Faedo-Galerkin approximation of mild solutions of nonlocal fractional functional differential equations

J. Vanterler da C. Sousa [1]
vanterler@ime.unicamp.br

Michal Fečkan [2]
Michal.Feckan@fmph.uniba.sk

E. Capelas de Oliveira [3]
capelas@ime.unicamp.br

[1,3] Department of Applied Mathematics, Imecc-Unicamp, 13083-859, Campinas, SP, Brazil.
[2] Department of Mathematical Analysis and Numerical Mathematics, Comenius University in Bratislava, Mlynská dolina, 842 48 Bratislava, Slovakia

Abstract

The existence, uniqueness and convergence of approximation of mild solutions for a class of nonlocal fractional functional differential equations in Hilbert separable space, will be investigated. To this end, the Gronwall inequality and Faedo-Galerkin approximation will be used.

Key words: Fractional differential equations, existence and uniqueness, mild solution, Faedo-Galerkin approximation, Gronwall inequality.
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1 Introduction

From the beginning of the fractional calculus, namely, on September 30, 1695, in a letter written by l'Hospital to his friend Leibniz, in which the meaning of a middle order derivative is proposed and discussed [30, 31, 32]. Leibniz's response to his friend, coupled with the contribution of countless brilliant mathematicians such as Lagrange, Laplace, Fourier, Liouville, among others, led to the first definitions of non integer orders fractional derivatives and integrals, that at the end of the nineteenth century, due primarily to the definitions proposed by Riemann-Liouville and Grünwald–Letnikov, seemed complete [25, 29, 51]. From then on, innumerable definitions of fractional derivatives and integrals were introduced by numerous researchers and scientists, each one with its own importance and relevance. Thus, countless incredible applications in various fields, such as mechanics, population dynamics, medicine, physics, engineering, among others, have been gaining strength over the years, making the theory well-established [34, 36, 49, 50]. But, an important question arise how do you know, what is the best fractional derivative to look at data for a given problem? One way to overcome this problem is to propose more general fractional derivatives and integrals, where the existing ones are particular cases. Then, in 2018, Sousa and Oliveira [41], introduced the so-called ψ-Hilfer fractional derivative, which contains as a particular case a wide class of fractional derivatives. To complete the ψ-Hilfer fractional derivative theory, in 2019, the same authors [42] introduced the two-part Leibniz-type rule, which, depending on the chosen parameter, gives the Leibniz rule and the Leibniz-type rule for their particular cases.

Also, another question, why study fractional differential equations? What are the advantages of the results obtained from them? In recent years, investigating fractional differential equations has attracted a great deal of attention from several researchers, for better describing physical phenomena and providing results more consistent with the reality compared to integer order differential equations [25, 27, 28, 29, 36, 43, 46, 47, 49]. On the other hand, investigating the existence, uniqueness, stability of Ulam-Hyers, attractivity, continuous dependence on data, among others, of fractional differential equations has been a very attractive field for researchers from various fields, specifically for mathematicians. To study these numerous solution properties, useful tools are needed, namely: fixed point theorem, Gronwall inequality, Arzelà-Ascoli theorem, Laplace transform, Fourier transform, measure of non compactness and others [2, 3, 4, 6, 12, 15, 16, 48, 52, 53].

The Faedo-Galerkin approach has been used by many researchers to investigate more regular solutions in fractional differential equations [11, 7, 23, 37, 38]. This approach can be used within a variational formulation to provide solutions of possibly weaker equations [17]. In this regard, in 2010 Muslim [40] did important work on the global existence and uniqueness of mild solutions of the fractional order integral equation in Banach space and also discussed these same properties in Hilbert separable space. In addition, through the Faedo-Galerkin approach, the approximate solution convergence was investigated. In 2013, Lizama and N'Guérékata [26] approached the existence of mild solutions for the fractional differential equation with nonlocal conditions and investigated the asymptotic behavior of mild solutions for abstract fractional relaxation equations towards the Caputo fractional derivative. On the other hand, we suggest other work on the existence and uniqueness of mild solutions for semilinear nonlocal fractional Cauchy problem, as discussed by Ghour and Omari [1]. In the literature there are numerous works on interesting properties of solutions of fractional differential equations, we refer some articles for a more detailed reading [14, 18, 19, 35, 44, 45].

On the other hand, the theme Faedo-Galerkin approximation, in fact, continues to be the subject of study by a class of researchers [6, 20, 21, 22, 24]. In 2016, Chaddha et al. [8] using the semi-group theory and the Banach fixed point theorem considered an impulsive fractional differential equation structured over a separable Hilbert space, and investigated the existence and uniqueness of solutions for each approximate integral equation. Also, using Faedo-Galerkin approximation the solution was investigated. In the same year, Chadha and Pandey [10], devoted a work on the Faedo-Galerkin approximation of the solution to a nonlocal neutral fractional differential equation with into separable Hilbert space.

Finally, in 2019, an interesting and important work on Faedo-Galerkin approximate solutions of a neutral stochastic finite delay fractional differential equation, performed by Chadha et al. [9], comes to highlight the importance of the theme in the academic community. In this paper, using Banach's fixed point theorem and semi group theory, the authors investigated the existence and uniqueness of mild solutions of a class of neutral stochastic fractional differential equations. Also, they showed the convergence of solutions using Faedo-Galerkin approximations. Other works on Faedo-Galerkin approximation can be found at [11, 13, 14, 37, 40]. Although there is a range of relevant and important work published so far, there are still many ways to go when it comes to mild solutions of fractional differential equations. We note that, the investigation of a mild solution to a fractional differential equation towards the ψ-Hilfer fractional derivative, as some properties and tools are still under discussion. Thus, through the work commented above, we were motivated to propose an investigation of the existence, uniqueness and convergence for a class of solutions of the nonlocal fractional functional differential equations, in order to contribute with new results that can be useful for future research.

So, we consider a class of abstract fractional functional differential equation with nonlocal condition in a separable Hilbert space $\mathscr{H}$

$$\begin{cases} {}^{H}\mathscr{D}_{0+}^{\mu,\nu}u(t) + \mathcal{A}u(t) &= f(t, u(t), u(b(t))),\ t \in (0, T_0] \\ \mathbf{I}_{0+}^{1-\gamma}u(0) + \displaystyle\sum_{k=1}^{p} C_k \mathbf{I}_{0+}^{1-\gamma}u(t_k) &= u_0 \end{cases} \tag{1.1}$$

where ${}^{H}\mathscr{D}_{0+}^{\mu,\nu}(\cdot)$ is the Hilfer fractional derivative of order $0 < \mu \leq 1$ and type $0 \leq \nu \leq 1$ [41], $\mathbf{I}_{0+}^{1-\gamma}(\cdot)$ is the Riemann-Liouville fractional integral of order $1-\gamma$ ($\gamma = \mu + \nu(1-\mu),\ 0 \leq \gamma \leq 1$) [41], $0 < t_1 < \cdots < t_p \leq T_0$, $I = [0, T_0]$, $-\mathcal{A}$ be the infinitesimal generator of a $(\mathbb{S}(t))_{t\geq 0}$ semigroup of bounded linear operators on a separable Hilbert space $\mathscr{H}$ and the nonlinear application $f : [0, T_0] \times \mathscr{H} \times \mathscr{H} \to \mathscr{H}$, $b \in C_{1-\gamma}(I, I)$, where $C_{1-\gamma}(I, I)$ the weighted space of all continuous functions from I into I, $c_k \neq 0$ for all $k = 1, 2, 3, \ldots, p$, $p \in \mathbb{N}$ and $u_0 \in \mathscr{H}$.

The extension to the scenario of Hilfer fractional derivative is not immediate and the evidence has important non trivial parts, which are worth highlighting and each step needs to be verified. In this sense, we will present some points that motivated to investigate a nonlocal fractional functional differential equation.

1. In semi group theory, we have $T(ts) = T(t)T(s)$. However, when this issue is addressed in the fractional context, such a property is not valid. For example, the case we are investigating here, $\mathbb{E}_\mu(ts) \neq \mathbb{E}_\mu(t)\mathbb{E}_\mu(s)$, where $\mathbb{E}_\mu(t)$, is the one parameter Mittag-Leffler function;

2. We present a class of an abstract fractional functional differential equations in the sense of Hilfer fractional derivative with nonlocal condition in a separable Hilbert space $\mathscr{H}$ and its respective class of mild solutions. In this sense, we have that from the choice of the limits $\nu \to 1$ and $\nu \to 0$, we have the problems with their respective solutions, for the Caputo and Riemann-Liouville fractional derivatives, respectively. The special case is the integer case when we choose $\mu = 1$;

3. Using Faedo-Galerkin approximation and Gronwall inequality, the existence, uniqueness and convergence of approximation for a class of mild solutions to abstract fractional functional differential equation will be investigated. As in item 2, here we can also obtain that all the results investigated here are also valid for their respective particular cases, since the properties are retained.

The article is organized as follows: In section 2, we present the idea of some function spaces with their respective norms, fundamental in the course of the work. In this sense, concepts of Riemann-Liouville fractional integral with respect to another function, the ψ-Hilfer fractional derivative, the one and two parameter Mittag-Leffler functions, and Gronwall inequality, are presented. To finish the section, some conditions about the Mittag-Leffler function, and the f function are discussed, and we show that the investigated problem is well-defined. In section 3, we will investigate the main results of the paper, approximation of solutions and convergence, i.e., we present results on existence and uniqueness of mild solutions for a class of abstract fractional functional differential equations. Finally, in section 4, we will use Galerkin approach to ensure the uniqueness of solutions.

2 Preliminaries

In this section, we present the spaces and their respective norms that will be very important for the elaboration of this article. In this sense, we introduce concepts of Riemann-Liouville fractional integral with respect to another function and the ψ-Hilfer fractional derivative. We discuss the mild solution of the nonlocal functional fractional differential equation with respect to the Mittag-Leffler functions.

