theta_n^{(2)}(s)\right\}_{n=1}^{\infty}\right)ds\right)+\widetilde{\varepsilon}\\
&\leq \frac{4}{\Gamma(\nu)}\int_0^t\Theta_\psi^{\nu-1}(s,t)\,\mu\left(u\left(s,\left\{\theta_n^{(2)}(s)\right\}_{n=1}^{\infty}\right)\right)ds+\widetilde{\varepsilon}\\
&\leq \frac{\left(4\overline{k}\right)^2\mu(Q_0)}{\Gamma(\nu)\Gamma(\nu+1)}\int_0^t\psi'(s)\left(\psi(t)-\psi(s)\right)^{\nu-1}\left(\psi(s)-\psi(0)\right)^{\nu}ds+\widetilde{\varepsilon}\\
&\quad (\text{let } u=\psi(s)-\psi(0))\\
&= \frac{\left(4\overline{k}\right)^2\mu(Q_0)}{\Gamma(\nu)\Gamma(\nu+1)}\int_0^{\psi(t)-\psi(0)}\left(\psi(t)-\psi(0)-u\right)^{\nu-1}u^{\nu}du+\widetilde{\varepsilon}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\left(4\overline{k}\right)^{2}\mu\left(Q_{0}\right)}{\Gamma\left(\nu\right)\Gamma\left(\nu+1\right)}\left(\psi\left(t\right)-\psi\left(0\right)\right)^{\nu-1}\int_{0}^{\psi(t)-\psi(0)}\left(1-\frac{u}{\psi\left(t\right)-\psi\left(0\right)}\right)^{\nu-1}u^{\nu}du+\widetilde{\varepsilon} \\
&\quad \left(\text{let } p=\frac{u}{\psi\left(t\right)-\psi\left(0\right)}\right) \\
&= \frac{\left(4\overline{k}\right)^{2}\mu\left(Q_{0}\right)}{\Gamma\left(\nu\right)\Gamma\left(\nu+1\right)}\left(\psi\left(t\right)-\psi\left(0\right)\right)^{2\nu-1}\int_{0}^{1}\left(1-p\right)^{\nu-1}p^{\nu}dp+\widetilde{\varepsilon} \\
&= \frac{\left(4\overline{k}\right)^{2}\mu\left(Q_{0}\right)}{\Gamma\left(2\nu+1\right)}\left(\psi\left(t\right)-\psi\left(0\right)\right)^{2\nu-1}+\widetilde{\varepsilon} \\
&\leq \frac{\left(4\overline{k}\right)^{2}\mu\left(Q_{0}\right)}{\Gamma\left(2\nu+1\right)}\left(\psi\left(t\right)-\psi\left(0\right)\right)^{2\nu}+\widetilde{\varepsilon}.
\end{aligned}
$$

By mathematical induction, for every $\widetilde{n}\in\mathbb{N}$, we have

$$
\mu\left(\mathcal{T}^{\widetilde{n}}\left(Q_{0}\left(t\right)\right)\right)\leq\frac{\left(4\overline{k}\right)^{\widetilde{n}}\left(\psi\left(t\right)-\psi\left(0\right)\right)^{\nu\widetilde{n}}}{\Gamma\left(\nu\widetilde{n}+1\right)}\mu\left(Q_{0}\right).
$$

Since

$$
\lim_{\widetilde{n}\to\infty}\frac{\left[4\overline{k}\left(\psi\left(a\right)-\psi\left(0\right)\right)^{\nu}\right]^{\widetilde{n}}}{\Gamma\left(\nu\widetilde{n}+1\right)}=0,
$$

there exists $m\in\mathbb{Z}_{+}$ such that

$$
\frac{\left(4\overline{k}\right)^{m}\left(\psi\left(t\right)-\psi\left(0\right)\right)^{\nu m}}{\Gamma\left(\nu m+1\right)}\leq\frac{\left[4\overline{k}\left(\psi\left(a\right)-\psi\left(0\right)\right)^{\nu}\right]^{m}}{\Gamma\left(\nu m+1\right)}=\widetilde{q}<1.
$$

Then

$$
\mu\left(\mathcal{T}^{m}\left(Q_{0}\left(t\right)\right)\right)\leq\widetilde{q}\mu\left(Q_{0}\right).
$$

We know from Property 2.5, $\mathcal{T}^{m}\left(Q_{0}\left(t\right)\right)$ is bounded and equicontinuous. Then, by Property 2.6, we get

$$
\mu\left(\mathcal{T}^{m}\left(Q_{0}\right)\right)=\max_{t\in[0,a]}\mu\left(\mathcal{T}^{m}\left(Q_{0}\left(t\right)\right)\right).
$$

Hence

$$
\mu\left(\mathcal{T}^{m}\left(Q_{0}\right)\right)\leq\widetilde{q}\mu\left(Q_{0}\right).
$$

We will prove that $\exists\widetilde{D}\subset Q_{\xi,\psi}$, such that $\mu\left(\mathcal{T}_{2}\left(\widetilde{D}\right)\right)=0$, i.e., $\mathcal{T}_{2}\left(\widetilde{D}\right)$ is relatively compact.

