# On some incomplete elliptic integrals and Black Holes-Wormholes formulas: new possible mathematical connections with $\phi$, $\zeta(2)$, and various parameters of Particle Physics. III

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## Abstract

*In this paper we have described some Ramanujan incomplete elliptic integrals and Black Holes-Wormholes formulas. Furthermore, we describe new possible mathematical connections with $\phi$, $\zeta(2)$, and various parameters of Particle Physics*

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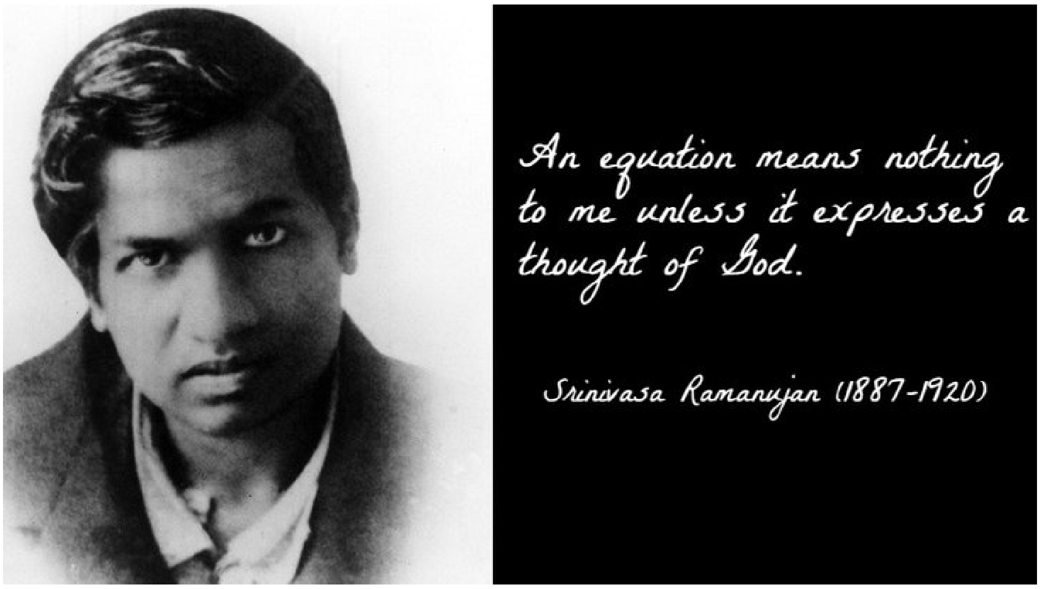



https://mobygeek.com/features/indian-mathematician-srinivasa-ramanujan-quotes-11012

We want to highlight that the development of the various equations was carried out according an our possible logical and original interpretation

From

**Black holes and naked singularities from Anton-Schmidt's fluids**
*Salvatore Capozziello, Rocco D'Agostino and Daniele Gregoris* - arXiv:2002.04875v1 [gr-qc] 12 Feb 2020

We have the following equations:

$$\mathcal{C}_2 = r_0 \left( \frac{8\pi\rho_* e^{-1/n}}{3} r_0^2 - 1 \right). \qquad (27)$$

$$S_{1,2} = \left[ \frac{e^{1/n}}{8\pi\rho_*} \left( \frac{3\mathcal{C}_2}{2} \pm \sqrt{\frac{9\mathcal{C}_2^2}{4} - \frac{e^{1/n}}{8\pi\rho_*}} \right) \right]^{1/3}. \qquad (21)$$

We take the following equation:

(**On a New Approach for Constructing Wormholes in Einstein-Born-Infeld Gravity -** *Jin Young Kim and Mu-In Park* - arXiv:1608.00445v3 [hep-th] 10 Oct 2016)

$$r_+^* = \sqrt{Q^2 - \frac{1}{4\beta^2}} \qquad \text{with } Q = 1, \beta = 1,, \quad r > r_0,$$

r = sqrt(1-1/4)

**Input:**

$$\sqrt{1 - \frac{1}{4}}$$

**Result:**

$$\frac{\sqrt{3}}{2}$$

**Decimal approximation:**

0.8660254037844386467637231707529361834714026269051903140279...

0.8660254037844…

If $r > r_0$ $r_0 < r$. We calculate as follows:

sqrt(1-1/4) – 1/4

**Input:**

$$\sqrt{1-\frac{1}{4}}-\frac{1}{4}$$

**Result:**

$$\frac{\sqrt{3}}{2}-\frac{1}{4}$$

**Decimal approximation:**

0.616025403784438646763723170752936183471402626905190314027...

0.6160254037844…

**Alternate form:**

$$\frac{1}{4}\left(2\sqrt{3}-1\right)$$

**Minimal polynomial:**

$$16x^2+8x-11$$

Thence $r_0 = 0.6160254....$

Now, we have:

$$\mathcal{C}_2 = r_0\left(\frac{8\pi\rho_* e^{-1/n}}{3}r_0^2-1\right).$$

where $\rho_*$ represents a reference density that has been interpreted as the Planck density.

From:

(**On a New Approach for Constructing Wormholes in Einstein-Born-Infeld Gravity -** *Jin Young Kim and Mu-In Park* - arXiv:1608.00445v3 [hep-th] 10 Oct 2016)

$$\rho = -p_r = -4\beta^2\left(1-\frac{1}{\sqrt{1-E^2/\beta^2}}\right), \qquad E(r)\le\beta.$$

for $E = 1/2 = 0.5$ and $\beta = 1$

we obtain:

-4(1-1/(sqrt(1-0.5^2)))

**Input:**

$$-4\left(1-\frac{1}{\sqrt{1-0.5^2}}\right)$$

**Result:**

0.618802153517006116073190244015659645180814010161015008148...

0.618802153517..... = ρ

For n = −1, which corresponds to a precise value of the so called dimensionless Gruneisen parameter $\gamma_G$ , $r_0$ = 0.6160254; 0.618802153517..... = ρ , we obtain:

$$C_2 = r_0\left(\frac{8\pi\rho_* e^{-1/n}}{3} r_0^2 - 1\right).$$

0.6160254(1/3(8Pi*0.618802153517*e)*0.6160254^2-1)

**Input interpretation:**

$$0.6160254\left(\frac{1}{3}(8\pi\times 0.618802153517\, e)\times 0.6160254^2 - 1\right)$$

**Result:**

2.678256510505069458004556559359756735542508689609839706738...

2.6782565105…. = $C_2$

**Alternative representations:**

$$0.616025\left(\frac{1}{3}(8\pi\, 0.6188021535170000\, e)\, 0.616025^2 - 1\right) = 0.616025\left(-1 + 297.0250336881600\,°\, e\, 0.616025^2\right)$$

$$0.616025\left(\frac{1}{3}(8\pi\, 0.6188021535170000\, e)\, 0.616025^2 - 1\right) = 0.616025\left(-1 - 1.650139076045333\, e\, i \log(-1)\, 0.616025^2\right)$$

$$0.616025\left(\frac{1}{3}(8\pi\, 0.6188021535170000\, e)\, 0.616025^2 - 1\right) = 0.616025\left(\frac{1}{3}(8\pi\, 0.6188021535170000 \exp(z))\, 0.616025^2 - 1\right) \text{ for } z = 1$$

**Series representations:**

$$0.616025\left(\frac{1}{3}(8\pi\, 0.6188021535170000\, e)\, 0.616025^2 - 1\right) = 1.54304\left(-0.399229 + \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{(-1)^{k_1}}{k_2!\,(1+2k_1)}\right)$$

$$0.616025\left(\frac{1}{3}(8\pi\, 0.6188021535170000\, e)\, 0.616025^2 - 1\right) = 1.54304\left(-0.399229 + \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{(-1)^{k_1}(-1+k_2)^2}{k_2!\,(1+2k_1)}\right)$$

$$0.616025\left(\frac{1}{3}(8\pi\, 0.6188021535170000\, e)\, 0.616025^2 - 1\right) = 0.385759\left(-1.59692 + \sum_{k_1=0}^{\infty}\sum_{k_2=1}^{\infty}\frac{4^{-k_2}\left(-1+3^{k_2}\right)\zeta(1+k_2)}{k_1!}\right)$$

**Integral representations:**

$$0.616025\left(\frac{1}{3}(8\pi\, 0.6188021535170000\, e)\, 0.616025^2 - 1\right) = -0.616025 + 0.771519\, e\int_0^{\infty}\frac{1}{1+t^2}\,dt$$

$$0.616025\left(\frac{1}{3}(8\pi\, 0.6188021535170000\, e)\, 0.616025^2 - 1\right) = -0.616025 + 1.54304\, e\int_0^{1}\sqrt{1-t^2}\,dt$$

$$0.616025\left(\frac{1}{3}(8\pi\, 0.6188021535170000\, e)\, 0.616025^2 - 1\right) = -0.616025 + 0.771519\, e\int_0^{\infty}\frac{\sin(t)}{t}\,dt$$

From

$$\Delta = 216\pi\rho_* e^{-1/n}(4 - 72\pi\rho_* e^{-1/n}C_2^2)\,. \tag{15}$$

For 2.6782565105…. = $C_2$ , $r_0$ = 0.6160254; 0.618802153517..... = ρ ,

we obtain:

216*Pi*0.618802153517*e (4-72*Pi*0.618802153517*e*2.6782565105^2)

**Input interpretation:**

$$216\pi\times 0.618802153517\,e\,(4+72\pi e\times 2.6782565105^2\times(-0.618802153517))$$

**Result:**

$$-3.1106127118\ldots\times 10^6$$

-3.1106127118…*$10^6$

**Alternative representations:**

$$216\pi\,0.6188021535170000\,e\,(4-72\pi\,0.6188021535170000\,e\,2.67825651050000^2)=24059.02772874096\,°\,e\,(4-8019.675909580320\,°\,e\,2.67825651050000^2)$$

$$216\pi\,0.6188021535170000\,e\,(4-72\pi\,0.6188021535170000\,e\,2.67825651050000^2)=-133.6612651596720\,e\,i\log(-1)\,(4+44.55375505322400\,e\,i\log(-1)\,2.67825651050000^2)$$

$$216\pi\,0.6188021535170000\,e\,(4-72\pi\,0.6188021535170000\,e\,2.67825651050000^2)=216\pi\,0.6188021535170000\exp(z)\,(4-72\pi\,0.6188021535170000\exp(z)\,2.67825651050000^2)\text{ for } z=1$$

**Series representations:**

$$216\pi\,0.6188021535170000\,e\,(4-72\pi\,0.6188021535170000\,e\,2.67825651050000^2)=-683461.73025766\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{1+2k}\right)\left(-0.0031290416827706\sum_{k=0}^{\infty}\frac{1}{k!}+1.00000000000000\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{1+2k}\right)\left(\sum_{k=0}^{\infty}\frac{1}{k!}\right)^2\right)$$

$$216\,\pi\,0.6188021535170000\,e\,(4-72\,\pi\,0.6188021535170000\,e\,2.67825651050000^2) = -\frac{1}{\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\right)^2}683461.7302577\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{1+2k}\right)\left(1.0000000000000\sum_{k=0}^{\infty}\frac{(-1)^k}{1+2k}-0.0031290416827706\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\right)$$

$$216\,\pi\,0.6188021535170000\,e\,(4-72\,\pi\,0.6188021535170000\,e\,2.67825651050000^2) = -683461.73025766\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{1+2k}\right)\left(-0.0031290416827706\sum_{k=0}^{\infty}\frac{(-1+k)^2}{k!}+1.00000000000000\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{1+2k}\right)\left(\sum_{k=0}^{\infty}\frac{(-1+k)^2}{k!}\right)^2\right)$$

**Integral representations:**

$$216\,\pi\,0.6188021535170000\,e\,(4-72\,\pi\,0.6188021535170000\,e\,2.67825651050000^2) = -170865.432564415\,e\left(-0.0062580833655412\int_0^{\infty}\frac{1}{1+t^2}\,dt+1.00000000000000\,e\left(\int_0^{\infty}\frac{1}{1+t^2}\,dt\right)^2\right)$$

$$216\,\pi\,0.6188021535170000\,e\,(4-72\,\pi\,0.6188021535170000\,e\,2.67825651050000^2) = -170865.432564415\,e\left(-0.0062580833655412\int_0^{\infty}\frac{\sin(t)}{t}\,dt+1.00000000000000\,e\left(\int_0^{\infty}\frac{\sin(t)}{t}\,dt\right)^2\right)$$

$$216\,\pi\,0.6188021535170000\,e\,(4-72\,\pi\,0.6188021535170000\,e\,2.67825651050000^2) = -683461.73025766\,e\left(-0.0031290416827706\int_0^{1}\sqrt{1-t^2}\,dt+1.00000000000000\,e\left(\int_0^{1}\sqrt{1-t^2}\,dt\right)^2\right)$$

From which, we obtain:

(-(216*Pi*0.618802153517*e (4-72*Pi*0.618802153517*e*2.6782565105^2)))^1/31

**Input interpretation:**

$$\sqrt[31]{-(216\,\pi\times 0.618802153517\,e\,(4+72\,\pi\,e\times 2.6782565105^2\times(-0.618802153517)))}$$

**Result:**

1.61974498251...

1.61974498251…

(-(216*Pi*0.618802153517*e (4-72*Pi*0.618802153517*e*2.6782565105^2)))^1/2-34

**Input interpretation:**

$$\sqrt{-(216\,\pi\times 0.618802153517\,e\,(4+72\,\pi\,e\times 2.6782565105^2\times(-0.618802153517)))}-34$$

**Result:**

1729.6929188...

1729.6929188…

((((-(216*Pi*0.618802153517*e (4-72*Pi*0.618802153517*e*2.6782565105^2)))^1/2-34)))^1/15

**Input interpretation:**

$$\left(\sqrt{-(216\,\pi\times 0.618802153517\,e\,(4+72\,\pi\,e\times 2.6782565105^2\times(-0.618802153517)))}-34\right)\hat{}\,(1/15)$$

**Result:**

1.64385913920...

1.64385913920...

(-(216*Pi*0.618802153517*e (4-72*Pi*0.618802153517*e*2.6782565105^2)))^1/3-7+1/golden ratio

**Input interpretation:**

$$\sqrt[3]{-(216\,\pi\times 0.618802153517\,e\,(4+72\,\pi\,e\times 2.6782565105^2\times(-0.618802153517)))}-7+\frac{1}{\phi}$$

$\phi$ is the golden ratio

**Result:**

139.59420937...

139.59420937…

**Alternative representations:**

$$(-216 (\pi\, 0.6188021535170000\, e\, (4 - 72 \pi\, 0.6188021535170000\, e\, 2.67825651050000^2)))\verb|^|(1/3) - 7 + \frac{1}{\phi} = -7 + \frac{1}{2 \cos\left(\frac{\pi}{5}\right)} + \sqrt[3]{-24059.02772874096^\circ\, e\, (4 - 8019.675909580320^\circ\, e\, 2.67825651050000^2)}$$

$$(-216 (\pi\, 0.6188021535170000\, e\, (4 - 72 \pi\, 0.6188021535170000\, e\, 2.67825651050000^2)))\verb|^|(1/3) - 7 + \frac{1}{\phi} = -7 + \sqrt[3]{-24059.02772874096^\circ\, e\, (4 - 8019.675909580320^\circ\, e\, 2.67825651050000^2)} + \frac{1}{\text{root of } -1 - x + x^2 \text{ near } x = 1.61803}$$

$$(-216 (\pi\, 0.6188021535170000\, e\, (4 - 72 \pi\, 0.6188021535170000\, e\, 2.67825651050000^2)))\verb|^|(1/3) - 7 + \frac{1}{\phi} = -7 + \sqrt[3]{-133.6612651596720\, e\, \pi\, (4 - 44.55375505322400\, e\, \pi\, 2.67825651050000^2)} + \frac{1}{\text{root of } -1 - x + x^2 \text{ near } x = 1.61803}$$

**Series representations:**

$$(-216(\pi\, 0.6188021535170000\, e\, (4 - 72\pi\, 0.6188021535170000\, e\, 2.67825651050000^2)))\,\hat{}\,(1/3) - 7 + \frac{1}{\phi} = \frac{1}{\phi} 8.116245709598 \left(0.12320967548055 - 0.86246772836385\,\phi + 1.00000000000000\,\phi \sqrt[3]{\left(\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{(-1)^{k_1}}{k_2!\,(1+2k_1)}\right)\left(-4 + 1278.34666505887\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{(-1)^{k_1}}{k_2!\,(1+2k_1)}\right)}\right)$$

$$(-216(\pi\, 0.6188021535170000\, e\, (4 - 72\pi\, 0.6188021535170000\, e\, 2.67825651050000^2)))\,\hat{}\,(1/3) - 7 + \frac{1}{\phi} = \frac{1}{\phi} 8.116245709598 \left(0.12320967548055 - 0.86246772836385\,\phi + 1.00000000000000\,\phi \left(\left(\sum_{k_1=1}^{\infty}\sum_{k_2=0}^{\infty}\frac{\tan^{-1}\left(\frac{1}{F_{1+2k_1}}\right)}{k_2!}\right)\left(-4 + 1278.34666505887\sum_{k_1=1}^{\infty}\sum_{k_2=0}^{\infty}\frac{\tan^{-1}\left(\frac{1}{F_{1+2k_1}}\right)}{k_2!}\right)\right)\hat{}\,(1/3)\right)$$

$$(-216(\pi\, 0.6188021535170000\, e\, (4 - 72\pi\, 0.6188021535170000\, e\, 2.67825651050000^2)))\,\hat{}\,(1/3) - 7 + \frac{1}{\phi} = \frac{1}{\phi} 8.116245709598 \left(0.12320967548055 - 0.86246772836385\,\phi + 1.00000000000000\,\phi \left(\left(\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{(-1)^{k_1}(-1+k_2)^2}{k_2!\,(1+2k_1)}\right)\left(-4 + 1278.34666505887\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{(-1)^{k_1}(-1+k_2)^2}{k_2!\,(1+2k_1)}\right)\right)\hat{}\,(1/3)\right)$$

**Integral representations:**

$$(-216 (\pi\, 0.6188021535170000\, e\, (4 - 72\, \pi\, 0.6188021535170000\, e\, 2.67825651050000^2)))\ \hat{}\ (1/3) - 7 + \frac{1}{\phi} = \frac{1}{\phi} 6.441868488724 \left(0.15523446368866 - 1.0866412458206\, \phi + 1.00000000000000\, \phi \sqrt[3]{e \left(\int_0^\infty \frac{1}{1+t^2}\, dt\right)\left(-4 + 639.17333252943\, e \int_0^\infty \frac{1}{1+t^2}\, dt\right)}\right)$$

$$(-216 (\pi\, 0.6188021535170000\, e\, (4 - 72\, \pi\, 0.6188021535170000\, e\, 2.67825651050000^2)))\ \hat{}\ (1/3) - 7 + \frac{1}{\phi} = \frac{1}{\phi} 6.441868488724 \left(0.15523446368866 - 1.0866412458206\, \phi + 1.00000000000000\, \phi \sqrt[3]{e \left(\int_0^\infty \frac{\sin(t)}{t}\, dt\right)\left(-4 + 639.17333252943\, e \int_0^\infty \frac{\sin(t)}{t}\, dt\right)}\right)$$

$$(-216 (\pi\, 0.6188021535170000\, e\, (4 - 72\, \pi\, 0.6188021535170000\, e\, 2.67825651050000^2)))\ \hat{}\ (1/3) - 7 + \frac{1}{\phi} = \frac{1}{\phi} 8.116245709598 \left(0.12320967548055 - 0.86246772836385\, \phi + 1.00000000000000\, \phi \sqrt[3]{e \left(\int_0^1 \sqrt{1-t^2}\, dt\right)\left(-4 + 1278.34666505887\, e \int_0^1 \sqrt{1-t^2}\, dt\right)}\right)$$

(-(216*Pi*0.618802153517*e (4-72*Pi*0.618802153517*e*2.6782565105^2)))^1/3-21+1/golden ratio

**Input interpretation:**

$$\sqrt[3]{-(216\, \pi \times 0.618802153517\, e\, (4 + 72\, \pi\, e \times 2.6782565105^2 \times (-0.618802153517)))} - 21 + \frac{1}{\phi}$$

$\phi$ is the golden ratio

**Result:**

125.59420937…

125.59420937…

**Alternative representations:**

$$(-216 (\pi\, 0.6188021535170000\, e\, (4 - 72 \pi\, 0.6188021535170000\, e\, 2.67825651050000^2)))\,\hat{}\,(1/3) - 21 + \frac{1}{\phi} = -21 + \frac{1}{2 \cos\left(\frac{\pi}{5}\right)} + \sqrt[3]{-24059.02772874096\,^\circ\, e\, (4 - 8019.675909580320\,^\circ\, e\, 2.67825651050000^2)}$$

$$(-216 (\pi\, 0.6188021535170000\, e\, (4 - 72 \pi\, 0.6188021535170000\, e\, 2.67825651050000^2)))\,\hat{}\,(1/3) - 21 + \frac{1}{\phi} = -21 + \sqrt[3]{-24059.02772874096\,^\circ\, e\, (4 - 8019.675909580320\,^\circ\, e\, 2.67825651050000^2)} + \frac{1}{\text{root of } -1 - x + x^2 \text{ near } x = 1.61803}$$

$$(-216 (\pi\, 0.6188021535170000\, e\, (4 - 72 \pi\, 0.6188021535170000\, e\, 2.67825651050000^2)))\,\hat{}\,(1/3) - 21 + \frac{1}{\phi} = -21 + \sqrt[3]{-133.6612651596720\, e\, \pi\, (4 - 44.55375505322400\, e\, \pi\, 2.67825651050000^2)} + \frac{1}{\text{root of } -1 - x + x^2 \text{ near } x = 1.61803}$$

