A Spacetime oddity: time dilation and length contraction for the amateur enthusiast

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Abstract

Special relativity is undoubtedly one of the pillars of modern physics where concepts such as time dilation and length contraction subtly play a role in various aspects of nature. Typically veiled under complex and difficult-to-fathom mathematical analysis, the path to understanding these phenomena can leave a novice student lost and confused. In this lecture notes, we attempt to explain and arrive at these concepts using physically intuitive methods and elementary mathematics without the use of advanced mathematical knowledge to make it easier for high school students and amateur enthusiasts to comprehend.

Contents

1 Introduction

It has been more than a hundred years since the dawn of modern physics. Yet, even today, surprisingly, high school chemistry and biology students have not been able to fully comprehend the concepts of modern physics. One of the main reasons for their failure to do so is the utilization of difficult mathematical tools such as calculus for the derivation of these concepts which in turn intimidate and stress the students and amateur enthusiasts.[1]

Both important and basic concepts of modern physics such as *time dilation* and *length contraction* play a significant role without us being aware of it in electromagnets,[2],[3] cathode ray tubes,[4] the yellow color of Gold[5] and liquidity of Mercury.[6] Ever wondered how GPS (Global Positioning System) technology works and maintains its accuracy? When pondering this question, the significance of special relativity usually is not taken into account. However, in order for the GPS to be applicable in our day-to-day life, it is important for it to maintain higher precision by $20-30$ nanoseconds which can only be achieved by taking special relativity into account[1] as the satellites carrying the atomic clocks are relatively in motion to the observers on earth. That is, relativity theory predicts that the observer on earth would see the clock in the satellites ticking in a slower rate than the ones on Earth because of the time dilation effect of their relative motion.[7] Time dilation accounts that for an observer at an inertial frame of reference, the clock moving relative to him will appear to tick slower than the one ticking in his own inertial frame of reference. The faster the relative velocity, the greater the time dilation. Similarly, the phenomenon of length contraction is observed in a moving object when it appears to be shorter than its actual length in its resting state (own inertial frame of reference), responsible in part for the working of electromagnets.[2]

After encountering the role of special relativity in these simple day-to-day examples, our minds are forced to delve deeper into understanding these phenomena in order to seek their origin. But in the quest to find these answers, an amateur enthusiast or a novice student might find themselves confused and lost in the complex, mathematically difficult derivations and analyses, ultimately serving nothing to their curiosity. Moreover, lecturers and teachers often struggle in their attempt to explain those concepts to such students in a simple and heuristic way without using advanced mathematical tools. It is high time we develop simple, pedagogic, intuitive and easily comprehensible methods to satisfy the curiosity of young minds, further fueling their desire to question the mysteries of nature and universe while encouraging them to quest for more. In [8] the reader can find a pedagogic exposition that supplements our approach.

In[1],[9] we have shown how to derive the *mass–energy equivalence*, $E = mc^2$, in a simple and pedagogic way. Here we continue our pedagogical approach by presenting the derivation of time dilation and length contraction. In this note we explain a simple, easily understood and physically intuitive method based on a thought experiment that uses Pythagorean theorem equipped with pictorial representations to derive the concept of time dilation and length contraction in a student-friendly way with the only assumptions here being the constancy of the speed of light, and that the magnitude of relative velocity, v, is the same in every reference frame, i.e., between two observers (though we do introduce the concept of inertial frame and the two postulates of special relativity). We will show how the derivation of time dilation and length contraction can be intuitively reached using a simple light-clock. *These lecture notes*

[1] As a matter of fact, General relativity should also be taken into account. The effect of gravitational frequency shift on the GPS is that a clock closer to a massive object will be slower than a clock farther away.

derive from the fact that they may aid the minds of students and amateur enthusiasts for an easy grasp. While they provide no original contribution to the field that has been widely discussed over the years, we hope that they may be of some help to teachers and curious students alike.

The next section briefly explains the theory of special relativity. It is not necessarily crucial to understanding fully the contents of this section for one to follow the derivation explained in the latter sections. All we need to keep in mind is that the speed of light is a constant regardless of the motion of the light source or observer and that the magnitude of relative velocity, v, is the same in either reference frame. The following section describes the phenomenon of time dilation using an easily perceivable derivation based on elementary mathematics. In the later sections we explain the origin of length contraction, again using a simple derivation followed by a subsequent and concise summary.