Let $D_{0}=Q_{\xi,\psi}, D_{1}=\overline{co}\left(\mathcal{T}^{m}\left(D\right)\right),..., D_{n}=\overline{co}\left(\mathcal{T}^{m}\left(D_{n-1}\right)\right),\ n=2,3,....$

So, we can get

1. $D_{0}\supset D_{1}\supset D_{2}\supset\cdots\supset D_{n-1}\supset D_{n}\supset\cdots;$

2. $\lim_{n\to\infty}\mu\left(D_{n}\right)=0.$

Then $\widetilde{D}=\bigcap_{n=0}^{\infty} D_n$ is a nonempty, compact and convex subset in $\mathcal{Q}_{\xi,\psi}$.

We will prove $\mathcal{T}_2\left(\widetilde{D}\right)\subset\widetilde{D}$. Firstly, we prove,

$$\mathcal{T}_2(D_n)\subset D_n,\ \ n=0,1,2,.... \tag{3.8}$$

From $\mathcal{T}^1(D_0)=\mathcal{T}_2(D_0)\subset D_0$, we know $\overline{co}\left(\mathcal{T}^1\left(D_0\right)\right)\subset D_0$.

Therefore $\mathcal{T}^2\left(D_0\right)=\mathcal{T}_2\left(\overline{co}\left(\mathcal{T}^1\left(D_0\right)\right)\right)\subset\mathcal{T}_2\left(D_0\right)=\mathcal{T}^1\left(D_0\right),\mathcal{T}^3\left(D_0\right)=\mathcal{T}_2\left(\overline{co}\left(\mathcal{T}^2\left(D_0\right)\right)\right)\subset\mathcal{T}_2\left(\overline{co}\left(\mathcal{T}^1\left(D_0\right)\right)\right)=\mathcal{T}^2\left(D_0\right),\mathcal{T}^4\left(D_0\right)=\mathcal{T}_2\left(\overline{co}\left(\mathcal{T}^3\left(D_0\right)\right)\right)\subset\mathcal{T}_2\left(\overline{co}\left(\mathcal{T}^2\left(D_0\right)\right)\right)=\mathcal{T}^3\left(D_0\right)$.

Performing this procedure, m-times, we have

$$\mathcal{T}^m\left(D_0\right)=\mathcal{T}_2\left(\overline{co}\left(\mathcal{T}^{m-1}\left(D_0\right)\right)\right)\subset\mathcal{T}_2\left(\overline{co}\left(\mathcal{T}^{m-2}\left(D_0\right)\right)\right)=\mathcal{T}^{m-1}\left(D_0\right).$$

Hence, $D_1=\overline{co}\left(\mathcal{T}^m\left(D_0\right)\right)\subset\overline{co}\left(\mathcal{T}^{m-1}\left(D_0\right)\right)$, so $\mathcal{T}\left(D_1\right)\subset\mathcal{T}\left(\overline{co}\left(\mathcal{T}^{m-1}\left(D_0\right)\right)\right)=\mathcal{T}^m\left(D_0\right)\subset\overline{co}\left(\mathcal{T}^m\left(D_0\right)\right)=D_1$.

Employing the same method, we can prove $\mathcal{T}_2\left(D_n\right)\subset D_n\left(n=0,1,2,...\right)$. By (3.8), we get

$$\mathcal{T}_2\left(\widetilde{D}\right)\subset\bigcap_{n=0}^{\infty}\mathcal{T}_2\left(D_n\right)\subset\bigcap_{n=0}^{\infty}D_n=\widetilde{D}.$$

Then $\mathcal{T}_2\left(\widetilde{D}\right)$ is compact. Hence, $\mu\left(\mathcal{T}_2\left(\widetilde{D}\right)\right)=0$, i.e., $\mathcal{T}_2\left(\widetilde{D}\right)$ is relatively compact.

