The problem of rationality of distances between a point on the plane and the four vertices of a rational square.

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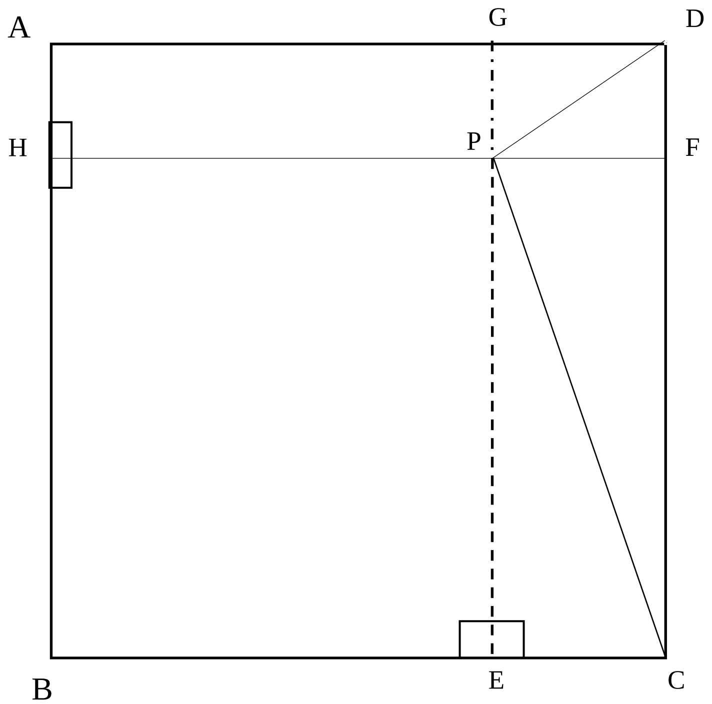
Note: rational values can be always expressed by integers and the two terms are used here interchangeably.

1) Abstract

This seemingly trivial problem has been apparently still unsolved [3].
If a point P is set in a plane's irrational place (be it inside or outside the square), then at least one of the four distances P to vertices must be irrational.
If the point P is inside the square and set in a plane's rational place and all four P to vertices distances are assumed rational, then these distances form hypotenuses of Pythagorean triangles. The distances are - at the same time - hypotenuses of other triangles: triangles formed by irrational legs which are "compatible" with the diagonals of the square (and, of course, not measurable with rational units). Calculation shows that these hypotenuses, if assumed rational, must be all even integers. Since primitive Pythagorean triangles must have odd hypotenuses [1], those triangles are not primitive and should be simplified by division by two. After the first (and all subsequent) divisions the situation doesn't change, the hypotenuses remain even integers and thus divisible by two. That infinite divisibility can be considered as reductio ad absurdum - - a kind of a proof of infinite descent introduced by Fermat [2]. For the point on the border the proof is rather trivial; for the (rationally set) point outside the square other sets of triangles are used to disprove by infinite descent the assumption that the distances can be all rational.

2 a) The point set in an irrational place - within the square
[rat. is shortcut for rational, irr. for irrational]



ABCD is a rational square; HP + PF = HF = BC, all rational sections.
GD = PF = EC, all rational.
{EP, PG} are irrational, but (EP + PG) = EG = CD, so (EP + PG) = rational.