Let $I=[0,T_0]$ $(0<T_0<\infty)$ be a finite interval and let $C([0,T_0],\mathscr{H}):=C_{T_0}$ a Banach space of all continuous functions with norm given by [44, 46, 47]

$$\|\Psi\|_{C[0,T_0]}:=\sup_{t\in[0,T_0]}\|\Psi(t)\|, \qquad \text{for all } \Psi\in C_{T_0}\cdot$$

The weighted space $C_{1-\gamma}([0,T_0],\mathscr{H})$ of continuous functions f on $(0,T_0]$ is defined by [44, 46, 47]

$$C_{1-\gamma}([0,T_0],\mathscr{H})=\left\{\Psi:(0,T_0]\to\mathscr{H};\quad t^{1-\gamma}\Psi(t)\in C([0,T_0],\mathscr{H})\right\}$$

with $0\le\gamma\le1$ and the norm given by

$$\|\Psi\|_{C_{1-\gamma}[0,T_0]}:=\sup_{t\in[0,T_0]}\left\|t^{1-\gamma}\Psi(t)\right\|_{C[0,T_0]}\cdot$$

Let (a,b) $(-\infty\le a<b\le\infty)$ be a finite or infinite interval of the real line $\mathbb{R}$ and $\mu>0$. Also let $\psi(x)$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $\psi'(x)$ on (a,b). The left and right-sided fractional integrals of a function f with respect to another function ψ on $[a,b]$ are defined by [29, 41]

$$\mathcal{I}^{\mu;\psi}_{a+}f(x)=\frac{1}{\Gamma(\mu)}\int_a^x\psi'(t)\left(\psi(x)-\psi(t)\right)^{\mu-1}f(t)\,\mathrm{d}t \tag{2.1}$$

and

$$\mathcal{I}^{\mu;\psi}_{b-}f(x)=\frac{1}{\Gamma(\mu)}\int_x^b\psi'(t)\left(\psi(t)-\psi(x)\right)^{\mu-1}f(t)\,\mathrm{d}t\,, \tag{2.2}$$

respectively.

Choosing $\psi(t)=t$ and replacing in Eq.(2.1) and Eq.(2.2), we have the Riemann-Liouville fractional integrals, given by [29, 41]

$$\mathcal{I}^{\mu}_{a+}f(x)=\frac{1}{\Gamma(\mu)}\int_a^x(x-t)^{\mu-1}f(t)\,\mathrm{d}t$$

and

$$\mathcal{I}^{\mu;\psi}_{b-}f(x)=\frac{1}{\Gamma(\mu)}\int_x^b(t-x)^{\mu-1}f(t)\,\mathrm{d}t\,,$$

respectively.

On the other hand, let $n-1<\mu<n$ with $n\in\mathbb{N}$, $I=[a,b]$ is the interval such that $-\infty\le a<b\le\infty$ and $f,\psi\in C^n([a,b],\mathbb{R})$ two functions such that ψ is increasing and $\psi'(x)\neq0$, for all $x\in I$. The left-sided and right-sided ψ-Hilfer fractional derivative of order μ and type $0\le\nu\le1$ of a function, denoted by ${}^H\mathscr{D}^{\mu,\nu;\psi}_{a+}(\cdot)$ are defined by [41, 42]

$${}^H\mathscr{D}^{\mu,\nu;\psi}_{a+}f(x)=\mathcal{I}^{\nu(n-\mu);\psi}_{a+}\left(\frac{1}{\psi'(x)}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n\mathcal{I}^{(1-\nu)(n-\mu);\psi}_{a+}f(x) \tag{2.3}$$

and

$${}^H\mathscr{D}^{\mu,\nu;\psi}_{b-}f(x)=\mathcal{I}^{\nu(n-\mu);\psi}_{b-}\left(-\frac{1}{\psi'(x)}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n\mathcal{I}^{(1-\nu)(n-\mu);\psi}_{b-}f(x)\,, \tag{2.4}$$

respectively.

Choosing $\psi(t) = t$ and replacing in Eq.(2.3) and Eq.(2.4), we obtain left-sided and right-sided Hilfer fractional derivative, which we use in the formulation of the nonlinear functional fractional differential equation according to Eq.(1.1), given by [41]

$$^{H}\mathscr{D}_{a+}^{\mu,\nu} f(x) = \mathcal{I}_{a+}^{\nu(n-\mu);\psi} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n} \mathcal{I}_{a+}^{(1-\nu)(n-\mu)} f(x)$$

and

$$^{H}\mathscr{D}_{b-}^{\mu,\nu} f(x) = \mathcal{I}_{b-}^{\nu(n-\mu);\psi} \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n} \mathcal{I}_{b-}^{(1-\nu)(n-\mu)} f(x),$$

respectively.

In what follows, let us state some properties of the special function M_ν also called Mainardi function. This function is a particular case of the Wright type function, introduced by Mainardi. More precisely, for $\xi \in (0,1)$, the entire function $M_\xi : \mathbb{C} \to \mathbb{C}$ is given by [5]

$$M_\xi(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(1-\xi(1+n))}.$$

Proposition 2.1 *[5] For $\xi \in (0,1)$ and $-1 < r < \infty$, when we restrict M_ξ to the positive real line, it holds that $M_\xi(t) \geq 0$ for all $t \geq 0$ and*

$$\int_0^\infty t^r M_\xi(t)\,\mathrm{d}t = \frac{\Gamma(r+1)}{\Gamma(\xi r+1)}.$$

In the sequence, we introduce the Mittag-Leffler operators. Then, for each $\xi \in (0,1)$, we define the Mittag-Leffler families $\left\{\mathbb{E}_\xi(-t^\xi\mathcal{A}) : t \geq 0\right\}$ and $\left\{\mathbb{E}_{\xi,\xi}(-t^\xi\mathcal{A}) : t \geq 0\right\}$, by [5]

$$\mathbb{E}_\xi(-t^\xi\mathcal{A}) = \int_0^\infty M_\xi(s)\mathbb{S}(st^\xi)\mathrm{d}s$$

and

$$\mathbb{E}_{\xi,\xi}(-t^\xi\mathcal{A}) = \int_0^\infty \xi\, s\, M_\xi(s)\mathbb{S}(st^\xi)\,\mathrm{d}s$$

respectively. The functions $\mathbb{E}_\xi(\cdot)$ and $\mathbb{E}_{\xi,\xi}(\cdot)$, are the one and two parameters, Mittag-Leffler functions, respectively.

To this end, let $\mathscr{H}$ be a Hilbert space and $-\mathcal{A} : D(\mathcal{A}) \subset \mathscr{H} \to \mathscr{H}$ be the infinitesimal generators of an $D(t), t \geq 0$. In order to investigate these results, we highlight some necessary conditions about the $\mathcal{A}$ operator and the f function, namely:

(H1) $\mathcal{A}$ is a closed, positive definite, self-adjoint linear operator $\mathcal{A} : D(\mathcal{A}) \subset \mathscr{H} \to \mathscr{H}$ such that $D(\mathcal{A})$ is dense in $\mathscr{H}$ and $\mathcal{A}$ has the pure point spectrum

$$0 < \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$$

and a corresponding complete orthonormal system of eigenfunctions $\{\phi_i\}$, i.e.,

$$\mathcal{A}\phi_i = \lambda_i\phi_i, \qquad \text{and} \qquad <\phi_i, \phi_j> = \tilde{\delta}_{ij}$$

where $\tilde{\delta}_{ij} = 1$ if $i = j$ and $\tilde{\delta}_{ij}$ otherwise.

It follows that the fractional powers $\mathcal{A}^\delta$ of $\mathcal{A}$ for $0 \leq \delta \leq 1$ are well defined

$$\mathcal{A}^\delta : D(\mathcal{A}^\delta) \subset \mathscr{H} \to \mathscr{H}.$$

Hence, for convenience, we suppose that $\left\|\mathbb{E}_\xi(-t^\xi\mathcal{A})\right\| \leq M$, for all $t \geq 0$ and $0 \in \rho(-\mathcal{A})$ where $\rho(-\mathcal{A})$ is the resolvent set of $-\mathcal{A}$. We can prove easily that $D(\mathcal{A}^\delta)$, denoted by H_δ, is the Banach space with the norm [39]

$$\|X\|_\delta = \left\|\mathcal{A}^\delta X\right\|, \qquad \text{for all } X \in D(\mathcal{A}^\delta)\cdot$$

Moreover $C^\delta_{1-\gamma} := C^\delta_{1-\gamma}([0,T_0], D(\mathcal{A}^\delta))$ $(0 \leq \delta \leq 1,$ and $\gamma = \mu+\nu(1-\mu)$, $0 \leq \gamma \leq 1)$, where $D(\mathcal{A}^\delta)$ is the domain of $\mathcal{A}^\delta$, is the Banach space of all weighted space of continuous functions with the norm

$$\|\Psi\|_{C^\delta_{1-\gamma}} := \sup_{t\in[0,T_0]} \left\|t^{1-\gamma}A^\delta\Psi(t)\right\|_{C[0,T_0]}\cdot$$

(H2) The nonlinear map $f : [0, T_0] \times \mathscr{H} \times \mathscr{H} \to \mathscr{H}$ is continuous with respect to the first variable on $[0, T_0]$, $b : [0, T_0] \to [0, T_0]$ is continuous and there exists a non decreasing continuous function $L_R : \mathbb{R}_+ \to \mathbb{R}_+$ depending on $R > 0$ such that

1. $\|f(t, x_1, x_2)\| \leq L_R(t)$
2. $\|f(t, x_1, x_2) - f(s, y_1, y_2)\| \leq L_R(t)\left(|t-s|^\mu + \|x_1 - y_1\|_{C^\delta_{1-\gamma}} + \|x_2 - y_2\|_{C^\delta_{1-\gamma}}\right)$

for all $t, s \in [0, T_0]$, $0 \leq \mu \leq 1$ and $x_i, y_i \in \mathrm{Br}(\mathscr{H}_\delta)$ for $i = 1, 2$.

For any Banach space Z and $r > 0$ we define $\mathrm{Br}(Z) = \{x \in Z, \|x\|_Z \leq r\}$. Throughout the paper we assume that there exists an operator $\mathcal{B}$ on $D(\mathcal{B}) = \mathscr{H}$ given by the formula

$$\mathcal{B} = \left(I + \sum_{k=1}^{p} c_k \mathbf{I}^{1-\gamma}_{0^+}\mathbb{E}_{\mu,\gamma}(-t_k^\mu\mathcal{A})t_k^{\gamma-1}\right)^{-1}$$

with

$$\mathrm{Br}(\mathscr{H}_\delta) := \mathrm{Br} = \mathrm{Br}(C^\delta_{T_0} \cap C^{\delta-1}_{T_0}, \widetilde{K}) = \{y \in C^\delta_{T_0} \cap C^{\delta-1}_{T_0} : \left\|y - \widetilde{K}\right\|_{C^{\delta,T}_{1-\gamma}} \leq R\}\cdot$$

Let $\left\{\mathbb{E}_\xi(-t^\xi\mathcal{A}); t \geq 0\right\}$ be a strongly continuous of operators on $\mathscr{H}$ such that

$$\left\|\mathbb{E}_\xi\left(-t_k^\xi\mathcal{A}\right)\right\| \leq \ell\left\|\mathbb{E}_\xi\left(-\delta t_k^\xi\mathcal{A}\right)\right\|$$

$k = 1, 2, \ldots, p$ where δ is a positive constant and ℓ is a constant satisfying the inequality $\ell \geq 1$ and if

$$\sum_{k=1}^{p} |c_k|\,\mathbb{E}_\xi\left(-t_k^\xi\mathcal{A}\right) < \frac{1}{\mathbb{E}_\xi\left(\delta t_0^\xi\mathcal{A}\right)}$$

then

$$\left\|\sum_{k=1}^{p} c_k\mathbb{E}_\xi\left(-t_k^\xi\mathcal{A}\right)\right\| < 1$$

hence the operator $\mathcal{B}$ exists.