On the other hand, for any $\theta_1,\theta_2\in\widetilde{D}$ and $t\in J$, we have

$$\begin{aligned}&\left|\left(\psi\left(t\right)-\psi\left(0\right)\right)^{\gamma}\left[\left(\mathcal{T}_1\theta_1\right)\left(t\right)-\left(\mathcal{T}_1\theta_2\right)\left(t\right)\right]\right|\\ =&\left|\left(\psi\left(t\right)-\psi\left(0\right)\right)^{\gamma}\left\{\left[\left(\psi\left(t\right)-\psi\left(0\right)\right)+\frac{1}{n}\right]^{\gamma-1}\frac{\theta_0}{\Gamma\left(\gamma\right)}-\left[\left(\psi\left(t\right)-\psi\left(0\right)\right)+\frac{1}{n}\right]^{\gamma-1}\frac{\theta_0}{\Gamma\left(\gamma\right)}\right\}\right|=0\end{aligned}$$

which implies that $\left\|\mathcal{T}_1\theta_1-\mathcal{T}_1\theta_2\right\|_{C_{\gamma,\psi}}=0$. Thus, we obtain that $\mu\left(\mathcal{T}_1\left(\widetilde{D}\right)\right)=0$.

So, we have

$$\mu\left(\mathcal{T}\left(\widetilde{D}\right)\right)\leq\mu\left(\mathcal{T}_2\left(\widetilde{D}\right)\right)+\mu\left(\mathcal{T}_1\left(\widetilde{D}\right)\right)=0$$

implies $\mu\left(\mathcal{T}\left(\widetilde{D}\right)\right)=0$, therefore $\mathcal{T}\left(\widetilde{D}\right)$ is relatively compact. By means of Arzelà-Ascoli theorem (see Lemma 2.3), $\mathcal{T}$ is relatively compact. Therefore, by Schauder's fixed point theorem guarantees that $\mathcal{T}$ has a fixed point $\theta_n\in\mathcal{Q}_{\xi,\psi}$ with $\theta_n(t)\to 0$ as $t\to\infty$. Using the idea as in Lemma 3.1, we know that $\{\theta_n(t)\}$ is uniformly bounded and equicontinuous on $[0,\infty)$ and for all $t\in[0,\infty)$, $\{\theta_n(t)\}$ is relatively compact.

As $\{(\mathcal{T}\theta)(t):\theta\in\mathcal{Q}_{\xi,\psi}\}$ is relatively compact for any $t\in[0,\infty)$, then, every sequence $\{\theta_n\}$ in $\mathcal{Q}_{\xi,\psi}$ admit a uniformly convergent subsequence $\{\theta_{n_k}\}$ in $C^0_{\gamma,\psi}(J,\Omega)$ ($\mathcal{Q}_{\xi,\psi}\subset C^0_{\gamma,\psi}(J,\Omega)$) by Arzelà-Ascoli theorem.

Furthermore, $\{\theta_{n_k}\}$ satisfies

$$\theta_{n_k}(t)=\left[(\psi(t)-\psi(0)+\frac{1}{n_k}\right]^{\gamma-1}\frac{\theta_0}{\Gamma(\gamma)}+\frac{1}{\Gamma(\nu)}\int_0^t\Theta_{\psi}^{\nu-1}\left(s,t\right)u(s,\theta_{n_k}(s))\,\mathrm{d}s, \tag{3.9}$$

with $t \in [0, \infty)$.

Let $\theta^*(t) = \lim_{k\to\infty} \theta_{n_k}(t)$ $(t \neq 0)$. The Lebesgue dominated convergence theorem with (3.9) yields

$$\theta^*(t) = \left[(\psi(t) - \psi(0) + \frac{1}{n}\right]^{\gamma-1} \frac{\theta_0}{\Gamma(\gamma)} + \frac{1}{\Gamma(\nu)} \int_0^t \Theta_\psi^{\nu-1}(s,t)\, u(s, \theta^*(s))\, ds,$$

with $t \in [0, \infty)$, so $\theta^*(t)$ is an attractive solution for the Cauchy fractional problem. □

Corollary 3.8 *Assume* (C1) *holds. Then, the Cauchy fractional problem* (1.1) *admits at least one attractive solution.*

We consider the following problem, a fractional differential equation and an initial condition, in $\mathbb{R}$

$$\begin{cases} {}^{\mathsf{H}}\mathscr{D}_{0+}^{\nu,\eta;\psi}\theta(t) &=& (\psi(t) - \psi(0))^{-\xi_1}, \quad t \in (0, \infty) \\ \mathcal{I}_{0+}^{1-\gamma;\psi}\theta(0) &=& 0 \end{cases} \tag{3.10}$$

with $0 < \nu < \xi_1 < 1$, $0 \leq \eta \leq 1$ and $\gamma = \nu + \eta(1 - \nu)$.

