On some equations concerning the M- Theory and Topological strings and the Gopakumar-Vafa formula applied in some sectors of String Theory and Number Theory

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Abstract

In the present paper we have described in the **Chapter 1** some equations concerning the M-Theory, the Topological strings and the Topological Gauge Theory, in the **Chapter 2** some equations concerning the Gopakumar-Vafa formula in Type IIA compactification to four dimensions on a Calabi-Yau manifold in terms of a counting of BPS states in M-theory. Finally, in the **Chapter 3**, we have described some possible methods of factorization and their various possible mathematical connections concerning the solutions for some equations regarding the above sectors of string theory

The BPS states

The **Bogomol'nyi–Prasad–Sommerfield bound** (named after Eugène Bogomolny, Manoj Prasad, and Charles Sommerfield) is a series of inequalities for solutions of partial differential equations depending on the homotopy class of the solution at infinity. This set of inequalities is very useful for solving soliton equations. Often, by insisting that the bound be satisfied (called "saturated"), one can come up with a simpler set of partial differential equations to solve, the Bogomol'nyi equations. Solutions saturating the bound are called **BPS states** and play an important role in field theory and string theory

In theoretical physics, **BPS states** are massive representations of an extended supersymmetry algebra with mass equal to the supersymmetry central charge Z. Quantum mechanically, if the supersymmetry is not broken, the mass is exactly equal to the modulus of Z. Their importance arises as the multiplets are shorter than for generic massive representations, the states are stable and the mass formula is exact.

A "BPS State" is a solution to the field equations that preserves some (but not all) of the supersymmetries of the field equations. Branes are BPS solutions of the supergravity equations under this definition.

In the context of supersymmetric theories exist some configurations, called BPS states, preserving a number of supercharges that are of particular importance in the study of extended objects known as branes.
The BPS states, that preserve a number of supersymmetries, acquire a greater importance in supergravity and M-theory solutions.

1. On some equations concerning the M-Theory, the Topological strings and the Topological Gauge Theory

With regard the M-theory and the topological strings, thence the Gopakumar-Vafa formula, we take $\mathcal{F}_g$ that is the partition function of the perturbative A-model topological closed string theory of genus g.
In the limit of large volume (radius) of the Calabi-Yau three-fold, the $\mathcal{F}_g$ admits a purely topological interpretation: It is roughly given by the worldsheet instanton sum

$$\mathcal{F}_g = \sum_{C_g} \exp(-A_C). \quad (1.1)$$

The sum is over Riemann surfaces (holomorphic curves) C_g of genus g embedded in the Calabi-Yau three-fold – the target space images of the worldsheet. And A_C denotes the complex area of C.
The appropriately weighted contribution to $\mathcal{F}_g$ from constant maps was found to be

$$\mathcal{F}_g = \frac{1}{2}\chi\kappa\int_{\mathcal{M}_g} c_{g-1}^3 + O(\exp(-A)). \quad (1.2)$$

Here $\chi\kappa$ denotes the Euler characteristic of K and c_{g-1} denotes the $(g-1)$-th chern class of the Hodge bundle over $\mathcal{M}_g$. We have that

$$\int_{\mathcal{M}_g} c_{g-1}^3 = \frac{B_g}{2g(2g-2)}\frac{B_{g-1}}{(2g-2)!} = (-1)^{g-1}\chi_g\frac{2\zeta(2g-2)}{(2\pi)^{2g-2}}. \quad (1.3)$$

Here $\chi_g = (-1)^{g-1}\dfrac{B_g}{2g(2g-2)}$ is the Euler characteristic of $\mathcal{M}_g$. We note that B_g are the Bernoulli numbers taken here to be all positive.
Let us consider the contribution of the one 4 dimensional $N = 2$ hypermultiplet to $\mathcal{F}_g$. Let Z denote the central terms in the supersymmetry algebra for this hypermultiplet, where the mass $m = |Z|$.
The contribution of this hypermultiplet to $\mathcal{F}_g$ is

$$\mathcal{F}_g = -\chi_g Z^{2-2g} \quad (1.4)$$

where $\chi_g = \chi(\mathcal{M}_g)$. After taking account of all the multiplets for a fixed Kaluza-Klein momentum around the circle, the net contribution turns out to be $h^{2,1}(K) - h^{1,1}(K) = -\chi(K)/2$ times the contribution of a single hypermultiplet. Using (4), with $Z=\frac{2\pi i n}{\lambda}$ and summing over the contribution of all Kaluza-Klein modes we find

$$\mathcal{F}_g=\chi_g\frac{\chi\kappa}{2}\sum_{n\in Z,n\neq 0}(2\pi i n)^{2-2g}\,. \quad (1.5)$$

Comparing this with the expected large radius behaviour of topological amplitude in equation (1.2) gives us

$$\int_{\mathcal{M}_g}c_{g-1}^3=(-1)^{g-1}\chi_g\frac{2\zeta(2g-2)}{(2\pi)^{2g-2}} \quad (1.6)$$

precisely agreeing, with the result (1.3). If we consider a constant self-dual graviphoton field, then in addition to the contribution to R_+^2 from the terms proportional to F^{2g-2}, there will be those from terms behaving like $e^{-\frac{1}{F}}$.

For a particle of BPS charge Z in a constant self-dual field F, the expression is

$$\mathcal{F}(Z)=\int_\varepsilon^\infty\frac{ds}{s^3}\left(\frac{s/2}{\sinh\frac{s}{2}}\right)^2 e^{-\frac{sZ}{F}}\,. \quad (1.7)$$

This has a perturbative expansion that agrees with (1.4)

$$\mathcal{F}(Z)=\sum_g\mathcal{F}_g F^{2g-2}+O\left(e^{-\frac{Z}{F}}\right).$$

In our case with $Z=\frac{2\pi i n}{\lambda}$, we can easily carry out the sum over $n\in Z,n\neq 0$, using

$$\sum_{n=-\infty}^{\infty}e^{in\theta}=2\pi\sum_{m=-\infty}^{\infty}\delta(\theta-2\pi m).$$

Also taking into account the factor of $\frac{\chi\kappa}{2}$, we have the complete zero brane contribution to be

$$\mathcal{F}(Z)=\frac{\chi\kappa}{8}\sum_{m=1}^{\infty}\frac{1}{m}\frac{1}{\left(\sinh\frac{m\lambda F}{2}\right)^2}, \quad (1.8)$$

after dropping an irrelevant field independent term.

Thence, connecting the eqs. (1.7) and (1.8), we obtain the following relationship:

$$\mathcal{F}(Z)=\int_{\varepsilon}^{\infty}\frac{ds}{s^3}\left(\frac{s/2}{\sinh\frac{s}{2}}\right)^2 e^{-\frac{sZ}{F}}=\frac{\chi\kappa}{8}\sum_{m=1}^{\infty}\frac{1}{m}\frac{1}{\left(\sinh\frac{m\lambda F}{2}\right)^2}, \quad (1.8b)$$

i.e.

$$\mathcal{F}(Z)=8\int_{\varepsilon}^{\infty}\frac{ds}{s^3}\left(\frac{s/2}{\sinh\frac{s}{2}}\right)^2 e^{-\frac{sZ}{F}}=\chi\kappa\sum_{m=1}^{\infty}\frac{1}{m}\frac{1}{\left(\sinh\frac{m\lambda F}{2}\right)^2};$$

$$8=\chi\kappa\sum_{m=1}^{\infty}\frac{1}{m}\frac{1}{\left(\sinh\frac{m\lambda F}{2}\right)^2}\cdot\frac{1}{\int_{\varepsilon}^{\infty}\frac{ds}{s^3}\left(\frac{s/2}{\sinh\frac{s}{2}}\right)^2 e^{-\frac{sZ}{F}}}. \quad (1.8c)$$

The topological string partition function defined as

$$F(\lambda,t_i)=\sum_{g}\lambda^{2g-2}F_g(t_i) \quad (1.9)$$

can be viewed as computing the correction of the form $F(\lambda,t_i)R_+^2$ in type IIA compactification on a Calabi-Yau threefold.
The free energy of a charged scalar (of mass m) in a constant self-dual field of magnitude F is

$$\mathcal{F}=\int_{\varepsilon}^{\infty}\frac{ds}{s}Tre^{-s\left(\Delta+m^2\right)}=\frac{1}{4}\int_{\varepsilon}^{\infty}\frac{ds}{s}\frac{1}{\sinh^2\frac{seF}{2}}e^{-sm^2}. \quad (1.10)$$