It follows that for $0 \leq \delta \leq 1$, $\mathcal{A}^\delta$ can be defined as a closed linear invertible operator with domain $D(\mathcal{A}^\delta)$ being dense in $\mathscr{H}$. We have $\mathscr{H}_\theta \to \mathscr{H}_\delta$ for $0 < \delta < \theta$ and the embedding is continuous.

We say that the function $u \in C^\delta_{1-\gamma}$ is called a mild solution of Eq.(1.1) on $[0, T_0]$ if it satisfies the equation

$$u(t) \;=\; \mathbb{E}_{\mu,\gamma}(-t^\mu\mathcal{A})\,\mathcal{B}\,u_0 + \int_0^t \mathbb{H}_\mu(t,s;\mathcal{A})\widetilde{f}_{s,u}b(s)\mathrm{d}s \tag{2.5}$$

$$-\mathbb{E}_{\mu,\mu}(-t^{\mu}\mathcal{A})\,\mathcal{B}\sum_{k=1}^{p}c_k\mathbf{I}_{0^+}^{1-\gamma}\int_0^{t_k}\mathbb{H}_\mu(t_k,s;\mathcal{A})\widetilde{f}_{s,u}b(s)\mathrm{d}s$$

with $\gamma=\mu+\nu(1-\mu)$, $t\in[0,T_0]$, $0<t_1<\cdots<t_p\leq T\leq T_0$, $\widetilde{f}_{s,u}b(s):=f(s,u(s),u(b(s)))$, $\mathbb{H}_\mu(t,s;\mathcal{A}):=(t-s)^{\mu-1}\mathbb{E}_{\mu,\mu}(-(t-s)^{\mu}\mathcal{A})$ and $\mathbb{H}_\mu(t_k,s;\mathcal{A}):=(t_k-s)^{\mu-1}\mathbb{E}_{\mu,\mu}(-(t_k-s)^{\mu}\mathcal{A})$.

Now, from Eq.(2.5), we get

$$u(0)=\mathbb{E}_{\mu,\gamma}(0)\,\mathcal{B}\,u_0-\mathbb{E}_{\mu,\gamma}(0)\,\mathcal{B}\sum_{k=1}^{p}c_k\mathbf{I}_{0^+}^{1-\gamma}\int_0^{t_k}\mathbb{H}_\mu(t_k,s;\mathcal{A})\widetilde{f}_{s,u}b(s)\mathrm{d}s \tag{2.6}$$

and

$$\begin{aligned} u(t_i) &= \mathbb{E}_{\mu,\gamma}\left(-t_i^{\mu}\mathcal{A}\,\mathcal{B}\,u_0+\int_0^{t_i}\mathbb{H}_\mu(t_i,s;\mathcal{A})\widetilde{f}_{s,u}b\,(s)\right)\mathrm{d}s \\ &\quad -\mathbb{E}_{\mu,\mu}(-t_i^{\mu}\mathcal{A})\,\mathcal{B}\sum_{k=1}^{p}c_k\mathbf{I}_{0^+}^{1-\gamma}\int_t^{t_k}\mathbb{H}_\mu(t_k,s;\mathcal{A})\widetilde{f}_{s,u}b(s)\mathrm{d}s. \end{aligned} \tag{2.7}$$

Hence, from Eq.(2.6) and Eq.2.7) and the definition of operator $\mathcal{B}$, we get

$$\begin{aligned} &\mathbf{I}_{0^+}^{1-\gamma}u(0)+\sum_{i=1}^{p}c_i\mathbf{I}_{0^+}^{1-\gamma}u(t_i) \\ =\;& \mathbf{I}_{0^+}^{1-\gamma}\mathbb{E}_{\mu,\gamma}(0)\,\mathcal{B}\,u_0-\mathbf{I}_{0^+}^{1-\gamma}\mathbb{E}_{\mu,\gamma}(0)\,\mathcal{B}\sum_{k=1}^{p}c_k\mathbf{I}_{0^+}^{1-\gamma}\int_0^{t_k}\mathbb{H}_\mu(t_k,s;\mathcal{A})\widetilde{f}_{s,u}b(s)\mathrm{d}s \\ &+\sum_{i=1}^{p}c_i\mathbf{I}_{0^+}^{1-\gamma}\mathbb{E}_{\mu,\gamma}(-t_i^{\mu}\mathcal{A})\,\mathcal{B}\,u_0+\sum_{i=1}^{p}c_i\mathbf{I}_{0^+}^{1-\gamma}\int_0^{t_i}\mathbb{H}_\mu(t_i,s;\mathcal{A})\widetilde{f}_{s,u}b(s)\mathrm{d}s \\ &-\sum_{i=1}^{p}c_i\mathbf{I}_{0^+}^{1-\gamma}\mathbb{E}_{\mu,\gamma}(-t_i^{\mu}\mathcal{A})\,\mathcal{B}\sum_{k=1}^{p}c_k\mathbf{I}_{0^+}^{1-\gamma}\int_0^{t_k}\mathbb{H}_\mu(t_k,s;\mathcal{A})\widetilde{f}_{s,u}b(s)\mathrm{d}s. \end{aligned}$$

$$\begin{aligned} &\mathbf{I}_{0^+}^{1-\gamma}u(0)+\sum_{i=1}^{p}c_i\mathbf{I}_{0^+}^{1-\gamma}u(t_i) \\ =\;& u_0-\mathcal{B}\sum_{k=1}^{p}c_k\mathbf{I}_{0^+}^{1-\gamma}\int_0^{t_k}\mathbb{H}_\mu(t_k,s;\mathcal{A})\widetilde{f}_{s,u}b(s)\mathrm{d}s \\ &+\sum_{i=1}^{p}c_i\mathbf{I}_{0^+}^{1-\gamma}\int_0^{t_i}\mathbb{H}_\mu(t_i,s;\mathcal{A})\widetilde{f}_{s,u}b(s)\mathrm{d}s \\ &-\sum_{i=1}^{p}c_i\mathbf{I}_{0^+}^{1-\gamma}\mathbb{E}_{\mu,\gamma}(-t_i^{\mu}\mathcal{A})\,\mathcal{B}\sum_{k=1}^{p}c_k\mathbf{I}_{0^+}^{1-\gamma}\int_0^{t_k}\mathbb{H}_\mu(t_k,s;\mathcal{A})\widetilde{f}_{s,u}b(s)\mathrm{d}s \\ =\;& u_0+\sum_{i=1}^{p}c_i\mathbf{I}_{0^+}^{1-\gamma}\int_0^{t_i}\mathbb{H}_\mu(t_i,s;\mathcal{A})\widetilde{f}_{s,u}b(s)\mathrm{d}s \\ &-\left(I+\sum_{i=1}^{p}c_k\mathbf{I}_{0^+}^{1-\gamma}\mathbb{E}_{\mu,\gamma}(-t_i^{\mu}\mathcal{A})\right)B\sum_{k=1}^{p}c_k\mathbf{I}_{0^+}^{1-\gamma}\int_0^{t_k}\mathbb{H}_\mu(t_k,s;\mathcal{A})\widetilde{f}_{s,u}b(s)\mathrm{d}s \\ =\;& u_0. \end{aligned}$$

Thus we have that the mild solution given by Eq.(2.5) satisfies the condition given in Eq.(1.1), is well defined.

A function $u:[0,T_0]\to\mathscr{H}$ is said to be a classical solution of the nonlocal fractional functional differential equation, Eq.(1.1) on $[0,T_0]$ if:

1. u is continuous on $[0, T_0]$ and continuously differentiable on $(0, T_0]$;

2. ${}^H\mathscr{D}_{0+}^{\mu,\nu} u(t) + \mathcal{A}u(t) = f(t, u(t), u(b(t)))$ for $t \in (0, T_0]$;

3. $I_{0+}^{1-\gamma} u(0) + \displaystyle\sum_{k=1}^{p} c_k \mathbf{I}_{0+}^{1-\gamma} u(t_k) = u_0$.

3 Main results

In this section, our main results, namely, the existence, uniqueness, and approximation solutions and convergence of a class of solutions of the nonlinear abstract fractional differential equation in the Hilbert space $\mathcal{H}$, are investigated.

3.1 Approximate solutions and convergence

Let $\mathscr{H}_n \subset \mathscr{H}$ the finite subspace covered by $\{\phi_0, \phi_1, \ldots, \phi_n\}$ and let $P^n : \mathscr{H} \to \mathscr{H}_n$ be the corresponding projection operator for $n = 0, 1, 2, \ldots$. Note that, for $0 < T_0, R < \infty$ fixed, choosing $0 < T \leq T_0$ such that

$$M \left\|\mathcal{B}\right\| \left\|u_0\right\| + \left(1 + M \left\|\mathcal{B}\right\| \widetilde{C} \sum_{k=1}^{p} |c_k|\right) T^{\mu-\delta\mu} C_\delta \frac{L_R(T_0)}{\mu + 1 - \delta\mu} \leq R \tag{3.1}$$

and

$$2L_R(T_0) \frac{C_{\delta\mu}}{1 - \delta\mu} \left\{ M \left\|\mathcal{B}\right\| \widetilde{C} \sum_{k=1}^{p} |c_k| + 1 \right\} T^{1-\delta\mu} := q < 1 \tag{3.2}$$

where C_δ is a positive constant depending on μ satisfying

$$\left\|\mathcal{A}^\delta \mathbb{E}_{\mu,\gamma}(-t^\mu \mathcal{A})\right\| \leq C_{\delta\mu} t^{-\delta\mu}, \qquad \text{for } t > 0 \cdot$$

We define

$$f_n : [0, T] \times \mathscr{H}_\mu \times \mathscr{H}_\mu \to \mathscr{H}$$

such that $f_n(t, x, y) = f(t, P^n x, P^n y)$ for all $t \in [0, T_0]$, $x, y \in \mathscr{H}_\mu \cdot$

Consider the following set $S = \left\{u \in C_{1-\gamma}^\delta ; \|u\|_{C_{1-\gamma}^\delta} \leq R\right\}$. Note that, clearly, S is a non empty, closed and bounded set. On the other hand, to facilitate the development of the article, we introduce the operator F_n on S as follows

$$\begin{aligned} F_n u(t) &= \mathbb{E}_{\mu,\gamma}(-t^\mu \mathcal{A})\, \mathcal{B}\, u_0 + \int_0^t \mathbb{H}_\mu(t, s; \mathcal{A}) \widetilde{f}_{n,s,u} b(s) \mathrm{d}s \\ &\quad - \mathbb{E}_{\mu,\gamma}(-t^\mu \mathcal{A})\, \mathcal{B} \sum_{k=1}^{p} c_k\, \mathbf{I}_{0+}^{1-\gamma} \int_0^{t_k} \mathbb{H}_\mu(t_k, s; \mathcal{A}) \widetilde{f}_{n,s,u} b(s) \mathrm{d}s \end{aligned}$$

with $t \in [0, T_0]$ for $u \in S$ and $n = 0, 1, 2, \ldots$ and $\widetilde{f}_{n,s,u} b(s) := f_n(s, u(s), u(b(s)))$.