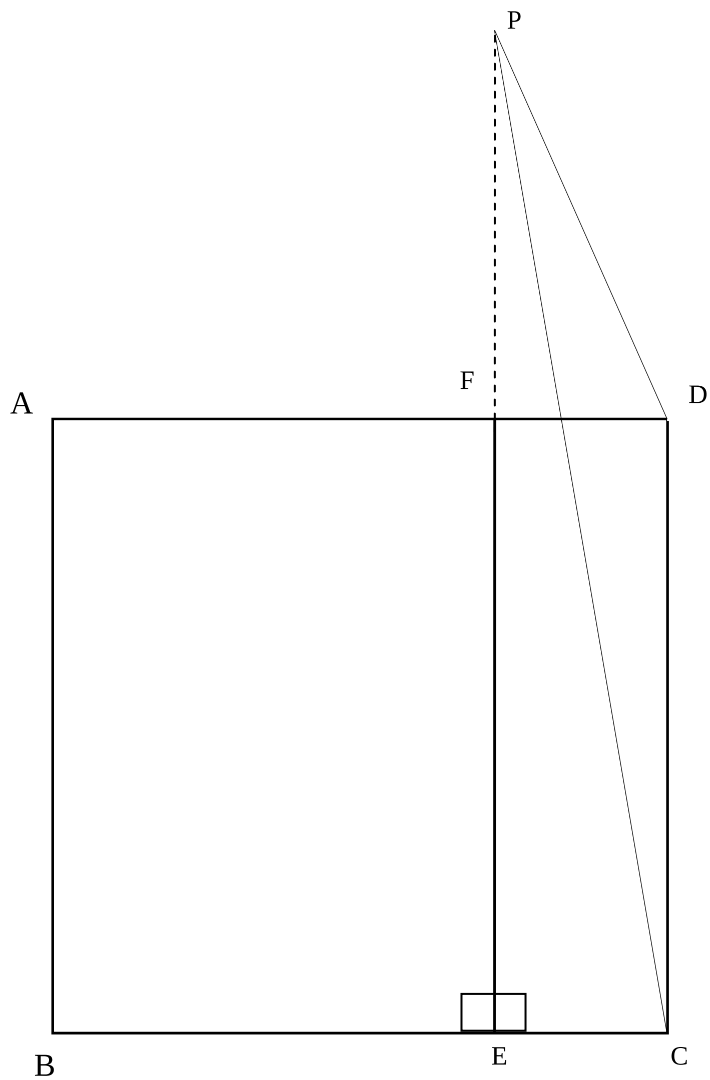
Note also that
$[(EP_{irr1}) + (PG_{irr2})]^2 = [(EP_{irr1})^2 + 2*(EP_{irr1})*(PG_{irr2}) + (PG_{irr2})^2] = (EG_{rat})^2$ = rational;
the whole expression $[(EP_{irr1})^2 + 2*(EP_{irr1})*(PG_{irr2}) + (PG_{irr2})^2]$ is rational if all three elements are present; so, if there are only two of them, then:
$(EP_{irr1})^2 + 2*(EP_{irr1})*(PG_{irr2})$ = irrational
$2*(EP_{irr1})*(PG_{irr2}) + (PG_{irr2})^2$ = irrational
$(EP_{irr1})^2 + (PG_{irr2})^2$ = irrational.
But if there is one: $(EP_{irr1})^2$ or $(PG_{irr2})^2$ - they can be rational (but not both together!);
only $2*(EP_{irr1})*(PG_{irr2})$ - if single or with a second factor - is always irrational.

Analysis of rectangle triangles Δ PDG and Δ PEC makes possible to determine when the hypotenuses (= distances point to vertex) PD and & PC are irrational.
If PG = irrational but $(\text{irr. } PG)^2$ = rational, it is still possible that the hypotenuse PD will be rational:
rat.$[(\text{irr. } PG)^2] + (\text{rat.} PF)^2 = (\text{rat. } PD)^2$ and $\sqrt{(\text{rat. } PD)^2}$ = rational.
But then, since rat.$[(\text{irr. } PG)^2]$ + irr.$[(\text{irr. } PE)^2]$ = irrational, the second Δ PEC has got an irrational hypotenuse:

irr.[(irr. PE)^2] + (rat. EC)^2 = irr.[(irr. PC)^2]
√ irr.[(irr. PC)^2] = irr. PC
Thus, if PD = rational, then PC cannot be.

Similarly, if {HP, PF} are irrational, but (HP + PF) = HF = AD = rational.
Then it could happen that PF^2 is rational and that GP^2 + PF^2 = rational PD^2
and √(PD^2) = rational hypotenuse PD. But then the hypotenuse PA must be irrational...