Note that $\mathcal{F}$ can be expanded in powers of $\frac{eF}{m^2}$ the only dimensionless combination. These terms are the quantum corrections to Maxwell's action due to the presence of charged particles. There are in addition, non-perturbative pieces which go like $\exp-\frac{m^2}{eF}$.
When the charged particle carries non-trivial spin, then the answer is modified to

$$\mathcal{F}=\int_{\varepsilon}^{\infty}\frac{ds}{s}Tr(-1)^F e^{-s\left(\Delta+m^2+2eJ\cdot F\right)}=\frac{1}{4}\int_{\varepsilon}^{\infty}\frac{ds}{s}\frac{1}{\sinh^2\frac{seF}{2}}e^{-sm^2}Tr(-1)^F e^{-2seJ\cdot F}, \quad (1.11)$$

where $J=J_R+J_L$ is the generator of spin angular momentum $J\cdot F\equiv J_{\mu\nu}F^{\mu\nu}$.
In the case of an isolated S^2, lightest states at strong coupling, other than the 0-branes, are wrapped 2-branes. The 0-branes as well as the S^2 wrapped 2-branes are BPS states in the $\left[\left(\frac{1}{2},0\right)\oplus 2(0,0)\right]$

representation of the Lorentz group $SO(4)=SU(2)\times SU(2)$. The contribution of such a hypermultiplet to $F_g R_+^2 F_+^{2g-2}$ is exactly that of a charged *scalar* to F_+^{2g-2} in the ordinary Schwinger computation. The contribution of such a hypermultiplet to $\mathcal{F}_g$ is that in (1.10)

$$\sum_{g=0} (Fg_s)^{2g-2} \mathcal{F}_g = \frac{1}{4} \int_\varepsilon^\infty \frac{ds}{s} \frac{1}{\sinh^2 \frac{sF}{2}} e^{-s\frac{Z}{g_s}}, \quad (1.12)$$

which gives

$$\mathcal{F}_g = -\chi_g Z^{2-2g} \quad (1.13)$$

where $\chi_g = \chi(\mathcal{M}_g)$ is the euler characteristic of the moduli space of genus g Riemann surfaces.
The contribution to the R_+^2 term from a general particle is given by the generalization of eq. (1.12)

$$\sum_{g=0} R_+^2 (g_s F)^{2g-2} \mathcal{F}_g = (-1)^{2j} R_+^2 \frac{1}{4} \int_\varepsilon^\infty \frac{ds}{s} \frac{Tr e^{-2sJ_3^L F}}{\sinh^2 \frac{sF}{2}} e^{-s\frac{Z}{g_s}}, \quad (1.14)$$

where $j = j_L + j_R$.
The partition function of the $c=1$ string at self-dual radius is given by

$$\mathcal{F}(\mu) = \sum_g \mu^{2-2g} \chi_g \quad (1.15)$$

where χ_g denotes the Euler characteristic of the moduli of genus g Riemann surfaces and is given by

$$\chi_g = \frac{B_g}{2g(2g-2)} \quad (1.16)$$

where B_g is the g-th Bernoulli number. This perturbative part is given by the large μ expansion of

$$\mathcal{F}(\mu) = \int^\infty \frac{ds}{s^3} e^{-is\mu} \left(\frac{s/2}{\sinh \frac{s}{2}} \right)^2. \quad (1.17)$$

This expression has imaginary terms too, like $e^{-2\pi i\mu}$, which correspond to one of many possible non-unitary, non-perturbative completions of the $c=1$ theory.
Furthermore, from the eqs. (1.15), (1.16) and (1.17), we obtain also the following relationship:

$$\mathcal{F}(\mu) = \sum_g \mu^{2-2g} \frac{B_g}{2g(2g-2)} = \int^\infty \frac{ds}{s^3} e^{-is\mu} \left(\frac{s/2}{\sinh \frac{s}{2}} \right)^2, \quad (1.18)$$

that is connected with the eq. (1.8b):

$$\mathcal{F}(Z) = \int_\varepsilon^\infty \frac{ds}{s^3}\left(\frac{s/2}{\sinh\frac{s}{2}}\right)^2 e^{-\frac{sZ}{F}} = \frac{\chi\kappa}{8}\sum_{m=1}^{\infty}\frac{1}{m}\frac{1}{\left(\sinh\frac{m\lambda F}{2}\right)^2},$$

thence:

$$\mathcal{F}(\mu) = \sum_g \mu^{2-2g}\frac{B_g}{2g(2g-2)} = \int^\infty \frac{ds}{s^3}e^{-is\mu}\left(\frac{s/2}{\sinh\frac{s}{2}}\right)^2 \Rightarrow$$

$$\Rightarrow \int_\varepsilon^\infty \frac{ds}{s^3}\left(\frac{s/2}{\sinh\frac{s}{2}}\right)^2 e^{-\frac{sZ}{F}} = \frac{\chi\kappa}{8}\sum_{m=1}^{\infty}\frac{1}{m}\frac{1}{\left(\sinh\frac{m\lambda F}{2}\right)^2} = 8\int_\varepsilon^\infty \frac{ds}{s^3}\left(\frac{s/2}{\sinh\frac{s}{2}}\right)^2 e^{-\frac{sZ}{F}} = \chi\kappa\sum_{m=1}^{\infty}\frac{1}{m}\frac{1}{\left(\sinh\frac{m\lambda F}{2}\right)^2};$$

(1.18b)

2. On some equations concerning the Gopakumar-Vafa formula in Type IIA compactification to four dimensions on a Calabi-Yau manifold in terms of a counting of BPS states in M-theory

For the bosonic variables that describe motion in the plane, the one-loop path integral equals

$$\frac{d^2x}{2\pi}\mathrm{T}\sum_{m=0}^{\infty}\exp\left(-\pi e^{\sigma}\mathrm{T}(1+2m)\right) = \frac{d^2x}{2\pi}\frac{\mathrm{T}e^{-\pi e^{\sigma}\mathrm{T}}}{1-e^{-2\pi e^{\sigma}\mathrm{T}}} = \frac{d^2x}{4\pi}\frac{\mathrm{T}}{\sinh\left(\pi e^{\sigma}\mathrm{T}\right)} \quad (2.1)$$

To express the result (2.1) in four-dimensional terms, we write $\mathrm{T} = \sqrt{\mathrm{T}^{-\mu\nu}\mathrm{T}^{-}_{\mu\nu}} = \frac{e^{-\sigma/2}}{4}\sqrt{\left(W^-\right)^2}$, and interpret W^- as the bottom component of a superfield $\mathcal{W}$. We also use $e^{3\sigma/2} = -i/2\chi^0$, and $d^4xd^4\psi^{(0)} = \frac{1}{4}d^4xd^4\theta\sqrt{g^E}e^{-\sigma}$. The resulting contribution to the 4d effective action is

$$-\int\frac{d^4xd^4\theta}{(2\pi)^4}\sqrt{g^E}\exp\left(2\pi i\sum_I q_I Z^I\right)\frac{\frac{1}{64}\pi^2\mathcal{W}^2}{\sin^2\left(\frac{\pi\sqrt{\mathcal{W}^2}}{8\chi^0}\right)} \quad (2.2)$$

The analog of eq. (2.2) with k-fold wrapping is obtained by multiplying $\sum_I q_I Z^I$ and e^{σ} by k, and dividing the whole formula by k. Summing over k gives the contribution of the given BPS state with any winding:

$$-\int \frac{d^4xd^4\theta}{(2\pi)^4}\sqrt{g^E}\sum_{k=1}^{\infty}\frac{1}{k}\exp\left(2\pi ik\sum_I q_I Z^I\right)\frac{\frac{1}{64}\pi^2\mathcal{W}^2}{\sin^2\left(\frac{\pi k\sqrt{\mathcal{W}^2}}{8\chi^0}\right)}. \quad (2.2b)$$

We can expand eq. (2.2b) in a power series in $\mathcal{W}$:

$$-\int \frac{d^4xd^4\theta}{(2\pi)^4}\left(\chi^0\right)^2\sum_{k=1}^{\infty}\frac{1}{k^3}\exp\left(2\pi ik\sum_I q_I Z^I\right)\left(1+\frac{\pi^2k^2\mathcal{W}^2}{64\cdot 3\left(\chi^0\right)^2}+\mathrm{O}\left(\mathcal{W}^4\right)\right). \quad (2.2c)$$

The formula (2.2c), even though it reflects a single wrapped M2-brane of genus (g) 0 and degree 1, is interpreted in perturbative string theory as a sum of contributions with all values $k \geq 1$ and $g \geq 0$. The contribution of BPS states of charges $\vec{q}$ propagating once around the circle to the Gopakumar-Vafa (GV) formula (that expresses certain couplings that arise in Type IIA compactification to four dimensions on a Calabi-Yau manifold in terms of a counting of BPS states in M-theory) is obtained by just including this trace in (2.2):

$$-\int \frac{d^4xd^4\theta}{(2\pi)^4}\sqrt{g^E}\,Tr_{V_{\vec{q}}}\left[(-1)^F\exp\left(-i\pi\partial_{\vec{q}}/4\chi^0\right)\right]\exp\left(2\pi i\sum_I q_I Z^I\right)\frac{\frac{1}{64}\pi^2\mathcal{W}^2}{\sin^2\left(\frac{\pi\sqrt{\mathcal{W}^2}}{8\chi^0}\right)}. \quad (2.3)$$

This can be extended as before to include multiple windings:

$$-\int \frac{d^4xd^4\theta}{(2\pi)^4}\sqrt{g^E}\sum_{k=1}^{\infty}\frac{1}{k}Tr_{V_{\vec{q}}}\left[(-1)^F\exp\left(-i\pi k\partial_{\vec{q}}/4\chi^0\right)\right]\exp\left(2\pi ik\sum_I q_I Z^I\right)\frac{\frac{1}{64}\pi^2\mathcal{W}^2}{\sin^2\left(\frac{\pi k\sqrt{\mathcal{W}^2}}{8\chi^0}\right)}. \quad (2.4)$$

To get the complete GV formula, we need to sum this formula over all possible charges $\vec{q}$. But states with $\zeta(\vec{q})<0$ do not contribute to the GV formula since they preserve the wrong supersymmetry. The complete GV formula is thus:

$$-\int \frac{d^4xd^4\theta}{(2\pi)^4}\sqrt{g^E}\sum_{q|\zeta(q)\geq 0}\sum_{k=1}^{\infty}\frac{1}{k}Tr_{V_{\vec{q}}}\left[(-1)^F\exp\left(-i\pi k\partial_{\vec{q}}/4\chi^0\right)\right]\exp\left(2\pi ik\sum_I q_I Z^I\right)\frac{\frac{1}{64}\pi^2\mathcal{W}^2}{\sin^2\left(\frac{\pi k\sqrt{\mathcal{W}^2}}{8\chi^0}\right)}. \quad (2.4b)$$

For a set of four states, we have that:

$$Tr(-1)^F\exp\left(-\frac{i\pi\partial}{8\chi^0}\right)=Tr(-1)^F\exp\left(2\pi e^{\sigma}J\right)=-4\sin^2\left(\frac{\pi\sqrt{W^2}}{8\chi^0}\right).$$