So, next the first main result of this article, that is, the solution $u_n \in S$ satisfying the approximate integral equation Eq.(3.3), is presented as a theorem.

Theorem 3.1 *Let us assume that the assumptions* (H1)-(H2) *hold and* $u_0 \in D(\mathcal{A})$. *Then, there exists a unique* $u_n \in S$ *such that* $F_n u_n = u_n$ *for each* $n = 0, 1, 2, \ldots$ *i.e.,* u_n *satisfies the approximate integral equation*

$$\begin{aligned} u_n(t) &= \mathbb{E}_{\mu,\gamma}(-t^\mu \mathcal{A})\, \mathcal{B}\, u_0 + \int_0^t \mathbb{H}_\mu(t,s;\mathcal{A}) \widetilde{f}_{n,s,u_n} b(s) \mathrm{d}s \\ & \quad \mathbb{E}_{\mu,\gamma}(-t^\mu A)\, \mathcal{B} \sum_{k=1}^{p} c_k \mathbf{I}_{0^+}^{1-\gamma} \int_0^{t_k} \mathbb{H}_\mu(t_k,s;\mathcal{A})) \widetilde{f}_{n,s,u_n} b(s) \mathrm{d}s \end{aligned} \tag{3.3}$$

with $t \in [0,T]$ *and* $\widetilde{f}_{n,s,u_n} b(s) := f_n(s, u_n(s), u_n(b(s)))$.

Proof:

Our goal here is to establish the uniqueness of solution of approximate integral equation, Eq.(3.3), on $[0,T]$. Two points are necessary and sufficient for the proof of this theorem, namely:

1. F_n is a mapping from S into S
2. F_n is a contraction mapping on S.

Then, we have

$$\begin{aligned} & \left\| F_n u\,(t+h) - F_n u\,(t) \right\|_{C^\delta} \\ = & \left\| \begin{array}{c} (\mathbb{E}_{\mu,\gamma}(-(t+h)^\mu \mathcal{A}) - \mathbb{E}_{\mu,\gamma}(-t^\mu \mathcal{A}))\, \mathcal{B}\, \mathcal{A}^\delta u_0 - \\ [\mathbb{E}_{\mu,\gamma}(-(t+h)^\mu \mathcal{A}) - \mathbb{E}_{\mu,\gamma}(-t^\mu \mathcal{A})]\, \mathcal{B} \sum_{k=1}^{p} c_k \mathbf{I}_{0^+}^{1-\gamma} \\ \times \displaystyle\int_0^{t_k} \mathbb{H}_\mu(t_k,s;\mathcal{A}) \mathcal{A}^\delta \widetilde{f}_{n,s,u}\, \mathrm{d}s - \int_t^{t+h} \mathbb{H}_\mu(t,h,s;\mathcal{A}) \mathcal{A}^\delta \widetilde{f}_{n,s,u} b(s) \mathrm{d}s \\ - \displaystyle\int_0^{t} \mathbb{H}_\mu(t,h,s;\mathcal{A}) \mathcal{A}^\delta \widetilde{f}_{n,s,u} b(s) \mathrm{d}s - \int_0^{t} \mathbb{H}_\mu(t,s;\mathcal{A}) \mathcal{A}^\delta \widetilde{f}_{n,s,u} b(s) \mathrm{d}s \end{array} \right\|_{C^\delta} \\ \leq & \ (\mathbb{E}_{\mu,\gamma}(-(t+h)^\mu \mathcal{A}) - \mathbb{E}_{\mu,\gamma}(-t^\mu \mathcal{A}))\ \mathcal{B}\, \mathcal{A}^\delta u_0 \\ & + \sum_{k=1}^{p} |c_k|\ [\mathbb{E}_{\mu,\gamma}(-(t+h)^\mu \mathcal{A}) - \mathbb{E}_{\mu,\gamma}(-t^\mu \mathcal{A})]\, \mathcal{B}\ \mathbf{I}_{0^+}^{1-\gamma} \times \\ & \times \int_0^{t_k} \mathbb{H}_\mu(t_k,s;\mathcal{A}) \mathcal{A}^\delta \widetilde{f}_{n,s,u} b(s) \mathrm{d}s \\ & + \int_0^{t} (t+h-s)^{\mu-1} \left[\mathbb{E}_{\mu,\mu}(-(t+h-s)^\mu \mathcal{A}) - \mathbb{E}_{\mu,\mu}(-(t-s)^\mu \mathcal{A} \right] \times \\ & \times \mathcal{A}^\delta \left\| \widetilde{f}_{n,s,u} b(s) \right\| \mathrm{d}s \end{aligned}$$

where $\mathbb{H}_\mu(t,h,s;\mathcal{A}) := (t+h-s)^{\mu-1} \mathbb{E}_{\mu,\mu}(-(t+h-s)^\mu \mathcal{A})$, for all $t \in [0,T]$, $h > 0$, $u \in S$. So we get,

$$F_n : C^\delta_{1-\gamma} \to C^\delta_{1-\gamma}.$$

On the other hand, for any $u \in S$, we get

$$\begin{aligned} & \| F_n u(t) \|_{C^\delta} \\ = & \left\| \mathbb{E}_{\mu,\gamma}(-t^\mu \mathcal{A})\, \mathcal{B} \mathcal{A}^\delta u_0 + \int_0^t \mathbb{H}_\mu(t,s;\mathcal{A}) \mathcal{A}^\delta \widetilde{f}_{n,s,u} b(s) \mathrm{d}s \right. \end{aligned}$$

$$
\begin{aligned}
&-\mathbb{E}_{\mu,\gamma}(-t^{\mu}\mathcal{A})\,\mathcal{B}\sum_{k=1}^{p}c_k\mathbf{I}_{0^+}^{1-\gamma}\int_0^{t_k}\mathbb{H}_{\mu}(t_k,s;\mathcal{A})\mathcal{A}^{\delta}\widetilde{f}_{n,s,u}b(s)\mathrm{d}s\Bigg\| \\
\leq\;& M\,\|\mathcal{B}\|\,\left\|\mathcal{A}^{\delta}u_0\right\| + \int_0^t (t-s)^{\mu-1}C_{\delta\mu}(t-s)^{-\delta\mu}L_R(s)\,\mathrm{d}s \\
&+\sum_{k=1}^{p}|c_k|\,M\,\|\mathcal{B}\|\,\widetilde{C}\int_0^{t_k}(t_k-s)^{\mu-1}C_{\delta\mu}(t_k-s)^{-\delta\mu}L_R(s)\,\mathrm{d}s \\
\leq\;& M\,\|\mathcal{B}\|\,\|u_0\|_{C^{\delta}} + L_R(T_0)C_{\delta\mu}\frac{T^{\mu-\delta\mu}}{\mu-\delta\mu} + M\,\|\mathcal{B}\|\,\widetilde{C}C_{\delta\mu}L_R(T_0)\sum_{k=1}^{p}|c_k|\,\frac{T^{\mu-\delta\mu}}{\mu-\delta\mu} \\
\leq\;& M\,\|\mathcal{B}\|\,\|u_0\|_{C^{\delta}} + \left(1+\widetilde{C}M\,\|\mathcal{B}\|\sum_{k=1}^{p}|c_k|\right)\frac{T^{\mu-\delta\mu}}{\mu-\delta\mu}C_{\delta\mu}L_R(T_0). \qquad (3.4)
\end{aligned}
$$

Therefore, from inequality (3.4), it follows that

$$
\begin{aligned}
&\|F_n u\|_{C^{\delta}_{1-\gamma}} \\
\leq\;& MT^{1-\gamma}\,\|\mathcal{B}\|\,\|u_0\|_{C^{\delta}} + \left(1+\widetilde{C}M\,\|\mathcal{B}\|\sum_{k=1}^{p}|c_k|\right)\frac{T^{\mu-\delta\mu+1-\gamma}C_{\delta\mu}L_R(T_0)}{\mu-\delta\mu} \leq R
\end{aligned}
$$

where R is given by Eq.(3.1). Hence $F_n : S \to S$.

Now, for any $u, v \in S$ and $t \in [0,T]$ we have

$$
\begin{aligned}
(F_n u)(t) - (F_n v)(t) \;&=\; -\mathbb{E}_{\mu,\gamma}(-t^{\mu}\mathcal{A})\mathcal{B}\sum_{k=1}^{p}c_k\mathbf{I}_{0^+}^{1-\gamma}\int_0^{t_k}\mathbb{H}_{\mu}(t_k,s;\mathcal{A})\Omega(u,v,s)\mathrm{d}s \\
&\quad+\int_0^t \mathbb{H}_{\mu}(t,s;\mathcal{A})\Omega(u,v,s)\,\mathrm{d}s
\end{aligned}
$$

with $t \in [0,T]$ and where to facilitate the development of the article, we have introduced $\Omega(u,v,s) := [f_n(s,u(s),u(b(s)) - f_n(s,v(s),v(b(s))]$.