2 b) The point set in an irrational place outside the square



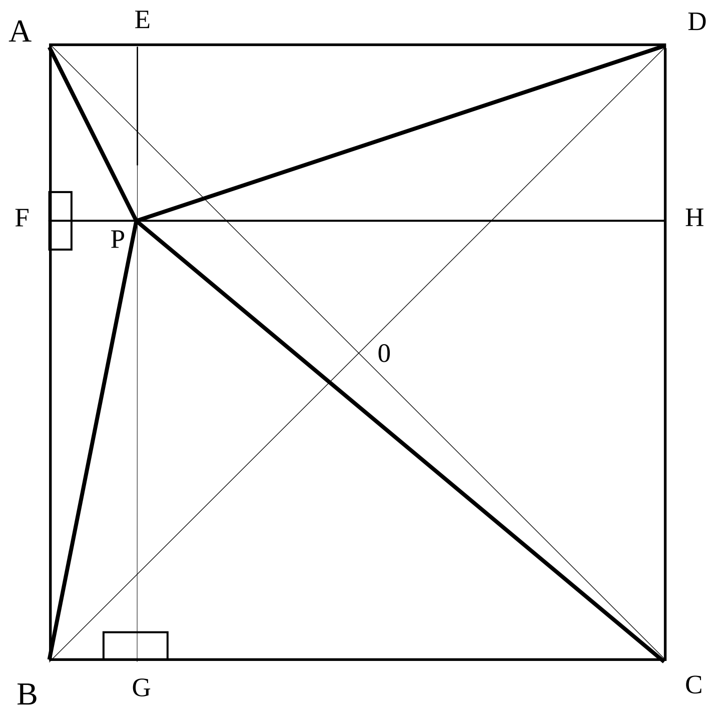
The section PF = irrational; {FD = EC} are both rational.
If PF^2 = rational, then, under circumstances, PD could be rational:
rat.[(irr. PF)^2] + (rat. FD)^2 = (rat. PD)^2 ; √(rat. PD)^2 = rational PD.
But then (PF + FE)^2 :
rat.[(irr. PF)^2] + irr.[2*(irr. PF)*(rat. FE)] + (rat. FE)^2 = irr.[(irr. PE)^2]

irr.[(irr. PE)^2] + (rat. EC)^2 = irr.[(irr. PC)^2]
√ irr.[(irr. PC)^2] = irrational hypotenuse PC. So, if PD happens to be rational, PC must be irrational.

The situation does not change basically when the point P is irrationally set not only on the axis y but also on x. Irrational distances on x and y (if squared) could produce one rational hypotenuse PD (= distance P to vertex D), but then hypotenuses PA and PC will be inevitably irrational.

It is clear that if the point P (within or outside the rational square) is located in an irrational position (in the Cartesian coordinates system) then the four distances between the point P and the vertices cannot be all rational. Therefore all farther discussion will be held for the points rationally set only.