2 Special relativity in a nutshell

Special relativity explains how space and time are linked for objects that are moving at a constant velocity.[10], [11], [12] This beautiful theory describes some remarkable features about space and time where moving clocks run slow and moving meter sticks are short (along the direction of motion).[13],[14] These phenomena have been proven by a lot of experiments. Special relativity is based on two postulates which can be simply written as:

The *two postulates of Special Relativity* can be simply written as:

- *The laws of physics are invariant (identical) in all inertial frames of reference (i.e. non-accelerating frames of reference).*
- *The speed of light in a vacuum is the same for all observers, regardless of the motion of the light source or observer.*

The first postulate upon which Einstein based the theory of special relativity relates to reference frames. One should keep in mind that the motion of a body can only be described relative to an observer, or a set of space-time coordinates. These are called frames of reference. For example, a car's motion is measured relative to its starting point. An *inertial frame of reference* is a reference frame in which a body at rest remains at rest and a body in motion moves at a constant speed in a straight line unless acted on by an outside force. While driving in a constant velocity (on a straight line) you cannot tell how fast your car is travelling without looking out the window. These are the simplest frame of references. In such a reference frame the physical laws should be the same, but that they may vary across non-inertial ones. The second assumption of special relativity is that light always travels at the same speed for every observer, regardless of that observer's velocity. We write the constant speed of light in c. The invariance of the speed of light leads to counter-intuitive phenomena, such as time dilation and length contraction. More precisely, the time interval in which two events occur at the same point in space appears longer to a moving observer. Thus, a moving clock appears to run slow. And an object appears shorter to an observer who is moving relative to the object. In the next two sections we will show how these phenomena can be intuitively explained.

3 Light-clock and Pythagorean theorem - a derivation of time dilation

As we explained in the introduction to this note, special relativity indicates that time dilation means that observers moving at a relative velocity do not measure the same elapsed time for an event. More precisely, for an observer in an inertial frame of reference, a clock that is moving relative to him will be measured to tick slower than a clock that is at rest in his frame of reference. Put simply, 'moving clocks run slow'.

To understand how time dilation comes into action, let us first build a light-clock consisting of two mirrors, M_1 and M_2, placed perfectly parallel to one another as in Fig. 1 and separated by a distance d. This clock will measure time in terms of the distance travelled between the mirrors by a light pulse at speed c. One period in this clock would be completed when the light pulse travelling from the mirror M_1 is reflected by M_2 ('tick') and reaches back to M_1 ('tock') as shown in Fig. 1. Hence, the time taken to complete one period covering twice the length d

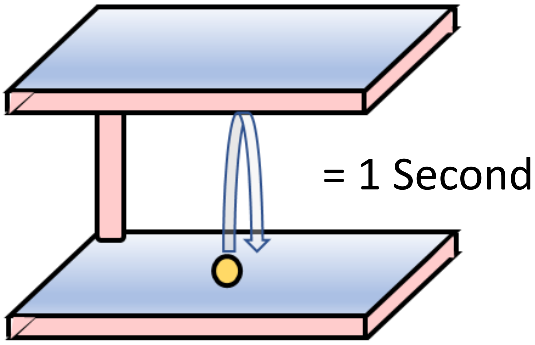

Figure 1: *The light clock consists of a light pulse bouncing between two parallel mirrors M_1 and M_2. The distance between the two mirrors is d and the clock period is $\Delta t_0 = 2d/c$.*

is given by:

$$\Delta t_0 = 2 \times \frac{\textit{distance}}{\textit{speed of light}} = \frac{2 \times d}{c} = \textit{1 second} \tag{1}$$

Consider the same light-clock placed in a train moving at a constant velocity v relative to a stationary observer on the platform (it does not matter whether the motion is to the right or to the left). Since the light-clock in the train is in a relative motion to the stationary clock at the platform, the light pulse in this case travels in a way as shown in Fig. 2. For the observer on

