$$Z(s) = \Gamma\left(\frac{s}{2}\right)\pi^{-s/2}\zeta(s)$$

$$Z(s) = \int_0^\infty \psi(x)\, x^{s/2-1}\, dx$$

where $\psi(x) = \sum_{n=1}^{\infty} e^{-n^2 \pi x}$

Neukirch, J. (1999) Algebraic Number Theory
Pg: 425 (Theorem 1.6)
Completed Zeta func. $Z(s)$ has an analytic continuation ~~to~~ $\mathbb{C} \setminus \{0, 1\}$ & satisfies

$$Z(s) = Z(1-s)$$

$$Z(s) = \int_0^\infty \psi(x)\, x^{s/2-1}\, dx$$

Let $(s = \sigma + it)$

$$Z(\sigma+it) = \int_0^\infty \psi(x)\, x^{\frac{\sigma-2}{2}+\frac{it}{2}}\, dx = 0 \quad \text{--- (1)}$$

$$Z(1-s) = Z(s) = 0$$

$$Z(1-\sigma-it) = 0$$

$$\int_0^\infty \psi(x)\, x^{-\left(\frac{\sigma+1}{2}\right)-\frac{it}{2}}\, dx = 0 \quad \text{--- (2)}$$

(1) − (2) gives

$$\int_0^\infty \psi(x)\left[x^{\frac{\sigma-2}{2}+\frac{it}{2}} - x^{-\left(\frac{\sigma+1}{2}\right)-\frac{it}{2}}\right] dx = 0 \quad \text{--- } (*)$$

① − ② gives

$$\int_0^\infty \psi(x)\left[x^{\frac{\sigma-2}{2}} x^{\frac{it}{2}} - x^{-\left(\frac{\sigma+1}{2}\right)-\frac{it}{2}}\right] dx = 0$$

$$\int_0^1 \psi(x)\left[x^{\frac{\sigma-2+it}{2}} - x^{-\left(\frac{\sigma+1+it}{2}\right)}\right] dx$$

$$+ \int_1^\infty \bar{\psi}(x)\left[x^{\frac{\sigma-2+it}{2}} - x^{-\left(\frac{\sigma+1+it}{2}\right)}\right] dx = 0 \quad ③$$

Let $I = \int_1^\infty \psi(x)\left[x^{\frac{\sigma-2+it}{2}} - x^{-\left(\frac{\sigma+1+it}{2}\right)}\right] dx$

$$x = \frac{1}{y} \qquad dx = -\frac{1}{y^2}\,dy$$

$$I = \int_0^1 \frac{\psi\left(\frac{1}{y}\right)}{y^2}\left[\frac{1}{y^{\frac{\sigma-2}{2}} \cdot y^{\frac{it}{2}}} - \frac{1}{y^{-\left(\frac{\sigma+1}{2}\right)} \cdot y^{\frac{-it}{2}}}\right]$$

$$I = \int_0^1 \psi\left(\frac{1}{y}\right)\left[\frac{1}{y^{\frac{\sigma+2}{2}} y^{\frac{it}{2}}} - \frac{1}{y^{\left(\frac{-\sigma+3}{2}\right)} y^{\frac{-it}{2}}}\right] dy$$

$$I = \int_0^1 \psi\left(\frac{1}{y}\right)\left[y^{-\left(\frac{\sigma+2}{2}\right)} y^{-\frac{it}{2}} - y^{\left(\frac{\sigma-3}{2}\right)} y^{\frac{it}{2}}\right] dy$$

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Putting value of I in (3),

$$\int_0^1 \psi(x)\left[x^{\frac{\sigma-2+it}{2}} - x^{-\left(\frac{\sigma+1+it}{2}\right)}\right]dx$$

$$+ \psi\left(\frac{1}{x}\right)\left[x^{-\left(\frac{\sigma+2+it}{2}\right)} - x^{\left(\frac{\sigma-3+it}{2}\right)}\right]dx = 0$$

$$\int_0^1 \psi(x)\, x^{\frac{\sigma-2+it}{2}}\left[1 - x^{\frac{1-2\sigma-2it}{2}}\right] - \psi\left(\frac{1}{x}\right) x^{\frac{\sigma-3+it}{2}}\left[1 - x^{\frac{1-2\sigma-2it}{2}}\right]dx = 0$$

$$\int_0^1 x^{\frac{\sigma-2+it}{2}}\left[1-x^{\frac{1-2\sigma-2it}{2}}\right]\left[\psi(x) - x^{-1/2}\psi\left(\frac{1}{x}\right)\right]dx = 0$$

$$\frac{1+2\psi(x)}{1+2\psi\left(\frac{1}{x}\right)} = \frac{1}{\sqrt{x}} \qquad (\text{Pg:- 15 Edwards Eq}^n\ (2))$$