3 a) The point P set in a rational place - within the square

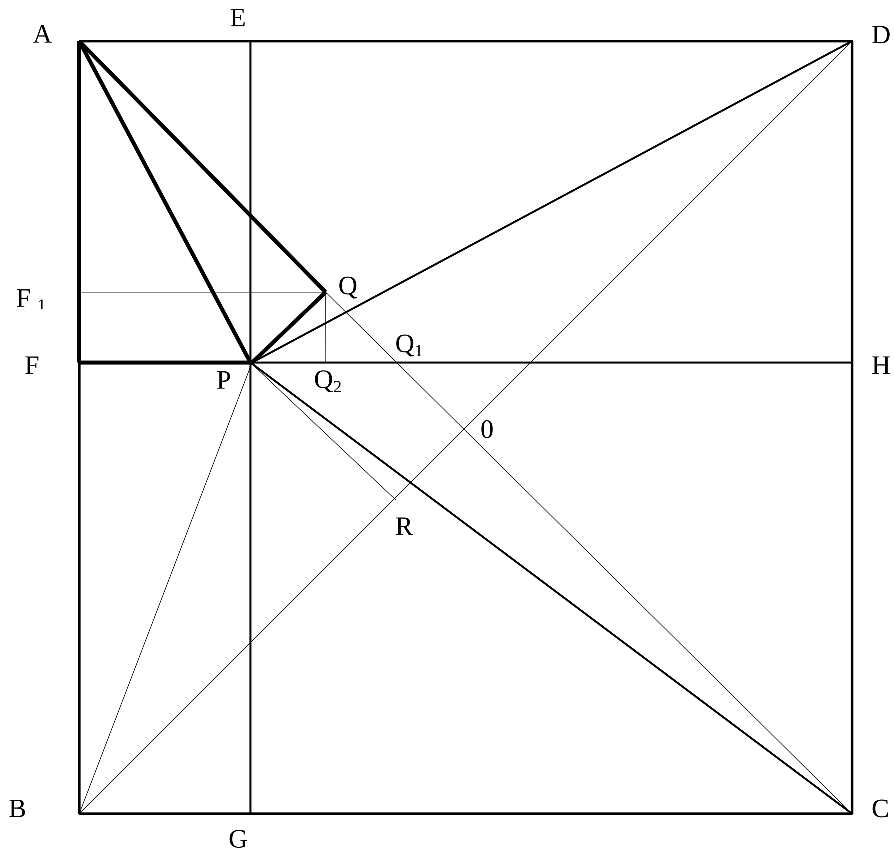


Assume that all 4 distances from the point P to the vertices are rational:
{PA, PB, PC, PD} are rational.

In Cartesian coordinates the point P is set in a rational position, as the four square vertices are. Therefore, perpendicular lines from the point P to the sides (of the square) are also of rational length: {FP, PH, PE, PG} are rational. By the same token, {AF, FB, BG, GC, CH, HD, DE, EA} are rational.

A glance at the drawing suffices now to realize that within the four rectangles (of the square) there are four "double" Pythagorean triangles:

1) Δ AFP (and its "double" Δ PEA);
2) Δ FBP (and Δ PBG)
3) Δ GCP (and Δ PCH)
4) Δ EPD (and Δ PHD)



A perpendicular line from P to the diagonal AC makes the point Q; a second perpendicular from P to the diagonal BD makes the point R.
It can be seen now that AP can be a hypotenuse of yet another (rectangle) triangle: Δ APQ.
The same happens with 3 other hypotenuses: PB is now the hypotenuse to the Δ PRB,
PC to the Δ PCQ, PD to the Δ PRD.

Thus:
$AP^2 = AF^2 + FP^2 = AQ^2 + QP^2$
$PB^2 = FP^2 + FB^2 = BR^2 + RP^2$
$PC^2 = PG^2 + GC^2 = PQ^2 + QC^2$
$PD^2 = PE^2 + ED^2 = PR^2 + RD^2$

Note that the new 4 pairs of triangle legs generating the old hypotenuses are all parts of diagonals or or are parallel to them.

On the example of the first triangle Δ APQ it could be shown that these legs can be measured with (rational units)*√2 - exactly like the diagonals. That makes also the hypotenuse AP, if it should be rational/integer, obligatorily even. The reasoning behind it is as follows:
rat. AF = rat. FQ_1
rat. FQ_1 = rat. FP + rat. PQ_1
(rat. PQ_1)/2 = rat. QQ_2 = rat. FF_1
rat. AF - rat.FF_1 = rat. AF_1
For simplicity, call : $AF_1 = f$; $FF_1 = g$
$AQ = f*\sqrt{2}$; $QP = QQ_1 = g*\sqrt{2}$
$AP^2 = AQ^2 + QP^2$
$AP^2 = (f*\sqrt{2})^2 + (g*\sqrt{2})^2$
$AP^2 = 2*f^2 + 2*g^2 = 2(f^2 + g^2)$;
By the assumption, AP must be integer/rational; to fulfill it, $(f^2 + g^2)$ must be even;
shortly: $AP^2 = 2(f^2 + g^2)$ = 2(even integer) = 4*(integer squared). Then
AP = √4*√(integer squared) = 2*integer = even integer. Thus, hypotenuse AP must be even.