From Corollary 3.8, the problem given in (3.10) has an attractive solution since (C1) is valid. Indeed, this can be proved directly, since the solution of (3.10) has an exact solution, given by

$$\theta(t) = \frac{\Gamma(1 - \xi_1)}{\Gamma(\nu + 1 - \xi_1)} (\psi(t) - \psi(0))^{\nu - \xi_1}. \tag{3.11}$$

Using (2.5), i.e.,

$$\theta(t) = \frac{(\psi(t) - \psi(0))^{\gamma-1}}{\Gamma(\gamma)} \mathcal{I}_{0+}^{1-\gamma,\psi}\theta(0) + \frac{1}{\Gamma(\nu)} \int_0^t \Theta_\psi^{\nu-1}(s,t)\, u(s, \theta(s))\, ds,$$

and taking $u(t, \theta(t)) = (\psi(t) - \psi(0))^{-\xi_1}$ we have

$$\theta(t) = \mathcal{I}_{0+}^{\nu,\psi}[(\psi(t) - \psi(0))^{-\xi_1}] = \frac{\Gamma(1 - \xi_1)}{\Gamma(\nu + 1 - \xi_1)} (\psi(t) - \psi(0))^{\nu - \xi_1}$$

which is attractively global.

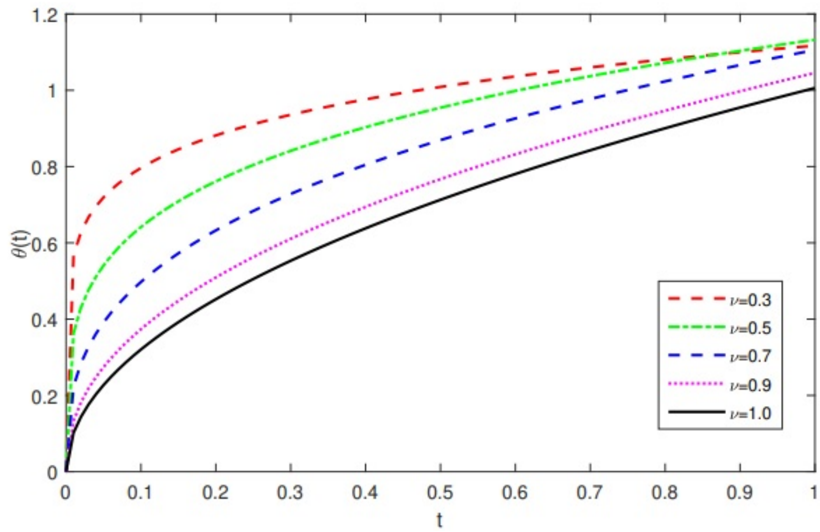
Remark 3.9 *As a particular case of (3.10), we take* $\psi(t) = t$ *and* $\eta \to 0$. *Then*

$$\theta(t) = \frac{\Gamma(1 - \xi_1)}{\Gamma(\nu + 1 - \xi_1)} t^{\nu - \xi_1} \tag{3.12}$$

which is the solution of (3.10), in the Riemann-Liouville fractional derivative sense.

Also, taking $\psi(t) = t^\rho$ $(\rho > 0)$ *and* $\eta \to 1$ *in (3.10), we get*

$$\theta(t) = \frac{\Gamma(1 - \xi_1)}{\Gamma(\nu + 1 - \xi_1)} t^{\rho(\nu - \xi_1)} \tag{3.13}$$



which is the solution of (3.10), in the Caputo-type fractional derivative sense.

The following graph is for (3.13). We choose $t \in [0,1]$, $\rho = 0.5$ *and* $\xi_1 = 0.06$.

Taking $\psi(t) = t$ *and* $\nu \to 1$ *on both sides of (3.10), we obtain the problem*

$$\begin{cases} \theta'(t) &= t^{-\xi}, \quad t \in (0,\infty) \\ \theta(0) &= = 0\,. \end{cases}$$

Consequently, the solution is given by

$$\theta(t) = \frac{\Gamma(1-\xi_1)}{\Gamma(2-\xi_1)} t^{1-\xi_1} = \frac{t^{1-\xi_1}}{1-\xi_1}$$

with $\theta(t) \to \infty$ *as* $t \to \infty$.

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