The full set of fermions $\tilde{\rho}_{A_\sigma}, \rho_{B_\sigma}$ consists of g copies of this spectrum, leading to

$$Tr(-1)^F \exp(2\pi e^\sigma J) = (-1)^g \left(4\sin^2\left(\frac{\pi\sqrt{W^2}}{8\chi^0}\right)\right)^g .$$

From (2.4), it then follows that the contribution to the GV formula of BPS states that arise from an M2-brane wrapped on $\sum$ is

$$\int \frac{d^4x d^4\theta}{(2\pi)^4} \sum_{k=1}^{\infty} \frac{1}{k} (-1)^{g-1} \exp\left(2\pi i k \sum_I q_I Z^I \right) \frac{\frac{1}{64}\pi^2 \mathcal{W}^2}{\sin^{2-2g}\left(\pi k \sqrt{\mathcal{W}^2}/8\chi^0\right)}. \quad (2.5)$$

Now we have the following superparticle action:

$$S = \int dt \left(-|z|^2 + \frac{\dot{x}^2}{4} + \frac{\bar{z}}{4} U_\mu \dot{x}^\mu + \frac{i}{2} \varepsilon^{ij} \varepsilon^{AB} \psi_{Ai} \nabla_t \psi_{Bj} \right), \quad (2.6)$$

the Euclidean version of this action is:

$$S_E = \int_0^\beta d\tau \left(|z|^2 + \frac{\dot{x}^2}{4} - i \frac{\bar{z}}{4} U_\mu \dot{x}^\mu + \frac{1}{2} \varepsilon^{ij} \varepsilon^{AB} \psi_{Ai} \nabla_\tau \psi_{Bj} \right). \quad (2.7)$$

We set $x^\mu(\tau) = x^\mu + z^\mu(\tau)$, where x^μ labels a point in M_4, and $z^\mu(0) = z^\mu(s) = 0$. The path integral over $x^\mu(\tau)$ splits as an integral over a field $z^\mu(\tau)$ that vanishes at $\tau = 0$ and an ordinary integral over x^μ . In these coordinates, the spin connection is

$$\omega_\mu^{ab}(z) = \frac{1}{2} z^\nu R_{\nu\mu}^{ab} + \mathrm{O}(z^2), \quad (2.8)$$

where the $\mathrm{O}(z^2)$ terms can be ignored as they are proportional to the covariant derivative of the Riemann tensor. Up to terms of order z^3, the part of the action that involves fermions is

$$\frac{1}{2} \varepsilon^{ij} \varepsilon^{AB} \psi_{Ai} \dot{\psi}_{Bj} - \frac{1}{16} \dot{z}^\mu z^\nu R_{\nu\mu}{}^{ab} \varepsilon^{ij} \psi_{Ai} \gamma_{ab}^{AB} \psi_{Bj} . \quad (2.9)$$

The action (2.9) contains a coupling $R\psi^{(0)}\psi^{(0)}$, which is the only coupling that can saturate the fermion zero-modes. Using this coupling to saturate the zero-modes gives an explicit factor of R^2 in the path integral, and as we do not wish to compute terms of higher order in R, we can drop the coupling of R to other fermion modes. The action then reduces to

$$S_E = \int_0^\beta d\tau \left[\frac{\dot{z}^2}{4} - i \frac{\bar{z}}{8} \left(W_{\nu\mu}^- - \frac{i}{2\bar{z}} R_{\nu\mu}^{-ab} \varepsilon^{ij} \psi_{Ai}^{(0)} \gamma_{ab}^{AB} \psi_{Bj}^{(0)} \right) z^\nu \dot{z}^\mu + \frac{1}{2} \varepsilon^{ij} \varepsilon^{AB} \psi_{Ai} \dot{\psi}_{Bj} + |z|^2 \right]. \quad (2.10)$$

Now we observe that replacing $iR\psi^{(0)}\psi^{(0)}/\bar{z}$ by $R\psi^{(0)}\psi^{(0)}$ has the effect of just multiplying the path integral by $-\bar{z}^2$. If we make this replacement, and also set $\psi^{(0)}_{Ai} = \sqrt{2}\theta_{Ai}$, and finally set $z^\mu = \sqrt{2}y^\mu$, then the action becomes

$$S_E = \int_0^s d\tau \left[\frac{\dot{y}^2}{2} - i\frac{\bar{z}}{4}\mathcal{W}^-_{\mu\nu} y^\nu \dot{y}^\mu + \frac{1}{2}\varepsilon^{ij}\varepsilon^{AB}\psi_{Ai}\dot{\psi}_{Bj} + |z|^2 \right], \quad (2.11)$$

where

$$\mathcal{W}_{\mu\nu}(x,\theta) = W^-_{\mu\nu}(x) + \ldots - R^-_{\mu\nu\lambda\rho}(x)\varepsilon_{ij}\bar{\theta}^i\sigma^{\lambda\rho}\theta^j + \ldots \quad (2.12)$$

is the superfield whose bottom component is $W^-_{\mu\nu}$. The constant term $|z|^2$ in the Lagrangian density just multiplies the path integral by $\exp\left(-s|z|^2\right)$. So

$$Tr_{\hat{\mathfrak{H}}}(-1)^F \exp(-sH) = -\frac{e^{-s|z|^2}}{\bar{z}^2}\int d^4y d^4\theta\sqrt{g}\int \mathcal{D}'y\mathcal{D}'\psi \exp\left(-\int_0^s d\tau \left(\frac{\dot{y}^2}{2} - i\frac{\bar{z}}{4}\mathcal{W}^-_{\mu\nu}y^\nu\dot{y}^\mu + \frac{1}{2}\varepsilon^{ij}\varepsilon^{AB}\psi_{Ai}\dot{\psi}_{Bj} \right)\right)$$
(2.13)

where $\mathcal{D}'$ represents a path integral over non-zero modes only. Apart from the decoupled fermions ψ_{Ai}, the remaining path integral describes a particle in a constant magnetic field $\bar{z}\mathcal{W}$. This is a very standard path integral and we finally learn that

$$Tr_{\hat{\mathfrak{H}}}(-1)^F \exp\left(-s\left(|z|^2 - D^2\right)\right) = -\frac{e^{-s|z|^2}}{(2\pi)^4}\int d^4x d^4\theta\sqrt{g}\, \frac{\pi^2\mathcal{W}^2/64}{\sinh^2\frac{s\bar{z}\sqrt{\mathcal{W}^2}}{8}}. \quad (2.14)$$

When this is inserted in the following expression:

$$-\int_0^\infty \frac{ds}{s}\left[(2Tr - Tr_L)\left(e^{-s\left(|z|^2 - D^2\right)}\right) - \mathcal{J}e^{-s\bar{z}} \right] \quad (2.14b)$$

we get

$$\int_0^\infty \frac{ds}{s}\left(\frac{e^{-s|z|^2}}{(2\pi)^4}\int d^4x d^4\theta\sqrt{g}\, \frac{\pi^2\mathcal{W}^2/64}{\sinh^2\frac{s\bar{z}\sqrt{\mathcal{W}^2}}{8}} + \mathcal{J}e^{-s\bar{z}} \right). \quad (2.15)$$

The integral converges for small s. In fact, with the help of the index theorem, (2.15) is equivalent to

$$\int_0^\infty \frac{ds}{s}\left(\frac{1}{(2\pi)^4}\int d^4x d^4\theta\sqrt{g}\left(e^{-s|z|^2}\frac{\pi^2\mathcal{W}^2/64}{\sinh^2\frac{s\bar{z}\sqrt{\mathcal{W}^2}}{8}} + \frac{\pi^2\mathcal{W}^2}{3\cdot 64}e^{-s\bar{z}} \right)\right). \quad (2.16)$$

Since z has non-negative real part, the integral also converges at large s. To establish holomorphy in z, we simply rescale $s \to s/\bar{z}z$, to get:

$$\int_0^\infty \frac{ds}{s}\left(\frac{1}{(2\pi)^4}\int d^4xd^4\theta\sqrt{g}\left(e^{-s}\frac{\pi^2\mathcal{W}^2/64}{\sinh^2\frac{s\sqrt{\mathcal{W}^2}}{8z}}+\frac{\pi^2\mathcal{W}^2}{3\cdot 64}e^{-s/z}\right)\right). \quad (2.17)$$

We would like to compare the hypermultiplet contribution to the effective action as computed in the field-based approach to the earlier particle-based result (2.2b). The field-based calculation involved a sum over states of definite momentum around the Kaluza-Klein circle, and the particle-based calculation involved a sum over orbits of definite winding number.
We define

$$S(\vec{q}) = 2\pi\left(e^\sigma M - iq_I\alpha^I\right) = -2\pi i q_I Z^I\ ,$$

so that the central charge of a particle of Kaluza-Klein momentum n is

$$z = e^{-3\sigma/2}\left(-in + S(\vec{q})/2\pi\right).$$

We also rescale the Schwinger parameter by $s \to sz$. The sum and integral to be performed are then

$$\int \frac{d^4xd^4\theta}{(2\pi)^4}\sqrt{g}\sum_{n\in Z}\int_0^\infty \frac{ds}{s}\exp\left(-se^{-3\sigma/2}S(\vec{q})/2\pi\right)\exp\left(inse^{-3\sigma/2}\right)\frac{\pi^2\mathcal{W}^2}{64}\frac{1}{\sinh^2\left(s\sqrt{\mathcal{W}^2}/8\right)}. \quad (2.18)$$

Upon using $\sum_{n\in Z} e^{in\theta} = 2\pi\sum_{k\in Z}\delta(\theta - 2\pi k)$, we get

$$\int \frac{d^4xd^4\theta}{(2\pi)^4}\sqrt{g}\sum_{k\in Z}\int_0^\infty \frac{ds}{s}\exp\left(-se^{-3\sigma/2}S(\vec{q})/2\pi\right)2\pi\delta\left(se^{-3\sigma/2} - 2\pi k\right)\frac{\pi^2\mathcal{W}^2}{64}\frac{1}{\sinh^2\left(s\sqrt{\mathcal{W}^2}/8\right)}. \quad (2.19)$$

Integrating over s with the help of the delta functions, promoting $S(\vec{q}) = -2\pi i q_I Z^I$ to a superfield $\mathcal{S}(\vec{q}) = -2\pi i q_I \mathcal{Z}^I$ and introducing again $\chi_0 = -ie^{-3\sigma/2}/2$, we recover the following result:

$$-\sum_{k=1}^\infty \frac{1}{k}\int \frac{d^4xd^4\theta}{(2\pi)^4}\sqrt{g}\exp\left(2\pi ik\sum_I q_I\mathcal{Z}^I\right)\frac{\pi^2\mathcal{W}^2}{64}\frac{1}{\sin^2\left(\frac{\pi k\sqrt{\mathcal{W}^2}}{8\chi^0}\right)}, \quad (2.20)$$

for the hypermultiplet contribution.