Three other hypotenuses PB, PC and PD follow the same pattern: their generating legs - those parallel to the diagonals - have the configuration (rat. *√2); if the hypotenuse is assumed rational/integer, it must present an even integer.

All primitive (i.e. not reducible) Pythagorean triangles must have an odd hypotenuse [1].
If it is even, it means that the triangle has been multiplied by an even integer. If so, all sides of the triangle must also be even integers. In consequence, all the Pythagorean triangles within the square (and the square side itself !) can be divided by 2. After the division, the legs parallel to the sides of the square will have value: (rational)/2 and the corresponding legs parallel to the diagonals: (rational* √2)/2. Since $(\text{rational}_1)/2 = (\text{rational}_2)$, it will results in values: - (rational_2) and $(\text{rational}_2 * \sqrt{2})$. Concluding, a new hypotenuse length will be again an even integer - thus once again divisible by 2.

In this way, the whole geometric construction can be divided by 2 infinitely - since the result of the division is similar to the first values. Also, the first Pythagorean triangles must have some definite value expressed in integers; those integers cannot be simplified infinitely; there is only a finite number of division to an integer.
Those arguments claim the absurdity of the assumption that all hypotenuses can be rational here, and, since the hypotenuses are also the distances from the point P to the vertices, the absurdity of the proposition that all these distances could be rational.

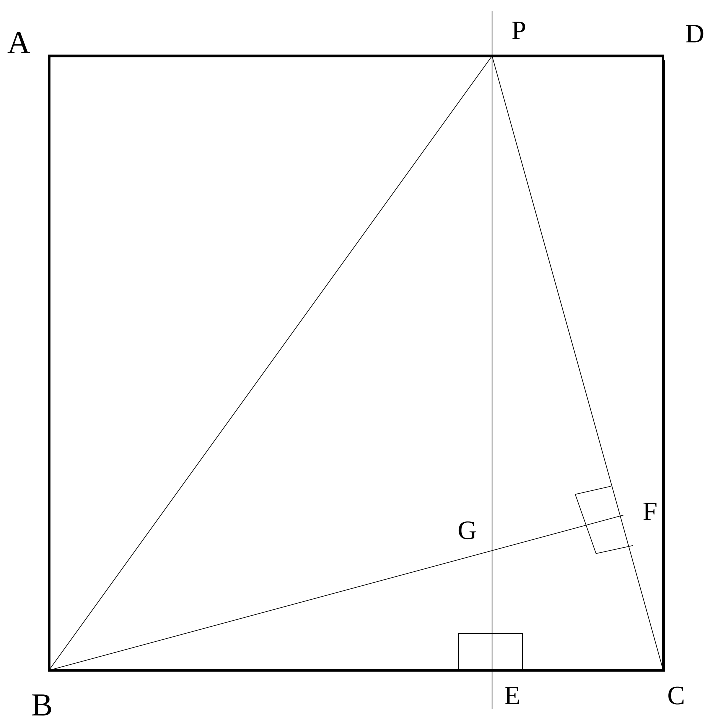
This type of reasoning - first time described by Pierre de Fermat (in 1659) - could be categorized as an exemplar of the "infinite descent" proof: an infinite repeating of a seemingly proper procedure leading to an absurd outcome.

3 b) The point P set in a rational place - on the square's border

If the point P is set on a vertex, it must be rational to the two nearest vertices but irrational to the vertex on the end of the diagonal.

If the point P is on the border of the square, rational to the two neighboring vertices on the same side, it must be in irrational distance to at least one of the two vertices on the opposite side.