**Series representations:**

$$(-216 (\pi\, 0.6188021535170000\, e\, (4 - 72 \pi\, 0.6188021535170000\, e\, 2.67825651050000^2)))\,\hat{}\,(1/3) - 21 + \frac{1}{\phi} = \frac{1}{\phi} 8.116245709598 \left(0.12320967548055 - 2.5874031850915\, \phi + 1.00000000000000\, \phi \sqrt[3]{\left(\sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{(-1)^{k_1}}{k_2!\,(1 + 2 k_1)}\right)\left(-4 + 1278.34666505887 \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{(-1)^{k_1}}{k_2!\,(1 + 2 k_1)}\right)}\right)$$

$$(-216 (\pi\, 0.6188021535170000\, e\, (4 - 72\, \pi\, 0.6188021535170000\, e\, 2.67825651050000^2)))\wedge(1/3) - 21 + \frac{1}{\phi} = \frac{1}{\phi} 8.116245709598 \left(0.12320967548055 - 2.5874031850915\, \phi + 1.00000000000000\, \phi \left(\left(\sum_{k_1=1}^{\infty} \sum_{k_2=0}^{\infty} \frac{\tan^{-1}\left(\frac{1}{F_{1+2 k_1}}\right)}{k_2!}\right) \left(-4 + 1278.34666505887 \sum_{k_1=1}^{\infty} \sum_{k_2=0}^{\infty} \frac{\tan^{-1}\left(\frac{1}{F_{1+2 k_1}}\right)}{k_2!}\right)\right)\wedge(1/3)\right)$$

$$(-216 (\pi\, 0.6188021535170000\, e\, (4 - 72\, \pi\, 0.6188021535170000\, e\, 2.67825651050000^2)))\wedge(1/3) - 21 + \frac{1}{\phi} = \frac{1}{\phi} 8.116245709598 \left(0.12320967548055 - 2.5874031850915\, \phi + 1.00000000000000\, \phi \left(\left(\sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{(-1)^{k_1} (-1 + k_2)^2}{k_2!\,(1 + 2 k_1)}\right) \left(-4 + 1278.34666505887 \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{(-1)^{k_1} (-1 + k_2)^2}{k_2!\,(1 + 2 k_1)}\right)\right)\wedge(1/3)\right)$$

## Integral representations:

$$(-216 (\pi\, 0.6188021535170000\, e\, (4 - 72\, \pi\, 0.6188021535170000\, e\, 2.67825651050000^2)))\wedge(1/3) - 21 + \frac{1}{\phi} = \frac{1}{\phi} 6.441868488724 \left(0.15523446368866 - 3.2599237374619\, \phi + 1.00000000000000\, \phi \sqrt[3]{e \left(\int_0^{\infty} \frac{1}{1+t^2}\, dt\right) \left(-4 + 639.17333252943\, e \int_0^{\infty} \frac{1}{1+t^2}\, dt\right)}\right)$$

$$(-216(\pi\, 0.6188021535170000\, e\, (4 - 72\pi\, 0.6188021535170000\, e\, 2.67825651050000^2)))\wedge(1/3) -$$

$$21 + \frac{1}{\phi} = \frac{1}{\phi} 6.441868488724 \Bigg(0.15523446368866 - 3.2599237374619\,\phi + 1.00000000000000\,\phi \sqrt[3]{e\left(\int_0^\infty \frac{\sin(t)}{t}\,dt\right)\left(-4 + 639.17333252943\, e \int_0^\infty \frac{\sin(t)}{t}\,dt\right)}\Bigg)$$

$$(-216(\pi\, 0.6188021535170000\, e\, (4 - 72\pi\, 0.6188021535170000\, e\, 2.67825651050000^2)))\wedge(1/3) -$$

$$21 + \frac{1}{\phi} = \frac{1}{\phi} 8.116245709598 \Bigg(0.12320967548055 - 2.5874031850915\,\phi + 1.00000000000000\,\phi \sqrt[3]{e\left(\int_0^1 \sqrt{1-t^2}\,dt\right)\left(-4 + 1278.34666505887\, e \int_0^1 \sqrt{1-t^2}\,dt\right)}\Bigg)$$

Now, we have that:

$$S_{1,2} = \left[\frac{e^{1/n}}{8\pi\rho_*}\left(\frac{3\mathcal{C}_2}{2} \pm \sqrt{\frac{9\mathcal{C}_2^2}{4} - \frac{e^{1/n}}{8\pi\rho_*}}\right)\right]^{1/3}.$$

For 2.6782565105…. = $C_2$ , $r_0$ = 0.6160254; 0.618802153517..... = ρ , we obtain:

[(1/e)/(8*Pi*0.61880215)*(((3/2*2.6782565+(((1/4*9*2.6782565^2-(1/e)/(8Pi*0.61880215))))^0.5)))]^(1/3)

**Input interpretation:**

$$\sqrt[3]{\frac{\frac{1}{e}}{8\pi\times 0.61880215}\left(\frac{3}{2}\times 2.6782565 + \sqrt{\frac{1}{4}\times 9\times 2.6782565^2 - \frac{\frac{1}{e}}{8\pi\times 0.61880215}}\right)}$$

**Result:**

0.57487843...

0.57487843…

Now, from:

$$C_2 = r_0 \left( \frac{8\pi\rho_* e^{-1/n}}{3} r_0^2 - 1 \right).$$

for $r_0 = r = 1.94973*10^{13}$ (SMBH 87); $\rho_* = 5.1 \times 10^{96}$ kg/m$^3$

we obtain:

(1.94973e+13)*((1/3*8*Pi*5.1e+96*e*(1.94973e+13)^2-1)

**Input interpretation:**

$1.94973 \times 10^{13} \left(\frac{1}{3} \times 8\,\pi \times 5.1 \times 10^{96}\, e\,(1.94973 \times 10^{13})^2 - 1\right)$

**Result:**

$8.60809... \times 10^{137}$

$8.60809...*10^{137} = C_2$

From:

$$S_{1,2} = \left[ \frac{e^{1/n}}{8\pi\rho_*} \left( \frac{3C_2}{2} \pm \sqrt{\frac{9C_2^2}{4} - \frac{e^{1/n}}{8\pi\rho_*}} \right) \right]^{1/3}.$$

we obtain:

[(1/e)/(8*Pi*5.1e+96)*(((3/2*8.60809e+137+(((1/4*9*(8.60809e+137)^2-(1/e)/(8Pi*5.1e+96))))^0.5)))]^(1/3)

**Input interpretation:**

$$\sqrt[3]{\frac{\frac{1}{e}}{8\pi\times 5.1\times 10^{96}}\left(\frac{3}{2}(8.60809\times 10^{137})+\sqrt{\frac{1}{4}\times 9(8.60809\times 10^{137})^2-\frac{\frac{1}{e}}{8\pi\times 5.1\times 10^{96}}}\right)}$$

**Result:**

$1.94973... \times 10^{13}$

$1.94973…*10^{13}$

From which:

55ln((([(1/e)/(8*Pi*5.1e+96)*(((3/2*8.60809e+137+(((1/4*9*(8.60809e+137)^2-(1/e)/(8Pi*5.1e+96))))^0.5)))]^(1/3))))+47-1/golden ratio

**Input interpretation:**

$$55\log\left(\left(\frac{\frac{1}{e}}{8\pi\times 5.1\times 10^{96}}\left(\frac{3}{2}(8.60809\times 10^{137})+\sqrt{\frac{1}{4}\times 9(8.60809\times 10^{137})^2-\frac{\frac{1}{e}}{8\pi\times 5.1\times 10^{96}}}\right)\right)\hat{}\,(1/3)\right)+47-\frac{1}{\phi}$$

$\log(x)$ is the natural logarithm

$\phi$ is the golden ratio

**Result:**

1729.4533...

1729.4533…

((55ln((([(1/e)/(8*Pi*5.1e+96)*(((3/2*8.60809e+137+(((1/4*9*(8.60809e+137)^2-(1/e)/(8Pi*5.1e+96))))^0.5)))]^(1/3))))+47-1/golden ratio))^1/15

**Input interpretation:**

$$\left(55 \log\left(\left(\frac{\frac{1}{e}}{8\pi\times 5.1\times 10^{96}}\left(\frac{3}{2}(8.60809\times 10^{137}) + \sqrt{\frac{1}{4}\times 9\,(8.60809\times 10^{137})^2 - \frac{\frac{1}{e}}{8\pi\times 5.1\times 10^{96}}}\right)\right)^{\wedge}(1/3)\right) + 47 - \frac{1}{\phi}\right)^{\wedge}(1/15)$$

log(x) is the natural logarithm

$\phi$ is the golden ratio

**Result:**

1.64384396...

1.64384396…

From

**On a New Approach for Constructing Wormholes in Einstein-Born-Infeld Gravity -** *Jin Young Kim and Mu-In Park* - arXiv:1608.00445v3 [hep-th] 10 Oct 2016

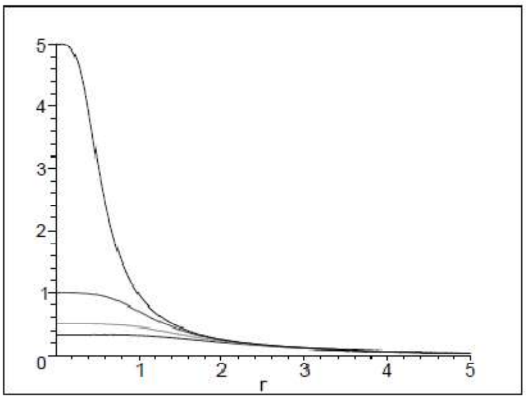



FIG. 1: The plots of $E(r)$ for varying $\beta$ with a fixed $Q$. In particular, we consider $\beta = 5, 1, 1/2, 1/3$, (top to bottom) with $Q = 1$.

With regard the small positive value of the cosmological constant or, equivalently, the energy density of the vacuum, the value is $\Lambda = 1.1056 * 10^{-52}$ $m^{-2}$

For $r_0 = r = 1.94973*10^{13}$ m (SMBH 87); $\Lambda = 1.1056 * 10^{-52}$ $m^{-2}$ ;

$h = 1.054571e\text{-}34$ kg $m^2$ $s^{-1}$ and $\beta = 1$

we obtain:

$$T_H \equiv \frac{\hbar}{4\pi} \frac{df}{dr}\bigg|_{r=r_+}$$

$$= \frac{\hbar}{4\pi} \left[ \frac{1}{r_+} - \Lambda r_+ + 2\beta^2 r_+ \left( 1 - \sqrt{1 + \frac{Q^2}{\beta r_+^4}} \right) \right].$$

1.054571e-34 / (4Pi) [1/(1.94973e+13)-(1.1056e-52*1.94973e+13)+2*1.94973e+13(1-sqrt(1+1/(1.94973e+13)^4)]

**Input interpretation:**

$$\frac{1.054571 \times 10^{-34}}{4\pi} \left( \frac{1}{1.94973 \times 10^{13}} - 1.1056 \times 10^{-52} \times 1.94973 \times 10^{13} + 2 \times 1.94973 \times 10^{13} \left( 1 - \sqrt{1 + \frac{1}{(1.94973 \times 10^{13})^4}} \right) \right)$$

**Result:**

$4.30419... \times 10^{-49}$

$4.30419 * 10^{-49}$

We have that:

4.3042e-49 kg = eV

**Input interpretation:**

convert $4.3042 \times 10^{-49}$ kg (kilograms) to electronvolts per speed of light squared

**Result:**

$2.4145 \times 10^{-13}$ eV/$c^2$

$2.4145*10^{-13}$

From which:

((2.41447×10^-13)) eV = K

**Input interpretation:**

convert $2.41447 \times 10^{-13}$ eV/$k_B$ (electronvolts per Boltzmann constant) to kelvins

**Result:**

$2.8019 \times 10^{-9}$ K (kelvins)

$2.8019*10^{-9}$ K = 4.37902e+31 Kg

**Additional conversions:**

−273.1499999971981 °C (degrees Celsius)

−459.6699999949566 °F (degrees Fahrenheit)

$5.0434 \times 10^{-9}$ °R (degrees Rankine)

−218.5199999977585 °Ré (degrees Réaumur)

−135.903749998529 °Rø (degrees Rømer)

**Comparisons as temperature:**

$7 \times 10^{-8}$ K below

temperature of a typical evaporation-cooled Bose-Einstein condensate ($7 \times 10^{-8}$ K)

$2.3519 \times 10^{-9}$ K above

lowest temperature sodium Bose-Einstein condensate gas ever achieved in the laboratory (at MIT) (450 pK)

$2.8019 \times 10^{-9}$ K above absolute zero (0 K)

**Interpretation:**

temperature

**Basic unit dimensions:**

[temperature]

**Corresponding quantity:**

Thermodynamic energy $E$ from $E = kT$:

$2.4 \times 10^{-13}$ eV (electronvolts)

SMBH 87 mass = $13.12806*10^{39}$

Wormhole mass = $4.37902*10^{31}$

From the ratio of the two masses, we obtain:

13.12806*10^39 / 4.37902*10^31

**Input interpretation:**

$$\frac{13.12806 \times 10^{39}}{4.37902 \times 10^{31}}$$

**Result:**

$2.997944745628017227598869153372215701229955560833324579... \times 10^8$

$2.9979447456...*10^8$ result practically equal to the Light speed 299792458 m/s

Inserting the above mass value, we have:

Mass = 4.37902e+31

Radius = 65035.7

Temperature = 2.80190e-9

From the Ramanujan-Nardelli mock formula, we obtain:

sqrt[[[[1/(((((((4*1.962364415e+19)/(5*0.0864055^2)))*1/(4.37902e+31)* sqrt[[-((((2.80190e-9 * 4*Pi*(65035.7)^3-(65035.7)^2))))) / ((6.67*10^-11))]]]]]

**Input interpretation:**

$$\sqrt{\frac{1}{\frac{4\times1.962364415\times10^{19}}{5\times0.0864055^2}\times\frac{1}{4.37902\times10^{31}}\sqrt{-\frac{2.80190\times10^{-9}\times4\pi\times65035.7^3-65035.7^2}{6.67\times10^{-11}}}}}$$

**Result:**

1.618078115508927166194986195275006000793607119920156681839...

1.6180781155089…..

Now, we have that:

$$r_+^* = \sqrt{\frac{\Lambda - 2\beta^2 \pm 2\sqrt{(\beta^2 - \Lambda/2)^2 + \Lambda(\Lambda - 4\beta^2)(\beta^2 Q^2 - 1/4)}}{\Lambda(\Lambda - 4\beta^2)}} \quad (22)$$

For $\Lambda = 1.1056 * 10^{-52}\ m^{-2}$ ; $\beta = Q = 1$ , we obtain:

sqrt[(((((1.1056e-52-2+2*(((1-(1.1056e-52)/2)^2+1.1056e-52(1.1056e-52-4)(1-1/4)))^0.5))))) / (((1.1056e-52(1.1056e-52-4))))]

**Input interpretation:**

$$\sqrt{\left(\left(1.1056\times 10^{-52} - 2 + 2\sqrt{\left(1 - \frac{1.1056\times 10^{-52}}{2}\right)^2 + 1.1056\times 10^{-52}\left(1.1056\times 10^{-52} - 4\right)\left(1 - \frac{1}{4}\right)}\right)\Big/ \left(1.1056\times 10^{-52}\left(1.1056\times 10^{-52} - 4\right)\right)\right)}$$

**Result:**

0.866025403784438646763723170752936183471402626905190373870...

0.8660254037844…. result equal to the previous:

$$r_+^* = \sqrt{Q^2 - \frac{1}{4\beta^2}}$$

r = sqrt(1-1/4)

$$\sqrt{1 - \frac{1}{4}}$$

$\frac{\sqrt{3}}{2}$

0.8660254037844386467637231707529361834714026269051903140 27...

0.8660254037844…

We note also that:

2sqrt[((((((1.1056e-52-2+2*(((1-(1.1056e-52)/2)^2+1.1056e-52(1.1056e-52-4)(1-1/4)))^0.5))))) / (((1.1056e-52(1.1056e-52-4))))]

**Input interpretation:**

$$2\sqrt{\left(\left(1.1056\times10^{-52}-2+2\sqrt{\left(1-\frac{1.1056\times10^{-52}}{2}\right)^2+1.1056\times10^{-52}\left(1.1056\times10^{-52}-4\right)\left(1-\frac{1}{4}\right)}\right)\Big/\left(1.1056\times10^{-52}\left(1.1056\times10^{-52}-4\right)\right)\right)}$$

**Result:**

1.732050807568877293527446341505872366942805253810380747740...

1.7320508075688….

**Input interpretation:**

1.732050807568877293527446341505872366942805253810 3807

**Rational approximation:**

$$\frac{101989905849209046397921927}{58883899596665430243946076}=1+\frac{43106006252543616153975851}{58883899596665430243946076}$$

**Possible closed forms:**

$\sqrt{3} \approx 1.7320508075688772935274463415058723669428052538103806280 5$

$\frac{9\,\mathcal{L}_{\text{si}}}{8} \approx 1.73205080756887729352744634150587236694280525381038062805$

$\frac{2\,\mathcal{T}_{\text{20VE}}}{3} \approx 1.73205080756887729352744634150587236694280525381038062805$

$\mathcal{T}_{\text{T}} \approx 1.73205080756887729352744634150587236694280525381038062805$

$\frac{8}{3\,\mathcal{L}_{\text{si}}} \approx 1.73205080756887729352744634150587236694280525381038062805$

$-\frac{2\,(249 + 73\,\pi)}{63 - 221\,\pi + 8\,\pi^2} \approx 1.73205080756887729 40377$

$\frac{4\,825\,528\,341\,\pi}{8\,752\,540\,237} \approx 1.7320508075688772935293912$

$\pi$ [root of $335\,x^5 + 1819\,x^4 + 483\,x^3 - 573\,x^2 - 76\,x - 50$ near $x = 0.551329$] $\approx$ 1.7320508075688772935 38289

$\pi$ [root of $592\,x^4 - 1315\,x^3 + 6523\,x^2 + 1333\,x - 2552$ near $x = 0.551329$] $\approx$ 1.7320508075688772935 19578

$\frac{13 + 233\,e + 373\,e^2}{-979 + 425\,e + 242\,e^2} \approx 1.7320508075688772932 5237$

$\frac{9\,863\,382\,151}{5\,694\,626\,340} \approx 1.7320508075688772935 36348$

$\mathcal{L}_{\text{si}}$ is Lieb's square ice constant

$\mathcal{T}_{\text{20VE}}$ is the twenty-vertex entropy constant

$\mathcal{T}_{\text{T}}$ is Theodorus's constant

We obtain also:

1/6((e/ (((sqrt[((((((1.1056e-52-2+2*(((1-(1.1056e-52)/2)^2+1.1056e-52(1.1056e-52-4)(1-1/4)))^0.5))))) / (((1.1056e-52(1.1056e-52-4))))]))))))^2

**Input interpretation:**

$$\frac{1}{6}\left(\frac{e}{\sqrt{\frac{1.1056\times 10^{-52}-2+2\sqrt{\left(1-\frac{1.1056\times 10^{-52}}{2}\right)^2+1.1056\times 10^{-52}\left(1.1056\times 10^{-52}-4\right)\left(1-\frac{1}{4}\right)}}{1.1056\times 10^{-52}\left(1.1056\times 10^{-52}-4\right)}}}\right)^2$$

**Result:**

$\tilde{\infty}$

$\tilde{\infty}$ is complex infinity

**Decimal approximation:**

1.642012466429033383828983880127779514040070126789299178426...

1.642012466429…

1/6((e/ (((sqrt[((((((1.1056e-52-2+2*(((1-(1.1056e-52)/2)^2+1.1056e-52(1.1056e-52-4)(1-1/4)))^0.5))))) / (((1.1056e-52(1.1056e-52-4))))]))))))^2-24/10^3

**Input interpretation:**

$$\frac{1}{6}\left(\frac{e}{\sqrt{\frac{1.1056\times 10^{-52}-2+2\sqrt{\left(1-\frac{1.1056\times 10^{-52}}{2}\right)^2+1.1056\times 10^{-52}\left(1.1056\times 10^{-52}-4\right)\left(1-\frac{1}{4}\right)}}{1.1056\times 10^{-52}\left(1.1056\times 10^{-52}-4\right)}}}\right)^2-\frac{24}{10^3}$$

**Result:**

$\tilde{\infty}$

$\tilde{\infty}$ is complex infinity

**Decimal approximation:**

1.618012466429033383828983880127779514040070126789299178426...

1.618012466429….

Now, we have:

$$E(r) = \frac{Q}{\sqrt{r^4 + \frac{Q^2}{\beta^2}}},$$

For $r_0 = r = 1.94973*10^{13}$ m, Q = 1 and β = 5, we obtain:

1/sqrt((1.94973e+13)^4+1/25)

**Input interpretation:**

$$\frac{1}{\sqrt{(1.94973 \times 10^{13})^4 + \frac{1}{25}}}$$

**Result:**

$2.63058... \times 10^{-27}$

$E(r) = 2.63058...*10^{-27}$

We have that the wormhole mass is equal to $4.37902*10^{31}$ . Performing the following calculations we have:

(((colog(((1/sqrt((1.94973e+13)^4+1/25) *1 / (4.37902e+31)))))))^1/10+11/10^3

**Input interpretation:**

$$\sqrt[10]{-\log\left(\frac{1}{\sqrt{(1.94973 \times 10^{13})^4 + \frac{1}{25}}} \times \frac{1}{4.37902 \times 10^{31}}\right)} + \frac{11}{10^3}$$

log(x) is the natural logarithm

**Result:**

1.6430362...

1.6430362...

(((colog((((1/sqrt((1.94973e+13)^4+1/25) *1 / (4.37902e+31)))))))^1/10 - (11+3)1/10^3

**Input interpretation:**

$$\sqrt[10]{-\log\left(\frac{1}{\sqrt{(1.94973\times 10^{13})^4+\frac{1}{25}}}\times\frac{1}{4.37902\times 10^{31}}\right)}-(11+3)\times\frac{1}{10^3}$$

log(x) is the natural logarithm

**Result:**

1.6180361641256422796511213925160292536874356057033835361 65...