If we turn on the continuous part of the *B*-field in the Type IIA description, then in the M-theory description we have to include the holonomy factor

$$\exp\left(2\pi i\sum_I \alpha^I Q_I\right).$$

Including this factor amounts to including in the trace that gives the GV formula the element exp$(2\pi i\sum_I \alpha^I Q_I)$ of the connected part of the global symmetry group $H^2(Y;U(1))$. If we want to turn on a discrete B-field in the Type IIA description, then in the M-theory description we have to proceed including in the trace an element of $H^2(Y;U(1))$ that might not be connected to the identity. If we pick a splitting of the following exact sequence

$$0 \to H^2(Y;R)/H^2(Y;Z) \to H^2(Y;U(1)) \to H^3_{tors}(Y;Z) \to 0,$$

this means that we multiply exp$(2\pi i\sum_I \alpha^I Q_I)$ by an element $x \in H^3_{tors}(Y;Z)$.

If a BPS state winds k times around the circle, its contribution is weighted by x^k as well as by a factor exp$(2\pi ik\sum_I \alpha^I Q_I)$ that is already present in eq. (2.4b). Thus, the generalization of eq. (2.4b) to include discrete B-fields is obtained simply by including an additional factor of x^k inside the trace:

$$-\int \frac{d^4xd^4\theta}{(2\pi)^4}\sqrt{g^E}\sum_{q|\zeta(q)\geq 0}\sum_{k=1}^{\infty}\frac{1}{k}Tr_{V_{\bar{q}}}\left[(-1)^F x^k \exp\left(-i\pi k\partial_{\bar{q}}/4\chi^0\right)\right]\exp\left(2\pi i\sum_I q_I z^I\right)\frac{\frac{1}{64}\pi^2 W^2}{\sin^2\left(\frac{\pi k\sqrt{W^2}}{8\chi^0}\right)}. \quad (2.21)$$

This is the GV formula with discrete B-fields.

3. On a possible method of factorization and various applications in number theory and some sectors of string theory

a) As we know increasing respectively of one and another value to the square the first and the second factor of any product, we have an increase of the values of R and Ti relative to it. These increments are best viewed starting from some particular starting products, different depending to the value of the square related to the second factor. Taking as square the number 9 we have that:

Q=9

1*1 / R=(1^2+2); Ti=3

1*2 / R=(1^2+1); Ti=3

-1*3 / R=(1^2) ; Ti=3

1*4 / R=(0^2+5); Ti=5

1*5 / R=(0^2+4); Ti=5

1*6 / R=(0^2+3); Ti=5

1*7 / R=(0^2+2); Ti=5

1*8 / R=(0^2+1); Ti=5

--1*9 / R=(0^2+0); Ti=5

1*10 / R=(1^2+5); Ti=7

1*11 / R=(1^2+4); Ti=7

...

1*14 / R=(1^2+1); Ti=7

-1*15 / R=(1^2+0); Ti=7

1*16 / R=(2^2+5); Ti=9

1*17 / R=(2^2+4); Ti=9

...

1*20 / R=(2^2+1); Ti=9

-1*21 / R=(2^2+0); Ti=9

1*22 /R=(3^2+5); Ti=11

1*23 /R=(3^2+4); Ti=11

...

Products marked by the dash are those of reference that vary depending on the value of Q. The one marked by the double dash is the product of origin of the sequence.
Thereby increasing the second factor, of any pair, 9 = 3^2 and the first of 1 there is an increase of R equal to the second number indicated in parentheses and an increase of Ti equal to 6 = 3*2.
Another way, more complete, to write the sequence is the following:
[For ease of presentation we start from the original product.]

--1*9 //{[0^2+(0+0)]; Ti=5} or {[1^2+(0+6)]; Ti=7} or {[2^2+(0+12)]; Ti=9} or {[3^2+(0+18)]; Ti=11} ...

1*10 // {[1^2+(5+0)]; Ti=7} or {[2^2+(5+6)]; Ti=9} or {[3^2+(5+12)]; Ti=11} or {[4^2+(5+18)]; Ti=13} ...

1*11 // {[1^2+(4+0)]; Ti=7} or {[2^2+(4+6)]; Ti=9} or {[3^2+(4+12)]; Ti=11} or {[4^2+(4+18)]; Ti=13} ...

...

1*14 // {[1^2+(1+0)]; Ti=7} or {[2^2+(1+6)]; Ti=9} or {[3^2+(1+12)]; Ti=11} or {[4^2+(1+18)]; Ti=13} ...

-1*15 // {[1^2+(0+0)]; Ti=7} or {[2^2+(0+6)]; Ti=9} or {[3^2+(0+12)]; Ti=11} or {[4^2+(0+18)]; Ti=13} ...

1*16 // {[2^2+(5+0)]; Ti=9} or {[3^2+(5+6)]; Ti=11} or {[4^2+(5+12)]; Ti=13} or {[5^2+(5+18)]; Ti=15} ...

1*17 // {[2^2+(4+0)]; Ti=9} or {[3^2+(4+6)]; Ti=11} or {[4^2+(4+12)]; Ti=13} or {[5^2+(4+18)]; Ti=15} ...

...

In this case, increasing the second factor, for any pair of 9 = 3^2 and the first of 1 there is an increase of R equal to the number indicated in parentheses, and an increase of Ti always equal to 6 = 3*2.
Given that the value of R, as can be deduced from this sequence, varies depending on the cases, we thought to adopt the same system experienced before, trying to reach the factorization of a number N by the square obtained from R, or rather from its increase.
We took then a starting product any, in this case:

1*17 , whose initial values of R and Ti are: {[2^2+(4+0)]; Ti=9};

and using a simple formula, we concluded that, taking as number N1 to factorize the second term of the various products obtained, the factorization of the first term that is in this case:

(N1-17)/9

allows us to calculate the factors of N1.

Then we have found a relationship between N'=[(N1-17)/9]+1 and N1.

In the following scheme the various products that appear in the left side and those, associated with them, that appear to the right, represent respectively the first and the second term of the pairs of the numbers obtained from:

1*17.

3* 15/29/43/57/71/...	7* 59/113/167/221/...
3* 27/49/71/93/...	11* 67/121/175/229/...
5* 29/55/81/107/...	13* 101/191/281/...
5* 41/75/109/143/...	17* 109/199/281/...
7* 43/81/119/157/...	19* 143/269/395/...
7* 55/101/147/...	23* 151/277/403/...

...

The trends of growth of various numbers are regular and constant also thereafter.

For example:

We have N'=7*147=1029; we obtain:

N1=(N'-1)*9+17=9269=23*403; or, being also:

N'=(N1-8)/9,

N1=(1029*9)+8=9269=23*403

We note that

8 = (23*403) – (1029*9) = (23*403) – (147*7*9) = 9269 – 9261;

$8=(23*13*31) – (3*7^2*7*3^2)$;

$64 = [(23*13*31) – (3*7^2*7*3^2)]^2$

$64 = (9269 – 9261)^2$ = 85914361 – 171680418 + 85766121.

$$64 = \left(\frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}\right)^2 = 85914361 - 171680418 + 85766121. \quad (3.1)$$

connected with the "modes" (8; $64 = 8^2$) that correspond to the physical vibrations of a superstring by the above Ramanujan function

Thence, we can factorize N1, provided that its factors are of the type shown above, factorizing in its place a number equal to about a ninth of N1 whose prime factors, it must be said, are not necessarily only two.

It also establishes, the possibility to easily go back to the factors of a number obtained by the sum of the other two. In this case:

9261=9*N'=9*7*147 and 8. (Indeed 9269 = 9261 + 8)

Moreover, the same relationship between N'=[(N1-17)/9]+1 e N1, illustrated in the scheme above, it has also between:

N''={[N2-(17+36)]/9+1} and N2.

Indeed:

3* 9/23/37/51/...	7* 41/95/149/...
3* 33/55/77/99/ ...	11*85/139/193/ ...
5*23/49/75/101/ ...	13* 83/173/263/ ...
5*47/81/115/	17* 127/217/307/ ...
7* 37/75/113/151/ ...	19* 125/251/377/ ...
7* 61/107/153/199/ ...	23* 169/295/421/ ...

...