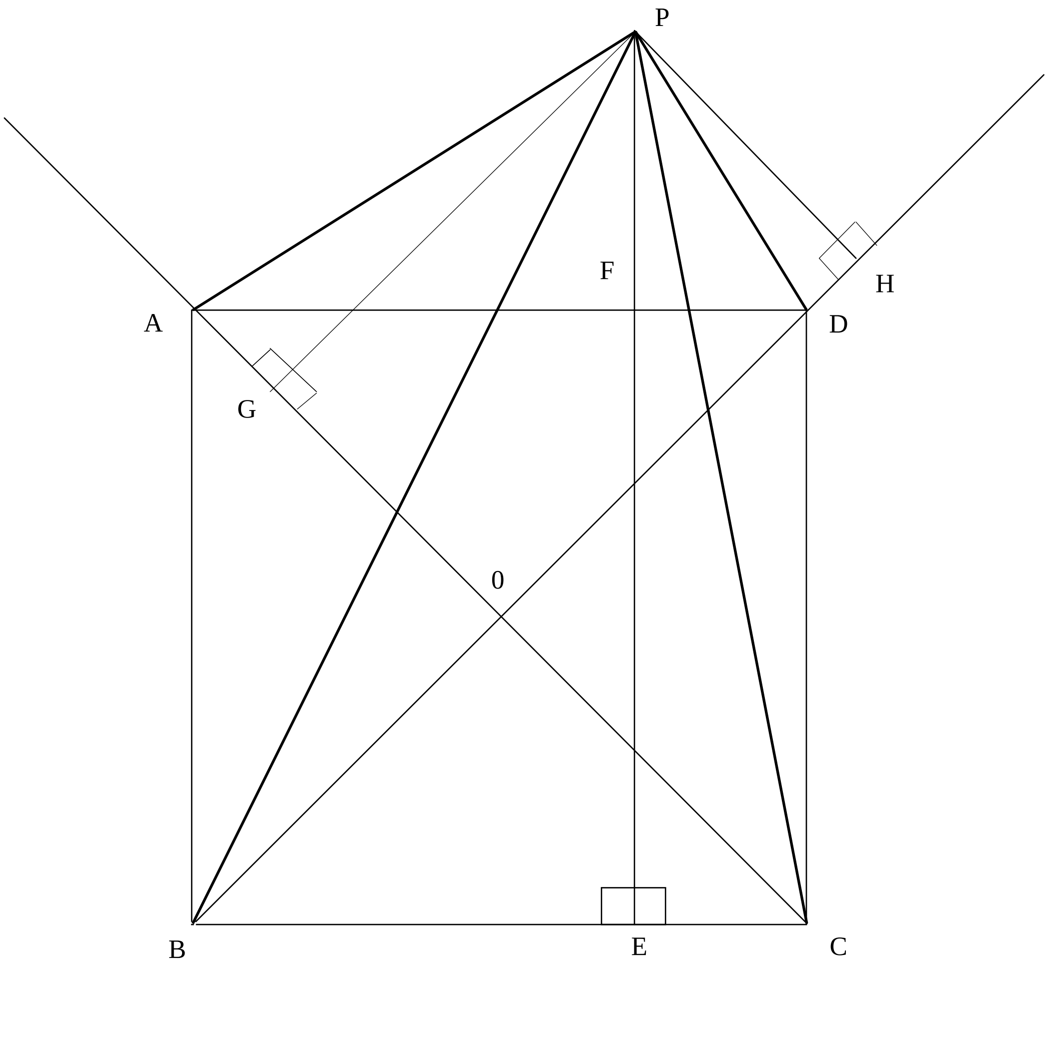
Lemma 1: if a rational square with the point P on its side is divided into two rational rectangles by a line going through this point P, at least one of the diagonals of the rectangles must be irrational.



{AB = BC = CD = DA} all sides are rational; {PD = EC} also rational.
Assume PC = rational.
Since BF is perpendicular to PC and BC to PE, the rectangle triangles ECP and BCF are similar; (also, they are similar to the triangles BEG and GFP).