1.61803616412…

Now, we have that:

$$M_0 = \frac{2}{3}\sqrt{\frac{\beta Q^3}{\pi}}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right),$$

$$f(r) = 1 - \frac{\Lambda}{3}r^2 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{Q^4}{20\beta^2 r^6} + \mathcal{O}(r^{-10}). \qquad (18)\text{-}(19)$$

For: $r_0 = r = 1.94973*10^{13}$ m, Q = 1 and β = 5, $M = 4.37902*10^{31}$ ,

$\Lambda = 1.1056 * 10^{-52}\ m^{-2}$ , we obtain:

2/3 sqrt(5/Pi) (gamma (1/4) gamma (5/4))

**Input:**

$$\frac{2}{3}\sqrt{\frac{5}{\pi}}\left(\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)\right)$$

Γ(x) is the gamma function

**Exact result:**

$$\frac{2}{3}\sqrt{\frac{5}{\pi}}\,\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)$$

**Decimal approximation:**

2.7638913425379026300435853765558687041396187306211517684 73...

2.7638913425379….. = $M_0$

**Alternate forms:**

$$-\frac{2\,\Gamma\left(-\frac{3}{4}\right)\Gamma\left(\frac{9}{4}\right)}{\sqrt{5\pi}}$$

$$\frac{32\times\frac{1}{4}!\times\frac{5}{4}!}{3\sqrt{5\pi}}$$

$$\frac{1}{6}\sqrt{\frac{5}{\pi}}\,\Gamma\left(\frac{1}{4}\right)^2$$

$n!$ is the factorial function

**Alternative representations:**

$$\frac{1}{3}\left(\sqrt{\frac{5}{\pi}}\left(\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)\right)\right)2 = \frac{2\,G\left(1+\frac{1}{4}\right)G\left(1+\frac{5}{4}\right)\sqrt{\frac{5}{\pi}}}{3\,G\left(\frac{1}{4}\right)G\left(\frac{5}{4}\right)}$$

$$\frac{1}{3}\left(\sqrt{\frac{5}{\pi}}\left(\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)\right)\right)2 = \frac{2}{3}\,e^{-\log G(1/4)+\log G(1+1/4)}\,e^{-\log G(5/4)+\log G(1+5/4)}\sqrt{\frac{5}{\pi}}$$

$$\frac{1}{3}\left(\sqrt{\frac{5}{\pi}}\left(\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)\right)\right)2 = \frac{2}{3}\left(-1+\frac{1}{4}\right)!\left(-1+\frac{5}{4}\right)!\sqrt{\frac{5}{\pi}}$$

**Series representations:**

$$\frac{1}{3}\left(\sqrt{\frac{5}{\pi}}\left(\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)\right)\right)2 = \frac{2\sqrt{\frac{5}{\pi}}}{3\left(\sum_{k=1}^{\infty}\left(\frac{5}{4}\right)^k c_k\right)\sum_{k=1}^{\infty}4^{-k}\,c_k}$$

for $\left(c_1 = 1 \text{ and } c_2 = 1 \text{ and } c_k = \frac{\gamma\,c_{-1+k}+\sum_{j=1}^{-2+k}(-1)^{1+j+k}\,c_j\,\zeta(-j+k)}{-1+k}\right)$

$$\frac{1}{3}\left(\sqrt{\frac{5}{\pi}}\left(\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)\right)\right)2 = \frac{2}{3}\sqrt{\frac{5}{\pi}}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{\left(\frac{1}{4}-z_0\right)^{k_1}\left(\frac{5}{4}-z_0\right)^{k_2}\Gamma^{(k_1)}(z_0)\,\Gamma^{(k_2)}(z_0)}{k_1!\,k_2!}$$

for $(z_0 \notin \mathbb{Z}$ or $z_0 > 0)$

$$\frac{1}{3}\left(\sqrt{\frac{5}{\pi}}\left(\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)\right)\right)2 = \left(2\sqrt{5}\,\pi^{3/2}\right)\Big/\left(3\left(\sum_{k=0}^{\infty}\left(\frac{1}{4}-z_0\right)^k\sum_{j=0}^{k}\frac{(-1)^j\,\pi^{-j+k}\sin\left(\frac{1}{2}(-j+k)\pi+\pi z_0\right)\Gamma^{(j)}(1-z_0)}{j!\,(-j+k)!}\right)\sum_{k=0}^{\infty}\left(\frac{5}{4}-z_0\right)^k\sum_{j=0}^{k}\frac{(-1)^j\,\pi^{-j+k}\sin\left(\frac{1}{2}(-j+k)\pi+\pi z_0\right)\Gamma^{(j)}(1-z_0)}{j!\,(-j+k)!}\right)$$

**Integral representations:**

$$\frac{1}{3}\left(\sqrt{\frac{5}{\pi}}\left(\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)\right)\right)2 = \frac{2}{3}\exp\left(-\frac{3\gamma}{2}+\int_0^1\frac{-2+\sqrt[4]{x}+x^{5/4}-\frac{\log(x)}{4}-\log(x^{5/4})}{(-1+x)\log(x)}\,dx\right)\sqrt{\frac{5}{\pi}}$$

$$\frac{1}{3}\left(\sqrt{\frac{5}{\pi}}\left(\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)\right)\right)2 = \frac{2}{3}\exp\left(\int_0^1\frac{\left(-1+\sqrt[4]{x}\right)\left(-1+\sqrt{x}+2x^{3/4}\right)}{2\left(1+\sqrt[4]{x}+\sqrt{x}+x^{3/4}\right)\log(x)}\,dx\right)\sqrt{\frac{5}{\pi}}$$

$$\frac{1}{3}\left(\sqrt{\frac{5}{\pi}}\left(\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)\right)\right)2 = \frac{2}{3}\sqrt{\frac{5}{\pi}}\left(\int_0^\infty\frac{e^{-t}}{t^{3/4}}\,dt\right)\int_0^\infty e^{-t}\sqrt[4]{t}\,dt$$

For β = 1, from

$$M_0 = \frac{2}{3}\sqrt{\frac{\beta Q^3}{\pi}}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right),$$

we obtain:

**Input:**

$$\frac{2}{3}\sqrt{\frac{1}{\pi}}\left(\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)\right)$$

$\Gamma(x)$ is the gamma function

**Exact result:**

$$\frac{2\,\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)}{3\sqrt{\pi}}$$

**Decimal approximation:**

1.2360497848675812789559002314635066974783992156811779370 57...

1.2360497848675….= $M_0$

**Alternate forms:**

$$\frac{\Gamma\left(\frac{1}{4}\right)^2}{6\sqrt{\pi}}$$

$$\frac{32\times\frac{1}{4}!\times\frac{5}{4}!}{15\sqrt{\pi}}$$

$$\frac{4\left(2+\sqrt{2}\right)K\left(\frac{\left(-2-2\sqrt{2}\right)^2}{\left(4+2\sqrt{2}\right)^2}\right)}{3\left(4+2\sqrt{2}\right)}$$

$n!$ is the factorial function

$K(m)$ is the complete elliptic integral of the first kind with parameter $m = k^2$

**Alternative representations:**

$$\frac{1}{3}\left(\sqrt{\frac{1}{\pi}}\left(\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)\right)\right)2 = \frac{2\,G\left(1+\frac{1}{4}\right)G\left(1+\frac{5}{4}\right)\sqrt{\frac{1}{\pi}}}{3\,G\left(\frac{1}{4}\right)G\left(\frac{5}{4}\right)}$$

$$\frac{1}{3}\left(\sqrt{\frac{1}{\pi}}\left(\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)\right)\right)2 = \frac{2}{3}\,e^{-\log G(1/4)+\log G(1+1/4)}\,e^{-\log G(5/4)+\log G(1+5/4)}\sqrt{\frac{1}{\pi}}$$

$$\frac{1}{3}\left(\sqrt{\frac{1}{\pi}}\left(\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)\right)\right)2 = \frac{2}{3}\left(-1+\frac{1}{4}\right)!\left(-1+\frac{5}{4}\right)!\sqrt{\frac{1}{\pi}}$$

**Series representations:**

$$\frac{1}{3}\left(\sqrt{\frac{1}{\pi}}\left(\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)\right)\right)2 = \frac{2}{3\sqrt{\pi}\left(\sum_{k=1}^{\infty}\left(\frac{5}{4}\right)^k c_k\right)\sum_{k=1}^{\infty}4^{-k}c_k}$$

$$\text{for}\left(c_1 = 1 \text{ and } c_2 = 1 \text{ and } c_k = \frac{\gamma\,c_{-1+k} + \sum_{j=1}^{-2+k}(-1)^{1+j+k}c_j\,\zeta(-j+k)}{-1+k}\right)$$

$$\frac{1}{3}\left(\sqrt{\frac{1}{\pi}}\left(\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)\right)\right)2 = \frac{2\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{\left(\frac{1}{4}-z_0\right)^{k_1}\left(\frac{5}{4}-z_0\right)^{k_2}\Gamma^{(k_1)}(z_0)\Gamma^{(k_2)}(z_0)}{k_1!k_2!}}{3\sqrt{\pi}}$$

for ($z_0 \notin \mathbb{Z}$ or $z_0 > 0$)

$$\frac{1}{3}\left(\sqrt{\frac{1}{\pi}}\left(\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)\right)\right)2 =$$

$$\left(2\pi^{3/2}\right)\Bigg/\left(3\left(\sum_{k=0}^{\infty}\left(\frac{1}{4}-z_0\right)^k\sum_{j=0}^{k}\frac{(-1)^j\pi^{-j+k}\sin\left(\frac{1}{2}(-j+k)\pi+\pi z_0\right)\Gamma^{(j)}(1-z_0)}{j!(-j+k)!}\right)\right.$$
$$\left.\sum_{k=0}^{\infty}\left(\frac{5}{4}-z_0\right)^k\sum_{j=0}^{k}\frac{(-1)^j\pi^{-j+k}\sin\left(\frac{1}{2}(-j+k)\pi+\pi z_0\right)\Gamma^{(j)}(1-z_0)}{j!(-j+k)!}\right)$$

$\zeta(s)$ is the Riemann zeta function

$\gamma$ is the Euler-Mascheroni constant

**Integral representations:**

$$\frac{1}{3}\left(\sqrt{\frac{1}{\pi}}\left(\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)\right)\right)2 = \frac{2\exp\left(-\frac{3\gamma}{2}+\int_0^1\frac{-2+\sqrt[4]{x}+x^{5/4}-\frac{\log(x)}{4}-\log\left(x^{5/4}\right)}{(-1+x)\log(x)}dx\right)}{3\sqrt{\pi}}$$

$$\frac{1}{3}\left(\sqrt{\frac{1}{\pi}}\left(\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)\right)\right)2 = \frac{2\exp\left(\int_0^1\frac{\left(-1+\sqrt[4]{x}\right)\left(-1+\sqrt{x}+2x^{3/4}\right)}{2\left(1+\sqrt[4]{x}+\sqrt{x}+x^{3/4}\right)\log(x)}dx\right)}{3\sqrt{\pi}}$$

$$\frac{1}{3}\left(\sqrt{\frac{1}{\pi}}\left(\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)\right)\right)2 = \frac{2\left(\int_0^\infty\frac{e^{-t}}{t^{3/4}}dt\right)\int_0^\infty e^{-t}\sqrt[4]{t}\,dt}{3\sqrt{\pi}}$$

From:

$$f(r) = 1-\frac{\Lambda}{3}r^2-\frac{2M}{r}+\frac{Q^2}{r^2}-\frac{Q^4}{20\beta^2r^6}+\mathcal{O}(r^{-10}).$$

For: $r_0 = r = 1.94973*10^{13}$ m, Q = 1 and β = 1, M = $4.37902*10^{31}$ ,

$\Lambda = 1.1056 * 10^{-52}$ m$^{-2}$ , we obtain:

1-1/3(1.1056e-52)(1.94973e+13)^2 – (2*4.37902e+31)/(1.94973e+13)+1/(1.94973e+13)^2-1/(20*(1.94973e+13)^6)+ (1.94973e+13)^-10

**Input interpretation:**

$$1-\frac{1}{3}\times 1.1056\times 10^{-52}\,(1.94973\times 10^{13})^2-\frac{2\times 4.37902\times 10^{31}}{1.94973\times 10^{13}}+\frac{1}{(1.94973\times 10^{13})^2}-\frac{1}{20\,(1.94973\times 10^{13})^6}+\frac{1}{(1.94973\times 10^{13})^{10}}$$

**Result:**

$-4.4919245228826555461783272555687197714555348803095\ldots \times 10^{18}$

$-4.49192452288\ldots*10^{18}$

From which:

(Pi-3) (((-(1-1/3(1.1056e-52)(1.94973e+13)^2 – (2*4.37902e+31)/(1.94973e+13)+1/(1.94973e+13)^2-1/(20*(1.94973e+13)^6)+ (1.94973e+13)^-10))))^1/2

**Input interpretation:**

$$(\pi-3)\sqrt{\left(-\left(1-\frac{1}{3}\times 1.1056\times 10^{-52}\,(1.94973\times 10^{13})^2-\frac{2\times 4.37902\times 10^{31}}{1.94973\times 10^{13}}+\frac{1}{(1.94973\times 10^{13})^2}-\frac{1}{20\,(1.94973\times 10^{13})^6}+\frac{1}{(1.94973\times 10^{13})^{10}}\right)\right)}$$

**Result:**

$3.0009374691893853972082431781485682218065665628490302 4\ldots \times 10^{8}$

$3.00094\ldots \times 10^{8}$

$3.00094\ldots*10^{8}$ result practically equal to the Light speed 299792458 m/s

and:

(((-(1-1/3(1.1056e-52)(1.94973e+13)^2 – (2*4.37902e+31)/(1.94973e+13)+1/(1.94973e+13)^2-1/(20*(1.94973e+13)^6)+ (1.94973e+13)^-10))))^1/9+7

**Input interpretation:**

$$\left(-\left(1-\frac{1}{3}\times 1.1056\times 10^{-52}\,(1.94973\times 10^{13})^2-\frac{2\times 4.37902\times 10^{31}}{1.94973\times 10^{13}}+\frac{1}{(1.94973\times 10^{13})^2}-\frac{1}{20\,(1.94973\times 10^{13})^6}+\frac{1}{(1.94973\times 10^{13})^{10}}\right)\right)\hat{}\,(1/9)+7$$

**Result:**

125.166...

125.166…

(((-(1-1/3(1.1056e-52)(1.94973e+13)^2 – (2*4.37902e+31)/(1.94973e+13)+1/(1.94973e+13)^2-1/(20*(1.94973e+13)^6)+(1.94973e+13)^-10))))^1/9+21

**Input interpretation:**

$$\left(-\left(1-\frac{1}{3}\times 1.1056\times 10^{-52}\,(1.94973\times 10^{13})^2-\frac{2\times 4.37902\times 10^{31}}{1.94973\times 10^{13}}+\frac{1}{(1.94973\times 10^{13})^2}-\frac{1}{20\,(1.94973\times 10^{13})^6}+\frac{1}{(1.94973\times 10^{13})^{10}}\right)\right)\hat{}\,(1/9)+21$$

**Result:**

139.166...

139.166…

27*1/2(((((-(1-1/3(1.1056e-52)(1.94973e+13)^2 – (2*4.37902e+31)/(1.94973e+13)+1/(1.94973e+13)^2-1/(20*(1.94973e+13)^6)+(1.94973e+13)^-10))))^1/9+7+Pi))-3

**Input interpretation:**

$$27\times\frac{1}{2}\left(\left(-\left(1-\frac{1}{3}\times 1.1056\times 10^{-52}\,(1.94973\times 10^{13})^2-\frac{2\times 4.37902\times 10^{31}}{1.94973\times 10^{13}}+\frac{1}{(1.94973\times 10^{13})^2}-\frac{1}{20\,(1.94973\times 10^{13})^6}+\frac{1}{(1.94973\times 10^{13})^{10}}\right)\right)\hat{}\,(1/9)+7+\pi\right)-3$$

**Result:**

1729.15...

1729.15…

and again:

[27*1/2(((((-(1-1/3(1.1056e-52)(1.94973e+13)^2 – (2*4.37902e+31)/(1.94973e+13)+1/(1.94973e+13)^2-1/(20*(1.94973e+13)^6)+ (1.94973e+13)^-10))))^1/9+7+Pi))-3]^1/15

**Input interpretation:**

$$\left(27\times\frac{1}{2}\left(\left(-\left(1-\frac{1}{3}\times 1.1056\times 10^{-52}\,(1.94973\times 10^{13})^2-\frac{2\times 4.37902\times 10^{31}}{1.94973\times 10^{13}}+\frac{1}{(1.94973\times 10^{13})^2}-\frac{1}{20\,(1.94973\times 10^{13})^6}+\frac{1}{(1.94973\times 10^{13})^{10}}\right)\right)\hat{}\,(1/9)+7+\pi\right)-3\right)\hat{}\,(1/15)$$

**Result:**

1.643825...

1.643825…

[27*1/2(((((-(1-1/3(1.1056e-52)(1.94973e+13)^2 – (2*4.37902e+31)/(1.94973e+13)+1/(1.94973e+13)^2-1/(20*(1.94973e+13)^6)+ (1.94973e+13)^-10))))^1/9+7+Pi))-3]^1/15-26/10^3

**Input interpretation:**

$$\left(27\times\frac{1}{2}\left(\left(-\left(1-\frac{1}{3}\times 1.1056\times 10^{-52}\,(1.94973\times 10^{13})^2-\frac{2\times 4.37902\times 10^{31}}{1.94973\times 10^{13}}+\frac{1}{(1.94973\times 10^{13})^2}-\frac{1}{20\,(1.94973\times 10^{13})^6}+\frac{1}{(1.94973\times 10^{13})^{10}}\right)\right)\hat{}\,(1/9)+7+\pi\right)-3\right)\hat{}\,(1/15)-\frac{26}{10^3}$$

**Result:**

1.617824884832180126697295658559294727774768704521229652321...

1.6178248848.....

From the ratio between eqs. (19) and (18), we obtain:

(-(1-1/3(1.1056e-52)(1.94973e+13)^2 – (2*4.37902e+31)/(1.94973e+13)+1/(1.94973e+13)^2-1/(20*(1.94973e+13)^6)+ (1.94973e+13)^-10)) / [2/3 sqrt(5/Pi) (gamma (1/4) gamma (5/4))]

**Input interpretation:**

$$-\frac{1}{\frac{2}{3}\sqrt{\frac{5}{\pi}}\left(\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)\right)}\left(1-\frac{1}{3}\times 1.1056\times 10^{-52}\left(1.94973\times 10^{13}\right)^2-\frac{2\times 4.37902\times 10^{31}}{1.94973\times 10^{13}}+\frac{1}{\left(1.94973\times 10^{13}\right)^2}-\frac{1}{20\left(1.94973\times 10^{13}\right)^6}+\frac{1}{\left(1.94973\times 10^{13}\right)^{10}}\right)$$

$\Gamma(x)$ is the gamma function

**Result:**

$1.62522... \times 10^{18}$

1.62522…*10^18

Or for β = 1,

(-(1-1/3(1.1056e-52)(1.94973e+13)^2 – (2*4.37902e+31)/(1.94973e+13)+1/(1.94973e+13)^2-1/(20*(1.94973e+13)^6)+ (1.94973e+13)^-10)) / [2/3 sqrt(1/Pi) (gamma (1/4) gamma (5/4))]

**Input interpretation:**

$$-\frac{1}{\frac{2}{3}\sqrt{\frac{1}{\pi}}\left(\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)\right)}\left(1-\frac{1}{3}\times 1.1056\times 10^{-52}\left(1.94973\times 10^{13}\right)^2-\frac{2\times 4.37902\times 10^{31}}{1.94973\times 10^{13}}+\frac{1}{\left(1.94973\times 10^{13}\right)^2}-\frac{1}{20\left(1.94973\times 10^{13}\right)^6}+\frac{1}{\left(1.94973\times 10^{13}\right)^{10}}\right)$$

$\Gamma(x)$ is the gamma function

**Result:**

$3.63410... \times 10^{18}$

3.63410…*10^18

From which:

((1627/4035)^(1/3)/π)(((-(1-1/3(1.1056e-52)(1.94973e+13)^2 - (2*4.37902e+31)/(1.94973e+13)+1/(1.94973e+13)^2-1/(20*(1.94973e+13)^6)+(1.94973e+13)^-10))/[2/3 sqrt(5/Pi) (gamma (1/4) gamma (5/4))]))^1/2

**Input interpretation:**

$$\left(\sqrt[3]{\frac{1627}{4035}} \times \frac{1}{\pi}\right) \sqrt{\left(-\frac{1}{\frac{2}{3}\sqrt{\frac{5}{\pi}}\left(\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)\right)} \left(1 - \frac{1}{3} \times 1.1056 \times 10^{-52}\left(1.94973 \times 10^{13}\right)^2 - \frac{2 \times 4.37902 \times 10^{31}}{1.94973 \times 10^{13}} + \frac{1}{\left(1.94973 \times 10^{13}\right)^2} - \frac{1}{20\left(1.94973 \times 10^{13}\right)^6} + \frac{1}{\left(1.94973 \times 10^{13}\right)^{10}}\right)\right)}$$

$\Gamma(x)$ is the gamma function

**Result:**

$2.99792... \times 10^8$

2.99792…*$10^8$ result practically equal to the Light speed 299792458 m/s

and for β = 1

((487/4035)^(1/3)/π)(((-(1-1/3(1.1056e-52)(1.94973e+13)^2 - (2*4.37902e+31)/(1.94973e+13)+1/(1.94973e+13)^2-1/(20*(1.94973e+13)^6)+(1.94973e+13)^-10))/[2/3 sqrt(1/Pi) (gamma (1/4) gamma (5/4))]))^1/2

**Input interpretation:**

$$\frac{\sqrt[3]{\frac{487}{4035}}}{\pi}\sqrt{\left(-\frac{1}{\frac{2}{3}\sqrt{\frac{1}{\pi}}\left(\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)\right)}\left(1-\frac{1}{3}\times 1.1056\times 10^{-52}\left(1.94973\times 10^{13}\right)^2-\frac{2\times 4.37902\times 10^{31}}{1.94973\times 10^{13}}+\frac{1}{\left(1.94973\times 10^{13}\right)^2}-\frac{1}{20\left(1.94973\times 10^{13}\right)^6}+\frac{1}{\left(1.94973\times 10^{13}\right)^{10}}\right)\right)}$$

$\Gamma(x)$ is the gamma function

**Result:**

$2.99877... \times 10^8$

2.99877…*$10^8$ result practically equal to the Light speed 299792458 m/s

From

$$f=1-\frac{2C}{r}-\frac{\Lambda}{3}r^2+\frac{2\beta^2}{r}\int_0^r r^2\left(1-\sqrt{1+\frac{Q^2}{\beta^2 r^4}}\right)dr, \tag{13}$$

where $C$ is an integration constant.