Also in this case the trends of growth of various numbers, also with respect to those of the first scheme, continue to be regular and constant

For example:

Let N''=5*49=245; we obtain:

N2=(N''-1)*9+(17+36)=2249=13*173; or, being:

N''=[N2-(8+36)]/9,

N2=245*9+8+36= 2205 + 8 + 36 = 2249=13*173

We note that

2249 = 8+36+(245*9)

8+36+(245*9) = 2249; 8 = 2249 – 36 – (49*5*9) or 8 = (13*173) – 36 – (49*5*9);

8 = (13*173) – (2^2*3^2) – (7^2*5*3^2)

And 8 is equal to the following expression:

$$8 = \frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]},$$

connected with the "modes" that correspond to the physical vibrations of a superstring by the above Ramanujan function. Thence, we have that:

$$8 = (13*173) - (2^2*3^2) - (7^2*5*3^2) = \frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (3.2)$$

As above is obtained the factorization of a number starting from the sum of the other two, in this case:

2205=9*N"=9*5*49 and (8+36)=44

It can obviously get infinite schemes of this type, however, according to our conclusions and, by adopting this system, must still an excessive number of attempts to factorize a number.
Moreover, similar schemes may be obtained also in cases where the square taken into consideration is different from 9 = 3^2, with the advantage of reach to the factorization of large numbers from those equal to a fraction still less than that of starting, depending exactly from the square chosen.

b) We analyze the following developments on the factorization

In essence, we have realized that instead of factorize N dividing it by the primes between 1 and $\sqrt{N}$ approximately, this can be done using the same primes to divide lower numbers.

Consider the following sequence:

1+11x	3+17x	15+23x	21+29x	28+31x	4+7x	14+13x	22+19x
19+41x	25+47x	53+53x	63+59x	76+61x	36+37x	62+43x	74+49x
57+71x	67+77x	111+83x	125+89x	144+91x	88+67x	130+73x	146+79x
115+101x	129+107x	189+113x	207+119x	232+121x	160+97x	218+103x	238+109x
193+131x	211+137x	287+143x	309+149x	340+151x	253+127x	326+133x	350+139x
291+161x	...						

All the numbers obtained from this sequence, multiplied by 90 and added to 53, give composite numbers divisible by the value associated to x.

For example:

n=3+17*3=54; implies: N=54*90+53=4913=17*289

Similar sequences can be obtained for all N, not only those of the type n*90+53.
In particular, the N of this type can be identified in the following way:

If N=8+y*9 e y has as last digit the 5, then N is obtained from the previous sequence.
Furthermore, given that $\sqrt{N} = \sqrt{4913}$ =70,092 , the value of n to search is lower than 57+71x.

Finally it must be said that all "n" excluded from the sequence give N primes.

n excluded: 2,5,6,7,8,9,10,13,15,16,17,24,26,29,30,31,34,35,42,43 ... (**see Appendix A**)

Furthermore, we can to obtain similar sequences, although proportionally complex, in which N is obtained by multiplying "n" for 900, 9000 and so on

Example 1:

N=9953;
(9953-8)/9=1105; 5 being the final digt of the value obtained we go to the calculation of n.
n=(9953-53)/90=110

Therefore, knowing that $\sqrt{N} = \sqrt{9953}$ =99,764, try "n" in the above sequence from 160+97x, because 97 is near at 99,764.

In conclusion, we have that:
n=36+37*2=110

Example 2:

N = 4823;
(4823 – 8) / 9 = 4815 / 9 = 535
n = (4823 – 53) / 90 = 4770 / 90 = 53
√N = √4823 = 69,4478 very near to (88+67x); (36+37x); (4+7x);
we take n = 4+7x = 4+ (7*7) = 4+49 = 53; 4823 /53 = 91; 4823 = 91*53 = 13*7*53

But
4823 = (535*9) + 8 = 4815 + 8;   (13*7*53) = (535*9) + 8;
8 = (13*7*53) – (5*107*3^2) = 4823 – 4815
64 = [(13*7*53) – (5*107*3^2)]^2 = $(4823 – 4815)^2$ = 23261329 – 46445490 + 23184225.

$$64 = \left(\frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}\right)^2 = 23261329 - 46445490 + 23184225. \quad (3.3)$$

connected with the "modes" (8; 64 = 8^2) that correspond to the physical vibrations of a superstring by the above Ramanujan function

c) Let's see if in the formula that we show below we can get something useful.

The formula is used for the factorization of N and is the following:

With d and n that are given, find:

A^2=4B^2d+4BCn+C^2

where A, B e C are unknowns, while "n" is the root of the square immediately following at N and "d" is the difference between n^2 and N.

Now, the formula to calculate the square which added to N gives as a result another square, is the following:

$4A^2=B^2+2B-4n^2+1+4d$

Indeed, if we have:

N=57
n=8
d=7

implies, with B=21:

$4A^2 = B^2+2B – 4*8^2+1+4*7$
$4A^2 = B^2+2B – 256+1+28$ for B=21, we have: $4*8^2$ = 441+42-256+1+28; 4*64 = 256;
256 = 256

therefore

$4A^2$=256 ; 4*64 = 256 ; 64 = 256/4; √64 = 8 whence,

A=8

this means that N added to 8^2 gives a square. In fact:

$57+64=11^2$

We note that:

$64 = 11^2 - 57 = 121 - 57; \; 8^2 = 11^2 - 3*19$

Now, if 8 is equal to the following expression:

$$8 = \frac{1}{3} \frac{4\left[anti\log \frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4} w'} \phi_{w'}(itw')} \right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4} \right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4} \right)} \right]},$$

connected with the "modes" that correspond to the physical vibrations of a superstring. Thence also $64 = 8^2$ is equal to:

$$64 = \left(\frac{1}{3} \frac{4\left[anti\log \frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4} w'} \phi_{w'}(itw')} \right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4} \right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4} \right)} \right]}\right)^2 = 11^2 - 3*19 = 121 - 57. \quad (3.4)$$

At this point we ask ourselves, the fact that in the first formula the result "A" depends on two variables, B and C, is necessarily a disadvantage?
It is possible use the factorization of "d" and "n" to calculate "A"?
We ask these questions because unlike the second formula, one that refers to the method of Fermat's factorization, whose efficacy depends on the distance between the factors, the first work in order to be not affected.
In other words, by varying in a gradual manner and regular the values of "A" and "B" in the second formula, we get a series equally regular of the values of "d" and "n", and then a regular series of values of N, for which the ratio between the factors follows a precise order.
Repeating instead the procedure with the first formula, we obtain values of N for which between the factors apparently does not exist any type of relationship.
So the first formula could work equally well both for numbers whose factors are distant from each other and both for those whose factors are near. To our humble opinion, are valid both formulas.

d) Another proposal of factorization: some new results recently developed.

1. These results are based on a simple argument about the possibilities of calculation which we believe allow the regularities at the base of formulas for the factorization which has been developed. In other words, using these rules and mechanisms, since any number can easily calculate and change the difference between a square and the number itself so that it assumes a value as desired.
The following proposal of factorization is based on the Pythagorean triples.
First we illustrate the way to calculate the value of the difference between a square and a given number, with known factors, using a scheme similar to that of the previous proposal, constructed taking as reference the square of three.

1*9 = 3^2-(0*1+0^2)	1*10 =4^2-(5*1+1^2)	1*11 = 4^2-(4*1+1^2) ...
2*18 = 6^2-(0*2+0^2)	2*19 =7^2-(5*2+1^2)	2*20 = 7^2-(4*2+1^2)
3*27 = 9^2-(0*3+0^2)	3*28 =10^2-(5*3+1^2)	3*29 = 10^2-(4*3+1^2)
...	...	...
1*15 = 4^2-(0*1+1^2)	1*16 = 5^2-(5*1+2^2)	1*17 = 5^2-(4*1+2^2)
2*24 = 7^2-(0*2+1^2)	2*25 = 8^2-(5*2+2^2)	2*26 = 8^2-(4*2+2^2)
3*33 = 10^2-(0*3+1^2)	3*34 = 11^2-(5*3+2^2)	3*35 = 11^2-(4*3+2^2)
...	...	...
1*21 = 5^2-(0*1+2^2)	1*22 = 6^2-(5*1+3^2)	1*23 = 6^2-(4*1+3^2)
2*30 =8^2-(0*2+2^2)	2*31 = 9^2-(5*2+3^2)	2*32 = 9^2-(4*2+3^2)
3*39 =11^2-(0*3+2^2)	3*40 = 12^2-(5*3+3^2)	3*41 = 12^2-(4*3+3^2)
...	...	...

A generalization of the scheme, valid for each square, is obtained with the following formulas:
The “key” points are:

1*q^2 =q^2-0^2
..
(1+n)*(q^2+n*q^2) =[q*(1+n)]^2-0^2

1*(q^2+x*2q) =(q+x)^2-x^2
...
(1+n)*(q^2+x*2q+n*q^2) =[q*(1+n)+x]^2-x^2

Furthermore, with y positive integer less (equal) to 2q we obtain the following and final generalization:

1*(q^2+x*2q-y) =(q+x)^2-(y*1+x^2)
...
(1+n)*(q^2+x*2q-y+n*q^2) =[q*(1+n)+x]^2-[y*(1+n)+x^2]

Given this premise we consider any number and try to place it in the scheme constructed starting from one of the possible squares.

N=12437
q^2=9

To understand in which columns there is the number taken for example, we perform the following operations:

12437=9*1380+17 ; thence:

1*17 =5^2-(4*1+2^2)
...
1381*12437 =4145^2-(4*1381+2^2)
___

1*26 =6^2-(1*1+3^2)
...
1380*12437 =4143^2-(1*1380+3^2)
___

1*35 =8^2-(4*1+5^2)
...
1379*12437 =4142^2-(4*1379+5^2)

After this first rapid passage we can conclude that:

1381*(12437+4)=4145^2-2^2 ; whence:

1381*12441=4143*4147=1381*3*(12441/3)
...

Result that obviously does not constitute advantage but that illustrates the principle used later.

For the purposes of our argument in fact we must consider only some cases, similar to the previous, i.e. those in which the given number is multiplied by a square:

1*12410 =2070^2-(1*1+2067^2)
...
4*12437 =2079^2-(1*4+2067^2)
___
1*12365 =2063^2-(4*1+2060^2)
...
9*12437 =2087^2-(4*9+2060^2)
___
1*12302 =2052^2-(1*1+2049^2)
...