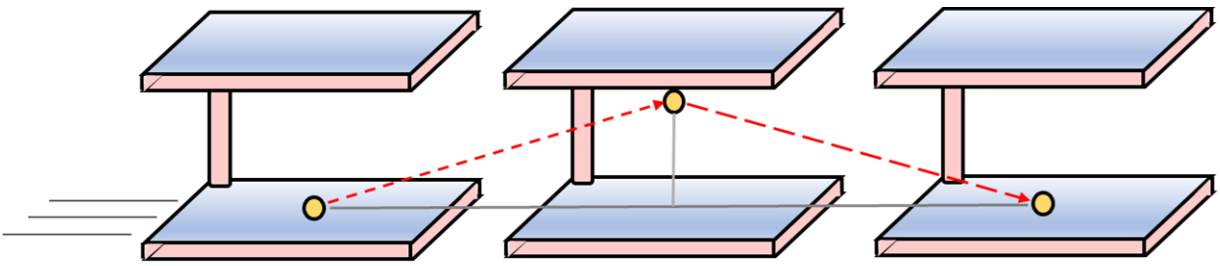

Figure 2: *Relative to an observer on the platform, the light-clock moves at velocity v.*

the platform that measure time on this light-clock, one period is again given by the distance travelled by the light pulse from M_1 to M_2 and back to M_1 divided by the speed of light, c, but since the light-clock is moving with the train at the velocity v, the distance covered by the light pulse is not the same as in the case of the stationary light-clock. The light pulse's path is actually longer and is described by the hypotenuse of the right triangle with sides given by d and l. The light pulse's path from mirror M_1 to mirror M_2 and back to mirror M_1 is given by $2 \times h$ (see Fig.3). Hence, the time period, Δt measured by the observer on the platform is given by,

$$\Delta t = \frac{2 \times h}{c}. \tag{2}$$

From the point of view of an observer on the platform, the distance the train covers as the light goes across it is given by $x = v\Delta t$, (see Fig.3):

$$\underbrace{\begin{pmatrix} \textit{distance the train covers} \\ \textit{as the light pulse goes across it} \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} \textit{relative} \\ \textit{velocity} \end{pmatrix}}_{v} = \underbrace{\begin{pmatrix} \textit{time as measured by an} \\ \textit{observer on the platform} \end{pmatrix}}_{\Delta t} \tag{3}$$

From this we see that the side l is equal to half the distance the train covers as the light goes across it (see Fig.3):

$$l = \frac{v\Delta t}{2}. \tag{4}$$

The hypotenuse, h, of this right triangle can thus be calculated using well-know Pythagorean theorem applied to our case as in Fig. 3,

$$h^2 = l^2 + d^2. \tag{5}$$

Thus, applying Pythagorean theorem, we can find the relation between the time Δt as measured by the observer on the platform (who observed the moving light-clock) and Δt_0, the proper time observed on the train (where the light-clock is at rest):

$$h = \sqrt{d^2 + l^2} = \sqrt{d^2 + \left(\frac{v\Delta t}{2}\right)^2}, \tag{6}$$

where we used (eqs.2-4).

The above equation enables us to relate Δt (eq.2), the time measured by an observer on the platform, to the proper time, Δt_0, (eq.1) as measured by the observer who stands on the train. By taking the square of these two equations we get expressions for $(\Delta t)^2$ $(\Delta t_0)^2$:

$$(\Delta t)^2 = \frac{4d^2}{c^2} + \frac{v^2}{c^2}(\Delta t)^2 \tag{7}$$

$$(\Delta t_0)^2 = \frac{4d^2}{c^2} \tag{8}$$

From this we get the relation between $(\Delta t)^2$ and $(\Delta t_0)^2$, namely