$$\psi\left(\frac{1}{x}\right) = \frac{\sqrt{x}\,(1+2\psi(x)) - 1}{2}$$

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$$\int_0^1 x^{\frac{\sigma-2+it}{2}}\left[1-x^{\frac{1-2\sigma-2it}{2}}\right]\left[\psi(x)-\left[\frac{(1+2\psi(x))-x^{-1/2}}{2}\right]\right]dx=0$$

$$\int_0^1 x^{\frac{\sigma-2+it}{2}}\left[1-x^{\frac{1-2\sigma-2it}{2}}\right]\left[x^{-1/2}-1\right]$$

$$\int_0^1 x^{\frac{\sigma-2+it}{2}}\left[1-x^{\frac{1-2\sigma-2it}{2}}\right]\left(\frac{1-\sqrt{x}}{\sqrt{x}}\right)$$

$$\int_0^1 x^{\frac{it}{2}}\left(1-x^{\frac{1-2\sigma-2it}{2}}\right)\left[x^{\frac{\sigma-2}{2}}\frac{(1-\sqrt{x})}{\sqrt{x}}\right]dx$$

$$\lim_{\substack{\xi\to 0^+\\ \eta\to 0^+}}\int_{\xi}^{1-\eta} x^{\frac{it}{2}}\left(1-x^{\frac{1-2\sigma-2it}{2}}\right)\left(x^{\frac{\sigma-2}{2}}\frac{(1-\sqrt{x})}{\sqrt{x}}\right)dx$$

let $f(x) = x^{\frac{it}{2}}\left(1-x^{\frac{1-2\sigma-2it}{2}}\right)$

$g(x) = x^{\frac{\sigma-2}{2}}\frac{(1-\sqrt{x})}{\sqrt{x}}$

f & g are Continuous on $[\xi, 1-\eta]$

$\Rightarrow f$ & g are integrable on $[\xi, 1-\eta]$

$$g(x) = x^{\frac{\sigma-2}{2}} \frac{(1-\sqrt{x})}{\sqrt{x}} > 0 \quad \forall \; x > 0$$

So, By Generalised 1^{st} mean value theorem $\exists \; c \in [\xi, 1-\eta]$ s.t

$$\int_a^b f(x) g(x)\, dx = f(c) \int_a^b g(x)\, dx$$

$$\lim_{\substack{\xi \to 0^+ \\ \eta \to 0^+}} \int_{\xi}^{1-\eta} x^{\frac{it}{2}} \left(1 - x^{\frac{1-2\sigma-2it}{2}}\right) x^{\frac{\sigma-2}{2}} \frac{(1-\sqrt{x})}{\sqrt{x}}\, dx$$

$$= \lim_{\substack{\xi \to 0^+ \\ \eta \to 0^+}} c^{\frac{it}{2}} \left(1 - c^{\frac{1-2\sigma-2it}{2}}\right) \int_{\xi}^{1-\eta} x^{\frac{\sigma-2}{2}} \frac{(1-\sqrt{x})}{\sqrt{x}}\, dx = 0$$

$$c^{\frac{it}{2}} \left(1 - c^{\frac{1-2\sigma-2it}{2}}\right) \int_0^1 x^{\frac{\sigma-2}{2}} \frac{(1-\sqrt{x})}{\sqrt{x}}\, dx = 0$$

$$0 < x < 1 \Rightarrow 0 < 1-\sqrt{x} < 1$$

$$x^{\frac{\sigma-2}{2}} \frac{(1-\sqrt{x})}{\sqrt{x}} > 0$$

$$\int_0^1 \frac{x^{\frac{\sigma-2}{2}}(1-\sqrt{x})}{\sqrt{x}}\,dx > 0$$

$$\Rightarrow \quad c^{\frac{it}{2}}\left(1-c^{\frac{1-2\sigma-2it}{2}}\right)=0$$

$c \in [\xi, 1-\eta]$ & $\xi \to 0^+$
$\eta \to 0^+$

$\Rightarrow c \neq 0$

$$1-c^{\frac{1-2\sigma-2it}{2}} = 0$$

$$c^{\frac{1-2\sigma}{2}} \cdot c^{it} = 1$$

$$c^{\frac{1-2\sigma}{2}} e^{it\ln c} = 1$$

$$\left|c^{\frac{1-2\sigma}{2}} e^{it\ln c}\right| = 1$$

$$c^{\frac{1-2\sigma}{2}} = 1 = c^0$$

$c \in [\xi, 1-\eta]$ $\eta \to 0^+$ so $c \neq 1$

$$\frac{1-2\sigma}{2} = 0 \Rightarrow \sigma = \frac{1}{2}$$