16*12437 =2097^2-(1*16+2049^2)

At this point, we can conclude that:

4*12437 =2079^2-(2^2+2067^2)

9*12437 =2087^2-(6^2+2060^2)

16*12437=2097^2-(4^2+2049^2)

...

Given the following equations, to factorize N so that its multiple corresponds to a difference between squares, we must be able to identify a Pythagorean triple in one of them.
Having established this, we can make use of some devices to vary at will the value of the squares. In fact, we can establish that:

1*(q^2+x*2q-y) =(q+x+/-z)^2-{[(y+/-2q*z)*1]+(x+/-z)^2}
...
(1+n)*(q^2+x*2q-y+n*q^2) =[q*(1+n)+x+/-z]^2-{[(y+/-2q*z)*(1+n)]+(x+/-z)^2}

Therefore, considering only the cases with (+z), we have:

4*12437=2079^2-(1*4+2067^2) =
=(2079+1)^2-[(1+6*1)*4+(2067+1)^2]=
=(2079+2)^2-[(1+6*2)*4+(2067+2)^2]=
= ...

9*12437=2087^2-(4*9+2060^2) =
=(2087+1)^2-[(4+6*1)*9+(2060+1)^2]=
=(2087+2)^2-[(4+6*2)*9+(2060+2)^2]=
= ...

16*12437=2097^2-(1*16+2049^2) =
= ...

whence:

4*12437 =2079^2-(1*4+2067^2) =2080^2-(7*4+2068^2) =2081^2-(13*4+2069^2) ...

9*12437 =2087^2-(4*9+2060^2) =2088^2-(10*9+2061^2) =2089^2-(16*9+2062^2) ...

16*12437 =2097^2-(1*16+2049^2) =2098^2-(7*16+2050^2) =2099^2-(13*16+2051^2) ...

...

In conclusion, we obtain:

4*12437 =2079^2-(1*4+2067^2) =2079^2-(2^2+2067^2)
4*12437 =2083^2-(25*4+2071^2) =2083^2-(10^2+2071^2)
4*12437 =2087^2-(49*4+2075^2) =2087^2-(14^2-2087^2)
...

9*12437 =2087^2-(4*9+2060^2) =2087^2-(6^2+2060^2)
9*12437 =2089^2-(16*9+2062^2) =2089^2-(12^2+2062^2)
9*12437 =2097^2-(64*9+2070^2) =2097^2-(24^2+2070^2)
...

16*12437 =2097^2-(1*16+2049^2) =2097^2-(4^2+2049^2)
16*12437 =2101^2-(25*16+2053^2) =2101^2-(20^2+2053^2)
16*12437 =2105^2-(49*16+2057^2) =2105^2-(28^2+2057^2)
...
We note that:

$$16*12437 = 2105^2 – (28^2 + 2057^2);$$

$$16 = [2105^2 – (28^2 + 2057^2)] / 12437;$$

$$8=\frac{1}{2}\cdot[2105^2 – (28^2 + 2057^2)] / 12437 =$$

$$=\frac{1}{2}\cdot[4431025– 784 – 4231249] / 12437 = \frac{1}{2}\cdot(198992 / 12437) = \frac{1}{2}\cdot 16 = 8 =$$

$$=\frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (3.4b)$$

This result shows how to change the value of the squares orderly between which identify the Pythagorean triple.
Furthermore, to the data obtained above must be added those relating to the cases with (-z) and again, those obtained starting from a different square, at the base of the scheme, and that can be associated with these.

Furthermore, we find that:

9(even)*12437=2087^2-(6^2+2060^2)
9 *12443=2088^2-(6^2+2061^2)
9 *12449=2089^2-(6^2+2062^2)
...

This is an unforeseen consequence and potentially useful that we have in any case decided to add

Now we go on to describe a series of examples.

N=2951=13*227
q=5

We calculate:

2951=8*25+2751

So, wanting to calculate y we do:

(2751-q^2+y)/(2^q) =(2751-25+y)/(10);

whence:

(2751-25+4)/10=273=x

therefore:

1*2751 =(q*1+x)^2-(4*1+x^2); i.e.;

1*2751=278^2-(4+273^2)
...
9*2951=(q*9+x)^2-(4*9+273^2); i.e.;

...
9*2951=318^2-(4*9+273^2)

In conclusion, we obtain:

N=2951
q=5

1*2751 =278^2-(4+273^2)
...
9*2951=318^2-(4*9+273^2) =
=(318+1)^2-[(4+10*1)*9+(273+1)^2] =
=(318+2)^2-[(4+10*2)*9+(273+2)^2] =
= ...

therefore:

9*2951 =318^2-(4*9+273^2) =319^2-(14*9+274^2) =320^2-(24*9+275^2) ...

whence:

9*2951=318^2-(4*9+273^2)   =318^2-(6^2+273^2)
9*2951=324^2-(64*9+279^2) =324^2-(24^2+279^2)
9*2951=332^2-(144*9+287^2)=332^2-(36^2+287^2)

By repeating the calculations with the other squares we get:

...

1*1751 =178^2-(4+173^2)
...
49*2951=418^2-(4*49+173^2)   =418^2-(14^2+173^2)
49*2951=424^2-(64*49+179^2) =424^2-(56^2+179^2)
49*2951=432^2-(144*49+187^2)=432^2-(84^2+187^2)*

*At this point we can say that:

84^2+187^2=205^2

therefore:

49*2951=432^2-205^2=227*637

whence:

2951=227*13 .

We note that:

2951 = 8*25+2751; 227*13 = 8*25 +2751
8 = [(227*13)-2751] / 25 = 200 / 25 = 8   and
$64 = [(227*13)-2751]^2 / 25^2 = 40000 / 625 = 64$

$$64 = \left(\frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}\right)^2 = 40000 / 625. \quad (3.5)$$

Example:

N=22471=23*977
q=20

1*3271   =92^2-(9+72^2)
...

49*22471=1052^2-(9*49+72^2)=1052^2-(21^2+72^2)

where:

21^2+72^2=75^2

therefore:

49*22471=1052^2-75^2=977*1127

whence:

22471=977*23

Example:

N=2375
q=3

1*863 =146^2-(4+143^2)
...
169*2375=650^2-(4*169+143^2) =
=(650+1)^2-[(4+6*1)*169+(143+1)^2]=
=(650+2)^2-[(4+6*2)*169+(143+2)^2]=
= ...

therefore:

169*2375=650^1-(4*169+143^2)=651^2-(10*169+144^2)=652^2+(16*169+145^2)= ...

whence:

169*2375=650^2-(4*169+143^2) = 650^2-(26^2+143^2)
169*2375=652^2-(16*169+145^2)=652^2-(52^2+145^2)
169*2375=660^2-(64*169+153^2)=660^2-(104^2+153^2)*

*At this point we can say that:

104^2+153^2=185^2

therefore:

169*2375=660^2-185^2=475*845

whence:

2375=475*5

We note that:

$660^2 = 24*18150$, therefore:

$24*18150 - 185^2 = 169*2375$

$$24 = \frac{169*2375+185^2}{18150} = \frac{401375+34225}{18150} = \frac{435600}{18150}$$

$$8\cdot 3 = \frac{435600}{18150}\,; \quad 8 = \frac{435600}{54450}\,;$$

$$64 = \frac{189747360000}{2964802500} = \left(\frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}\right)^2. \quad (3.6)$$

In conclusion, we believe that the fact of being able to vary at will the difference between a square and a multiple of N is an enormous advantage that it is worth taking into consideration. In this sense, we believe that this thesis can be a first useful result to achieve the purpose, and also an interesting starting point for further investigation.

Naturally it is necessary to make it effective, in this case, to be able to find a way to quickly identify the Pythagorean triple varying the squares through a thorough knowledge of the rules that generate them.

Furthermore, an important aspect of this method consists in the fact that it can be applied to large numbers without that the progress of the sequence of square to be altered.
In fact, in these cases, starting from small squares in generating the initial schema, also getting very large starting values in the equations, remains very low their vary, proportionally to the square chosen.

2. Up to this point we had considered only the schemes of numbers in which, starting from a pair of factors, the second could increase of an any square while the first could increase only of the square of one.
However also the first factor of each starting pair originated, could be increased of an any square.

The general formula is the following:
(Q1^2*n+2Q1*x-a)*(Q2^2*n+2Q2*y-b)=
=(Q1*Q2*n+Q1*y+Q2*x)^2- [(Q1^2*b+Q2^2*a)*n+(Q1*y-Q2*x)^2+2*b*x*Q1+2*a*y*Q2-a*b]

The second term in brackets of the first line of the formula represents the number to be factorized.

As an example we consider:

N=176149=(Q2^2*n+2Q2*y-b)

(The values of "a" and "b" must be less respectively at "2Q1" e "2Q2".)

Now we choose any square, possibly small, according to which to calculate the values in the brackets.

Furthermore, we choose a value of "n" equal to a square.

Q2=11;
whence:
$Q2^2=121$

$n=32^2=1024$

Then calculate:

$176149=Q2^2*1024+52245$;
whence:
52245=2Q2*y-b=22*2375-5;
whence:
y=2375
b=5

At this point we examine the formula written above.

The terms of the second factor in brackets of the first line are known.
We only need to choose the appropriate values to be attributed to the terms of the first factor in brackets to easily perform the operations and get a Pythagorean triple between the terms of the second and third line of the formula.