$$(\Delta t)^2 - \frac{v^2}{c^2}(\Delta t)^2 = (\Delta t_0)^2. \tag{9}$$

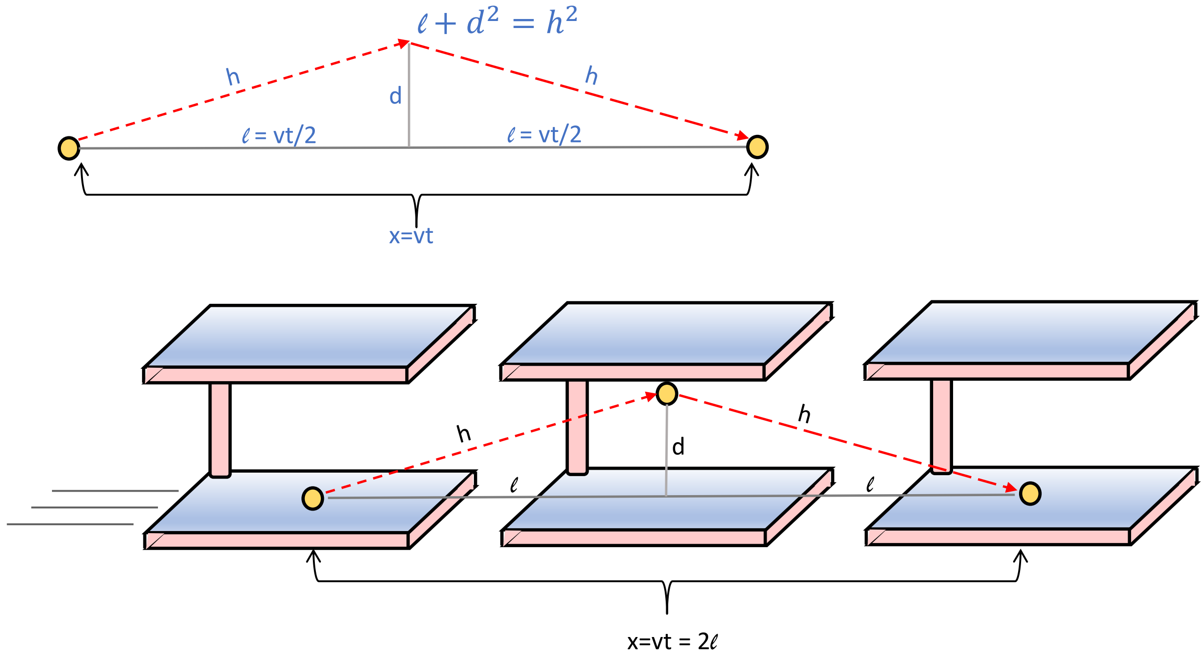

Figure 3: *Pythagorean theorem: For a right-angled triangle the square of the hypotenuse is equal to the sum of the squares of the other two sides,* $h^2 = l^2 + d^2$.

The above equation can be simply written as,

$$(\Delta t)^2 \left(1 - \frac{v^2}{c^2}\right) = (\Delta t_0)^2. \tag{10}$$

By taking the square root of this last equation and introducing the γ factor

$$\gamma = \sqrt{1 - \frac{v^2}{c^2}}, \tag{11}$$

we finally get:

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma \Delta t_0 \tag{12}$$

In plain English, (eq.12) tell us,

$$\underbrace{\begin{pmatrix}\textit{time as measured by an} \\ \textit{observer on the platform}\end{pmatrix}}_{\Delta t} = \gamma \cdot \underbrace{\begin{pmatrix}\textit{proper time, as measured by an} \\ \textit{observer on the train}\end{pmatrix}}_{\Delta t_0} \tag{13}$$

We should pay attention that for velocities much smaller than the speed of light c, $v << c$, the γ factor (eq.11) is almost 1. As such, in order to feel time dilation, the velocity v in the γ factor should be large enough, close to c. But this cannot be achieved in day-to-day life. Thus, time dilation is infinitesimal and hardly noticeable by current scientific instruments.[15] However, satellite clocks make use of this equation in order to maintain accuracy in Earth clocks . Due to their constant motion relative to the clocks on Earth, the atomic clocks on the satellites fall 7 microseconds (10^{-6}) behind per day than the ones on Earth. This difference must be rectified in order to maintain the accuracy of the GPS system.[7]

This equation for Δt is truly remarkable. The theory of relativity states that time is not absolute and being relative as per coordinates and speed, it undergoes dilation. Most intriguingly, time-dilation is valid not only for a light-clock but for all other clocks in the universe, be it biological clocks, sand clocks and likewise, thus extending its observance to every asset encapsulated by the universe. Time itself is not absolute, time passes slower for an observer who is moving relative to another observer.