With a straightforward integration, one can express the solution in a compact form in terms of the incomplete elliptic integral of the first kind,

We note that the solution is in terms of the "incomplete elliptic integral of the first kind" (see Ramanujan incomplete elliptic integrals in the next Section)

$$f=1-\frac{2C}{r}-\frac{\Lambda}{3}r^2+\frac{2\beta^2}{3}r^2\left(1-\sqrt{1+\frac{Q^2}{\beta^2 r^4}}\right)-\frac{4}{3r}\sqrt{-i\beta Q^3}\ \text{EllipticF}\left(r\sqrt{\frac{i\beta}{Q}},i\right)$$

We have that:

$$M \equiv C+M_0,$$

$C = M - M_0 = 4.37902*10^{31}$ -2.7638913425379 = $4.3790199999999999999999999999997236108657462 1\times 10^{31}$; for Q = 1; β = 5:

1-2*(4.3790199999e+31)/(1.94973e+13)- 1/3(1.1056e-52)(1.94973e+13)^2+50/3*(1.94973e+13)^2*(1-(1+1/(25*(1.94973e+13)^4))^0.5)-4/(3*1.94973e+13)*sqrt(-5i)

**Input interpretation:**

$$1-2\times\frac{4.3790199999\times10^{31}}{1.94973\times10^{13}}-\frac{1}{3}\times1.1056\times10^{-52}\,(1.94973\times10^{13})^2+\frac{50}{3}\,(1.94973\times10^{13})^2\left(1-\sqrt{1+\frac{1}{25\,(1.94973\times10^{13})^4}}\right)-\frac{4}{3\times1.94973\times10^{13}}\sqrt{-5\,i}$$

$i$ is the imaginary unit

**Result:**

$-4.49192\ldots\times10^{18}+$
$1.08127\ldots\times10^{-13}\,i$

**Polar coordinates:**

$r=4.49192\times10^{18}$ (radius), $\theta=180°$ (angle)

**Alternate form:**

$-4.49192\times10^{18}$

**-4.49192 * $10^{18}$**

From which:

(((1-2*(4.3790199999e+31)/(1.94973e+13)- 1/3(1.1056e-52)(1.94973e+13)^2+50/3*(1.94973e+13)^2*(1-(1+1/(25*(1.94973e+13)^4))^0.5)-4/(3*1.94973e+13)*sqrt(-5i))))^1/89-2/10^3

**Input interpretation:**

$$\left(1-2\times\frac{4.3790199999\times10^{31}}{1.94973\times10^{13}}-\frac{1}{3}\times1.1056\times10^{-52}\,(1.94973\times10^{13})^2+\frac{50}{3}\,(1.94973\times10^{13})^2\left(1-\sqrt{1+\frac{1}{25\,(1.94973\times10^{13})^4}}\right)-\frac{4}{3\times1.94973\times10^{13}}\sqrt{-5\,i}\right)\text{^}(1/89)-\frac{2}{10^3}$$

$i$ is the imaginary unit

**Result:**

$1.617225\ldots+$
$0.05718045\ldots\,i$

**Polar coordinates:**

$r = 1.61824$ (radius), $\theta = 2.02497°$ (angle)

1.61824

We have also, for r = 0.6160254037844… β = Q = 1

1-2*(4.3790199999e+31)/(0.6160254)- 1/3(1.1056e-52)(0.6160254)^2+2/3*(0.6160254)^2*((1-(1+1/(0.6160254)^4))^0.5)-4/(3*0.6160254)*sqrt(-i)

**Input interpretation:**

$$1 - 2\times\frac{4.3790199999\times10^{31}}{0.6160254} - \frac{1}{3}\times1.1056\times10^{-52}\times0.6160254^2 + \frac{2}{3}\times0.6160254^2\sqrt{1-\left(1+\frac{1}{0.6160254^4}\right)} - \frac{4}{3\times0.6160254}\sqrt{-i}$$

$i$ is the imaginary unit

**Result:**

$-1.421701... \times 10^{32} + 2.197138... i$

**Alternate form:**

$-1.4217\times10^{32}$

$-1.4217 * 10^{32}$

From which:

(((1-2*(4.3790199999e+31)/(0.6160254)- 1/3(1.1056e-52)(0.6160254)^2+2/3*(0.6160254)^2*((1-(1+1/(0.6160254)^4))^0.5)-4/(3*0.6160254)*sqrt(-i))))^1/154

**Input interpretation:**

$$\left(1 - 2\times\frac{4.3790199999\times10^{31}}{0.6160254} - \frac{1}{3}\times1.1056\times10^{-52}\times0.6160254^2 + \frac{2}{3}\times0.6160254^2\sqrt{1-\left(1+\frac{1}{0.6160254^4}\right)} - \frac{4}{3\times0.6160254}\sqrt{-i}\right)\hat{}\,(1/154)$$

$i$ is the imaginary unit

**Result:**

1.61694071... +
0.0329900899... $i$

**Polar coordinates:**

$r = 1.61728$ (radius), $\theta = 1.16883°$ (angle)

1.61728

Now, we have that:

$$\rho + p_\theta = \rho + p_\phi = 4\beta^2 \left( \frac{1}{\sqrt{1 - E^2/\beta^2}} - \sqrt{1 - E^2/\beta^2} \right), \quad (43)$$

$$\rho - p_\theta = 4\beta^2 \left( \frac{1}{\sqrt{1 - E^2/\beta^2}} + \sqrt{1 - E^2/\beta^2} - 2 \right), \quad (44)$$

$$\rho + \sum_i p_i = 2p_\theta = 4\beta^2 \left( 1 - \sqrt{1 - E^2/\beta^2} \right), \quad (45)$$

E(r) = 2.63058e-27 β = 5, we obtain

4*25[1/sqrt((1-(2.63058e-27)^2) / (25)) – sqrt((1-(2.63058e-27)^2)/(25))]

**Input interpretation:**

$$4 \times 25 \left( \frac{1}{\sqrt{\frac{1}{25}\left(1 - \left(2.63058 \times 10^{-27}\right)^2\right)}} - \sqrt{\frac{1}{25}\left(1 - \left(2.63058 \times 10^{-27}\right)^2\right)} \right)$$

**Result:**

480.0000000000000000000000000000000000000000000000000017991...

480

4*25[1/sqrt((1-(2.63058e-27)^2) / (25)) + sqrt((1-(2.63058e-27)^2)/(25))-2]

**Input interpretation:**

$$4\times 25\left(\frac{1}{\sqrt{\frac{1}{25}\left(1-\left(2.63058\times 10^{-27}\right)^2\right)}}+\sqrt{\frac{1}{25}\left(1-\left(2.63058\times 10^{-27}\right)^2\right)}-2\right)$$

**Result:**

320.0000000000000000000000000000000000000000000000000016607...

320

4*25[1 - sqrt((1-(2.63058e-27)^2)/(25))]

**Input interpretation:**

$$4\times 25\left(1-\sqrt{\frac{1}{25}\left(1-\left(2.63058\times 10^{-27}\right)^2\right)}\right)$$

**Result:**

80.00000000000000000000000000000000000000000000000000006919...

80

From the sum of the three results, we obtain:

**Input interpretation:**

$$4\times 25\left(\frac{1}{\sqrt{\frac{1}{25}\left(1-\left(2.63058\times 10^{-27}\right)^2\right)}}+\sqrt{\frac{1}{25}\left(1-\left(2.63058\times 10^{-27}\right)^2\right)}-2\right)+ 4\times 25\left(1-\sqrt{\frac{1}{25}\left(1-\left(2.63058\times 10^{-27}\right)^2\right)}\right)$$

**Result:**

400.0000000000000000000000000000000000000000000000000017299...

400 (partial result)

4*25[1/sqrt((1-(2.63058e-27)^2) / (25)) – sqrt((1-(2.63058e-27)^2)/(25))]+400

**Input interpretation:**

$$4\times 25\left(\frac{1}{\sqrt{\frac{1}{25}\left(1-\left(2.63058\times 10^{-27}\right)^2\right)}}-\sqrt{\frac{1}{25}\left(1-\left(2.63058\times 10^{-27}\right)^2\right)}\right)+400$$

**Result:**

880.00000000000000000000000000000000000000000000000000017991...

880

2*(((4*25[1/sqrt((1-(2.63058e-27)^2) / (25)) – sqrt((1-(2.63058e-27)^2)/(25))]+400)))-29-2

**Input interpretation:**

$$2\left(4\times 25\left(\frac{1}{\sqrt{\frac{1}{25}\left(1-\left(2.63058\times 10^{-27}\right)^2\right)}}-\sqrt{\frac{1}{25}\left(1-\left(2.63058\times 10^{-27}\right)^2\right)}\right)+400\right)-29-2$$

**Result:**

1729.00000000000000000000000000000000000000000000000000003598...

1729

(((2*(((4*25[1/sqrt((1-(2.63058e-27)^2) / (25)) – sqrt((1-(2.63058e-27)^2)/(25))]+400)))-29-2)))^1/15

**Input interpretation:**

$$\left(2\left(4\times 25\left(\frac{1}{\sqrt{\frac{1}{25}\left(1-\left(2.63058\times 10^{-27}\right)^2\right)}}-\sqrt{\frac{1}{25}\left(1-\left(2.63058\times 10^{-27}\right)^2\right)}\right)+400\right)-29-2\right)^{\wedge}(1/15)$$

**Result:**

1.643815228748728130580088031324769514329283143699940172873...

1.6438152287....

(((2*(((4*25[1/sqrt((1-(2.63058e-27)^2) / (25)) – sqrt((1-(2.63058e-27)^2)/(25))]+400)))-29-2)))^1/15-26/10^3

**Input interpretation:**

$$\left(2\left(4\times 25\left(\frac{1}{\sqrt{\frac{1}{25}\left(1-\left(2.63058\times 10^{-27}\right)^2\right)}}-\sqrt{\frac{1}{25}\left(1-\left(2.63058\times 10^{-27}\right)^2\right)}\right)+400\right)-29-2\right)\hat{}\,(1/15)-\frac{26}{10^3}$$

**Result:**

1.617815228748728130580088031324769514329283143699940172873...

1.6178152287487....

Now, we have also:

1/7*(((4*25[1/sqrt((1-(2.63058e-27)^2) / (25)) – sqrt((1-(2.63058e-27)^2)/(25))]+400)))

**Input interpretation:**

$$\frac{1}{7}\left(4\times 25\left(\frac{1}{\sqrt{\frac{1}{25}\left(1-\left(2.63058\times 10^{-27}\right)^2\right)}}-\sqrt{\frac{1}{25}\left(1-\left(2.63058\times 10^{-27}\right)^2\right)}\right)+400\right)$$

**Result:**

125.7142857142857142857142857142857142857142857142857145427...

125.7142857....

and:

1/7*(((4*25[1/sqrt((1-(2.63058e-27)^2) / (25)) – sqrt((1-(2.63058e-27)^2)/(25))]+400)))+13+1/golden ratio

**Input interpretation:**

$$\frac{1}{7}\left(4\times 25\left(\frac{1}{\sqrt{\frac{1}{25}\left(1-\left(2.63058\times 10^{-27}\right)^2\right)}}-\sqrt{\frac{1}{25}\left(1-\left(2.63058\times 10^{-27}\right)^2\right)}\right)+400\right)+13+\frac{1}{\phi}$$

$\phi$ is the golden ratio

**Result:**

139.33231970303560913391887254865135240343459489409147740 48...

139.3323197…

From:

**INTEGRALS ASSOCIATED WITH RAMANUJAN AND ELLIPTIC FUNCTIONS**
*BRUCE C. BERNDT*

Now, we have that:

**Theorem 2.1.** *We have*

$$\int_{-\infty}^{\infty}\frac{dx}{\cos\sqrt{x}+\cosh\sqrt{x}}=\frac{\pi}{4}\frac{\Gamma^2(\frac{1}{4})}{\Gamma^2(\frac{3}{4})}. \tag{2.1}$$

((Pi(gamma^2 (1/4)))) /  ((4(gamma^2 (3/4))))

**Input:**

$$\frac{\pi\,\Gamma\left(\frac{1}{4}\right)^2}{4\,\Gamma\left(\frac{3}{4}\right)^2}$$

$\Gamma(x)$ is the gamma function

**Decimal approximation:**

6.875185818020372827490095779810557197900856451819160896274...

6.875185818...

**Alternate forms:**

$$\frac{\Gamma\left(\frac{1}{4}\right)^4}{8\pi}$$

$$\frac{4\pi\,\Gamma\left(\frac{5}{4}\right)^2}{\Gamma\left(\frac{3}{4}\right)^2}$$

$$\frac{9\pi\left(\frac{1}{4}!\right)^2}{4\left(\frac{3}{4}!\right)^2}$$

$n!$ is the factorial function

**Alternative representations:**

$$\frac{\pi\,\Gamma\left(\frac{1}{4}\right)^2}{4\,\Gamma\left(\frac{3}{4}\right)^2} = \frac{\pi\left(\left(-1+\frac{1}{4}\right)!\right)^2}{4\left(\left(-1+\frac{3}{4}\right)!\right)^2}$$

$$\frac{\pi\,\Gamma\left(\frac{1}{4}\right)^2}{4\,\Gamma\left(\frac{3}{4}\right)^2} = \frac{\pi\,\Gamma\left(\frac{1}{4},0\right)^2}{4\,\Gamma\left(\frac{3}{4},0\right)^2}$$

$$\frac{\pi\,\Gamma\left(\frac{1}{4}\right)^2}{4\,\Gamma\left(\frac{3}{4}\right)^2} = \frac{\pi\left(\frac{G\left(1+\frac{1}{4}\right)}{G\left(\frac{1}{4}\right)}\right)^2}{4\left(\frac{G\left(1+\frac{3}{4}\right)}{G\left(\frac{3}{4}\right)}\right)^2}$$

**Series representations:**

$$\frac{\pi\,\Gamma\left(\frac{1}{4}\right)^2}{4\,\Gamma\left(\frac{3}{4}\right)^2} = \frac{\pi\left(\sum_{k=1}^{\infty}\left(\frac{3}{4}\right)^k c_k\right)^2}{4\left(\sum_{k=1}^{\infty}4^{-k}c_k\right)^2}$$

$$\text{for }\left(c_1 = 1 \text{ and } c_2 = 1 \text{ and } c_k = \frac{\gamma\, c_{-1+k} + \sum_{j=1}^{-2+k}(-1)^{1+j+k}\, c_j\, \zeta(-j+k)}{-1+k}\right)$$

$$\frac{\pi\,\Gamma\left(\frac{1}{4}\right)^2}{4\,\Gamma\left(\frac{3}{4}\right)^2} = \frac{9\,\pi\left(\sum_{k=0}^{\infty}\frac{4^{-k}\,\Gamma^{(k)}(1)}{k!}\right)^2}{4\left(\sum_{k=0}^{\infty}\frac{\left(\frac{3}{4}\right)^k\,\Gamma^{(k)}(1)}{k!}\right)^2}$$

$$\frac{\pi\,\Gamma\left(\frac{1}{4}\right)^2}{4\,\Gamma\left(\frac{3}{4}\right)^2} = \frac{\pi\left(\sum_{k=0}^{\infty}\frac{\left(\frac{1}{4}-z_0\right)^k\,\Gamma^{(k)}(z_0)}{k!}\right)^2}{4\left(\sum_{k=0}^{\infty}\frac{\left(\frac{3}{4}-z_0\right)^k\,\Gamma^{(k)}(z_0)}{k!}\right)^2} \quad \text{for } (z_0 \notin \mathbb{Z} \text{ or } z_0 > 0)$$

$$\frac{\pi\,\Gamma\left(\frac{1}{4}\right)^2}{4\,\Gamma\left(\frac{3}{4}\right)^2} = \frac{\pi\left(\sum_{k=0}^{\infty}\left(\frac{3}{4}-z_0\right)^k\sum_{j=0}^{k}\frac{(-1)^j\,\pi^{-j+k}\sin\left(\frac{1}{2}(-j+k)\pi+\pi\,z_0\right)\Gamma^{(j)}(1-z_0)}{j!\,(-j+k)!}\right)^2}{4\left(\sum_{k=0}^{\infty}\left(\frac{1}{4}-z_0\right)^k\sum_{j=0}^{k}\frac{(-1)^j\,\pi^{-j+k}\sin\left(\frac{1}{2}(-j+k)\pi+\pi\,z_0\right)\Gamma^{(j)}(1-z_0)}{j!\,(-j+k)!}\right)^2}$$

**Integral representations:**

$$\frac{\pi\,\Gamma\left(\frac{1}{4}\right)^2}{4\,\Gamma\left(\frac{3}{4}\right)^2} = \frac{1}{4}\exp\left(\gamma + \int_0^1 \frac{2\sqrt[4]{x} - 2\,x^{3/4} + \log(x)}{(-1+x)\log(x)}\,dx\right)\pi$$

$$\frac{\pi\,\Gamma\left(\frac{1}{4}\right)^2}{4\,\Gamma\left(\frac{3}{4}\right)^2} = \frac{1}{4}\,e^{\int_0^1 \frac{\left(-1+\sqrt[4]{x}\right)^2}{\left(1+\sqrt{x}\right)\log(x)}\,dx}\,\pi$$

$$\frac{\pi\,\Gamma\left(\frac{1}{4}\right)^2}{4\,\Gamma\left(\frac{3}{4}\right)^2} = \frac{\pi\left(\int_0^1 \frac{1}{\log^{3/4}\left(\frac{1}{t}\right)}\,dt\right)^2}{4\left(\int_0^1 \frac{1}{\sqrt[4]{\log\left(\frac{1}{t}\right)}}\,dt\right)^2}$$

We have that:

$$\int_0^\infty \frac{x^5\,dx}{\cos x + \cosh x} = -\frac{\pi^6}{16}\left(\frac{\sqrt{\pi}}{\Gamma^2(\frac{3}{4})}\right)^6\left\{1 - \frac{16}{4}\right\}\frac{1}{2}$$
$$= \frac{3\pi^9}{32\Gamma^{12}(\frac{3}{4})} = \frac{3\pi^3}{256}\frac{\Gamma^6(\frac{1}{4})}{\Gamma^6(\frac{3}{4})},$$

(3Pi^3 (gamma^6(1/4)))/(256(gamma^6(3/4)))

**Input:**

$$\frac{3\pi^3\,\Gamma\left(\frac{1}{4}\right)^6}{256\,\Gamma\left(\frac{3}{4}\right)^6}$$

$\Gamma(x)$ is the gamma function

**Decimal approximation:**

243.73314075132068520019472519777166534319832265637343917 76...

243.73314075132…..

**Alternate forms:**

$$\frac{3\,\Gamma\left(\frac{1}{4}\right)^{12}}{2048\,\pi^3}$$

$$\frac{48\,\pi^3\,\Gamma\left(\frac{5}{4}\right)^6}{\Gamma\left(\frac{3}{4}\right)^6}$$

$$\frac{2187\,\pi^3\left(\frac{1}{4}!\right)^6}{256\left(\frac{3}{4}!\right)^6}$$

Now, we have that:

$$\int_0^\infty \frac{x^9\,dx}{\cos x+\cosh x}=\frac{\pi^{10}}{2^6}\left(\frac{\sqrt{\pi}}{\Gamma^2(\frac{3}{4})}\right)^{10}\left\{1-\frac{1232}{4}+\frac{7936}{16}\right\}\frac{1}{2}$$
$$=\frac{189\pi^{15}}{2^7\Gamma^{20}(\frac{3}{4})}=\frac{189\pi^{15}}{2^7\Gamma^{10}(\frac{3}{4})}\cdot\frac{\Gamma^{10}(\frac{1}{4})}{(\pi\sqrt{2})^{10}}=\frac{3^3\cdot 7\pi^5}{2^{12}}\frac{\Gamma^{10}(\frac{1}{4})}{\Gamma^{10}(\frac{3}{4})}.$$

[3^3 * (7Pi^5) * (gamma^10(1/4))] / [2^12 * (gamma^10(3/4))]

**Input:**

$$\frac{3^3\,(7\,\pi^5)\,\Gamma\left(\frac{1}{4}\right)^{10}}{2^{12}\,\Gamma\left(\frac{3}{4}\right)^{10}}$$

$\Gamma(x)$ is the gamma function

**Exact result:**

$$\frac{189\,\pi^5\,\Gamma\left(\frac{1}{4}\right)^{10}}{4096\,\Gamma\left(\frac{3}{4}\right)^{10}}$$

**Decimal approximation:**

725811.7845430244874980537425854957142684872912626410861573...