For convenience we decide to assume:

x=0
a=0

(It is one of the possible choices.)

whence:

(Q1^2*n)*(Q2^2*n+2Q2*y-b)= (Q1*Q2*n+Q1*y)^2-[(Q1^2*b)*n+(Q1*y)^2]

For example, if establish:
Q1=1;
we obtain:

$(1*1024)*(176149)=(1*11*1024+1*2375)^2-[(1*5*1024)+(1*2375)^2]$
i.e.:
$1024*176149=(11*1024+2375)^2-(5*1024+2375^2)$

We note that 1024 = 64*16 = 8^3*2

We have:

64*16*176149 = $(11*1024+2375)^2$ – $(5*1024+2375^2)$

$$64 = \frac{(11264+2375)^2 - (5120+5640625)}{2818384} = \frac{(126877696+53504000+5640625)-5645745}{2818384} =$$

$$= \frac{186022321-5645745}{2818384} = \frac{180376576}{2818384} = 64 = \left(\frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}\right)^2 \quad (3.7)$$

The same result is obtained if, using the previous formulas, we replace "Q2" to "q", given that in them Q1, or better Q1^2, ie the square relative to the first term of each pair of factors, was always equal to one.

Instead, using this formula, we can choose any value attributable to Q1 and then get new schemes.

Also in this case must first make a small change to the formula, to facilitate the operations, valid only when "x" and "a" are equal to "0", as has been placed above.

(Q1^2*n)*(Q2^2*n+2Q2*y-b)=(Q1*Q2*n+Q1*y+z)^2-[(Q1^2*b+2*Q1*Q2*z)*n+(Q1*y+z)^2]

(With "z" integer positive or negative)
As an example we choose:

Q1=5
and obviously:
x=0
a=0

Furthermore:
N=176149
Q2=11
n=32^2=1024
y=2375
b=5

we obtain:
(5^2*32^2)*(11^2*32^2+2*11*2375-5)=
=(5*11*32^2+5*2375+z)^2-
-[(5^2*5+2*5*11*z)*32^2+(5*2375+z)^2]

thence:

25600*176149=(68195+z)^2-[(125+110*z)*32^2+(11875+z)^2]

Now we need to calculate "z" in such a way as to obtain an ordered series of sums of squares of which identify a Pythagorean triple.

In particular we obtain:
125+110*z=q

whence:
125+110*10=35^2
125+110*50=75^2

125+110*190=145^2
125+110*310=185^2

125+110*590=255^2
125+110*790=295^2

...

Therefore is:
z=10/50;190/310;590/790; ...

whence:
25600*176149=68205^2-[(35^2*32^2)+11885^2]
i.e.:
25600*176149=68205^2-(1120^2+11885^1)

25600*176149=68245^2-[(75^2*32^2)+11925^2]
i.e.:
25600*176149=68245^2-(2400^2+11925^2)

25600*176149=68385^2-[(145^2*32^2)+12065^2]
i.e.:
25600*176149=68385^2-(4640^2+12065^2)

...

We note that 25600 = 400*64 and 4640 = 8*580; $4640^2 = 64*580^2$
$= 8^3*5^2*2$

Thence, we obtain:

$400*64*176149 = 68385^2 - (64*580^2 + 12065^2)$;

$$64 = \frac{68385^2 - \left(4640^2 + 12065^2\right)}{400 * 176149} = \frac{4676508225 - (21529600 + 145564225)}{70459600} = \frac{4676508225 - 167093825}{70459600} =$$

$$= \frac{4509414400}{70459600} = 64 = \left(\frac{1}{3} \frac{4\left[anti\log \frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi^2}{4}w'} \phi_{w'}(itw')}\right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}\right)^2 \quad (3.8)$$

Naturally, the same procedure applies to each "Q2", "Q1" and "n" that we want prefix.

Finally, as we specified earlier, it must be said that to use these conclusions need detailed knowledge of the mechanisms and rules that generate the Pythagorean triples, through which, for example, obtain ordered series similar to those exposed.

e)Mathematical connections

Now we consider various mathematical connections between some equations of Chapter 1 and 2 and some relationship concerning the Chapter 3.

We have the eq. (1.8c):

$$8 = \chi\kappa \sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{\left(\sinh \frac{m\lambda F}{2}\right)^2} \cdot \frac{1}{\int_\varepsilon^\infty \frac{ds}{s^3} \left(\frac{s/2}{\sinh \frac{s}{2}}\right)^2 e^{-\frac{sZ}{F}}},$$

We note easily that this equation can be related with the relationships (3.2) and (3.4b):

$$8 = (13*173) - (2^2*3^2) - (7^2*5*3^2) = \frac{1}{3} \frac{4\left[anti\log \frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi^2}{4}w'} \phi_{w'}(itw')}\right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} =$$

$$= \chi\kappa \sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{\left(\sinh \frac{m\lambda F}{2}\right)^2} \cdot \frac{1}{\int_\varepsilon^\infty \frac{ds}{s^3} \left(\frac{s/2}{\sinh \frac{s}{2}}\right)^2 e^{-\frac{sZ}{F}}}, \quad (3.9)$$

$$\frac{1}{2}\cdot[4431025-784-4231249]/12437=\frac{1}{2}\cdot(198992/12437)=\frac{1}{2}\cdot 16=8=$$

$$=\frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}=$$

$$=\chi\kappa\sum_{m=1}^{\infty}\frac{1}{m}\frac{1}{\left(\sinh\frac{m\lambda F}{2}\right)^2}\cdot\frac{1}{\int_\varepsilon^\infty\frac{ds}{s^3}\left(\frac{s/2}{\sinh\frac{s}{2}}\right)^2e^{-\frac{sZ}{F}}}. \quad (3.10)$$

Also the eq. (1.18b) can be related with the above expressions. Indeed, we have:

From $$8\int_\varepsilon^\infty\frac{ds}{s^3}\left(\frac{s/2}{\sinh\frac{s}{2}}\right)^2e^{-\frac{sZ}{F}}=\chi\kappa\sum_{m=1}^{\infty}\frac{1}{m}\frac{1}{\left(\sinh\frac{m\lambda F}{2}\right)^2};$$

$$(13*173)-(2^2*3^2)-(7^2*5*3^2)=\frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}\times$$

$$\times\int_\varepsilon^\infty\frac{ds}{s^3}\left(\frac{s/2}{\sinh\frac{s}{2}}\right)^2e^{-\frac{sZ}{F}}=\chi\kappa\sum_{m=1}^{\infty}\frac{1}{m}\frac{1}{\left(\sinh\frac{m\lambda F}{2}\right)^2}; \quad (3.11)$$

$$\frac{1}{2}\cdot[4431025-784-4231249]/12437=\frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}\times$$

$$\times\int_\varepsilon^\infty\frac{ds}{s^3}\left(\frac{s/2}{\sinh\frac{s}{2}}\right)^2e^{-\frac{sZ}{F}}=\chi\kappa\sum_{m=1}^{\infty}\frac{1}{m}\frac{1}{\left(\sinh\frac{m\lambda F}{2}\right)^2}. \quad (3.12)$$

Now, the eqs. (2.2b), (2.4b), (2.5) and (2.11) can be related all with the expressions concerning the number 64, thence with the relationships (3.3), (3.4), (3.5), (3.6), (3.7) and (3.8).
For example, from the eq. (2.4b):

$$-\int \frac{d^4xd^4\theta}{(2\pi)^4}\sqrt{g^E}\sum_{q|\zeta(q)\geq 0}\sum_{k=1}^{\infty}\frac{1}{k}Tr_{V_{\vec{q}}}\left[(-1)^F \exp\left(-i\pi k\partial_{\vec{q}}/4\chi^0\right)\right]\exp\left(2\pi ik\sum_I q_I Z^I\right)\frac{\frac{1}{64}\pi^2 w^2}{\sin^2\left(\frac{\pi k\sqrt{w^2}}{8\chi^0}\right)}$$

where 64 is given from the following expression (3.7):

$$\frac{186022321-5645745}{2818384}=\frac{180376576}{2818384}=64=\left(\frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}\right)^2,$$

thence, we have the following mathematical connection with the eq. (3.7):

$$-\int \frac{d^4xd^4\theta}{(2\pi)^4}\sqrt{g^E}\sum_{q|\zeta(q)\geq 0}\sum_{k=1}^{\infty}\frac{1}{k}Tr_{V_{\vec{q}}}\left[(-1)^F \exp\left(-i\pi k\partial_{\vec{q}}/4\chi^0\right)\right]\exp\left(2\pi ik\sum_I q_I Z^I\right)\frac{\frac{1}{64}\pi^2 w^2}{\sin^2\left(\frac{\pi k\sqrt{w^2}}{8\chi^0}\right)}\Rightarrow$$

$$\Rightarrow\frac{186022321-5645745}{2818384}=\frac{180376576}{2818384}=64=\left(\frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}\right)^2. \quad (3.13)$$

From the eq. (2.5):

$$\int \frac{d^4xd^4\theta}{(2\pi)^4}\sum_{k=1}^{\infty}\frac{1}{k}(-1)^{g-1}\exp\left(2\pi ik\sum_I q_I Z^I\right)\frac{\frac{1}{64}\pi^2 w^2}{\sin^{2-2g}\left(\pi k\sqrt{w^2}/8\chi^0\right)},$$

where 64 is given from the following expression (3.1):

$$64 = \left(\frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}\right)^2 = 85914361 - 171680418 + 85766121,$$

thence, we have the following mathematical connection with the eq. (3.1):

$$\int\frac{d^4xd^4\theta}{(2\pi)^4}\sum_{k=1}^{\infty}\frac{1}{k}(-1)^{g-1}\exp\left(2\pi ik\sum_I q_I Z^I\right)\frac{\frac{1}{64}\pi^2 \mathcal{W}^2}{\sin^{2-2g}\left(\pi k\sqrt{\mathcal{W}^2}/8\chi^0\right)}\Rightarrow$$

$$\Rightarrow 64 = \left(\frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}\right)^2 = 85914361 - 171680418 + 85766121. \quad (3.14)$$