Δ ECP ~ Δ BCF ~ Δ BEG ~ Δ GFP
The Δ ECP is a Pythagorean triangle and all the other similar triangles are too.
AP and AB are rational and $AP^2 + AB^2 = BP^2$.
Also BF and FP are rational and $BF^2 + FP^2 = BP^2$.
Any Pythagorean triangle is determined by an unique set of values: a pair of rational legs can produce only one rational hypotenuse; this hypotenuse cannot be brought about by a different pair of rational legs. Thus, the BP hypotenuse determined by the two different pairs of rational legs cannot be possibly rational. That concludes the proof of correctness of *Lemma 1*.

3 c) The point P set in a rational place - outside the square



A careful reader of the point 3a of this essay may himself (herself) apply its arguments and the logic of reasoning to this section; it is nothing new here. I reiterate, however, the proof - to complete orderly this point and adopt it to slightly different drawing.

Straight lines connect point P set outside the square with 4 vertices of a rational square ABCD. [The vertices of the square and the point P are set in rational places of the plane; they can make therefore easily Pythagorean triangles.]
An assumption is made that all lines {PA, PB, PC, PD} are rational. They form 4 hypotenuses of Pythagorean triangles whose legs are parallel to the sides of the square; the triangles are (respectively): Δ PFA, Δ PEB, Δ PEC, Δ PFD.

PA, PB, PC, PD are also hypotenuses of other rectangle triangles whose legs are parallel to the diagonals of the square: PA in the Δ PAG, PB in the Δ PHB, PC in the Δ PCG, PD in the Δ PHD. The units of measurement of legs of the Pythagorean triangles are (rational) or (rat.); units of measurement of the legs parallel to diagonals are (rational*√2) or (rat.* √2). [So, they are, in fact, irrational.]

$(\text{rat. PF})^2 + (\text{rat. FA})^2 = (\text{rat. PA})^2$
$(\text{rat. }*\sqrt{2}\text{ AG})^2 + (\text{rat.}*\sqrt{2}\text{ GP})^2 = (\text{rat. PA})^2$

$(\text{rat. BE})^2 + (\text{rat. EP})^2 = (\text{rat. PB})^2$
$(\text{rat. }*\sqrt{2}\text{ BH})^2 + (\text{rat.}*\sqrt{2}\text{ HP})^2 = (\text{rat. PB})^2$

$(\text{rat. CE})^2 + (\text{rat. EP})^2 = (\text{rat. PC})^2$
$(\text{rat. }*\sqrt{2}\text{ CG})^2 + (\text{rat.}*\sqrt{2}\text{ GP})^2 = (\text{rat. PC})^2$

$(\text{rat. PF})^2 + (\text{rat. FD})^2 = (\text{rat. PD})^2$
$(\text{rat. }*\sqrt{2}\text{ PH})^2 + (\text{rat.}*\sqrt{2}\text{ HD})^2 = (\text{rat. PD})^2$

$(\text{Hypotenuse})^2 = (\text{rat.}_1\ \sqrt{2})^2 + (\text{rat.}_2\ \sqrt{2})^2 = 2*(\text{rat}_1^2 + \text{rat}_2^2) = 2*(\text{rat}_3)$.
$(\text{Hypotenuse})^2 = 2*(\text{rat.})$. If hypotenuse = rational/integer, then it has to be an even integer.

Since primitive Pythagorean triangles have always odd hypotenuses, the triangles with even hypotenuses must have been multiplied; now they could and should be simplified by dividing by 2. After the division, the legs which are parallel to sides of the square are still rational and the legs parallel to the diagonal are still (rational*√2).

This and every next division doesn't change nature of the hypotenuses; they remain all the time even integers which is nonsense: no integer can be divided infinitely. This proves falsehood of the assumption that all distances from the point P to vertices are rational.

Conclusion: Distances from a point P to four vertices of a rational square can never be all rational.

4) Literature

1] Wacław Sierpiński, Pythagorean Triangles [Trójkąty pitagorejskie]; in English: Dover Publications, 2003
2] Albert H. Beiler, Recreations in the Theory of Numbers, Dover Publications, 1966
3] http://unsolvedproblems.org/index_files/Prizes.htm