725811.78454302…

**Alternate forms:**

$$\frac{189\,\Gamma\left(\frac{1}{4}\right)^{20}}{131\,072\,\pi^5}$$

$$\frac{48\,384\,\pi^5\,\Gamma\left(\frac{5}{4}\right)^{10}}{\Gamma\left(\frac{3}{4}\right)^{10}}$$

$$\frac{11\,160\,261\,\pi^5\left(\frac{1}{4}!\right)^{10}}{4096\left(\frac{3}{4}!\right)^{10}}$$

$n!$ is the factorial function

**Alternative representations:**

$$\frac{3^3\left((7\,\pi^5)\,\Gamma\left(\frac{1}{4}\right)^{10}\right)}{2^{12}\,\Gamma\left(\frac{3}{4}\right)^{10}} = \frac{189\,\pi^5\left(\left(-1+\frac{1}{4}\right)!\right)^{10}}{2^{12}\left(\left(-1+\frac{3}{4}\right)!\right)^{10}}$$

$$\frac{3^3\left((7\,\pi^5)\,\Gamma\left(\frac{1}{4}\right)^{10}\right)}{2^{12}\,\Gamma\left(\frac{3}{4}\right)^{10}} = \frac{189\,\pi^5\,\Gamma\left(\frac{1}{4},0\right)^{10}}{2^{12}\,\Gamma\left(\frac{3}{4},0\right)^{10}}$$

$$\frac{3^3\left((7\,\pi^5)\,\Gamma\left(\frac{1}{4}\right)^{10}\right)}{2^{12}\,\Gamma\left(\frac{3}{4}\right)^{10}} = \frac{189\,\pi^5\left(\frac{G\left(1+\frac{1}{4}\right)}{G\left(\frac{1}{4}\right)}\right)^{10}}{2^{12}\left(\frac{G\left(1+\frac{3}{4}\right)}{G\left(\frac{3}{4}\right)}\right)^{10}}$$

## Series representations:

$$\frac{3^3\left((7\pi^5)\Gamma\left(\frac{1}{4}\right)^{10}\right)}{2^{12}\Gamma\left(\frac{3}{4}\right)^{10}} = \frac{189\pi^5\left(\sum_{k=1}^{\infty}\left(\frac{3}{4}\right)^k c_k\right)^{10}}{4096\left(\sum_{k=1}^{\infty}4^{-k}c_k\right)^{10}}$$

$$\text{for }\left(c_1 = 1 \text{ and } c_2 = 1 \text{ and } c_k = \frac{\gamma c_{-1+k} + \sum_{j=1}^{-2+k}(-1)^{1+j+k}c_j\,\zeta(-j+k)}{-1+k}\right)$$

$$\frac{3^3\left((7\pi^5)\Gamma\left(\frac{1}{4}\right)^{10}\right)}{2^{12}\Gamma\left(\frac{3}{4}\right)^{10}} = \frac{11160261\pi^5\left(\sum_{k=0}^{\infty}\frac{4^{-k}\Gamma^{(k)}(1)}{k!}\right)^{10}}{4096\left(\sum_{k=0}^{\infty}\frac{\left(\frac{3}{4}\right)^k\Gamma^{(k)}(1)}{k!}\right)^{10}}$$

$$\frac{3^3\left((7\pi^5)\Gamma\left(\frac{1}{4}\right)^{10}\right)}{2^{12}\Gamma\left(\frac{3}{4}\right)^{10}} = \frac{189\pi^5\left(\sum_{k=0}^{\infty}\frac{\left(\frac{1}{4}-z_0\right)^k\Gamma^{(k)}(z_0)}{k!}\right)^{10}}{4096\left(\sum_{k=0}^{\infty}\frac{\left(\frac{3}{4}-z_0\right)^k\Gamma^{(k)}(z_0)}{k!}\right)^{10}} \quad \text{for } (z_0 \notin \mathbb{Z} \text{ or } z_0 > 0)$$

$$\frac{3^3\left((7\pi^5)\Gamma\left(\frac{1}{4}\right)^{10}\right)}{2^{12}\Gamma\left(\frac{3}{4}\right)^{10}} = \frac{189\pi^5\left(\sum_{k=0}^{\infty}\left(\frac{3}{4}-z_0\right)^k\sum_{j=0}^{k}\frac{(-1)^j\pi^{-j+k}\sin\left(\frac{1}{2}(-j+k)\pi+\pi z_0\right)\Gamma^{(j)}(1-z_0)}{j!(-j+k)!}\right)^{10}}{4096\left(\sum_{k=0}^{\infty}\left(\frac{1}{4}-z_0\right)^k\sum_{j=0}^{k}\frac{(-1)^j\pi^{-j+k}\sin\left(\frac{1}{2}(-j+k)\pi+\pi z_0\right)\Gamma^{(j)}(1-z_0)}{j!(-j+k)!}\right)^{10}}$$

## Integral representations:

$$\frac{3^3\left((7\pi^5)\Gamma\left(\frac{1}{4}\right)^{10}\right)}{2^{12}\Gamma\left(\frac{3}{4}\right)^{10}} = \frac{189\exp\left(5\gamma+\int_0^1\frac{5\left(2\sqrt[4]{x}-2x^{3/4}+\log(x)\right)}{(-1+x)\log(x)}dx\right)\pi^5}{4096}$$

$$\frac{3^3\left((7\pi^5)\Gamma\left(\frac{1}{4}\right)^{10}\right)}{2^{12}\Gamma\left(\frac{3}{4}\right)^{10}} = \frac{189\exp\left(\int_0^1\frac{5\left(-1+\sqrt[4]{x}\right)^2}{\left(1+\sqrt{x}\right)\log(x)}dx\right)\pi^5}{4096}$$

$$\frac{3^3\left((7\pi^5)\Gamma\left(\frac{1}{4}\right)^{10}\right)}{2^{12}\Gamma\left(\frac{3}{4}\right)^{10}} = \frac{189\pi^5\left(\int_0^1\frac{1}{\log^{3/4}\left(\frac{1}{t}\right)}dt\right)^{10}}{4096\left(\int_0^1\frac{1}{\sqrt[4]{\log\left(\frac{1}{t}\right)}}dt\right)^{10}}$$

Dividing the three results, adding 16 and multiplying by $10^{16}$ , we obtain:

((((725811.784543024487 / 243.73314075132 *1 / 6.8751858)+16)))*10^16

**Input interpretation:**

$$\left(\frac{725811.784543024487}{243.73314075132} \times \frac{1}{6.8751858} + 16\right) \times 10^{16}$$

**Result:**

$4.4913670767056819660736572571571251815104539581838480... \times 10^{18}$

$4.4913670767056...*10^{18}$ value practically equal to the previous results concerning the below equations:

$$\left(1 - \frac{1}{3} \times 1.1056 \times 10^{-52} (1.94973 \times 10^{13})^2 - \frac{2 \times 4.37902 \times 10^{31}}{1.94973 \times 10^{13}} + \frac{1}{(1.94973 \times 10^{13})^2} - \frac{1}{20 (1.94973 \times 10^{13})^6} + \frac{1}{(1.94973 \times 10^{13})^{10}}\right) \Rightarrow$$

$$\Rightarrow \left(1 - 2 \times \frac{4.3790199999 \times 10^{31}}{1.94973 \times 10^{13}} - \frac{1}{3} \times 1.1056 \times 10^{-52} (1.94973 \times 10^{13})^2 + \frac{50}{3} (1.94973 \times 10^{13})^2 \left(1 - \sqrt{1 + \frac{1}{25 (1.94973 \times 10^{13})^4}}\right) - \frac{4}{3 \times 1.94973 \times 10^{13}} \sqrt{-5 i}\right) =$$

$= -4.49192 * 10^{18}$

From

**Incomplete Elliptic Integrals in Ramanujan's Lost Notebook**
*Dan Schultz* - March 17, 2015

We have that:

- On pages 51-53 of his lost notebook, Ramanujan recorded intriguing identities between $\eta$ functions and incomplete elliptic integrals.
- These identities take the form

$$\int_0^q \text{product of } \eta \text{ functions } dq = \int_{L(q)}^{U(q)} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

where $k \in \mathbb{C}$ is fixed.

We have:

- level 15

$$\int_{i\infty}^{\tau} \eta_1\eta_3\eta_5\eta_{15} 2\pi i d\tau = \frac{1}{5}\int_{2\tan^{-1}\left(\frac{1}{\sqrt{5}}\sqrt{\frac{1-11x-x^2}{1+x-x^2}}\right)}^{2\tan^{-1}\left(\frac{1}{\sqrt{5}}\right)} \frac{d\theta}{\sqrt{1-\frac{9}{25}\sin^2\theta}}.$$

where

$$x = \frac{\eta_1^3\eta_{15}^3}{\eta_3^3\eta_5^3}$$

- level 10

$$5^{3/4}\int_{i\infty}^{\tau} \eta_1^2\eta_5^2 2\pi i d\tau = 2\int_{\cos^{-1}\sqrt{\epsilon^5 y}}^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-\epsilon^{-5}5^{-3/2}\sin^2\theta}}$$

$$= \int_0^{2\tan^{-1}5^{3/4}x} \frac{d\theta}{\sqrt{1-\epsilon^{-5}5^{-3/2}\sin^2\theta}}$$

where

$$x = \frac{\eta_5^3}{\eta_1^3}, \quad y = \frac{\eta_{1,5}^5}{\eta_{2.5}^5}, \quad \epsilon = \frac{1+\sqrt{5}}{2}$$

These identities are established first by obtaining a differential equation, and then integrating both sides.

- level 15

$$\eta_1\eta_3\eta_5\eta_{15} 2\pi i d\tau = \frac{dx}{\sqrt{(x^2-x-1)(x^2+11x-1)}}$$

where

$$x = \frac{\eta_1^3\eta_{15}^3}{\eta_3^3\eta_5^3}$$

- level 10

$$\eta_1^2\eta_5^2 2\pi i d\tau = \frac{dy}{\sqrt{y(1-11y+y^2)}}$$

$$= \frac{2dx}{\sqrt{1+22x^2+125x^4}}$$

where

$$x = \frac{\eta_5^3}{\eta_1^3}, \quad y = \frac{\eta_{1,5}^5}{\eta_{2,5}^5}$$

For the identity of level 15, we have:

$$\frac{1}{2\pi i}\frac{dx}{d\tau} = \eta_1\eta_3\eta_5\eta_{15}\sqrt{(x^2-x-1)(x^2+11x-1)}$$

$$x = \frac{\eta_1^3\eta_{15}^3}{\eta_3^3\eta_5^3}$$

$$R = \frac{\eta_1^2\eta_5^2}{\eta_3^2\eta_{15}^2}, \quad P = \frac{\eta_1^6}{\eta_5^6}, \quad Q = \frac{\eta_3^6}{\eta_{15}^6}$$

$$R + 5 + \frac{9}{R} = \frac{1}{x} - x$$

$$P + \frac{125}{P} = R - 4 + \frac{135}{R} + \frac{486}{R^2} + \frac{729}{R^3}$$

$$Q + \frac{125}{Q} = R^3 + 6R^2 + 15R - 4 + \frac{9}{R}$$

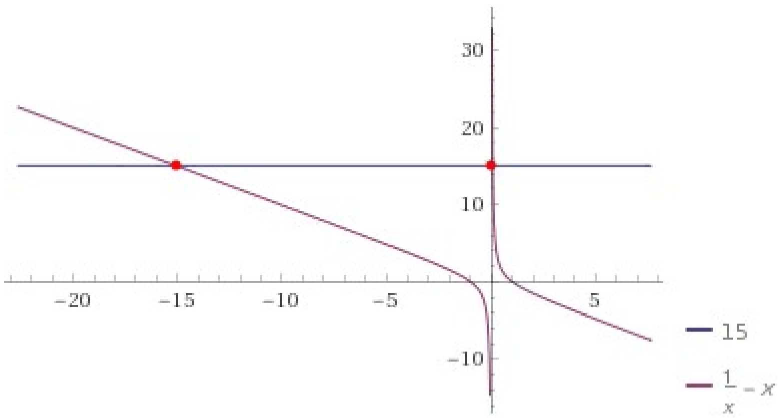
$$1 + 6\sum_k \sigma_1(k)q^k - 30\sum_k \sigma_1(k)q^{5k} = \sqrt{\frac{\eta_1^{12} + 22\eta_1^6\eta_5^6 + 125\eta_5^{12}}{\eta_1^2\eta_5^2}}$$

for 1/x – x = 15 and R = 1, we obtain:

**Input:**

$$15 = \frac{1}{x} - x$$

**Plot:**




**Alternate forms:**

$$x + 15 = \frac{1}{x}$$

$$\frac{4}{229}\left(x+\frac{15}{2}\right)^2 = 1 \quad (\text{for } x \neq 0)$$

$$15 = -\frac{(x-1)(x+1)}{x}$$

**Alternate form assuming x is positive:**

$$x(x+15) = 1 \quad (\text{for } x \neq 0)$$

**Solutions:**

$x \approx -15.066$

$x \approx 0.066373$

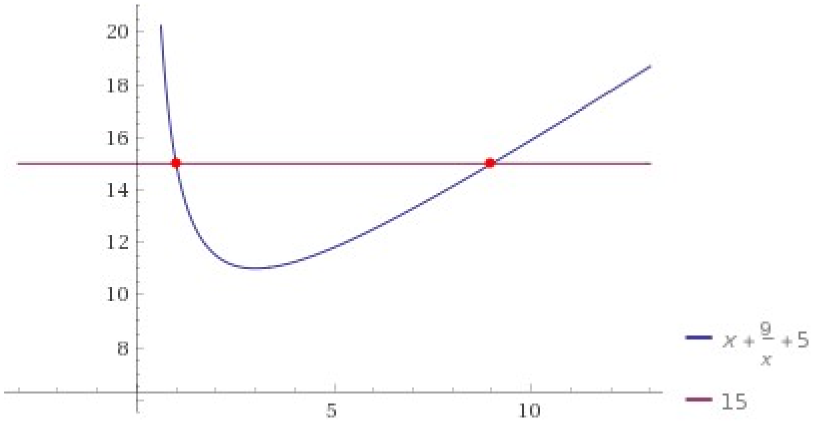
x = -15.066

R + 5 + 9/R = 15 , we obtain:

**Input:**

$$x + 5 + \frac{9}{x} = 15$$

**Plot:**




**Alternate forms:**

$$x + \frac{9}{x} = 10$$

$$\frac{x^2 + 5x + 9}{x} = 15$$

**Alternate form assuming x is positive:**

$$x^2 + 9 = 10x \quad (\text{for } x \neq 0)$$

**Solutions:**

$x = 1$

$x = 9$

R = 1  or  R = 9

For R = 9, we obtain:

$$P + \frac{125}{P} = R - 4 + \frac{135}{R} + \frac{486}{R^2} + \frac{729}{R^3}$$

x+125/x = 9-4+135/9+486/9^2+729/9^3

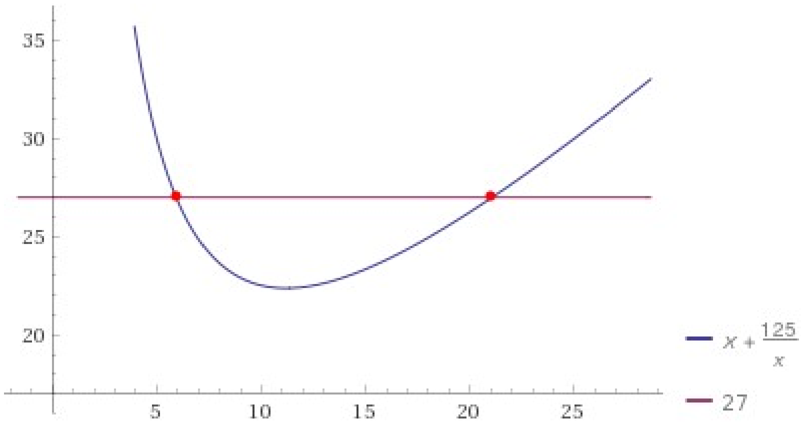
**Input:**

$$x + \frac{125}{x} = 9 - 4 + \frac{135}{9} + \frac{486}{9^2} + \frac{729}{9^3}$$

**Result:**

$$x + \frac{125}{x} = 27$$

**Plot:**




**Alternate forms:**

$$\frac{4}{229}\left(x - \frac{27}{2}\right)^2 = 1 \quad (\text{for } x \neq 0)$$

$$\frac{x^2 + 125}{x} = 27$$

**Alternate form assuming x is positive:**

$x^2 + 125 = 27x$ (for $x \neq 0$)

**Solutions:**

$x = \frac{27}{2} - \frac{\sqrt{229}}{2}$

$x = \frac{27}{2} + \frac{\sqrt{229}}{2}$

**Solutions:**

$x \approx 5.9336$

$x \approx 21.066$

P = 5.9336 or P = 21.066

From:

$$Q + \frac{125}{Q} = R^3 + 6R^2 + 15R - 4 + \frac{9}{R}$$

for R = 9 , we obtain:

x+125/x = 9^3+6*9^2+15*9-4+9/9

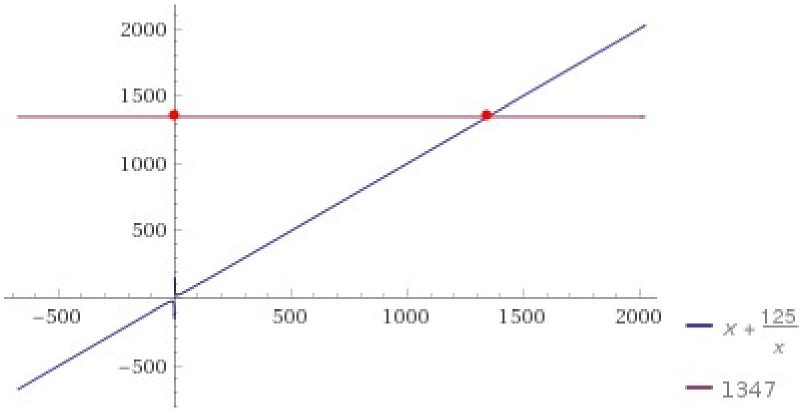
**Input:**

$x + \frac{125}{x} = 9^3 + 6 \times 9^2 + 15 \times 9 - 4 + \frac{9}{9}$

**Result:**

$x + \frac{125}{x} = 1347$

**Plot:**




**Alternate forms:**

$$\frac{4\left(x-\frac{1347}{2}\right)^2}{1813909}=1 \quad (\text{for } x \neq 0)$$

$$\frac{x^2+125}{x}=1347$$

**Alternate form assuming x is positive:**

$$x^2+125=1347\,x \quad (\text{for } x \neq 0)$$

**Solutions:**

$$x=\frac{1347}{2}-\frac{89\sqrt{229}}{2}$$

$$x=\frac{1347}{2}+\frac{89\sqrt{229}}{2}$$

**Solutions:**

$x \approx 0.092805$

$x \approx 1346.9$

Q = 0.092805 or Q = 1346.9

Thence, we obtain:

P = 21.066 Q = 1346.9 R = 9 x = -15.066

$$R+5+\frac{9}{R}=\frac{1}{x}-x$$

9+5+9/9 = (1/(-15.066))+15.066

**Input:**

$$9+5+\frac{9}{9}$$

**Exact result:**

15

15

and:

**Input interpretation:**

$$-\frac{1}{15.066}+15.066$$

**Result:**

14.99962538165405548918093720961104473649276516660029204832...

14.999625381... ≈ 15

$$\frac{1}{2\pi i}\frac{dx}{d\tau}=\eta_1\eta_3\eta_5\eta_{15}\sqrt{(x^2-x-1)(x^2+11x-1)}$$

-1/(2Pi*i) = sqrt((-(-15.066^2+15.066-1)*-(15.066^2+11*15.066-1))) x

**Input interpretation:**

$$-\frac{1}{2\pi i}=\sqrt{-(-15.066^2+15.066-1)(-(15.066^2+11\times 15.066-1))}\ x$$

$i$ is the imaginary unit

**Result:**

$$\frac{i}{2\pi}=(288.795\,i)\,x$$

**Alternate form:**

$$\frac{i}{2\pi}-(288.795\,i)\,x=0$$

**Alternate form assuming x is real:**

$$\frac{i}{2\pi}=0+i\,(288.795\,x+0)$$

**Real solution:**

$x \approx 0.000551101$

**Complex solution:**

$x = 0.000551101$

0.000551101

or:

1/(2Pi*i) = sqrt(((-15.066^2+15.066-1)(15.066^2+11*15.066-1))) x

**Input interpretation:**

$$\frac{1}{2\pi i} = \sqrt{(-15.066^2 + 15.066 - 1)(15.066^2 + 11 \times 15.066 - 1)}\ x$$

$i$ is the imaginary unit

**Result:**

$$-\frac{i}{2\pi} = (288.795\, i)\, x$$

**Alternate form:**

$$(-288.795\, i)\, x - \frac{i}{2\pi} = 0$$

**Alternate form assuming x is real:**

$$-\frac{i}{2\pi} = 0 + i\,(288.795\, x + 0)$$

**Real solution:**

$x \approx -0.000551101$

-0.000551101

**Complex solution:**

$x = -0.000551101$

We note that:

(1/(2Pi)) 1/-0.000551101  or -(1/(2Pi)) 1/0.000551101

**Input interpretation:**

$$\frac{1}{2\pi}\left(-\frac{1}{0.000551101}\right)$$

**Result:**

-288.795...

-288.795…

**Alternative representations:**

$$-\frac{1}{0.000551101\,(2\pi)} = -\frac{1}{0.000551101\,(360\,°)}$$

$$-\frac{1}{0.000551101\,(2\pi)} = -\frac{1}{0.000551101\,(-2\,i\log(-1))}$$

$$-\frac{1}{0.000551101\,(2\pi)} = -\frac{1}{0.000551101\,(2\cos^{-1}(-1))}$$

**Series representations:**

$$-\frac{1}{0.000551101\,(2\pi)} = -\frac{226.819}{\sum_{k=0}^{\infty}\frac{(-1)^k}{1+2k}}$$

$$-\frac{1}{0.000551101\,(2\pi)} = -\frac{453.637}{-1+\sum_{k=1}^{\infty}\frac{2^k}{\binom{2k}{k}}}$$

$$-\frac{1}{0.000551101\,(2\pi)} = -\frac{907.275}{\sum_{k=0}^{\infty}\frac{2^{-k}(-6+50k)}{\binom{3k}{k}}}$$

**Integral representations:**

$$-\frac{1}{0.000551101\,(2\pi)} = -\frac{453.637}{\int_0^{\infty}\frac{1}{1+t^2}\,dt}$$

$$-\frac{1}{0.000551101\,(2\pi)} = -\frac{226.819}{\int_0^{1}\sqrt{1-t^2}\,dt}$$

$$-\frac{1}{0.000551101\,(2\pi)} = -\frac{453.637}{\int_0^{\infty}\frac{\sin(t)}{t}\,dt}$$

From which:

-6* (((1/(2Pi)) 1/-0.000551101)) - 3 -1/golden ratio

**Input interpretation:**

$$-6\left(\frac{1}{2\pi}\left(-\frac{1}{0.000551101}\right)\right)-3-\frac{1}{\phi}$$

$\phi$ is the golden ratio

**Result:**

1729.15...