Finally, from the eq. (2.21):

$$-\int\frac{d^4xd^4\theta}{(2\pi)^4}\sqrt{g^E}\sum_{q|\zeta(q)\geq 0}\sum_{k=1}^{\infty}\frac{1}{k}Tr_{V_{\bar q}}\left[(-1)^F x^k \exp\left(-i\pi k\partial_{\bar q}/4\chi^0\right)\right]\exp\left(2\pi i\sum_I q_I Z^I\right)\frac{\frac{1}{64}\pi^2\mathcal{W}^2}{\sin^2\left(\frac{\pi k\sqrt{\mathcal{W}^2}}{8\chi^0}\right)},$$

where 64 is given from the following expression (3.8):

$$\frac{4509414400}{70459600} = 64 = \left(\frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}\right)^2,$$

thence, we have the following mathematical connection with the eq. (3.8):

$$-\int \frac{d^4 x d^4 \theta}{(2\pi)^4} \sqrt{g^E} \sum_{q|\zeta(q)\geq 0} \sum_{k=1}^{\infty} \frac{1}{k} Tr_{V_{\bar{q}}} \left[(-1)^F x^k \exp\left(-i\pi k \partial_{\bar{q}} / 4\chi^0\right) \right] \exp\left(2\pi i \sum_I q_I Z^I \right) \frac{\frac{1}{64} \pi^2 w^2}{\sin^2\left(\frac{\pi k \sqrt{w^2}}{8\chi^0} \right)} \Rightarrow$$

$$\Rightarrow \frac{4509414400}{70459600} = 64 = \left(\frac{1}{3} \frac{4\left[anti\log \frac{\int_0^{\infty} \frac{\cos \pi x w'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi^2}{4} w'} \phi_{w'}(itw')} \right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4} \right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4} \right)} \right]} \right)^2 . \quad (3.15)$$

Appendix A

Numerical analysis of the Servi's numbers

Table 1

Comparison of Servi's number with partition numbers, Fibonacci's numbers and triangular numbers

Servi's numbers	partitions	Fibonacci	Triangular	*Observations*
2	**2**	**2**		
5	**5**	**5**		
6			**6**	
7	**7**			
8		**8**		
9				**9 square**
10			**10**	
13		**13**		
15	**15**		**15**	
16	**.**	**.**	**.**	**16 square**
17		**Mean between 13 and 21**		
24	**22**			
26	**.**	**.**	**.**	**26-1 = 25 square**
29			***28***	
30	**30**			
31				**~Mean between the squares 25 and 36**
34		**34**		
35			***36***	
42	**42**			

43			*45*	
…				

We note that the lines without any numerical value are connected to the squares 9, 16, 26-1, 31 = 25+6 = 36 -5, i.e. in half between the following squares 25 and 36.

The ***triangular numbers in cursive*** are very near to the Servi's numbers, differ only in one or two Units.

Finally it must be said that all "n" excluded from the sequence give N primes.

n excluded: **2,5,6,7,8,9,10,13,15,16,17,24,26,29,30,31,34,35,42,43 ...**

From Wikipedia we know that:

In number theory, the **partition function** $p(n)$ represents the number of possible partitions of a natural number n, which is to say the number of distinct ways of representing n as a sum of natural numbers (with order irrelevant). By convention $p(0) = 1$, $p(n) = 0$ for n negative.

The first few values of the partition function are (starting with $p(0)=1$):

1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231, 297, 385, 490, 627, 792,…

The partition numbers that interest us are those less than 56 and we have 2, 5, 7, 15, 30 and 42, i.e. six numbers of a total of nine, wanting to exclude the unit

From wikipedia:

List of triangular numbers

The first triangular numbers are:

1, **3**, **6**, **10**, **15**, **21**, **28**, **36**, **45**, 55, 66, 78, 91, 105, 120, 136, 153, 171, 190, 210, 231, 253, 276, 300, 325, 351, 378, 406, 435, 465, 496, 528, 561, 595, 630, 666, 703, 741, 780, 820, 861, 903, 946, 990, 1035, 1081, 1128, 1176, 1225, 1275, 1326, 1378, 1431, 1485, 1540, 1596, 1653, 1711, 1770, 1830, 1891, 1953, 2016, 2080, 2145, 2211, 2278, 2346, 2415, 2485, 2556, 2628, 2701, 2775, 2850, 2926, 3003, 3081, 3160, 3240 ecc.

The numbers in bold type, up to 45, are those regarding our table

Prime numbers in the numbers of Servi are 2, 5, 7, 13, 17, 29, 31, 43 and, except 2 (= 6*0 + 2), are, as all the prime numbers except 2 and 3 initials, of the form 6k + 1, and precisely (attention to the values of k):

5 = 6*1 – 1

7 = 6*1 +1

13 = 6*2 +1

17 = 6*3 -1

29 = 6*5 -1

31 =6*5+1

43 = 6*7 +1

…

As we can easily notice, the numbers in red (successive values of k), namely 0, 1, 2, 3, 5 (repeated twice) are all Fibonacci's numbers, while 7 is ≈ 8 = next Fibonacci's number.

And this is a further interesting result: a possible connection between the Servi's numbers and the Fibonacci's numbers, both in the above table,both in the prime numbers of the same, using the values of k.

Now we analyze some connections with the Pythagorean triples.

The smaller Pythagorean triples are

(3, 4, 5)	**(5, 12, 13)**	**(7, 24, 25)**	**(8, 15, 17)**
(9, 40, 41)	**(11, 60, 61)**	**(12, 35, 37)**	**(13, 84, 85)**
(16, 63, 65)	**(20, 21, 29)**	**(28, 45, 53)**	**(33, 56, 65)**
(36, 77, 85)	**(39, 80, 89)**	**(48, 55, 73)**	**(65, 72, 97)**
(20, 99, 101)	**(60, 91, 109)**	**(15, 112, 113)**	**(44, 117, 125)**
(88, 105, 137)	(17, 144, 145)	(24, 143, 145)	(51, 140, 149)
(85, 132, 157)	(119, 120, 169)	(52, 165, 173)	(19, 180, 181)

The numbers in red are Servi's numbers:

2,5,6,7,8,9,10,13,15,16,17,24,26,29,30,31,34,35,42,43 ...

In the first 20 triples smaller, is present at least a number of Servi in 10 triples.

Inevitable the numbers 8 and 24 that are related to the modes that correspond to the vibrations of bosonic strings and of the superstrings by the following relationships:

$$8 = \frac{1}{3} \frac{4\left[anti\log \frac{\int_0^\infty \frac{\cos \pi x w'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi^2}{4} w'} \phi_{w'}(itw')} \right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log \left[\sqrt{\left(\frac{10 + 11\sqrt{2}}{4} \right)} + \sqrt{\left(\frac{10 + 7\sqrt{2}}{4} \right)} \right]},$$

$$24 = \frac{4\left[anti\log \frac{\int_0^\infty \frac{\cos \pi x w'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi^2}{4} w'} \phi_{w'}(itw')} \right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log \left[\sqrt{\left(\frac{10 + 11\sqrt{2}}{4} \right)} + \sqrt{\left(\frac{10 + 7\sqrt{2}}{4} \right)} \right]}.$$

Now we see some possible connections with the congruent numbers

From wikipedia:

Congruent Number

In mathematics, a **congruent number** is a positive integer that is the area of a right triangle with three rational number sides.[1] A more general definition includes all positive rational numbers with this property.[2]

The sequence of integer congruent numbers starts with

> 5,6,7,13,14,15,20,21,22,23,24,28,29,30,31,34,37,38,39,41,45,46,47,52,53,54,55,56,60…
> (sequence A003273 in OEIS)

For example, 5 is a congruent number because it is the area of a 20/3, 3/2, 41/6 triangle. Similarly, 6 is a congruent number because it is the area of a 3,4,5 triangle. 3 is not a congruent number.

If q is a congruent number then s^2q is also a congruent number for any natural number s (just by multiplying each side of the triangle by s), and vice versa

The **congruent numbers** in red are also numbers of Servi. We note two groups of three consecutive numbers of Servi (5,6,7 and 29,30,31) in groups of 4 numbers of Servi: 5,6,7,13 and 29,30,31,34, and two isolated number of Servi, 15 and 24, respectively 15= 4^2 -1 and 24=5^2 -1, thence connected at two squares, a partial confirmation of the previous connection with the squares 16=4^2 and 26 = 5^2 +1, of the initial Table. Also 42 and 43 are very near to 41 and 45. Indeed, 42 = 41 +1 and 43 = 45 – 2 or 43 = 41 + 2 , with 1 and 2 that are primenumbers and Fibonacci's Numbers.

Further, we have the congruent numbers 5, 13, 21, 34 and 55 that are also Fibonacci's numbers

Now we see the attempt with the subsequent ratio, assuming the numbers of Servi as an **almost geometric progression** with ratio some roots of phi = 1.618

TABLE 2

SUBSEQUENT RATIO BETWEEN THE SERVI'S NUMBERS

Servi's series : 2,5,6,7,8,9,10,13,15,16,17,24,26,29,30,31,34,35,42,43 ...

Numbers of Servi	Subsequent ratio	Values
2	-	
5	5/2	2,50
6	6/5	1,20
7	7/6	1,16
8	8/7	1,14
9	9/8	1,12
10	10/9	1,11
13	13/10	1,30
15	15/13	1,15
16	16/15	1,06

17	**17/16**	**1,06**
24	**24/17**	**1,41**
26	**26/24**	**1,08**
29	**29/26**	**1,11**
30	**30/29**	**1,03**
31	**31/30**	**1,03**
34	**34/31**	**1,09**
35	**35/34**	**1,02**
42	**42/35**	**1,20**
43	**43/42**	**1,02**
…	**…**	**…**

Total = 22,79

Mean = 22,79/20 = 1,1395 ≈ 1,1278 = √√ 1,618

Mean ratio = 1,13 ≈ 1,1278 = √√ 1,618

Thence, also in this aspect there is a connection with Phi, and in particular with its fourth root very near to the mean ratio.

References

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2) https://www.physicsforums.com/threads/what-is-the-meaning-of-bps-d-branes.457328/

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