Riemann Hypothesis

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1 Abstract

The Riemann Zeta function is defined as

$\zeta(s) = \sum_{n=1}^{\infty} 1/n^s, \; Re(s) > 1$

$The\ Zeta\ function\ is\ holomorphic\ in\ the\ complex\ plane\ except\ for\ a\ pole\ at$

$s = 1. The\ trivial\ zeros\ of\ \zeta(s)\ are\ -2, -4, -6, ... Its\ non\ trivial\ zeros\ lie\ in$

$the\ critical\ strip\ 0 < Re(s) < 1.$

$The\ Riemann\ Hypothesis\ states\ that\ all\ the\ non\ trivial\ zeros\ lie\ on\ the\ critical\ line$

$Re(s) = 1/2.$

2 Proof

Riemann Hypothesis is equivalent to the integral equation (see [3, p.136, Corollary 8.7)

$\int_{-\infty}^{\infty} \frac{log|\zeta(1/2+it)|}{1+4t^2} dt = 0$

$Let, I = \int_{-\infty}^{\infty} \frac{log|\zeta(1/2+it)|}{1+4t^2} dt$

$I = \int_{-\infty}^{1} \frac{log|\zeta(1/2+it)|}{1+4t^2} dt + \int_{1}^{\infty} \frac{log|\zeta(1/2+it)|}{1+4t^2} dt$

$Let,\ I = J + K,\ where$

$J = \int_{-\infty}^{1} \frac{log|\zeta(1/2+it)|}{1+4t^2} dt$

$K = \int_{1}^{\infty} \frac{log|\zeta(1/2+it)|}{1+4t^2} dt$

$We\ have,$

$J = \int_{-\infty}^{1} \frac{log|\zeta(1/2+it)|}{1+4t^2} dt$

$Substitute\ t = 1/u.$

$dt = -1/u^2 du.$

$J = -\int_{0}^{1} \frac{log|\zeta(1/2+i/u)|}{u^2(1+4/u^2)} du$

$J = -\int_{0}^{1} \frac{log|\zeta(1/2+i/u)|}{u^2+4} du$

$We\ have,$

$K = \int_{1}^{\infty} \frac{log|\zeta(1/2+it)|}{1+4t^2} dt$

$Substitute\ t = 1/v.$

$dt = -1/v^2 dv.$

$K = -\int_{1}^{0} \frac{log|\zeta(1/2+i/v)|}{v^2(1+4/v^2)} dv$

$K = \int_0^1 \frac{log|\zeta(1/2+i/v)|}{v^2+4} dv$

$I = J + K$

$I = -\int_0^1 \frac{log|\zeta(1/2+i/u)|}{u^2+4} du + \int_0^1 \frac{log|\zeta(1/2+i/v)|}{v^2+4} dv$

$Since\ in\ definite\ integral\ \int_a^b f(x)dx = \int_a^b f(t)dt$

$I = -\int_0^1 \frac{log|\zeta(1/2+i/u)|}{u^2+4} du + \int_0^1 \frac{log|\zeta(1/2+i/u)|}{u^2+4} du$

$I = 0.$

$\int_{-\infty}^{\infty} \frac{log|\zeta(1/2+it)|}{1+4t^2} dt = 0$

$which\ proves\ the\ Riemann\ Hypothesis.$

3 References

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