Through inequality (3.2), we have

$$
\begin{aligned}
&\|(F_n u)(t) - (F_n v)(t)\|_{C^{\delta}} \\
=\;& \left\| \begin{array}{c} \displaystyle\int_0^t \mathbb{H}_{\mu}(t,s;\mathcal{A})\Omega(u,v,s)\,\mathrm{d}s - \\ \displaystyle\mathbb{E}_{\mu,\gamma}(-t^{\mu}\mathcal{A})\mathcal{B}\sum_{k=1}^{p}c_k\mathbf{I}_{0^+}^{1-\gamma}\int_0^{t_k}\mathbb{H}_{\mu}(t_k,s;\mathcal{A})\Omega(u,v,s)\,\mathrm{d}s \end{array} \right\|_{C^{\delta}}.
\end{aligned}
$$

Using the precedent we can write

$$
\begin{aligned}
&\|(F_n u)(t) - (F_n v)(t)\|_{C^{\delta}} \\
\leq\;& \int_0^t (t-s)^{\mu-1}\,\|\mathbb{E}_{\mu,\mu}(-(t-s)^{\mu}\mathcal{A})\|\,\|\Omega(u,v,s)\|\,\mathrm{d}s + \\
&+\|\mathbb{E}_{\mu,\gamma}(-t^{\mu}\mathcal{A})\|\,\|\mathcal{B}\|\sum_{k=1}^{p}|c_k|\,\left\|\mathbf{I}_{0^+}^{1-\gamma}\right\| \times \\
&\times\int_0^{t_k}(t_k-s)^{\mu-1}\,\|\mathbb{E}(-(t_k-s)^{\mu}\mathcal{A})\|\,\|\Omega(u,v,s)\|\,\mathrm{d}s
\end{aligned}
$$

$$\begin{aligned}
&\leq \int_0^t (t-s)^{\mu-1} C_{\delta\mu}(t-s)^{-\delta\mu} L_R(s)\times \\
&\quad \times \left(\|u(s)-v(s)\|_{C^\delta_{1-\gamma}} + \|u(b(s))-v(b(s))\|_{C^\delta_{1-\gamma}}\right) \mathrm{d}s \\
&\quad + M\,\|\mathcal{B}\| \sum_{k=1}^{p} |c_k| \widetilde{C} \int_t^{t_k} (t_k-s)^{\mu-1} C_{\delta\mu}(t_k-s)^{-\delta\mu} L_R(s)\times \\
&\quad \times \left(\|u(s)-v(s)\|_{C^\delta_{1-\gamma}} + \|u(b(s))-v(b(s))\|_{C^\delta_{1-\gamma}}\right) \mathrm{d}s \\
&= 2C_{\delta\mu} L_R(T_0)\,\|u(t)-v(t)\|_{C^\delta_{1-\gamma}} \int_0^t (t-s)^{\mu-1-\delta\mu}\,\mathrm{d}s + \\
&\quad 2M\,\|\mathcal{B}\|\, L_R(T_0)\widetilde{C} C_{\delta\mu}\,\|u(t)-v(t)\|_{C^\delta_{1-\gamma}} \sum_{k=1}^{p} |c_k| \int_0^{t_k} (t_k-s)^{\mu-1-\delta\mu}\,\mathrm{d}s \\
&\leq 2C_{\delta\mu} L_R(T_0)\,\|u(t)-v(t)\|_{C^\delta_{1-\gamma}} \frac{T^{\mu-\delta\mu}}{\mu-\delta\mu} \\
&\quad + 2M\,\|\mathcal{B}\|\, L_R(T_0)\widetilde{C} C_{\delta\mu}\,\|u(t)-v(t)\|_{C^\delta_{1-\gamma}} \frac{T^{\mu-\delta\mu}}{\mu-\delta\mu} \sum_{k=1}^{p} |c_k| \\
&= \frac{2C_{\delta\mu} L_R(T_0)}{\mu(1-\delta)} T^{\mu-\delta\mu} \left(1 + M\,\|\mathcal{B}\|\,\widetilde{C} \sum_{k=1}^{p} |c_k|\right) \|u(t)-v(t)\|_{C^\delta_{1-\gamma}}.
\end{aligned}$$

Then, we have

$$\|F_n u - F_n v\|_{C^\delta_{1-\gamma}} \leq q\,\|u-v\|_{C^\delta_{1-\gamma}}$$

where

$$q := \frac{2C_{\delta\mu} L_R(T_0)}{\mu(1-\delta)} T^{\mu-\delta\mu+1-\gamma} \left(1 + M\,\|\mathcal{B}\|\,\widetilde{C} \sum_{k=1}^{p} |c_k|\right) < 1$$

for $u, v \in S$.

Thus, the operator F_n, as defined, has a unique fixed point that is $F_n u_n = u_n$, for $u_n \in S$ given by

$$\begin{aligned}
u_n(t) &= \mathbb{E}_{\mu,\gamma}(-t^\mu \mathcal{A})\,\mathcal{B}\,u_0 + \int_0^t \mathbb{H}_\mu(t,s;\mathcal{A}) \widetilde{f}_{n,s,u_n} b(s)\mathrm{d}s \\
&\quad - \mathbb{E}_{\mu,\gamma}(-t^\mu \mathcal{A})\mathcal{B} \sum_{k=1}^{p} c_k \mathbf{I}_{0^+}^{1-\gamma} \int_0^{t_k} \mathbb{H}_\mu(t_k,s;\mathcal{A}) \widetilde{f}_{n,s,u_n} b(s)\mathrm{d}s
\end{aligned}$$

with $t \in [0,T]$. □

The following result Corollary 3.2, we will not present the demonstration, however, we suggest the article which contains the following proof. (see Lemma 3.2 [10]).

Corollary 3.2 *If all the hypothesis of the* Theorem 3.1 *hold then* $u_n(t) \in D(\mathcal{A}^\theta)$ *for all* $t \in [0,T]$ *with* $0 \leq \theta < 1$.

Corollary 3.3 *If all the hypothesis of the* Theorem 3.1 *hold then there exist a constant* M_0 *independent on* n*, such that*

$$\|u_n\|_{C^\theta_{1-\gamma}} := \left\|\mathcal{A}^\theta u_n(t)\right\|_{C_{1-\gamma}} \leq M_0$$

for all $0 \leq t \leq T$ *and* $0 \leq \theta < 1$.

Proof: In fact, by means of the Eq.(3.3), we get

$$
\begin{aligned}
&\left\|\mathcal{A}^{\theta}u_n\left(t\right)\right\| \\
= &\left\|\begin{array}{c} \mathcal{A}^{\theta}\mathbb{E}_{\mu,\gamma}(-t^{\mu}\mathcal{A})\,\mathcal{B}\,u_0+\mathcal{A}^{\theta}\int_0^t \mathbb{H}_{\mu}(t,s;\mathcal{A})\widetilde{f}_{n,s,u_n}b(s)\mathrm{d}s \\ -\mathcal{A}^{\theta}\mathbb{E}_{\mu,\gamma}(-t^{\mu}\mathcal{A})\,\mathcal{B}\sum_{k=1}^{p}c_k\mathbf{I}_{0+}^{1-\gamma}\int_0^{t_k}\mathbb{H}_{\mu}(t_k,s;\mathcal{A})\widetilde{f}_{n,s,u_n}b(s)\mathrm{d}s \end{array}\right\| \\
\leq &\left\|\mathcal{A}^{\theta}\mathbb{E}_{\mu,\gamma}(-t^{\mu}\mathcal{A})\right\|\left\|\mathcal{B}\right\|\left\|u_0\right\| \\
&+\int_0^t (t-s)^{\mu-1}\left\|\mathcal{A}^{\theta}\mathbb{E}_{\mu,\mu}(-(t-s)^{\mu}\mathcal{A})\right\|_{C_{1-\gamma}}\left\|\widetilde{f}_{n,s,u_n}b(s)\right\|_{C_{1-\gamma}}\mathrm{d}s \\
&+\sum_{k=1}^{p}|c_k|\left\|\mathbb{E}_{\mu,\gamma}(-t^{\mu}\mathcal{A})\right\|\left\|\mathcal{B}\right\|\left\|\mathbf{I}_{0+}^{1-\gamma}\right\|\times \\
&\times\int_0^{t_k}(t_k-s)^{\mu-1}\left\|\mathcal{A}^{\theta}\mathbb{E}_{\mu,\mu}(-(t_k-s)^{\mu}\mathcal{A})\right\|\left\|\widetilde{f}_{n,s,u_n}b(s)\right\|\mathrm{d}s \\
\leq &M\left\|\mathcal{B}\right\|\left\|u_0\right\|+\int_0^t (t-s)^{\mu-1}C_{\theta\mu}(t-s)^{-\theta\mu}L_R(s)\,\mathrm{d}s \\
&+\sum_{k=1}^{p}|c_k|\,M\,\left\|\mathcal{B}\right\|\widetilde{C}\int_0^{t_k}(t_k-s)^{\mu-1}C_{\theta\mu}(t_k-s)^{-\theta\mu}L_R(s)\,\mathrm{d}s \\
\leq &M\left\|\mathcal{B}\right\|\left\|u_0\right\|+C_{\theta\mu}L_R(T_0)\frac{T^{\mu-\theta\mu}}{\mu-\theta\mu}+M\left\|\mathcal{B}\right\|\widetilde{C}C_{\theta\mu}L_R(T_0)\sum_{k=1}^{p}|c_k|\frac{T^{\mu-\theta\mu}}{\mu-\theta\mu} \\
= &M\left\|\mathcal{B}\right\|\left\|u_0\right\|+\left(MM\left\|\mathcal{B}\right\|\widetilde{C}\sum_{k=1}^{p}|c_k|+1\right)T^{\mu-\mu\theta}\frac{C_{\theta\mu}L_R(T_0)}{\mu-\mu\theta}
\end{aligned}
$$

for $t\in[0,T]$.

Then, we have

$$
\left\|u_n\right\|_{C^{\theta}_{1-\gamma}}\leq M\left\|\mathcal{B}\right\|\left\|u_0\right\|+\left(M\left\|\mathcal{B}\right\|\widetilde{C}\sum_{k=1}^{p}|c_k|+1\right)T^{\mu-\mu\theta+1-\gamma}\frac{C_{\theta\mu}L_R(T_0)}{\mu-\mu\theta}
$$

with $0\leq\theta<1$, which conclude the proof □

Theorem 3.4 *The sequence $\{u_n\}\subset S$ is a Cauchy sequence and therefore converges to a unique function $u\in S$ if the assumptions* (H1)-(H2) *hold and $u\in D(\mathcal{A})$.*

Proof:

In fact, for $n\geq m\geq n_0$ where n_0 is large enough, $n,m,n_0\in\mathbb{N}$ and $t\in[0,T]$, we get

$$
\begin{aligned}
&\left\|\mathcal{A}^{\delta}\left(u_n(t)-u_m(t)\right)\right\| \\
= &\left\|\begin{array}{c} \mathcal{A}^{\delta}\int_0^t \mathbb{H}_{\mu}(t,s;\mathcal{A})\Omega(n,m,s)\mathrm{d}s-\mathcal{A}^{\delta}\mathbb{E}_{\mu,\gamma}(-t^{\mu}\mathcal{A})\,\mathcal{B}\sum_{k=1}^{p}c_k\mathbf{I}_{0+}^{1-\gamma}\times \\ \times\int_0^{t_k}\mathbb{H}_{\mu}(t_k,s;\mathcal{A})\Omega(n,m,s)\,\mathrm{d}s \end{array}\right\|
\end{aligned}
\tag{3.5}
$$

where $\Omega\left(n,m,s\right)=f_n\left(s,u_n(s),u_n(b(s))\right)-f_m\left(s,u_m(s),u_m(b(s))\right)$.