1729.15…

**Alternative representations:**

$$-\frac{6}{(2\pi)(-0.000551101)}-3-\frac{1}{\phi}=-3--\frac{6}{0.000551101\,(2\pi)}--\frac{1}{2\cos(216\,°)}$$

$$-\frac{6}{(2\pi)(-0.000551101)}-3-\frac{1}{\phi}=-3--\frac{6}{0.000551101\,(360\,°)}--\frac{1}{2\cos(216\,°)}$$

$$-\frac{6}{(2\pi)(-0.000551101)}-3-\frac{1}{\phi}=-3--\frac{6}{0.000551101\,(2\pi)}-\frac{1}{2\cos\left(\frac{\pi}{5}\right)}$$

**Series representations:**

$$-\frac{6}{(2\pi)(-0.000551101)}-3-\frac{1}{\phi}=-3-\frac{1}{\phi}+\frac{1360.91}{\sum_{k=0}^{\infty}\frac{(-1)^k}{1+2k}}$$

$$-\frac{6}{(2\pi)(-0.000551101)}-3-\frac{1}{\phi}=-3-\frac{1}{\phi}+\frac{2721.82}{-1+\sum_{k=1}^{\infty}\frac{2^k}{\binom{2k}{k}}}$$

$$-\frac{6}{(2\pi)(-0.000551101)}-3-\frac{1}{\phi}=-3-\frac{1}{\phi}+\frac{5443.65}{\sum_{k=0}^{\infty}\frac{2^{-k}(-6+50k)}{\binom{3k}{k}}}$$

**Integral representations:**

$$-\frac{6}{(2\pi)(-0.000551101)}-3-\frac{1}{\phi}=-3-\frac{1}{\phi}+\frac{2721.82}{\int_0^{\infty}\frac{1}{1+t^2}\,dt}$$

$$-\frac{6}{(2\pi)(-0.000551101)}-3-\frac{1}{\phi}=-3-\frac{1}{\phi}+\frac{1360.91}{\int_0^{1}\sqrt{1-t^2}\,dt}$$

$$-\frac{6}{(2\pi)(-0.000551101)} - 3 - \frac{1}{\phi} = -3 - \frac{1}{\phi} + \frac{2721.82}{\int_0^\infty \frac{\sin(t)}{t}\, dt}$$

Thence:

$\eta_1\eta_3\eta_5\eta_{15}$

$x \approx -0.000551101$

-0.0005511001

From P = 21.066   Q = 1346.9   R = 9   x = -15.066

$$P = \frac{\eta_1^6}{\eta_5^6},$$

21.066 = x^6 / y^6

**Input interpretation:**

$$21.066 = \frac{15.06525^6}{y^6}$$

**Result:**

$$21.066 = \frac{1.16912\times 10^7}{y^6}$$

**Alternate form assuming y is positive:**

$y = 9.06524$ (for $y \neq 0$)

**Input interpretation:**

$$\frac{15.06525^6}{9.06524^6}$$

**Result:**

21.06600676052446135220422687266461009277312832972210597695...

Indeed:

-15.06525^6 / -9.06524^6

**Input interpretation:**

$$\frac{-15.06525^6}{-9.06524^6}$$

**Result:**

21.06600676052446135220422687266461009277312832972210597695...

21.06600676052...

$\eta_1 = -15.06525;\ \eta_5 = -9.06524$

From:

$$\sqrt{\frac{\eta_1^{12} + 22\eta_1^6\eta_5^6 + 125\eta_5^{12}}{\eta_1^2\eta_5^2}}$$

we obtain:

sqrt((((-15.06525^12+22*(-15.06525^6)*(-9.06524^6)+125*(-9.06524^12))/( -15.06525^2*(-9.06524^2)))))

**Input interpretation:**

$$\sqrt{-\frac{-15.06525^{12} + 22\,(-15.06525^6)\,(-9.06524^6) + 125\,(-9.06524^{12})}{15.06525^2\,(-9.06524^2)}}$$

**Result:**

41704.7... $i$

**Polar coordinates:**

$r = 41\,704.7$ (radius), $\theta = 90°$ (angle)

41704.7

From the ratio between this expression and (1/(2Pi)) 1/-0.000551101

$$\frac{1}{2\pi}\left(-\frac{1}{0.000551101}\right)$$

$-288.795...$

we obtain:

sqrt((((-15.06525^12+22*(-15.06525^6)*(-9.06524^6)+125*(-9.06524^12))/( -15.06525^2*(-9.06524^2)))))/-((((1/(2Pi)) 1/-0.000551101)))

**Input interpretation:**

$$-\frac{\sqrt{-\frac{-15.06525^{12}+22\left(-15.06525^{6}\right)\left(-9.06524^{6}\right)+125\left(-9.06524^{12}\right)}{15.06525^{2}\left(-9.06524^{2}\right)}}}{\frac{1}{2\pi}\left(-\frac{1}{0.000551101}\right)}$$

**Result:**

$144.410...\, i$

**Polar coordinates:**

$r = 144.41$ (radius), $\theta = 90°$ (angle)

144.41

**Series representations:**

$$-\frac{\sqrt{-\frac{-15.0653^{12}+\left(22\left(-9.06524^{6}\right)\right)(-1)\,15.0653^{6}+125\,(-1)\,9.06524^{12}}{15.0653^{2}\,(-1)\,9.06524^{2}}}}{\frac{1}{(2\pi)(-0.000551101)}} = 0.0011022\,\pi\,\sqrt{-1.73928\times10^{9}}\sum_{k=0}^{\infty}\left(-1.73928\times10^{9}\right)^{-k}\binom{\frac{1}{2}}{k}$$

$$-\frac{\sqrt{-\frac{-15.0653^{12}+\left(22\left(-9.06524^{6}\right)\right)(-1)\,15.0653^{6}+125\,(-1)\,9.06524^{12}}{15.0653^{2}\,(-1)\,9.06524^{2}}}}{\frac{1}{(2\pi)(-0.000551101)}} = 0.0011022\,\pi\,\sqrt{-1.73928\times10^{9}}\sum_{k=0}^{\infty}\frac{e^{-21.2767k}\left(-\frac{1}{2}\right)_k}{k!}$$

$$-\frac{\sqrt{-\frac{-15.0653^{12}+(22(-9.06524^6))(-1)\,15.0653^6+125(-1)\,9.06524^{12}}{15.0653^2(-1)\,9.06524^2}}}{\frac{1}{(2\pi)(-0.000551101)}} = \frac{0.000551101\,\pi\sum_{j=0}^{\infty}\operatorname{Res}_{s=-\frac{1}{2}+j}\left(-1.73928\times10^9\right)^{-s}\Gamma\left(-\frac{1}{2}-s\right)\Gamma(s)}{\sqrt{\pi}}$$

and:

[sqrt((((-15.06525^12+22*(-15.06525^6)*(-9.06524^6)+125*(-9.06524^12))/( -15.06525^2*(-9.06524^2)))))/-((((1/(2Pi)) 1/-0.000551101)))]^1/10

**Input interpretation:**

$$\sqrt[10]{-\frac{\sqrt{-\frac{-15.06525^{12}+22(-15.06525^6)(-9.06524^6)+125(-9.06524^{12})}{15.06525^2(-9.06524^2)}}}{\frac{1}{2\pi}\left(-\frac{1}{0.000551101}\right)}}$$

**Result:**

1.62398... +
0.257212... $i$

**Polar coordinates:**

$r = 1.64422$ (radius), $\theta = 9.°$ (angle)

1.64422

12sqrt((((-15.06525^12+22*(-15.06525^6)*(-9.06524^6)+125*(-9.06524^12))/( -15.06525^2*(-9.06524^2)))))/-((((1/(2Pi)) 1/-0.000551101)))-4i

**Input interpretation:**

$$12\left(-\frac{\sqrt{-\frac{-15.06525^{12}+22(-15.06525^6)(-9.06524^6)+125(-9.06524^{12})}{15.06525^2(-9.06524^2)}}}{\frac{1}{2\pi}\left(-\frac{1}{0.000551101}\right)}\right)-4i$$

$i$ is the imaginary unit

**Result:**

1728.91... $i$

**Polar coordinates:**

$r = 1728.91$ (radius), $\theta = 90°$ (angle)

1728.91 ≈ 1729

**Series representations:**

$$-\frac{12\sqrt{-\frac{-15.0653^{12}+22(-15.0653^6)(-9.06524^6)+125(-9.06524^{12})}{15.0653^2(-9.06524^2)}}}{\frac{1}{(2\pi)(-0.000551101)}} - i\,4 = -4\,i + 0.0132264\,\pi\sqrt{-1.73928\times 10^9}\sum_{k=0}^{\infty}(-1.73928\times 10^9)^{-k}\binom{\frac{1}{2}}{k}$$

$$-\frac{12\sqrt{-\frac{-15.0653^{12}+22(-15.0653^6)(-9.06524^6)+125(-9.06524^{12})}{15.0653^2(-9.06524^2)}}}{\frac{1}{(2\pi)(-0.000551101)}} - i\,4 = -4\,i + 0.0132264\,\pi\sqrt{-1.73928\times 10^9}\sum_{k=0}^{\infty}\frac{e^{-21.2767k}\left(-\frac{1}{2}\right)_k}{k!}$$

$$-\frac{12\sqrt{-\frac{-15.0653^{12}+22(-15.0653^6)(-9.06524^6)+125(-9.06524^{12})}{15.0653^2(-9.06524^2)}}}{\frac{1}{(2\pi)(-0.000551101)}} - i\,4 = -4\,i + \frac{0.00661321\,\pi\sum_{j=0}^{\infty}\text{Res}_{s=-\frac{1}{2}+j}(-1.73928\times 10^9)^{-s}\,\Gamma\left(-\frac{1}{2}-s\right)\Gamma(s)}{\sqrt{\pi}}$$

Now, we have that:

When $\mathbb{H}/\Gamma_0(N)$ has $g = 1$:

- There a function $x(\tau)$ of order 2, which can usually be constructed as an $\eta$ quotient
- There is a cusp form $f(\tau)$ of weight 2, which can usually be constructed as an $\eta$ quotient
- There is then necessarily an identity of the form

$$f(\tau)2\pi i d\tau = \frac{dx}{\sqrt{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4}}$$

for constants $a_i$.

## Example: $\Gamma_0(15)$

First,

$$x(\tau) = \frac{\eta_1^3 \eta_{15}^3}{\eta_3^3 \eta_5^3} = q - 3q^2 + 8q^4 - 9q^5 + \cdots$$

has simple poles at $\tau = 1/3$ and $\tau = 1/5$ and simple zeros at $\tau = 1/1$ and $\tau = 1/15$.

Next,

$$\eta_1 \eta_3 \eta_5 \eta_{15} = q - q^2 - q^3 - q^4 + q^5 + \cdots$$

is a cusp form.

Therefore,

$$\eta_1 \eta_3 \eta_5 \eta_{15} 2\pi i d\tau = \frac{dx}{\sqrt{a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0}}$$

$$q = x + 3x^2 + 18x^3 + 127x^4 + O(x^5)$$

$$\left(\frac{1}{\eta_1 \eta_3 \eta_5 \eta_{15}} \frac{1}{2\pi i} \frac{dx}{d\tau}\right)^2 = 1 - 10x - 13x^2 + 10x^3 + x^4 + O(x^5)$$
$$= (x^2 - x - 1)(x^2 + 11x - 1)$$

We have obtained previously x = -15.066 , thence:

$$\left(\frac{1}{\eta_1\eta_3\eta_5\eta_{15}}\frac{1}{2\pi i}\frac{dx}{d\tau}\right)^2 = 1 - 10x - 13x^2 + 10x^3 + x^4 + O(x^5)$$
$$= (x^2 - x - 1)(x^2 + 11x - 1)$$

(-15.066^2+15.066-1)((-15.066^2+11*(-15.066)-1)

**Input interpretation:**

$(-15.066^2 + 15.066 - 1)(-15.066^2 + 11\times(-15.066) - 1)$

**Result:**

83828.161739694736

83828.161739694736

From

$$\left(1 - \frac{1}{3}\times 1.1056\times 10^{-52}\,(1.94973\times 10^{13})^2 - \frac{2\times 4.37902\times 10^{31}}{1.94973\times 10^{13}} + \frac{1}{(1.94973\times 10^{13})^2} - \frac{1}{20\,(1.94973\times 10^{13})^6} + \frac{1}{(1.94973\times 10^{13})^{10}}\right) \Rightarrow$$

$$\Rightarrow \left(1 - 2\times\frac{4.3790199999\times 10^{31}}{1.94973\times 10^{13}} - \frac{1}{3}\times 1.1056\times 10^{-52}\,(1.94973\times 10^{13})^2 + \frac{50}{3}\,(1.94973\times 10^{13})^2\left(1 - \sqrt{1 + \frac{1}{25\,(1.94973\times 10^{13})^4}}\right) - \frac{4}{3\times 1.94973\times 10^{13}}\sqrt{-5\,i}\right) =$$

$= \quad -4.49192 * 10^{18}$

Performing some calculations, and taking the absolute value, we obtain:

(((7/9 + 8/(81 e) + (10 e)/27))) (4.49192e+18)^1/4

**Input interpretation:**

$$\left(\frac{7}{9} + \frac{8}{81 e} + \frac{10 e}{27}\right) \sqrt[4]{4.49192 \times 10^{18}}$$

**Result:**

83828.16286777860277823431376748163785784059631184832077279...

83828.1628677786...

But, from

$$(x^2 - x - 1)(x^2 + 11x - 1)$$

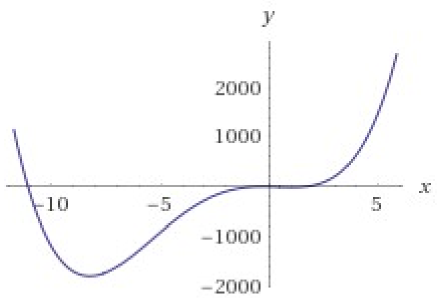
we have also:

(x^2-x-1)((x^2+11*x-1))

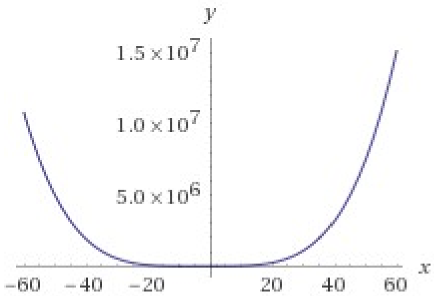
**Input:**

$$(x^2 - x - 1)(x^2 + 11 x - 1)$$

**Plots:**




(x from -11.7 to 5.8)




(x from −60 to 60)

**Alternate forms:**

$x^4 + 10x^3 - 13x^2 - 10x + 1$

$x(x(x(x+10)-13)-10)+1$

$((x-1)x-1)(x(x+11)-1)$

**Roots:**

$x = -\frac{11}{2} - \frac{5\sqrt{5}}{2}$

$x = \frac{1}{2} - \frac{\sqrt{5}}{2}$

$x = \frac{1}{2} + \frac{\sqrt{5}}{2}$

$x = \frac{5\sqrt{5}}{2} - \frac{11}{2}$

**Roots:**

$x \approx -11.0901699437495$

$x \approx -0.618033988749895$

$x \approx 1.61803398874989$

$x \approx 0.0901699437494742$

1.61803398…;   -0.61803398…

**Polynomial discriminant:**

$\Delta = 12960000$

**Properties as a real function:**

**Domain**

$\mathbb{R}$ (all real numbers)

**Range**

$\{y \in \mathbb{R} : y \geq -1783.97\}$

$\mathbb{R}$ is the set of real numbers

**Derivative:**

$$\frac{d}{dx}((x^2 - x - 1)(x^2 + 11x - 1)) = 4x^3 + 30x^2 - 26x - 10$$

**Indefinite integral:**

$$\int (-1 - x + x^2)(-1 + 11x + x^2)\,dx = \frac{x^5}{5} + \frac{5x^4}{2} - \frac{13x^3}{3} - 5x^2 + x + \text{constant}$$

**Local maximum:**

$$\max\{(x^2 - x - 1)(x^2 + 11x - 1)\} \approx 2.5699 \text{ at } x \approx -0.29081$$

**Global minimum:**

$$\min\{(x^2 - x - 1)(x^2 + 11x - 1)\} \approx -1784.0 \text{ at } x \approx -8.2511$$

for x ≈ 1.61803398874989 ,

$$(x^2 - x - 1)(x^2 + 11x - 1)$$

we obtain:

((1.61803398)^2-(1.61803398)-1) (1.61803398^2 + 11*1.61803398 - 1)

**Input interpretation:**

$$(1.61803398^2 - 1.61803398 - 1)(1.61803398^2 + 11 \times 1.61803398 - 1)$$

**Result:**

$$-3.7988899958182719972268784 \times 10^{-7}$$

$-3.79888... * 10^{-7}$

From the previous expression, for β =1:

$$T_H \equiv \frac{\hbar}{4\pi}\frac{df}{dr}\bigg|_{r=r_+}$$

$$= \frac{\hbar}{4\pi}\left[\frac{1}{r_+} - \Lambda r_+ + 2\beta^2 r_+\left(\mathbf{1} - \sqrt{1+\frac{Q^2}{\beta r_+^4}}\right)\right].$$

1.054571e-34 / (4Pi) [1/(1.94973e+13)-(1.1056e-52*1.94973e+13)+2*1.94973e+13(1-sqrt(1+1/(1.94973e+13)^4)]

**Input interpretation:**

$$\frac{1.054571\times 10^{-34}}{4\pi}\left(\frac{1}{1.94973\times 10^{13}} - 1.1056\times 10^{-52}\times 1.94973\times 10^{13} + 2\times 1.94973\times 10^{13}\left(1-\sqrt{1+\frac{1}{(1.94973\times 10^{13})^4}}\right)\right)$$

**Result:**

$4.30419... \times 10^{-49}$

4.30419 * $10^{-49}$

we obtain:

Pi ((((1.054571e-34 / (4Pi) ((1/(1.94973e+13)-(1.1056e-52*1.94973e+13)+2*1.94973e+13(1-sqrt(1+1/(1.94973e+13)^4)))))))^1/7

**Input interpretation:**

$$\pi\left(\frac{1.054571\times 10^{-34}}{4\pi}\left(\frac{1}{1.94973\times 10^{13}} - 1.1056\times 10^{-52}\times 1.94973\times 10^{13} + 2\times 1.94973\times 10^{13}\left(1-\sqrt{1+\frac{1}{(1.94973\times 10^{13})^4}}\right)\right)\right)^{\wedge}(1/7)$$

**Result:**

$3.86995... \times 10^{-7}$

3.86995…*$10^{-7}$

Now, we have that:

## Example: $\Gamma_0(24)$

First,

$$x(\tau) = \frac{\eta_4 \eta_6 \eta_1^2 \eta_{24}^2}{\eta_2 \eta_{12} \eta_3^2 \eta_8^2} = q - 2q^2 + 2q^4 + \cdots$$

has simple poles at $\tau = 1/3$ and $\tau = 1/8$ and simple zeros at $\tau = 1/1$ and $\tau = 1/24$.

Next,

$$\eta_2 \eta_4 \eta_6 \eta_{12} = q - q^3 - 2q^5 + \cdots$$

is a cusp form.