4 Length contraction - distance depends on an observer's motion

In the last section, we showed that clocks measure different time between two different observers that are in relative motion. Even though our light-clock measures different time in relative frames (the train relative to observer on the platform), the relative speed, v, i.e., the distance divided by time elapsed, remains constant. How is that possible? What this means is that the other quantity, i.e., distance or length must change as well for it to be possible.

Suppose a rod of proper length L_0 is placed at rest on the platform and the light-clock is placed at rest on the train moving parallel to the rod with speed v, see Fig. 4. The principle

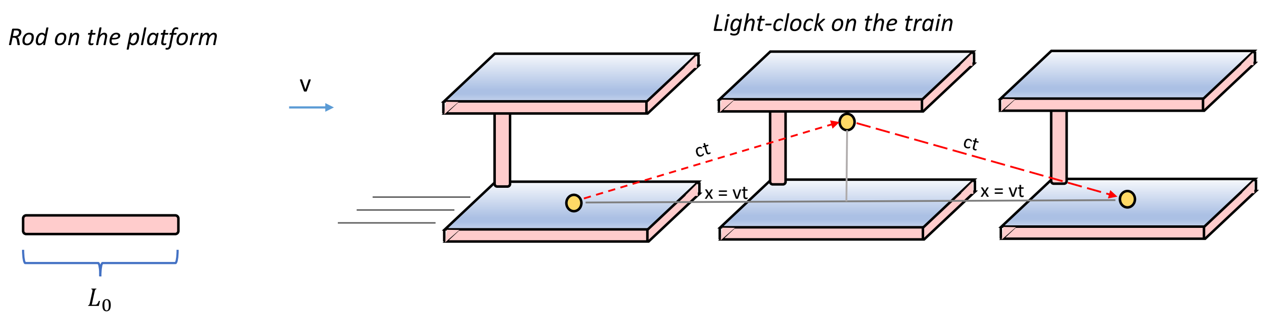

Figure 4: *A rod of proper length L_0 is placed at rest on the platform and the light-clock is placed at rest on the train moving parallel to the rod with speed v.*

of relativity states that the velocity v is the same for the two observers. The observer on the platform sees the train moving at speed v, while the observer on the train sees the the platform (and the rod) passing in the opposite direction at speed v.

What will be the travel time of the light-clock (at rest on the train) from one end of the rod to the other? From the point of view of the observer on the platform, due to time dilation, it will simply be,

$$\Delta t = L_0/v = \frac{\textit{proper length of the rod as measured on the platform}}{\textit{train's velocity}} \tag{14}$$

The observer on the train (where the light-clock is at rest) does not see any time dilation and his light-clock time period is (eq.1). As such he will measure travel time of the light-clock (at rest on the train) from one end of the rod to the other as,

$$\Delta t_0 = L/v = \frac{\textit{length of the rod as measured on the train}}{\textit{rod's velocity}}. \tag{15}$$

We used to think that the length of the rod, measured either on the train or on the platform, should be one and the same. But as we see from the above equations, the elapsed time is different! The expression for the length of the rod (the two end points of the rod) on the platform (where the rod is at rest) is given by,

$$L_0 = \Delta t \times v. \tag{16}$$

However, from the point of view of the observer on the train, the length of the moving rod is:

$$L = \Delta t_0 \times v. \tag{17}$$

Surely the length of the rod is different between the two observers. By using the expression for time dilation (eq.12), i.e.,

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma \Delta t_0 \tag{18}$$

we can see that the ratio between the length on the platform,L_0, and the length on the train, L, is given by,

$$\frac{L}{L_0} = \frac{(\Delta t_0)v}{(\Delta t)v} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{\gamma}, \tag{19}$$

where we used the expression for the γ factor, (eq.11). We therefore conclude that the observer on the train measure the length of the moving rod to be

$$L = \frac{L_0}{\sqrt{1 - \frac{v^2}{c^2}}} = L_0/\gamma. \tag{20}$$

In words, (eq.12) tell us,

$$\underbrace{\begin{pmatrix} \textit{length of the rod} \\ \textit{as measured on the train} \end{pmatrix}}_{L} = \frac{1}{\gamma} \cdot \underbrace{\begin{pmatrix} \textit{proper length, as measured} \\ \textit{by an observer on the platform} \end{pmatrix}}_{L_0} \tag{21}$$