So, Eq.(3.5) can be written as follows

$$
\begin{aligned}
&\left\|\mathcal{A}^{\delta}\left(u_n(t)-u_m(t)\right)\right\| \\
\leq\ & \int_0^t (t-s)^{\mu-1}\left\|\mathcal{A}^{\delta}\mathbb{E}_{\mu,\mu}(-(t-s)^{\mu}\mathcal{A})\right\|\|\Omega(n,m,s)\|\,\mathrm{d}s+ \\
& \sum_{k=1}^{p}|c_k|\,\|\mathbb{E}_{\mu,\gamma}(-t^{\mu}\mathcal{A})\|\,\|\mathcal{B}\|\left\|\mathbf{I}_{0^+}^{1-\gamma}\right\|\times \\
& \int_0^{t_k}(t_k-s)^{\mu-1}\left\|A^{\delta}\mathbb{E}_{\mu,\mu}(-(t_k-s)^{\mu}\mathcal{A})\right\|\|\Omega(n,m,s)\|\,\mathrm{d}s \\
\leq\ & \int_0^t (t-s)^{\mu-1}C_{\delta\mu}(t-s)^{-\mu\delta}\,\|\Omega(n,m,s)\|\,\mathrm{d}s+ \\
& \sum_{k=1}^{p}|c_k|\,M\,\|\mathcal{B}\|\,\widetilde{C}\int_0^{t_k}(t_k-s)^{\mu-1}C_{\delta\mu}(-(t_k-s)^{-\delta\mu}\,\|\Omega(n,m,s)\|\,\mathrm{d}s
\end{aligned}
\tag{3.6}
$$

with $t\in[0,T]$.

Note that, for $0<\delta<\theta<1$, we get

$$
\begin{aligned}
\|\Omega(n,m,s)\| &\leq \|f_n(s,u_n(s),u_n(b(s)))-f_n(s,u_m(s),u_m(b(s)))\| \\
&\quad +\|f_n(s,u_m(s),u_m(b(s)))-f_m(s,u_m(s),u_m(b(s)))\| \\
&\leq \left(\|u_n(s)-u_m(s)\|_{C^{\delta}_{1-\gamma}}+\|u_n(b(s))-u_m(b(s))\|_{C^{\delta}_{1-\gamma}}\right)L_R(T_0) \\
&\quad +\|f_n(s,u_m(s),u_m(b(s)))\|+\|f_m(s,u_m(s),u_m(b(s)))\| \\
&\leq 2L_R(T_0)\,\|u_n-u_m\|_{C^{\delta,s}_{1-\gamma}}+2L_R(T_0)\frac{M_0}{\lambda_m^{\theta-\delta}}
\end{aligned}
\tag{3.7}
$$

where M_0 is the same as in Corollary 3.3.

Using the inequality (3.7) in inequality (3.6), we have

$$
\begin{aligned}
&\left\|\mathcal{A}^{\delta}(u_n(t)-u_m(t))\right\| \\
\leq\ & \left(1+M\,\|\mathcal{B}\|\,\widetilde{C}\sum_{k=1}^{p}|c_k|\right)C_{\delta\mu}\int_0^t (t-s)^{\mu(1-\delta)-1}\,\|\Omega\left(n,m,s\right)\|\,\mathrm{d}s \\
\leq\ & \left(1+M\,\|\mathcal{B}\|\,\widetilde{C}\sum_{k=1}^{p}|c_k|\right)C_{\delta\mu}\int_0^t (t-s)^{\mu(1-\delta)-1}\times \\
& \times\left(2L_R(T_0)\,\|u_n-u_m\|_{C^{\delta,s}_{1-\gamma}}+2L_R(T_0)\frac{M_0}{\lambda_m^{\theta-\delta}}\right)\mathrm{d}s \\
\leq\ & \left(1+M\,\|\mathcal{B}\|\,\widetilde{C}\sum_{k=1}^{p}|c_k|\right)C_{\delta\mu}2L_R(T_0)\int_0^t (t-s)^{\mu(1-\delta)-1}\,\|u_n-u_m\|_{C^{\delta,s}_{1-\gamma}}\,\mathrm{d}s \\
& +\left(1+M\,\|\mathcal{B}\|\,\widetilde{C}\sum_{k=1}^{p}|c_k|\right)C_{\delta\mu}\frac{T^{\mu(1-\delta)}}{\mu(1-\delta)}2L_R(T_0)\frac{M_0}{\lambda_m^{\theta-\delta}} \\
=\ & \frac{C_1}{\lambda_m^{\theta-\delta}}+C_2\int_0^t (t-s)^{\mu(1-\delta)-1}\,\|u_n-u_m\|_{C^{\delta,s}_{1-\gamma}}\,\mathrm{d}s
\end{aligned}
\tag{3.8}
$$

where

$$
C_1:=\left(1+M\,\|\mathcal{B}\|\,\widetilde{C}\sum_{k=1}^{p}|c_k|\right)C_{\delta\mu}\frac{T^{\mu(1-\delta)}}{\mu(1-\delta)}2M_0L_R(T_0)
$$

and

$$C_2 := \left(1 + M\ \|\mathcal{B}\|\ \widetilde{C} \sum_{k=1}^{p} |c_k|\right) C_{\delta\mu} 2L_R(T_0)\,.$$

Considering t_0' such that $0 < t_0' < t < T$, we have

$$\begin{aligned}
&\left\|\mathcal{A}^{\delta}(u_n(t) - u_m(t))\right\| \\
\leq\ & \frac{C_1}{\lambda_m^{\theta-\delta}} + C_2 \left(\int_0^{t_0'} + \int_{t_0'}^{t}\right) (t-s)^{\mu(1-\delta)-1} \|u_n - u_m\|_{C^{\delta,s}_{1-\gamma}}\ \mathrm{d}s \\
\leq\ & \frac{C_1}{\lambda_m^{\theta-\delta}} + 2C_2 L_R(T_0) M_0 \int_0^{t_0'} (t-s)^{\mu(1-\delta)-1} \mathrm{d}s \\
& + C_2 \int_{t_0'}^{t} (t-s)^{\mu(1-\delta)-1} \|u_n - u_m\|_{C^{\delta,s}_{1-\gamma}}\ \mathrm{d}s.
\end{aligned}$$

Integrating and introducing the notation $N_R = 2L_R(T_0)M_0$ we can write

$$\begin{aligned}
\left\|A^{\delta}\left(u_n\left(t\right) - u_m\left(t\right)\right)\right\| \ \leq\ & \frac{C_1}{\lambda_m^{\theta-\delta}} + \frac{C_2 N_R}{\mu(1-\delta)} \left((T - t_0')^{\mu(1-\delta)-1} t_0'\right) \\
& + C_2 \int_{t_0'}^{t} (t-s)^{\mu(1-\delta)-1} \|u_n - u_m\|_{C^{\delta,s}_{1-\gamma}}\ \mathrm{d}s.
\end{aligned}$$

Taking the following change $t = t + \tilde{\theta}$ in inequality (3.8), where $\tilde{\theta} \in [t_0' - t, 0]$, we obtain

$$\begin{aligned}
&\left\|u_n(t+\widetilde{\theta}) - u_m(t+\widetilde{\theta})\right\|_{C^{\delta}} \\
\leq\ & \frac{C_1}{\lambda_m^{\theta-\delta}} + C_2 \int_{t_0'}^{t+\widetilde{\theta}} (t + \widetilde{\theta} - s)^{\mu(1-\delta)-1} \|u_n - u_m\|_{C^{\delta,s}_{1-\gamma}}\ \mathrm{d}s \\
& + \frac{C_2 N_R}{\mu(1-\delta)} \left((T - t_0')^{\mu(1-\delta)-1} t_0'\right). \qquad (3.9)
\end{aligned}$$

Introducing $s - \widetilde{\theta} = \widetilde{\gamma}$ in inequality (3.9), we get

$$\begin{aligned}
&\left\|u_n(t+\widetilde{\theta}) - u_m(t+\widetilde{\theta})\right\|_{C^{\delta}} \\
\leq\ & \frac{C_1}{\lambda_m^{\theta-\delta}} + C_2 \int_{t_0'-\widetilde{\omega}}^{t} (t - \widetilde{\gamma})^{\mu(1-\delta)-1} \|u_n - u_m\|_{C^{\delta,\widetilde{\gamma}}_{1-\gamma}}\ \mathrm{d}\widetilde{\gamma} \\
& + \frac{C_2 N_R}{\mu(1-\delta)} \left((T - t_0')^{\mu(1-\delta)-1} t_0'\right). \\
\leq\ & \frac{C_1}{\lambda_m^{\theta-\delta}} + C_2 \int_{t_0'}^{t} (t - \widetilde{\gamma})^{\mu(1-\delta)-1} \|u_n - u_m\|_{C^{\delta,\widetilde{\gamma}}_{1-\gamma}}\ \mathrm{d}\widetilde{\gamma} \\
& + \frac{C_2 N_R}{\mu(1-\delta)} \left((T - t_0')^{\mu(1-\delta)-1} t_0'\right).
\end{aligned}$$

Thus, we have

$$\sup_{t_0'-t\leq\widetilde{\theta}\leq 0} \left\|u_n(t+\widetilde{\theta}) - u_m(t+\widetilde{\theta})\right\|_{C^{\delta}} \leq \frac{C_1}{\lambda_m^{\theta-\delta}} + C_2 \int_{t_0'}^{t} (t - \widetilde{\gamma})^{\mu(1-\delta)-1} \|u_n - u_m\|_{C^{\delta,\widetilde{\gamma}}_{1-\gamma}}\ \mathrm{d}\widetilde{\gamma}. \qquad (3.10)$$

For $t+\widetilde{\theta} \leq 0$, we have $u_n(t+\widetilde{\theta}) = K(t+\widetilde{\theta})$ for all $n \geq n_0$. Thus, we get

$$\begin{aligned}\sup_{-t\leq\widetilde{\theta}\leq 0}\left\|u_n(t+\widetilde{\theta})-u_m(t+\widetilde{\theta})\right\|_{C^\delta} &\leq \sup_{0\leq\widetilde{\theta}+t\leq t_0'}\left\|u_n(t+\widetilde{\theta})-u_m(t+\widetilde{\theta})\right\|_{C^\delta}\\ &\quad+\sup_{t_0'-t\leq\widetilde{\theta}\leq 0}\left\|u_n(t+\widetilde{\theta})-u_m(t+\widetilde{\theta})\right\|_{C^\delta}.\end{aligned} \tag{3.11}$$

Then, for each $t \in (0, t_0']$, we have

$$\left\|u_n(t+\widetilde{\theta})-u_m(t+\widetilde{\theta})\right\|_{C^\delta} \leq \frac{C_1}{\lambda_m^{\theta-\delta}}+\frac{C_2 N_R}{\mu\,(1-\delta)}\left((T-t_0')^{\mu(1-\delta)-1}\,t_0'\right). \tag{3.12}$$

