Riemann Hypothesis

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1 Abstract

The Riemann Zeta function is defined as

$\zeta(s) = \sum_{n=1}^{\infty} 1/n^s, \ Re(s) > 1$

The Zeta function is holomorphic in the complex plane except for a pole at

$s = 1$. *The trivial zeros of* $\zeta(s)$ *are* $-2, -4, -6, ...$ *Its non trivial zeros lie in*

the critical strip $0 < Re(s) < 1$.

The Riemann Hypothesis states that all the non trivial zeros lie on the critical line

$Re(s) = 1/2$.

2 Proof

Analytic continuation of $\zeta(s)$ *is defined as* [*see* 1, *p*.14 , *Eq*. 2.1.4]

$$\zeta(s) = s\int_1^\infty \frac{[x]-x+\frac{1}{2}}{x^{s+1}}dx + \frac{1}{s-1} + \frac{1}{2} \tag{1}$$

Here [.] *denotes the Greatest Integer Function.*

Let, $s = \sigma + it$, $0 < \sigma < 1$.

By, [1, *p*.14],

$\frac{s}{2}\int_1^\infty \frac{1}{x^{s+1}}dx = \frac{1}{2}$.

So, using $\frac{1}{2} = \frac{s}{2}\int_1^\infty \frac{1}{x^{s+1}}dx$ *in* (1),

$$\zeta(s) = s\int_1^\infty \frac{[x]-x+\frac{1}{2}}{x^{s+1}}dx + \frac{1}{s-1} + \frac{s}{2}\int_1^\infty \frac{1}{x^{s+1}}dx$$

$$\zeta(s) = s\int_1^\infty \frac{[x]-x+\frac{1}{2}}{x^{s+1}}dx + \frac{s}{2}\int_1^\infty \frac{1}{x^{s+1}}dx + \frac{1}{s-1}$$

$$\zeta(s) = s\int_1^\infty \frac{[x]-x+1}{x^{s+1}}dx + \frac{1}{s-1}$$

Let, ρ *be a non trivial zero of the Riemann Zeta Function.*

Then, $\zeta(\rho) = 0$; $0 < Re(\rho) < 1$.

$$\zeta(\rho) = \rho\int_1^\infty \frac{[x]-x+1}{x^{\rho+1}}dx + \frac{1}{\rho-1} = 0;\ 0 < Re(\rho) < 1$$

$$\int_1^\infty \frac{[x]-x+1}{x^{\rho+1}}dx = \frac{1}{\rho(1-\rho)};\ 0 < Re(\rho) < 1 \tag{2}$$

The functional equation of the Riemann Zeta function is [*see* [1], *p*.22, 2.6.4],

$$\Gamma(s/2)\pi^{-s/2}\zeta(s) = \Gamma((1-s)/2)\pi^{-(1-s)/2}\zeta(1-s).$$

So, by functional equation if ρ *is a zero of the Riemann Zeta function then* $1-\rho$ *is also a zero* [*see* [1], *p*.30].

$$\zeta(\rho) = 0 \Rightarrow \zeta(1-\rho) = 0.$$

$\zeta(1-\rho) = (1-\rho)\int_1^\infty \frac{[x]-x+1}{x^{2-\rho}}dx - \frac{1}{\rho} = 0;\ 0 < Re(\rho) < 1$

$$\int_1^\infty \frac{[x]-x+1}{x^{2-\rho}}dx = \frac{1}{\rho(1-\rho)};\ 0 < Re(\rho) < 1 \qquad (3)$$

Equating left sides of equation (2) *and* (3),

$$\int_1^\infty \frac{[x]-x+1}{x^{\rho+1}}dx = \int_1^\infty \frac{[x]-x+1}{x^{2-\rho}}dx$$

$$\int_1^\infty ([x]-x+1)\left(\frac{1}{x^{\rho+1}} - \frac{1}{x^{2-\rho}}\right)dx = 0 \qquad (4)$$

$Let, \rho = \sigma + it\ ; 0 < \sigma < 1$

$Since,\ 0 < \sigma < 1$ *so we discuss* 2 *cases*

$1/2 \leq \sigma < 1$ *and* $0 < \sigma \leq 1/2.$

$Case1 : 1/2 \leq \sigma < 1.$

Putting, $\rho = \sigma + it$ *in equation* (4),

$$\int_1^\infty ([x]-x+1)\left(\frac{1}{x^{\sigma+1+it}} - \frac{1}{x^{2-\sigma-it}}\right)dx = 0.$$

$$\int_1^\infty ([x]-x+1)\left(\frac{x^{-it}}{x^{\sigma+1}} - \frac{x^{it}}{x^{2-\sigma}}\right)dx = 0.$$

$$\int_1^\infty ([x]-x+1)\left(\frac{e^{-it(lnx)}}{x^{\sigma+1}} - \frac{e^{it(lnx)}}{x^{2-\sigma}}\right)dx = 0.$$

$$\int_1^\infty ([x]-x+1)\left(\frac{cos(tlnx)}{x^{\sigma+1}} - \frac{cos(t(lnx))}{x^{2-\sigma}}\right)dx +$$

$$i\int_1^\infty ([x]-x+1)\left(\frac{-sin(tlnx)}{x^{\sigma+1}} - \frac{sin(t(lnx))}{x^{2-\sigma}}\right)dx = 0$$