Therefore,

$$\begin{aligned}\eta_2 \eta_4 \eta_6 \eta_{12} 2\pi i d\tau &= \frac{dx}{\sqrt{a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0}} \\ &= \frac{dx}{\sqrt{x^4 + 8x^3 + 2x^2 - 8x + 1}}\end{aligned}$$

$$\int_{i\infty}^{\tau} \eta_2 \eta_4 \eta_6 \eta_{12} 2\pi i d\tau = \int_{\sin^{-1}\left(\frac{1}{\sqrt[4]{3}}\sqrt{\frac{\sqrt{2}-1-(\sqrt{2}+\sqrt{3})x}{\sqrt{2}+\sqrt{3}+(\sqrt{2}-1)x}}\right)}^{\sin^{-1}\left(\frac{1}{\sqrt[4]{3}}\sqrt{\frac{\sqrt{2}-1}{\sqrt{2}+\sqrt{3}}}\right)} \frac{d\theta}{\sqrt{1 + 3\sin^2\theta}},$$

From

$$\begin{aligned}\eta_2 \eta_4 \eta_6 \eta_{12} 2\pi i d\tau &= \frac{dx}{\sqrt{a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0}} \\ &= \frac{dx}{\sqrt{x^4 + 8x^3 + 2x^2 - 8x + 1}}\end{aligned}$$

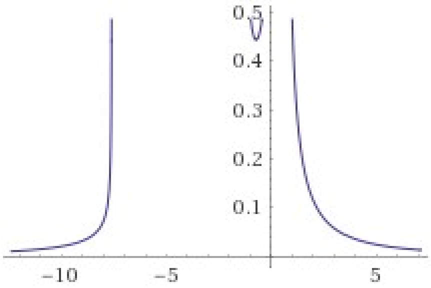
we obtain:

1/(sqrt(x^4+8x^3+2x^2-8x+1))

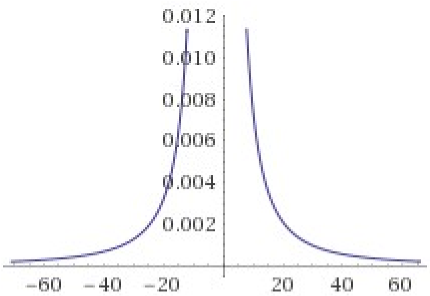
**Input:**

$$\frac{1}{\sqrt{x^4 + 8x^3 + 2x^2 - 8x + 1}}$$

**Plots:**




(x from -12.4 to 7.1)




(x from -70.9 to 65.6)

**Alternate form:**

$$\frac{1}{\sqrt{x(x(x(x+8)+2)-8)+1}}$$

**Series expansion at x = 0:**

$$1 + 4x + 23x^2 + 144x^3 + 953x^4 + O(x^5)$$

(Taylor series)

**Derivative:**

$$\frac{d}{dx}\left(\frac{1}{\sqrt{x^4 + 8x^3 + 2x^2 - 8x + 1}}\right) = -\frac{2(x^3 + 6x^2 + x - 2)}{(x^4 + 8x^3 + 2x^2 - 8x + 1)^{3/2}}$$

**Indefinite integral:**

$$\int \frac{1}{\sqrt{x^4+8x^3+2x^2-8x+1}}\,dx =$$

$$-\left(\left(2\sqrt{\frac{x-\sqrt{5-2\sqrt{6}}-\sqrt{6}+2}{x-\sqrt{5+2\sqrt{6}}+\sqrt{6}+2}}\sqrt{\frac{x+\sqrt{5-2\sqrt{6}}-\sqrt{6}+2}{x-\sqrt{5+2\sqrt{6}}+\sqrt{6}+2}}\left(x-\sqrt{5+2\sqrt{6}}+\sqrt{6}+2\right)^2 \sqrt{\frac{x-\boxed{\text{root of } x^4+8x^3+2x^2-8x+1 \text{ near } x=-7.59575}}{x-\sqrt{5+2\sqrt{6}}+\sqrt{6}+2}}\right.\right.$$

$$F\left(\sin^{-1}\left(\sqrt{\left(\left(\left(2\sqrt{6}+\sqrt{5-2\sqrt{6}}-\sqrt{5+2\sqrt{6}}\right)\left(x-\boxed{\text{root of } x^4+8x^3+2x^2-8x+1 \text{ near } x=-7.59575}\right)\right)\Big/\left(\left(x-\sqrt{5+2\sqrt{6}}+\sqrt{6}+2\right)\left(-2+\sqrt{6}+\sqrt{5-2\sqrt{6}}-\boxed{\text{root of } x^4+8x^3+2x^2-8x+1 \text{ near } x=-7.59575}\right)\right)\right)}\right)\right|$$

$$\left(\left(-2\sqrt{6}+\sqrt{5-2\sqrt{6}}+\sqrt{5+2\sqrt{6}}\right)\left(2-\sqrt{6}-\sqrt{5-2\sqrt{6}}+\boxed{\text{root of } x^4+8x^3+2x^2-8x+1 \text{ near } x=-7.59575}\right)\right)\Big/$$

$$\left.\left.\left(\left(-2\sqrt{6}-\sqrt{5-2\sqrt{6}}+\sqrt{5+2\sqrt{6}}\right)\left(2-\sqrt{6}+\sqrt{5-2\sqrt{6}}+\boxed{\text{root of } x^4+8x^3+2x^2-8x+1 \text{ near } x=-7.59575}\right)\right)\right)\right)\Big/$$

$$\left.\left(\sqrt{\left(-2\sqrt{6}-\sqrt{5-2\sqrt{6}}+\sqrt{5+2\sqrt{6}}\right)(x^4+8x^3+2x^2-8x+1)}\left(\boxed{\text{root of } x^4+8x^3+2x^2-8x+1 \text{ near } x=-7.59575}+2-\sqrt{6}+\sqrt{5-2\sqrt{6}}\right)\right)\right) + \text{constant}$$

Now, from

$$\frac{1}{\sqrt{x^4 + 8x^3 + 2x^2 - 8x + 1}}$$

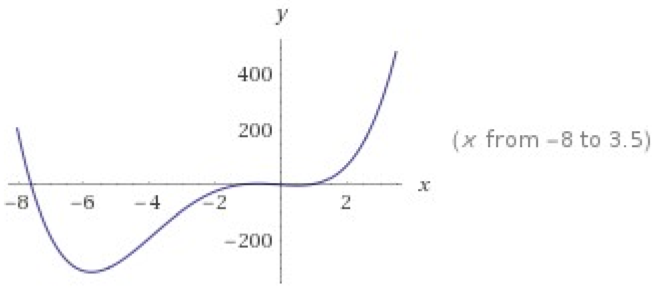
we perform the following calculation:

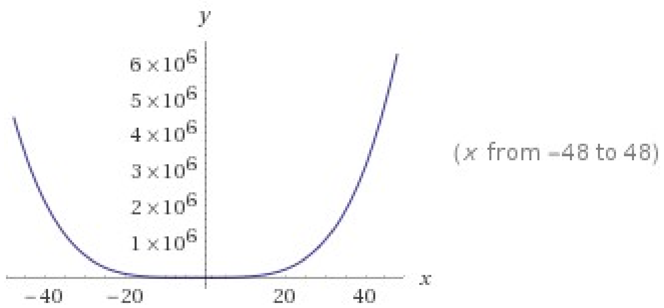
(x^4+8x^3+2x^2-8x+1)

**Input:**

$x^4 + 8x^3 + 2x^2 - 8x + 1$

**Plots:**




**Alternate forms:**

$x(x(x(x+8)+2)-8)+1$

$-\left(-x^2 + \left(2\sqrt{6} - 4\right)x + 2\sqrt{6} - 5\right)\left(x^2 + \left(4 + 2\sqrt{6}\right)x + 2\sqrt{6} + 5\right)$

$$\left(-x-\left(5+2\sqrt{6}\right)^{3/2}+10\sqrt{5+2\sqrt{6}}+\sqrt{6}-2\right)$$
$$\left(-x+\left(5+2\sqrt{6}\right)^{3/2}-10\sqrt{5+2\sqrt{6}}+\sqrt{6}-2\right)$$
$$\left(x-\sqrt{5+2\sqrt{6}}+\sqrt{6}+2\right)\left(x+\sqrt{5+2\sqrt{6}}+\sqrt{6}+2\right)$$

**Roots:**

$$x = -2+\sqrt{6}-\sqrt{5-2\sqrt{6}}$$

$$x = -2+\sqrt{6}+\sqrt{5-2\sqrt{6}}$$

$$x = -2-\sqrt{6}-\sqrt{5+2\sqrt{6}}$$

$$x = -2-\sqrt{6}+\sqrt{5+2\sqrt{6}}$$

**Polynomial discriminant:**

$\Delta = 589\,824$

**Properties as a real function:**

**Domain**

$\mathbb{R}$ (all real numbers)

**Range**

$\{y \in \mathbb{R} : y \geq -314.638\}$

$\mathbb{R}$ is the set of real numbers

**Derivative:**

$$\frac{d}{dx}(x^4+8x^3+2x^2-8x+1) = 4(x^3+6x^2+x-2)$$

**Indefinite integral:**

$$\int (1-8x+2x^2+8x^3+x^4)\,dx = \frac{x^5}{5}+2x^4+\frac{2x^3}{3}-4x^2+x+\text{constant}$$

**Local maximum:**

$\max\{x^4+8x^3+2x^2-8x+1\} \approx 5.0797$ at $x \approx -0.71718$

**Global minimum:**

$\min\{x^4+8x^3+2x^2-8x+1\} \approx -314.64$ at $x \approx -5.7664$

Now, from

$$x = -2 + \sqrt{6} + \sqrt{5 - 2\sqrt{6}}$$

we have, for

$$\frac{1}{\sqrt{x^4 + 8x^3 + 2x^2 - 8x + 1}}$$

1/(-2 + sqrt(6) + sqrt(5 - 2 sqrt(6)))^1/2

**Input:**

$$\frac{1}{\sqrt{-2 + \sqrt{6} + \sqrt{5 - 2\sqrt{6}}}}$$

**Decimal approximation:**

1.141588968430058786133608839087698612098785192658239643502...

1.1415889684…

**Alternate forms:**

$$\frac{1}{\sqrt{-2 - \sqrt{2} + \sqrt{3} + \sqrt{6}}}$$

$$\sqrt{2 + \sqrt{6} - \sqrt{5 + 2\sqrt{6}}}$$

$$\sqrt{-2 + \sqrt{6} + \sqrt{5 - 2\sqrt{6}}} \left(2 + \sqrt{6} - 5\sqrt{5 - 2\sqrt{6}} - 2\sqrt{6\left(5 - 2\sqrt{6}\right)}\right)$$

**Minimal polynomial:**

$$x^8 - 8x^6 + 2x^4 + 8x^2 + 1$$

Indeed:

integrate ((((1/(-2 + sqrt(6) + sqrt(5 - 2 sqrt(6)))^1/2)))dx x = infinity*i..1/24

**Definite integral:**

$$\int_{\infty i}^{\frac{1}{24}} \frac{1}{\sqrt{-2+\sqrt{6}+\sqrt{5-2\sqrt{6}}}}\,dx = \frac{i(-\infty)+\frac{1}{24}}{\sqrt{-2+\sqrt{6}+\sqrt{5-2\sqrt{6}}}}$$

**Indefinite integral:**

$$\int \frac{1}{\sqrt{-2+\sqrt{6}+\sqrt{5-2\sqrt{6}}}}\,dx = \frac{x}{\sqrt{-2+\sqrt{6}+\sqrt{5-2\sqrt{6}}}} + \text{constant}$$

$$\int \frac{1}{\sqrt{-2+\sqrt{6}+\sqrt{5-2\sqrt{6}}}}\,dx \approx \text{constant} + 1.14159\,x$$

1.14159

Thence:

$$\int_{i\infty}^{\tau} \eta_2\eta_4\eta_6\eta_{12} 2\pi i\, d\tau = \int_{\sin^{-1}\left(\frac{1}{\sqrt[4]{3}}\sqrt{\frac{\sqrt{2}-1-(\sqrt{2}+\sqrt{3})x}{\sqrt{2}+\sqrt{3}+(\sqrt{2}-1)x}}\right)}^{\sin^{-1}\left(\frac{1}{\sqrt[4]{3}}\sqrt{\frac{\sqrt{2}-1}{\sqrt{2}+\sqrt{3}}}\right)} \frac{d\theta}{\sqrt{1+3\sin^2\theta}},$$

$$\int_{\infty i}^{\frac{1}{24}} \frac{1}{\sqrt{-2+\sqrt{6}+\sqrt{5-2\sqrt{6}}}}\,dx = \frac{i(-\infty)+\frac{1}{24}}{\sqrt{-2+\sqrt{6}+\sqrt{5-2\sqrt{6}}}}$$

$$\int \frac{1}{\sqrt{-2+\sqrt{6}+\sqrt{5-2\sqrt{6}}}}\,dx \approx \text{constant} + 1.14159\,x$$

From which:

1/2+((((1/(-2 + sqrt(6) + sqrt(5 - 2 sqrt(6)))^1/2))))

**Input:**

$$\frac{1}{2}+\frac{1}{\sqrt{-2+\sqrt{6}+\sqrt{5-2\sqrt{6}}}}$$

**Decimal approximation:**

1.641588968430058786133608839087698612098785192658239643502...

1.6415889684...

**Alternate forms:**

$$\frac{1}{2}+\frac{1}{\sqrt{-2-\sqrt{2}+\sqrt{3}+\sqrt{6}}}$$

$$\frac{1}{2}\left(1+2\sqrt{2+\sqrt{6}-\sqrt{5+2\sqrt{6}}}\right)$$

$$\frac{2+\sqrt{-2-\sqrt{2}+\sqrt{3}+\sqrt{6}}}{2\sqrt{-2-\sqrt{2}+\sqrt{3}+\sqrt{6}}}$$

**Minimal polynomial:**

$$256x^8-1024x^7-256x^6+4352x^5-6048x^4+3648x^3+1008x^2-1936x+769$$

and:

1/2+((((1/(-2 + sqrt(6) + sqrt(5 - 2 sqrt(6)))^1/2)))) - (21+2)1/10^3

**Input:**

$$\frac{1}{2}+\frac{1}{\sqrt{-2+\sqrt{6}+\sqrt{5-2\sqrt{6}}}}-(21+2)\times\frac{1}{10^3}$$

**Exact result:**

$$\frac{477}{1000}+\frac{1}{\sqrt{-2+\sqrt{6}+\sqrt{5-2\sqrt{6}}}}$$

**Decimal approximation:**

1.618588968430058786133608839087698612098785192658239643502...

1.61858896843…

**Alternate forms:**

$$\frac{477}{1000}+\frac{1}{\sqrt{-2-\sqrt{2}+\sqrt{3}+\sqrt{6}}}$$

$$\frac{477+1000\sqrt{2+\sqrt{6}-\sqrt{5+2\sqrt{6}}}}{1000}$$

$$\frac{1000+477\sqrt{-2-\sqrt{2}+\sqrt{3}+\sqrt{6}}}{1000\sqrt{-2-\sqrt{2}+\sqrt{3}+\sqrt{6}}}$$

**Minimal polynomial:**

$$1\,000\,000\,000\,000\,000\,000\,000\,000\,x^8 - 3\,816\,000\,000\,000\,000\,000\,000\,000\,x^7 - 1\,629\,188\,000\,000\,000\,000\,000\,000\,x^6 + 16\,818\,245\,352\,000\,000\,000\,000\,000\,x^5 - 21\,679\,618\,791\,130\,000\,000\,000\,000\,x^4 + 12\,166\,147\,842\,695\,208\,000\,000\,000\,x^3 + 4\,847\,827\,905\,877\,192\,892\,000\,000\,x^2 - 7\,359\,886\,287\,750\,824\,288\,424\,000\,x + 2\,832\,218\,565\,262\,629\,120\,197\,281$$

We have that:

Ramanujan has also given the example

$$\int_{i\infty}^{\tau} \eta_1\eta_5\eta_7\eta_{35} 2\pi i d\tau - \int_0^x \frac{x\,dx}{\sqrt{1+x-x^2}\sqrt{1-5x-9x^3-6x^5-x^6}}$$

where

$$x = \frac{\eta_1\eta_{35}}{\eta_5\eta_7}$$

which belong to $\Gamma_0(35)$.

We perform as follows:

1/((sqrt(1+x-x^2) sqrt(1-5x-9x^3-6x^5-x^6)))^2

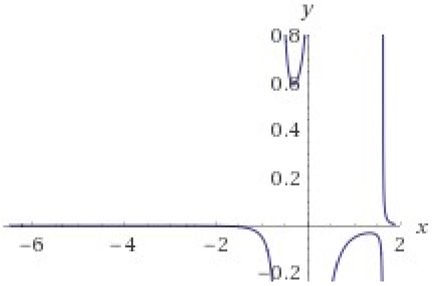
**Input:**

$$\frac{1}{\left(\sqrt{1+x-x^2}\,\sqrt{1-5x-9x^3-6x^5-x^6}\right)^2}$$

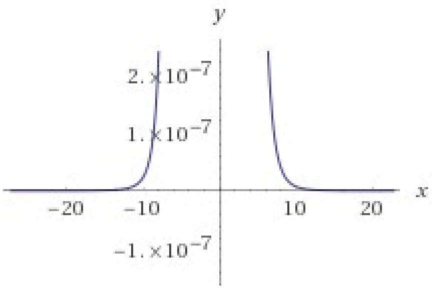
**Result:**

$$\frac{1}{(-x^2+x+1)(-x^6-6x^5-9x^3-5x+1)}$$

**Plots:**




(x from −6.5 to 1.9)




(x from −27.3 to 22.7)

**Alternate forms:**

$$\frac{1}{((x-1)x-1)(x^6+6x^5+9x^3+5x-1)}$$

$$\frac{1}{(x^2-x-1)(x^6+6x^5+9x^3+5x-1)}$$

$$\frac{1}{x(x(x(x(x(x(x(x+5)-7)+3)-9)-4)-6)-4)+1}$$

**Partial fraction expansion:**

$$\frac{61x-92}{869(x^2-x-1)}+\frac{-61x^5-335x^4+156x^3-728x^2+256x-777}{869(x^6+6x^5+9x^3+5x-1)}$$

## Expanded form:

$$\frac{1}{x^8+5x^7-7x^6+3x^5-9x^4-4x^3-6x^2-4x+1}$$

## Roots:

(no roots exist)

## Properties as a real function:

### Domain

$\{x \in \mathbb{R} : -0.618034 < x < 0.187791\}$

### Range

$\{y \in \mathbb{R} : y \geq 0.594003\}$

$\mathbb{R}$ is the set of real numbers

## Series expansion at x = 0:

$$1+4x+22x^2+116x^3+621x^4+O(x^5)$$

(Taylor series)

## Series expansion at x = ∞:

$$\left(\frac{1}{x}\right)^8-\frac{5}{x^9}+\frac{32}{x^{10}}-\frac{198}{x^{11}}+O\left(\left(\frac{1}{x}\right)^{12}\right)$$

(Laurent series)

## Derivative:

$$\frac{d}{dx}\left(\frac{1}{\left(\sqrt{1+x-x^2}\ \sqrt{1-5x-9x^3-6x^5-x^6}\right)^2}\right)=$$

$$\frac{-8x^7-35x^6+42x^5-15x^4+36x^3+12x^2+12x+4}{(-x^2+x+1)^2(-x^6-6x^5-9x^3-5x+1)^2}$$

## Indefinite integral:

$$\int\frac{1}{\left(\sqrt{1+x-x^2}\ \sqrt{1-5x-9x^3-6x^5-x^6}\right)^2}\,dx=$$

$$\frac{1}{8690}\Bigg(-10\sum_{\{\omega:\,\omega^6+6\omega^5+9\omega^3+5\omega-1=0\}}\frac{1}{6\omega^5+30\omega^4+27\omega^2+5}$$
$$(61\omega^5\log(x-\omega)+335\omega^4\log(x-\omega)-156\omega^3\log(x-\omega)+$$
$$728\omega^2\log(x-\omega)-256\omega\log(x-\omega)+777\log(x-\omega))+$$
$$(305-123\sqrt{5})\log(-2x+\sqrt{5}+1)+(305+123\sqrt{5})$$
$$\log(2x+\sqrt{5}-1)\Bigg)+\text{constant}$$

(assuming a complex-valued logarithm)

log(x) is the natural logarithm

## Local maxima:

$$\max\left\{\frac{1}{\left(\sqrt{1+x-x^2}\,\sqrt{1-5x-9x^3-6x^5-x^6}\right)^2}\right\} = 0 \text{ at } x \approx -5.4317$$

$$\max\left\{\frac{1}{\left(\sqrt{1+x-x^2}\,\sqrt{1-5x-9x^3-6x^5-x^6}\right)^2}\right\} \approx -0.031087 \text{ at } x \approx 1.3326$$

## Local minimum:

$$\min\left\{\frac{1}{\left(\sqrt{1+x-x^2}\,\sqrt{1-5x-9x^3-6x^5-x^6}\right)^2}\right\} \approx 0.59400 \text{ at } x \approx -0.31377$$

## Limit:

$$\lim_{x\to\pm\infty} \frac{1}{(1+x-x^2)(1-5x-9x^3-6x^5-x^6)} = 0$$

## Series representations:

$$\frac{1}{\left(\sqrt{1+x-x^2}\,\sqrt{1-5x-9x^3-6x^5-x^6}\right)^2} = \frac{1}{\left(\sum_{k=0}^{\infty} \frac{(-1)^k(-(-1+x)x)^k\left(-\frac{1}{2}\right)_k}{k!}\right)^2 \left(\sum_{k=0}^{\infty} \frac{(-1)^k(-x(5+9x^2+6x^4+x^5))^k\left(-\frac{1}{2}\right)_k}{k!}\right)^2}$$

for $(|(-1+x)x| < 1 \text{ and } |5x+9x^3+6x^5+x^6| < 1)$

$$\frac{1}{\left(\sqrt{1+x-x^2}\,\sqrt{1-5x-9x^3-6x^5-x^6}\right)^2} = 1\Bigg/\left(\sqrt{x-x^2}^{\,2}\sqrt{-x(5+9x^2+6x^4+x^5)}^{\,2} \left(\sum_{k=0}^{\infty}(-(-1+x)x)^{-k}\binom{\frac{1}{2}}{k}\right)^2 \left(\sum_{k=0}^{\infty}(-x(5+9x^2+6x^4+x^5))^{-k}\binom{\frac{1}{2}}{k}\right)^2\right)$$

for $(|(-1+x)x| > 1 \text{ and } |5x+9x^3+6x^5+x^6| > 1)$

$$\frac{1}{\left(\sqrt{1+x-x^2}\,\sqrt{1-5x-9x^3-6x^5-x^6}\right)^2} = \frac{1}{\left(\sum_{k=0}^{\infty} \frac{(-1)^k(-(-1+x)x)^k\left(-\frac{1}{2}\right)_k}{k!}\right)^2 \left(\sum_{k=0}^{\infty} \frac{(-1)^k(-x(5+9x^2+6x^4+x^5))^k\left(-\frac{1}{2}\right)_k}{k!}\right)^2}$$

for $(|x-x^2| < 1 \text{ and } |-5x-9x^3-6x^5-x^6| < 1)$

From

$$\frac{1}{(-x^2+x+1)(-x^6-6x^5-9x^3-5x+1)}$$

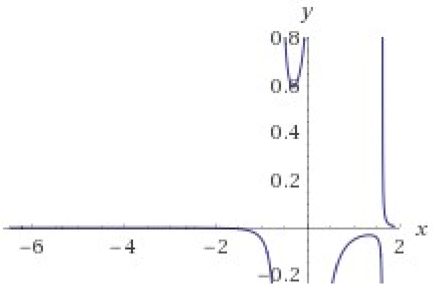
we obtain:

1/((-1 + (-1 + x) x) (-1 + 5 x + 9 x^3 + 6 x^5 + x^6))

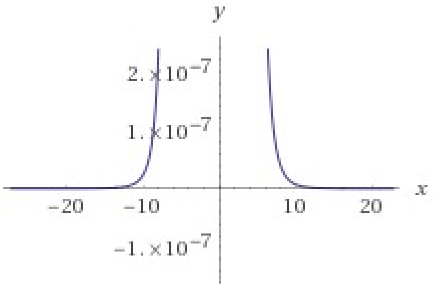
**Input:**

$$\frac{1}{(-1+(-1+x)x)(-1+5x+9x^3+6x^5+x^6)}$$

**Plots:**




(x from -6.5 to 1.9)




(x from -27.3 to 22.7)

**Alternate forms:**

$$\frac{1}{(x^2-x-1)(x^6+6x^5+9x^3+5x-1)}$$

$$\frac{1}{x(x(x(x(x(x(x(x+5)-7)+3)-9)-4)-6)-4)+1}$$

(ignoring removable singularities)

**Partial fraction expansion:**

$$\frac{61x-92}{869(x^2-x-1)}+\frac{-61x^5-335x^4+156x^3-728x^2+256x-777}{869(x^6+6x^5+9x^3+5x-1)}$$

**Expanded form:**

$$\frac{1}{x^8+5x^7-7x^6+3x^5-9x^4-4x^3-6x^2-4x+1}$$

**Roots:**

(no roots exist)

**Properties as a real function:**

**Domain**

$\{x \in \mathbb{R} : x \neq -6.23493$ and $x \neq -0.618034$ and $x \neq 0.187791$ and $x \neq 1.61803\}$