Using Eq.(3.10), Eq.(3.11) and Eq.(3.12), we have

$$\begin{aligned}&\sup_{0\leq t+\widetilde{\theta}\leq t}\left\|t^{1-\gamma}\left(u_n(t+\widetilde{\theta})-u_m(t+\widetilde{\theta})\right)\right\|_{C^\delta}\\ &\leq \frac{2C_1}{\lambda_m^{\theta-\delta}}+C_2\int_{t_0'}^{t}(t-\widetilde{\gamma})^{\mu(1-\delta)-1}\,\|u_n-u_m\|_{C^{\delta,\widetilde{\gamma}}_{1-\gamma}}\,\mathrm{d}\widetilde{\gamma}\\ &\quad+\frac{C_2 N_R}{\mu\,(1-\delta)}\left((T-t_0')^{\mu(1-\delta)-1}\,t_0'\right).\end{aligned}$$

Then, we can write

$$\begin{aligned}\|u_n-u_m\|_{C^{\delta,t}_{1-\gamma}} &\leq \frac{2C_1}{\lambda_m^{\theta-\delta}}+C_2\int_{t_0'}^{t}(t-\widetilde{\gamma})^{\mu(1-\delta)-1}\,\|u_n-u_m\|_{C^{\delta,\widetilde{\gamma}}_{1-\gamma}}\,d\widetilde{\gamma}\\ &\quad+\frac{C_2 N_R}{\mu\,(1-\delta)}\left((T-t_0')^{\mu(1-\delta)-1}\,t_0'\right).\end{aligned}$$

Now, using the Gronwall inequality, we have

$$\begin{aligned}\|u_n-u_m\|_{C^{\delta,t}_{1-\gamma}} &\leq \left(\frac{2C_1}{\lambda_m^{\theta-\delta}}+\frac{C_2 N_R}{\mu\,(1-\delta)}\left((T-t_0')^{\mu(1-\delta)-1}\,t_0'\right)\right)\times\\ &\quad\times\mathbb{E}_\mu\left(C_2\Gamma(\mu(1-\delta))(T-t_0')^{\mu(1-\delta)}\right),\end{aligned}$$

where $\mathbb{E}_\mu(\cdot)$ is an one parameter Mittag-Leffler function. Since t_0' is arbitrary and taking $m \to \infty$, therefore the right hand side can be made as small as desired by taking t_0' sufficiently small. This complete the proof. □

Theorem 3.5 *Suppose that* (H1)-(H(2) *hold and* $u_0 \in D(\mathcal{A})$. *Then, there exist a unique function* $u_n \in C_{1-\gamma}([0,T],\mathscr{H}_R)$ *and another one* $u \in C_{1-\gamma}([0,T],\mathscr{H}_R)$ *satisfying*

$$\begin{aligned}u_n(t) &= \mathbb{E}_{\mu,\gamma}(-t^\mu\mathcal{A})\,\mathcal{B}\,u_0+\int_0^t\mathbb{H}_\mu(t,s;\mathcal{A})\widetilde{f}_{n,s,u_n}b(s)\mathrm{d}s\\ &\quad-\mathbb{E}_{\mu,\gamma}(-t^\mu\mathcal{A})\,\mathcal{B}\sum_{k=1}^{p}c_k\mathbf{I}_{0^+}^{1-\gamma}\int_0^{t_k}\mathbb{H}_\mu(t_k,s;\mathcal{A})\,\widetilde{f}_{n,s,u_n}b(s)\mathrm{d}s\end{aligned} \tag{3.13}$$

with $t \in [0,T]$, *and*

$$u(t) = \mathbb{E}_{\mu,\gamma}(-t^\mu\mathcal{A})\,\mathcal{B}\,u_0+\int_0^t\mathbb{H}_\mu(t,s;\mathcal{A})\widetilde{f}_{s,u}b(s)\mathrm{d}s \tag{3.14}$$

$$-\mathbb{E}_{\mu,\gamma}(-t^{\mu}\mathcal{A})\,\mathcal{B}\sum_{k=1}^{p}c_k\mathbf{I}_{0^+}^{1-\gamma}\int_0^{t_k}\mathbb{H}_{\mu}(t_k,s;\mathcal{A})\,\widetilde{f}_{s,u}b(s)\mathrm{d}s$$

with $t\in[0,T]$*, such that* $u_n\to u$ *in* $C_{1-\gamma}([0,T],\mathscr{H}_{B_R})$ *as* $n\to\infty$*, where* f_n *is as defined earlier.*

Proof: Let $u_0\in D(\mathcal{A})$. For $t\in(0,T]$, it follows that there exists $X_n\in B_R$ such that $\mathcal{A}^{\delta}u_n(t)\to\mathcal{A}^{\delta}u(t)\in B_R$ as $n\to\infty$. Also, for $t\in[0,T]$, we have $\mathcal{A}^{\delta}u_n(t)\to\mathcal{A}^{\delta}u(t)$ as $n\to\infty$ in $\mathscr{H}$. Since $X_n\in B_R$, therefore it follows that $X\in B_R$ and

$$\lim_{n\to\infty}\sup_{t_0\le t\le T}\|X_n(t)-X(t)\|_{C^{\delta}}=0,\quad t_0\in(0,T].$$

Also, we have

$$\begin{aligned}
&\sup_{t\in[t_0,T]}\|f_n(t,u_n(t),u_n(b(t)))-f(t,u(t),u(b(t)))\|\\
\le\;& L_R(t)\left(\|u_n(t)-u(t)\|_{C^{\delta}_{1-\gamma}}+\|u_n(b(t))-u(b(t))\|_{C^{\delta}_{1-\gamma}}\right)\to 0
\end{aligned}$$

as $n\to\infty$.

For $0<t_0<t$ we rewrite Eq.(3.13) as

$$\begin{aligned}
u_n(t)\;=\;&\mathbb{E}_{\mu,\gamma}(-t^{\mu}\mathcal{A})\mathcal{B}\,u_0+\left(\int_0^{t_0}+\int_{t_0}^{t}\right)\mathbb{H}_{\mu}(t,s;\mathcal{A})\widetilde{f}_{n,s,u_n}b(s)\mathrm{d}s\\
&-\mathbb{E}_{\mu,\gamma}(-t^{\mu}\mathcal{A})\mathcal{B}\sum_{k=1}^{p}c_k\mathbf{I}_{0^+}^{1-\gamma}\left(\int_0^{t_0}+\int_{t_0}^{t_k}\right)\mathbb{H}_{\mu}(t_k,s;\mathcal{A})\widetilde{f}_{n,s,u_n}b(s)\mathrm{d}s\\
=\;&\mathbb{E}_{\mu,\gamma}(-t^{\mu}\mathcal{A})\,\mathcal{B}\,u_0+\underbrace{\int_0^{t_0}\mathbb{H}_{\mu}(t,s;\mathcal{A})\widetilde{f}_{n,s,u_n}b(s)\mathrm{d}s}_{\text{(I)}}\\
&+\int_{t_0}^{t}\mathbb{H}_{\mu}(t,s;\mathcal{A})\widetilde{f}_{n,s,u_n}b(s)\mathrm{d}s\\
&-\mathbb{E}_{\mu,\gamma}(-t^{\mu}\mathcal{A})\,\mathcal{B}\sum_{k=1}^{p}c_k\mathbf{I}_{0^+}^{1-\gamma}\underbrace{\int_0^{t_0}\mathbb{H}_{\mu}(t_k,s;\mathcal{A})\,\widetilde{f}_{n,s,u_n}b(s)\mathrm{d}s}_{\text{(II)}}-\\
&\mathbb{E}_{\mu,\gamma}(-t^{\mu}\mathcal{A})\,\mathcal{B}\sum_{k=1}^{p}c_k\mathbf{I}_{0^+}^{1-\gamma}\int_{t_0}^{t_k}\mathbb{H}_{\mu}(t_k,s;\mathcal{A})\,\widetilde{f}_{n,s,u_n}b(s)\mathrm{d}s.
\end{aligned}$$

Now, we obtain the estimate for integrals (I) and (II), i.e.,

$$\begin{aligned}
&\left\|\int_0^{t_0}\mathbb{H}_{\mu}(t,s;\mathcal{A})\mathcal{A}^{\delta}\widetilde{f}_{n,s,u_n}b(s)ds\right\|\\
\le\;&\int_0^{t_0}(t-s)^{\mu-1}\left\|\mathcal{A}^{\delta}\mathbb{E}_{\mu,\mu}(-(t-s)^{\mu}\mathcal{A})\right\|\left\|\widetilde{f}_{n,s,u_n}b(s)\right\|ds\\
\le\;&\int_0^{t_0}(t-s)^{\mu-1}C_{\delta\mu}(t-s)^{-\delta\mu}L_R(s)\,ds\\
\le\;&L_R(T_0)C_{\delta\mu}\frac{(t-t_0)^{\mu(1-\delta)}-t^{\mu(1-\delta)}}{\mu(1-\delta)}\\
\le\;&L_R(T_0)C_{\delta\mu}\frac{T^{\mu(1-\delta)}\,t_0}{\mu(1-\delta)}
\end{aligned}$$

and

$$
\begin{aligned}
&\left\| \sum_{k=1}^{p} c_k \, \mathbb{E}_{\mu,\gamma}(-t^{\mu}\mathcal{A}) \, \mathcal{B} \mathbf{I}_{0^+}^{1-\gamma} \int_0^{t_0} \mathbb{H}_{\mu}(t_k, s; \mathcal{A}) \mathcal{A}^{\delta} \widetilde{f}_{n,s,u_n} b(s) \mathrm{d}s \right\| \\
\leq\ & \sum_{k=1}^{p} |c_k| \, \|\mathbb{E}_{\mu,\gamma}(-t^{\mu}\mathcal{A})\| \, \|\mathcal{B}\| \left\|\mathbf{I}_{0^+}^{1-\gamma}\right\| \times \\
& \times \int_0^{t_0} (t_k - s)^{\mu-1} \left\|\mathcal{A}^{\delta}\mathbb{E}_{\mu,\mu}(-(t_k - s)^{\mu}\mathcal{A})\right\| \left\||\widetilde{f}_{n,s,u_n} b(s)\right\| \mathrm{d}s \\
\leq\ & L_R(T_0) M \widetilde{C} C_{\delta\mu} \, \|\mathcal{B}\| \sum_{k=1}^{p} |c_k| \int_0^{t_0} (t_k - s)^{\mu(1-\delta)-1} \, \mathrm{d}s \\
\leq\ & \frac{L_R(T_0) M \widetilde{C}}{\mu(1-\delta)} C_{\delta\mu} \, \|\mathcal{B}\| \sum_{k=1}^{p} |c_k| [t_k^{\mu(1-\delta)} - (t_k - t_0)^{\mu(1-\delta)}] \\
\leq\ & \frac{L_R(T_0) M \widetilde{C} C_{\delta\mu} \, \|\mathcal{B}\| \, T^{\mu(1-\delta)} t_0}{\mu(1-\delta)} \sum_{k=1}^{p} |c_k| \,,
\end{aligned}
$$

respectively.