Equating Real part to zero,

$$\int_1^\infty ([x]-x+1)\left(\frac{cos(tlnx)}{x^{\sigma+1}} - \frac{cos(t(lnx))}{x^{2-\sigma}}\right)dx = 0$$

$$\int_1^\infty ([x]-x+1)\left(\frac{1}{x^{\sigma+1}} - \frac{1}{x^{2-\sigma}}\right)cos(tlnx)dx = 0 \qquad (5)$$

$$Let, I = \int_1^\infty ([x] - x + 1)(\frac{1}{x^{\sigma+1}} - \frac{1}{x^{2-\sigma}}) cos(tlnx) dx = 0 \qquad (6)$$

$$0 \leq x - [x] < 1$$

$$0 < [x] - x + 1 \leq 1.$$

$$I \leq \int_1^\infty (\frac{1}{x^{\sigma+1}} - \frac{1}{x^{2-\sigma}}) cos(tlnx) dx$$

$$cos(tlnx) \leq 1.$$

$$I \leq \int_1^\infty (\frac{1}{x^{\sigma+1}} - \frac{1}{x^{2-\sigma}}) dx$$

$$I \leq -\frac{1}{\sigma x^\sigma} - \frac{1}{(\sigma-1)x^{1-\sigma}}\Bigg|_1^\infty, \ 1/2 \leq \sigma < 1$$

$$I \leq \frac{1}{\sigma} + \frac{1}{(\sigma-1)}$$

Since by equation (6), $I = 0$, *so*

$$0 \leq \frac{1}{\sigma} + \frac{1}{(\sigma-1)}$$

$$0 \leq \frac{2\sigma-1}{\sigma(\sigma-1)}$$

$$0 \leq \frac{1-2\sigma}{\sigma(1-\sigma)} \qquad (7)$$

Also by Case 1 *we had* $1/2 \leq \sigma < 1$.

$$\Rightarrow \frac{1-2\sigma}{\sigma(1-\sigma)} \leq 0. \qquad (8)$$

Combining equations (7) *and* (8),

$$0 \leq \frac{1-2\sigma}{\sigma(1-\sigma)} \leq 0.$$

$$\frac{1-2\sigma}{\sigma(1-\sigma)} = 0.$$

$$1 - 2\sigma = 0.$$

$$\sigma = 1/2.$$

Now we proceed to Case 2

Case 2: $0 < \sigma \leq 1/2$.

$\rho = \sigma + it.$

$Let,\ \zeta(\rho) = 0$

$The\ functional\ equation\ of\ the\ Riemann\ Zeta\ function\ is$ [see [1], $p.22, 2.6.4$],

$$\Gamma(s/2)\pi^{-s/2}\zeta(s) = \Gamma((1-s)/2)\pi^{-(1-s)/2}\zeta(1-s).$$

$So,\ by\ functional\ equation\ if\ \rho = \sigma+it\ is\ a\ zero\ of\ the\ Riemann\ Zeta\ function\ then$
$1-\rho = 1-\sigma-it \quad is\ also\ a\ zero\ and\ then$
$1-\bar{\rho} = 1-\sigma+it \ \ is\ also\ a\ zero$ [see [1], $p.30$].

$\zeta(\rho) = 0 \Rightarrow \zeta(1-\rho) = 0.$

$\zeta(1-\rho) = 0 \Rightarrow \zeta(1-\bar{\rho}) = 0$

$Since,\ \rho = \sigma + it.$

$\zeta(1-\bar{\rho}) = 0 \Rightarrow \zeta(1-\sigma+it) = 0.$

$0 < \sigma \leq 1/2 \Rightarrow 1/2 \leq 1-\sigma < 1.$

$\zeta(1-\sigma+it) = 0, 1/2 \leq 1-\sigma < 1.$

$So,\ by\ case\ (1),$

$(1-\sigma) = 1/2.$

$\sigma = 1/2.$

$So,\ by\ the\ above\ two\ cases\ we\ get\ that$

$\zeta(\rho) = 0\ ; 0 < Re(\rho) < 1 \Rightarrow Re(\rho) = 1/2\ , which\ proves\ the\ R.H.$

3 References

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