According to (eq.20), the rod from the point of view of the train is shorter than the rod on the platform. Length, then, is not an absolute concept, it depends on the observe. Since the light-clock's travel time across the rod's endpoints is longer on the platform than on the train, the rod's length is also longer on the platform than on the train (length contraction observed on the train). As a matter of fact, the observer on the platform will see the two parallel mirrors of the light-clock shorter than an observer on the train where the light-clock is at rest, see Fig. 5. We need to keep in mind that length contraction is observed in the direction of the relative motion of the body. Also, as with time dilation, it is not felt in the day-to-day speed, being only perceivable at speeds closer to the speed of light c.[16]

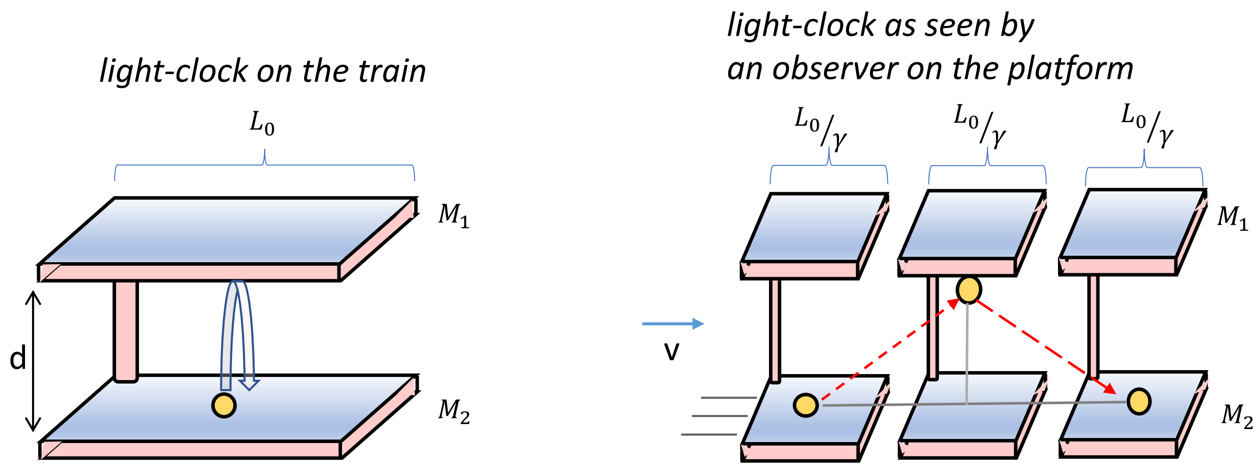

Figure 5: *An observer on the platform will see the two parallel mirrors of the light-clock shorter than an observer on the train (where the light-clock is at rest).*

5 Summary

In this lecture note we continue our pedagogic presentation of introducing concepts in special relativity for amateur enthusiasts.[1],[9] We have introduced a simple, memoryless and physically intuitive line of approach to derive the concepts of time dilation and length contraction to enable amateur enthusiasts and beginners to fully comprehend and appreciate modern physics. Firstly, a simple derivation for time dilation with a pictorial presentation was given. This method used basic Pythagorean theorem to derive the concept of time dilation along with the fact that the speed of light is constant. The following section explained in a similar pedagogic way the concept of length contraction using the time dilation derivation and the fact that the magnitude of relative velocity, v, is the same in either reference frame (i.e., between two observers). We avoid the use of the relativity of simultaneity concept and didn't introduce Lorentz transformation; we have discarded their use completely. We hope that our approach will help mathematically ill-equipped students to understand better such complex concepts without the prior knowledge of advanced mathematical tools and analysis techniques. We believe that the treatment presented here has a great pedagogic merit as students will be able to grasp it easily and gracefully. We also hope that such kind of treatments to explain and derive complex theories will keep satisfy students curiosity and awaken their thirst for more.

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