Thus, we deduce that

$$
\begin{aligned}
&\left\| \begin{array}{c} u_n(t) - \mathbb{E}_{\mu,\gamma}(-t^{\mu}\mathcal{A}) \, \mathcal{B} u_0 - \int_{t_0}^{t} \mathbb{H}_{\mu}(t, s; \mathcal{A}) \widetilde{f}_{n,s,u_n} b(s) \mathrm{d}s \\ + \mathbb{E}_{\mu,\gamma}(-t^{\mu}\mathcal{A}) \mathcal{B} \sum_{k=1}^{p} c_k \mathbf{I}_{0^+}^{1-\gamma} \int_{t_0}^{t_k} \mathbb{H}_{\mu}(t_k, s; \mathcal{A}) \, \widetilde{f}_{n,s,u_n} b(s) \mathrm{d}s \end{array} \right\|_{C^{\delta}} \\
\leq\ & L_R(T_0) C_{\delta\mu} \frac{T^{\mu(1-\delta)}}{\mu(1-\delta)} t_0 + \frac{L_R(T_0) M \widetilde{C} C_{\delta\mu} \, \|\mathcal{B}\|}{\mu(1-\delta)} T^{\mu(1-\delta)} t_0 \sum_{k=1}^{p} |c_k| \\
=\ & \frac{L_R(T_0) C_{\delta\mu} T^{\mu(1-\delta)}}{\mu(1-\delta)} \left(1 + M \widetilde{C} \, \|\mathcal{B}\| \sum_{k=1}^{p} |c_k| \right) t_0. \qquad (3.15)
\end{aligned}
$$

Taking the limit $n \to \infty$ on both sides of the inequality Eq.(3.15), we have

$$
\begin{aligned}
&\left\| \begin{array}{c} u(t) - \mathbb{E}_{\mu,\gamma}(-t^{\mu}\mathcal{A}) \, \mathcal{B} u_0 - \int_{t_0}^{t} \mathbb{H}_{\mu}(t, s; \mathcal{A}) \widetilde{f}_{s,u} b(s) \mathrm{d}s \\ + \mathbb{E}_{\mu,\gamma}(-t^{\mu}\mathcal{A}) \, \mathcal{B} \sum_{k=1}^{p} c_k \mathbf{I}_{0^+}^{1-\gamma} \int_{t_0}^{t_k} \mathbb{H}_{\mu}(t_k, s; \mathcal{A}) \, \widetilde{f}_{s,u} b(s) \mathrm{d}s \end{array} \right\|_{C^{\delta}} \\
\leq\ & \frac{L_R(T_0) C_{\delta\mu} T^{\mu(1-\delta)}}{\mu(1-\delta)} \left(1 + M \widetilde{C} \, \|\mathcal{B}\| \sum_{k=1}^{p} |c_k| \right) t_0.
\end{aligned}
$$

Since t_0 is arbitrary we conclude that $u(\cdot)$ satisfies Eq.(2.5), which complete the proof $\square$

3.2 Faedo-Galerkin approximation

In this section, we investigate the Faedo-Galerkin approximations and some convergence results.

Before investigating the two main results of this section, namely Theorem 3.6 and Theorem 3.7, we have from the previous sections that a uniqueness $u \in C_T^{\delta}$ satisfies the integral equation,

$$
u(t) \;=\; \mathbb{E}_{\mu,\gamma}(-t^{\mu}\mathcal{A}) \, \mathcal{B} \, u_0 + \int_{t_0}^{t} \mathbb{H}_{\mu}(t, s; \mathcal{A}) \widetilde{f}_{s,u} b(s) \mathrm{d}s
$$

$$-\mathbb{E}_{\mu,\gamma}(-t^{\mu}\mathcal{A})\,\mathcal{B}\sum_{k=1}^{p}c_k\mathbf{I}_{0^+}^{1-\gamma}\int_0^{t_k}\mathbb{H}_{\mu}(t_k,s;\mathcal{A})\,\widetilde{f}_{s,u}b(s)\mathrm{d}s$$

with $t\in[0,T]$. On the other hand, there is a single $u_n\in C_T^{\delta}$ function that satisfies the integral equation approximation.

$$\begin{aligned}
u_n(t) &= \mathbb{E}_{\mu,\gamma}(-t^{\mu}\mathcal{A})\,\mathcal{B}u_0+\int_{t_0}^{t}\mathbb{H}_{\mu}(t,s;\mathcal{A})\widetilde{f}_{n,s,u_n}b(s)\mathrm{d}s\\
&\quad -\mathbb{E}_{\mu,\gamma}(-t^{\mu}\mathcal{A})\mathcal{B}\sum_{k=1}^{p}c_k\mathbf{I}_{0^+}^{1-\gamma}\int_0^{t_k}\mathbb{H}_{\mu}(t_k,s;\mathcal{A})\,\widetilde{f}_{n,s,u_n}b(s)\mathrm{d}s
\end{aligned}$$

with $t\in[0,T]$.

Now, Faedo-Galerkin approximation is given by $\overline{u}_n=P^n u_n$, satisfying

$$\begin{aligned}
\overline{u}_n(t) &= \mathbb{E}_{\mu,\gamma}(-t^{\mu}\mathcal{A})\,\mathcal{B}\,P^n\,u_0+\int_{t_0}^{t}\mathbb{H}_{\mu}(t,s;\mathcal{A})P^n\,\widetilde{f}_{n,s,u_n}b(s)\mathrm{d}s\\
&\quad -\mathbb{E}_{\mu,\gamma}(-t^{\mu}\mathcal{A})\,\mathcal{B}\sum_{k=1}^{p}c_k\mathbf{I}_{0^+}^{1-\gamma}\int_0^{t_k}\mathbb{H}_{\mu}(t_k,s;\mathcal{A})\,P^n\,\widetilde{f}_{n,s,u_n}b(s)\mathrm{d}s
\end{aligned}$$

with $t\in[0,T]$, where f_n as before. On the other hand, if exist $u(t)$ the solution given by Eq.(3.14) in $[0,T]$, so it has the following representation

$$u(t)=\sum_{i=0}^{\infty}\delta_i(t)\phi_i\,,\qquad \delta_i(t)=(u(t),\phi_i),\quad i=1,2,\ldots$$

and

$$\overline{u}_n(t)=\sum_{i=0}^{\infty}\delta_i^n(t)\phi_i\,,\qquad \delta_i^n(t)=(\overline{u}_n(t),\phi_i),\quad i=1,2,\ldots.$$

Finally, we investigate Theorem 3.6, as a direct consequence of Theorem 3.1 and Theorem 3.4, and finally Theorem 3.7. So we start with the following theorem:

Theorem 3.6 *Suppose that* (H1)-(H2) *hold and* $u_0\in D(A)$. *Then, there exists a unique function* $\overline{u}_n\in C([0,T],\mathscr{H})$ *and* $u\in C([0,T],\mathscr{H})$ *satisfying*

$$\begin{aligned}
\overline{u}_n(t) &= \mathbb{E}_{\mu,\gamma}(-t^{\mu}\mathcal{A})\,\mathcal{B}\,P^n\,u_0+\int_{t_0}^{t}\mathbb{H}_{\mu}(t,s;\mathcal{A})P^n\widetilde{f}_{n,s,u_n}b(s)\mathrm{d}s\\
&\quad -\mathbb{E}_{\mu,\gamma}(-t^{\mu}\mathcal{A})\,\mathcal{B}\sum_{k=1}^{p}c_k\mathbf{I}_{0^+}^{1-\gamma}\int_0^{t_k}\mathbb{H}_{\mu}(t_k,s;\mathcal{A})\,P^n\,\widetilde{f}_{n,s,u_n}b(s)\mathrm{d}s
\end{aligned}$$

with $t\in[0,T]$, *and*

$$\begin{aligned}
u(t) &= \mathbb{E}_{\mu,\gamma}(-t^{\mu}\mathcal{A})\mathcal{B}u_0+\int_{t_0}^{t}\mathbb{H}_{\mu}(t,s;\mathcal{A})\widetilde{f}_{s,u}b(s)\,\mathrm{d}s\\
&\quad -\mathbb{E}_{\mu,\gamma}(-t^{\mu}\mathcal{A})\,\mathcal{B}\sum_{k=1}^{p}c_k\mathbf{I}_{0^+}^{1-\gamma}\int_0^{t_k}\mathbb{H}_{\mu}(t_k,s;\mathcal{A})\,\widetilde{f}_{s,u}b(s)\mathrm{d}s
\end{aligned}$$

with $t\in[0,T]$, *such that* $\overline{u}_n(t)\to u$ *in* $C([0,T],Br)$ *as* $n\to\infty$ *where* f_n *is as before.*

Proof: We have

$$
\begin{aligned}
\|\overline{u}_n(t) - u(t)\|_{C^\delta} &= \|P^n\overline{u}_n(t) - u(t)\|_{C^\delta} \\
&= \|P^n\overline{u}_n(t) - P^n u(t) + P^n u(t) - u(t)\|_{C^\delta} \\
&\leq \|P^n(u_n(t) - u(t))\|_{C^\delta} + \|(P - I)u(t)\|_{C^\delta}.
\end{aligned}
$$

By means of Theorem 3.5, we can write

$$
\lim_{n\to\infty} \sup_{t\in[0,T]} \|u_n(t) - u(t)\|_{C^\delta} = 0
$$

which completes the proof. □

Theorem 3.7 *Suppose the statements* (H1)-(H2) *hold. If* $u_0 \in D(A)$*, then for any* $0 \leq t \leq T \leq T_0$*, we have*

$$
\lim_{n\to\infty} \sup_{t\in 0\leq t\leq T} \left[\sum_{i=0}^{n} \lambda_i^{2\delta} \left\{\delta_i(t) - \delta_i^n(t)\right\}^2\right] = 0.
$$

Proof: In fact, using Eq.(1.1) and Eq.(2.5), we obtain

$$
\begin{aligned}
\mathcal{A}^\delta[u(t) - \overline{u}_n(t)] &= \mathcal{A}^\delta\left[\sum_{i=0}^{\infty} (\delta_i(t) - \delta_i^n(t))\,\phi_i\right] \\
&= \mathcal{A}^\delta\left[\sum_{i=0}^{n} (\delta_i(t) - \delta_i^n(t))\,\phi_i\right] + \mathcal{A}^\delta \sum_{i=n+1}^{\infty} \delta_i(t)\phi_i \\
&= \sum_{i=0}^{n} A^\delta\,(\delta_i(t) - \delta_i^n(t))\,\phi_i + \sum_{i=n+1}^{\infty} \mathcal{A}^\delta \delta_i(t)\phi_i.
\end{aligned}
$$

Thus, we get

$$
\left\|\mathcal{A}^\delta(u(t) - \overline{u}_n(t))\right\|^2 \geq \sum_{i=0}^{n} \lambda_i^{2\delta}\left(\delta_i(t) - \delta_i^n(t)\right)^2.
$$

Through the Theorem 3.6, we conclude the result □

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