**Range**

$\{y \in \mathbb{R} : y \leq -7.08814\times 10^{-6}$ or $y > 0\}$

$\mathbb{R}$ is the set of real numbers

**Series expansion at x = 0:**

$$1+4x+22x^2+116x^3+621x^4+O(x^5)$$

(Taylor series)

**Series expansion at x = ∞:**

$$\left(\frac{1}{x}\right)^8-\frac{5}{x^9}+\frac{32}{x^{10}}-\frac{198}{x^{11}}+O\left(\left(\frac{1}{x}\right)^{12}\right)$$

(Laurent series)

**Derivative:**

$$\frac{d}{dx}\left(\frac{1}{(-1+(-1+x)x)(-1+5x+9x^3+6x^5+x^6)}\right)=$$
$$\frac{-8x^7-35x^6+42x^5-15x^4+36x^3+12x^2+12x+4}{(x^2-x-1)^2(x^6+6x^5+9x^3+5x-1)^2}$$

**Indefinite integral:**

$$\int \frac{1}{(-1+(-1+x)x)(-1+5x+9x^3+6x^5+x^6)}\,dx =$$
$$\frac{1}{8690}\Bigg(-10\sum_{\{\omega:\,\omega^6+6\omega^5+9\omega^3+5\omega-1=0\}}\frac{1}{6\omega^5+30\omega^4+27\omega^2+5}(61\omega^5\log(x-\omega)+$$
$$335\omega^4\log(x-\omega)-156\omega^3\log(x-\omega)+728\omega^2\log(x-\omega)-$$
$$256\omega\log(x-\omega)+777\log(x-\omega))+(305-123\sqrt{5})$$
$$\log(-2x+\sqrt{5}+1)+(305+123\sqrt{5})\log(2x+\sqrt{5}-1)\Bigg)+\text{constant}$$

(assuming a complex-valued logarithm)

$\log(x)$ is the natural logarithm

**Local maxima:**

$$\max\left\{\frac{1}{(-1+(-1+x)\,x)\left(-1+5\,x+9\,x^3+6\,x^5+x^6\right)}\right\} = 0 \text{ at } x \approx -5.4317$$

$$\max\left\{\frac{1}{(-1+(-1+x)\,x)\left(-1+5\,x+9\,x^3+6\,x^5+x^6\right)}\right\} \approx -0.031087 \text{ at } x \approx 1.3326$$

**Local minimum:**

$$\min\left\{\frac{1}{(-1+(-1+x)\,x)\left(-1+5\,x+9\,x^3+6\,x^5+x^6\right)}\right\} \approx 0.59400 \text{ at } x \approx -0.31377$$

**Limit:**

$$\lim_{x\to\pm\infty} \frac{1}{(-1+(-1+x)\,x)\left(-1+5\,x+9\,x^3+6\,x^5+x^6\right)} = 0$$

From

$$\frac{1}{\left(-x^2+x+1\right)\left(-x^6-6\,x^5-9\,x^3-5\,x+1\right)}$$

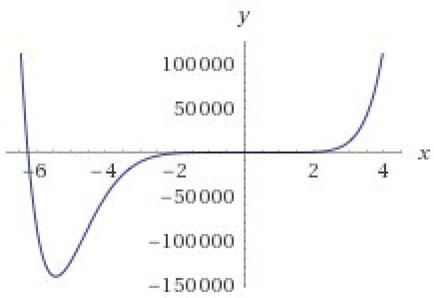
we obtain:

((-1 + (-1 + x) x) (-1 + 5 x + 9 x^3 + 6 x^5 + x^6))

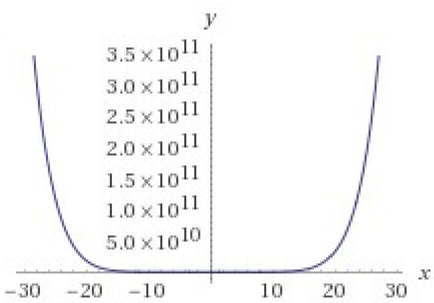
**Input:**

$$(-1+(-1+x)\,x)\left(-1+5\,x+9\,x^3+6\,x^5+x^6\right)$$

**Plots:**




(x from -6.6 to 4.3)




(x from -30 to 30)

## Alternate forms:

$x(x(x(x(x(x(x(x+5)-7)+3)-9)-4)-6)-4)+1$

$(x^2-x-1)(x^6+6x^5+9x^3+5x-1)$

## Alternate form assuming x>0:

$(x-1)x^7+6(x-1)x^6-x^6-6x^5+9(x-1)x^4-9x^3+5(x-1)x^2-(x-1)x-5x+1$

## Expanded form:

$x^8+5x^7-7x^6+3x^5-9x^4-4x^3-6x^2-4x+1$

## Real roots:

$x = \frac{1}{2} - \frac{\sqrt{5}}{2}$

$x = \frac{1}{2} + \frac{\sqrt{5}}{2}$

$x =$ root of $x^6+6x^5+9x^3+5x-1$ near $x = -6.23493$

$x =$ root of $x^6+6x^5+9x^3+5x-1$ near $x = 0.187791$

## Complex roots:

$x \approx -0.310345 - 0.83398\,i$

$x \approx -0.310345 + 0.83398\,i$

$x \approx 0.333915 - 0.983411\,i$

$x \approx 0.333915 + 0.983411\,i$

## Polynomial discriminant:

$\Delta = 9619368712066960$

## Properties as a real function:

### Domain

$\mathbb{R}$ (all real numbers)

### Range

$\{y \in \mathbb{R} : y \geq -141081.\}$

$\mathbb{R}$ is the set of real numbers

**Derivative:**

$$\frac{d}{dx}\left(\left(-1+(-1+x)\,x\right)\left(-1+5x+9x^3+6x^5+x^6\right)\right) = 8x^7+35x^6-42x^5+15x^4-36x^3-12x^2-12x-4$$

**Indefinite integral:**

$$\int \left(-1+(-1+x)\,x\right)\left(-1+5x+9x^3+6x^5+x^6\right)dx = \frac{x^9}{9}+\frac{5x^8}{8}-x^7+\frac{x^6}{2}-\frac{9x^5}{5}-x^4-2x^3-2x^2+x+\text{constant}$$

**Local maximum:**

$$\max\left\{\left(-1+(-1+x)\,x\right)\left(-1+5x+9x^3+6x^5+x^6\right)\right\} \approx 1.6835 \text{ at } x \approx -0.31377$$

**Global minimum:**

$$\min\left\{\left(-1+(-1+x)\,x\right)\left(-1+5x+9x^3+6x^5+x^6\right)\right\} = -141081 \text{ at } x \approx -5.4317$$

From

$$x = \frac{1}{2}+\frac{\sqrt{5}}{2}$$

we obtain:

sqrt[1/(((1/2 + sqrt(5)/2)))]

**Input:**

$$\sqrt{\frac{1}{\frac{1}{2}+\frac{\sqrt{5}}{2}}}$$

**Result:**

$$\frac{1}{\sqrt{\frac{1}{2}+\frac{\sqrt{5}}{2}}}$$

**Decimal approximation:**

0.7861513777574232860695585858429589295231220578377232376 64...

0.786151377757…

**Alternate forms:**

$$\frac{1}{2}\sqrt{\left(\sqrt{5}-1\right)2}$$

$$\sqrt{\frac{1}{2}\left(\sqrt{5}-1\right)}$$

$$\sqrt{\frac{2}{1+\sqrt{5}}}$$

**Minimal polynomial:**

$x^4 + x^2 - 1$

**All 2nd roots of 1/(1/2 + sqrt(5)/2):**

$$\frac{e^0}{\sqrt{\frac{1}{2}+\frac{\sqrt{5}}{2}}} \approx 0.78615 \text{ (real, principal root)}$$

$$\frac{e^{i\pi}}{\sqrt{\frac{1}{2}+\frac{\sqrt{5}}{2}}} \approx -0.7862 \text{ (real root)}$$

Indeed:

integrate((((sqrt[1/(((1/2 + sqrt(5)/2))]))))dx x = infinity*i..1/24

**Definite integral:**

$$\int_{\infty i}^{\frac{1}{24}} \sqrt{\frac{1}{\frac{1}{2}+\frac{\sqrt{5}}{2}}}\, dx = i(-\infty) + \frac{1}{12\sqrt{2\left(1+\sqrt{5}\right)}}$$

**Indefinite integral:**

$$\int \sqrt{\frac{1}{\frac{1}{2}+\frac{\sqrt{5}}{2}}}\, dx = \frac{x}{\sqrt{\frac{1}{2}+\frac{\sqrt{5}}{2}}} + \text{constant}$$

$$\int \sqrt{\frac{1}{\frac{1}{2}+\frac{\sqrt{5}}{2}}}\, dx \approx \text{constant} + 0.786151\, x$$

0.786151

From

$$\frac{1}{\sqrt{\frac{1}{2}+\frac{\sqrt{5}}{2}}}$$

we obtain:

[1/(((sqrt[1/(((1/2 + sqrt(5)/2)))]))))]^2

**Input:**

$$\left(\frac{1}{\sqrt{\frac{1}{\frac{1}{2}+\frac{\sqrt{5}}{2}}}}\right)^2$$

**Result:**

$$\frac{1}{2}+\frac{\sqrt{5}}{2}$$

**Decimal approximation:**

1.6180339887498948482045868343656381177203091798057628621354...

**1.6180339887…**

$\phi \approx$ 1.618033988749894848204586834365638117720309179805762862135448 62

**Alternate form:**

$$\frac{1}{2}\left(1+\sqrt{5}\right)$$

**Minimal polynomial:**

$$x^2 - x - 1$$

Furthermore, we obtain also:

[1/(((sqrt[1/(((1/2 + sqrt(5)/2)))])))]^2+(21+5)1/10^3

**Input:**

$$\left(\frac{1}{\sqrt{\frac{1}{\frac{1}{2}+\frac{\sqrt{5}}{2}}}}\right)^2 + (21+5)\times\frac{1}{10^3}$$

**Result:**

$$\frac{263}{500}+\frac{\sqrt{5}}{2}$$

**Decimal approximation:**

1.644033988749894848204586834365638117720309179805762862135...

1.6440339887…

**Alternate form:**

$$\frac{1}{500}\left(263+250\sqrt{5}\right)$$

**Minimal polynomial:**

$$250\,000\,x^2 - 263\,000\,x - 243\,331$$

We have that:

$\mathbb{H}/\Gamma_0(30)$ has genus 3.
There is a function of order 2 on $\mathbb{H}/\Gamma_0(30)$ given by

$$x = 2\frac{\eta_6\eta_{10}}{\eta_1\eta_{15}} = 2 + 2q + 4q^2 + 6q^3 + \cdots$$

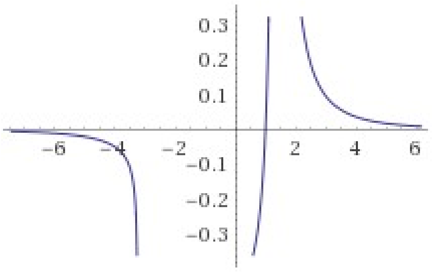
$$\eta_1\eta_3\eta_5\eta_{15}2\pi i d\tau = \frac{2(x-1)dx}{\sqrt{(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)}},$$
$$\eta_3\eta_5\eta_6\eta_{10}2\pi i d\tau = \frac{x(x-1)dx}{\sqrt{(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)}},$$
$$\eta_1\eta_2\eta_{15}\eta_{30}2\pi i d\tau = \frac{(x-2)dx}{\sqrt{(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)}}.$$

(2(x-1)) / [(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)]^1/2

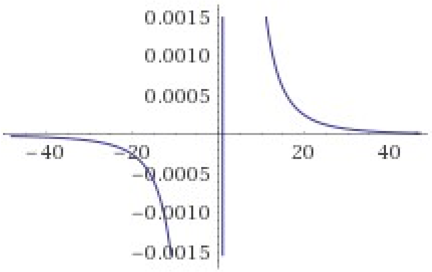
**Input:**

$$\frac{2(x-1)}{\sqrt{(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)}}$$

**Plots:**




(x from -7.4 to 6.1)




(x from -48.1 to 46.8)

**Alternate forms:**

$$\frac{2(x-1)}{\sqrt{((x-1)x-1)(x(x+2)-4)((x-2)x((x-1)x+3)+4)}}$$

$$\frac{2(x-1)}{\sqrt{x^8-2x^7-5x^6+22x^5-39x^4+44x^3-20x^2-16x+16}}$$

**Expanded form:**

$$\frac{2x\sqrt{(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)}}{(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)} - \frac{2\sqrt{(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)}}{(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)}$$

**Alternate forms assuming x>0:**

$$\frac{2(x-1)(-1)^{\lfloor -(\arg(x^2-x-1)+\arg(x^2+2x-4)-\pi)/(2\pi)\rfloor}}{\sqrt{x^2-x-1}\sqrt{x^2+2x-4}\sqrt{x^4-3x^3+5x^2-6x+4}}$$

$$\frac{2x\exp\left(i\pi\left\lfloor -\frac{\arg(x^2-x-1)}{2\pi}-\frac{\arg(x^2+2x-4)}{2\pi}+\frac{1}{2}\right\rfloor\right)}{\sqrt{x^2-x-1}\sqrt{x^2+2x-4}\sqrt{x^4-3x^3+5x^2-6x+4}} - \frac{2\exp\left(i\pi\left\lfloor -\frac{\arg(x^2-x-1)}{2\pi}-\frac{\arg(x^2+2x-4)}{2\pi}+\frac{1}{2}\right\rfloor\right)}{\sqrt{x^2-x-1}\sqrt{x^2+2x-4}\sqrt{x^4-3x^3+5x^2-6x+4}}$$

**Series expansion at x = 0:**

$$-\frac{1}{2}+\frac{x}{4}-\frac{x^2}{4}+\frac{9x^3}{16}-\frac{21x^4}{32}+O(x^5)$$

(Taylor series)

**Series expansion at x = -1 - sqrt(5):**

$$-\frac{2+\sqrt{5}}{2\left(\sqrt[4]{5}\sqrt{6\left(38+17\sqrt{5}\right)}\sqrt{-x-\sqrt{5}-1}\right)}-\frac{\left(11002+4921\sqrt{5}\right)\left(x+\sqrt{5}+1\right)}{96\left(5^{3/4}\sqrt{6}\left(38+17\sqrt{5}\right)^{3/2}\sqrt{-x-\sqrt{5}-1}\right)}-\frac{\left(252550660+112944089\sqrt{5}\right)\left(x+\sqrt{5}+1\right)^2}{15360\left(5^{3/4}\sqrt{6}\left(9+4\sqrt{5}\right)^2\left(38+17\sqrt{5}\right)^{3/2}\sqrt{-x-\sqrt{5}-1}\right)}-\frac{7\left(3770014184+1686001599\sqrt{5}\right)\left(x+\sqrt{5}+1\right)^3}{2211840\left(\sqrt[4]{5}\left(9+4\sqrt{5}\right)^4\sqrt{6\left(38+17\sqrt{5}\right)}\sqrt{-x-\sqrt{5}-1}\right)}-\frac{\left(136327471816423+60967498836384\sqrt{5}\right)\left(x+\sqrt{5}+1\right)^4}{424673280\left(\sqrt[4]{5}\left(2+\sqrt{5}\right)^3\left(9+4\sqrt{5}\right)^4\sqrt{6\left(38+17\sqrt{5}\right)}\sqrt{-x-\sqrt{5}-1}\right)}+O\left(\left(x+\sqrt{5}+1\right)^5\right)$$

(generalized Puiseux series)

**Series expansion at x = 1/2 (1 - sqrt(5)):**

$$-\frac{1+\sqrt{5}}{\sqrt[4]{5}\sqrt{6\left(2+\sqrt{5}\right)}\sqrt{2x+\sqrt{5}-1}}-\frac{\left(92+37\sqrt{5}\right)\left(x-\frac{1}{2}\left(1-\sqrt{5}\right)\right)}{12\left(5^{3/4}\sqrt{6}\left(2+\sqrt{5}\right)^{3/2}\sqrt{2x+\sqrt{5}-1}\right)}-\frac{\left(30265+13509\sqrt{5}\right)\left(x-\frac{1}{2}\left(1-\sqrt{5}\right)\right)^2}{240\left(5^{3/4}\sqrt{6}\left(1+\sqrt{5}\right)\left(2+\sqrt{5}\right)^{3/2}\left(3+\sqrt{5}\right)^2\sqrt{2x+\sqrt{5}-1}\right)}+\frac{7\left(66538+29781\sqrt{5}\right)\left(x-\frac{1}{2}\left(1-\sqrt{5}\right)\right)^3}{4320\sqrt[4]{5}\sqrt{6\left(2+\sqrt{5}\right)}\left(3+\sqrt{5}\right)^4\sqrt{2x+\sqrt{5}-1}}+\frac{7\left(45870577+20532075\sqrt{5}\right)\left(x-\frac{1}{2}\left(1-\sqrt{5}\right)\right)^4}{6635520\sqrt[4]{5}\sqrt{6}\left(2+\sqrt{5}\right)^{3/2}\left(47+21\sqrt{5}\right)\sqrt{2x+\sqrt{5}-1}}+O\left(\left(x-\frac{1}{2}\left(1-\sqrt{5}\right)\right)^5\right)$$

(generalized Puiseux series)

**Derivative:**

$$\frac{d}{dx}\left(\frac{2(x-1)}{\sqrt{\left(x^2-x-1\right)\left(x^2+2x-4\right)\left(x^4-3x^3+5x^2-6x+4\right)}}\right)=-\frac{2\left(3x^8-9x^7-3x^6+48x^5-94x^4+100x^3-66x^2+28x-8\right)}{\left(x^8-2x^7-5x^6+22x^5-39x^4+44x^3-20x^2-16x+16\right)^{3/2}}$$

((((2(x-1)) / [(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)]^1/2)))^2

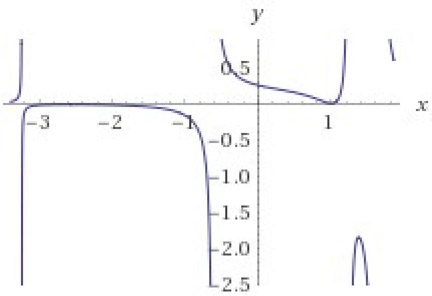
**Input:**

$$\left(\frac{2(x-1)}{\sqrt{(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)}}\right)^2$$

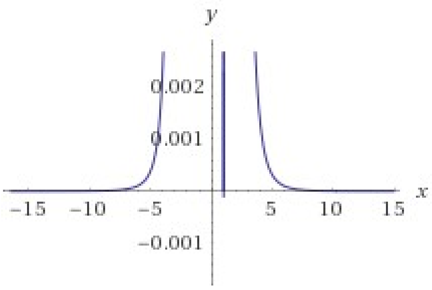
**Result:**

$$\frac{4(x-1)^2}{(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)}$$

**Plots:**




(x from -3.4 to 1.8)




(x from -16.5 to 14.9)

**Alternate forms:**

$$\frac{4(x-1)^2}{((x-1)x-1)(x(x+2)-4)((x-2)x((x-1)x+3)+4)}$$

$$\frac{x(4x-8)+4}{x(x(x(x(x(x((x-2)x-5)+22)-39)+44)-20)-16)+16}$$

**Partial fraction expansion:**

$$\frac{-x-3}{12(x^2+2x-4)}+\frac{x}{3(x^2-x-1)}+\frac{-x^3+2x^2-4x+3}{4(x^4-3x^3+5x^2-6x+4)}$$

**Expanded form:**

$$\frac{4x^2-8x+4}{x^8-2x^7-5x^6+22x^5-39x^4+44x^3-20x^2-16x+16}$$

**Root:**

$x = 1$

**Properties as a real function:**

**Domain**

$\{x \in \mathbb{R} : x + \sqrt{5} + 1 < 0 \text{ or } \frac{1}{2}\left(1 - \sqrt{5}\right) < x < \sqrt{5} - 1 \text{ or } 2x > 1 + \sqrt{5}\}$

**Range**

$\{y \in \mathbb{R} : y \geq 0\}$ (all non-negative real numbers)

$\mathbb{R}$ is the set of real numbers

**Series expansion at x = 0:**

$$\frac{1}{4} - \frac{x}{4} + \frac{5x^2}{16} - \frac{11x^3}{16} + x^4 + O(x^5)$$

(Taylor series)

**Series expansion at x = ∞:**

$$\frac{4}{x^6} + \frac{24}{x^8} - \frac{40}{x^9} + O\left(\left(\frac{1}{x}\right)^{10}\right)$$

(Laurent series)

**Derivative:**

$$\frac{d}{dx}\left(\left(\frac{2(x-1)}{\sqrt{(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)}}\right)^2\right) = -\frac{8(x-1)(3x^8-9x^7-3x^6+48x^5-94x^4+100x^3-66x^2+28x-8)}{(x^2-x-1)^2(x^2+2x-4)^2(x^4-3x^3+5x^2-6x+4)^2}$$

**Indefinite integral:**

$$\int\left(\frac{2(x-1)}{\sqrt{(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)}}\right)^2 dx =$$

$$\frac{1}{120}\left(-30\sum_{\{\omega:\,\omega^4+\omega^3+2\omega^2-\omega+1=0\}}\frac{\omega^3\log(x-\omega-1)+\omega^2\log(x-\omega-1)+3\omega\log(x-\omega-1)}{4\omega^3+3\omega^2+4\omega-1} + 4(5+\sqrt{5})\log(-2x+\sqrt{5}+1) - (5+2\sqrt{5})\log(-x+\sqrt{5}-1) + (2\sqrt{5}-5)\log(x+\sqrt{5}+1) - 4(\sqrt{5}-5)\log(2x+\sqrt{5}-1)\right) + \text{constant}$$

(assuming a complex-valued logarithm)

$\log(x)$ is the natural logarithm

**Local maxima:**

$$\max\left\{\left(\frac{2(x-1)}{\sqrt{(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)}}\right)^2\right\} \approx -0.016731$$

at $x \approx -2.5823$

$$\max\left\{\left(\frac{2(x-1)}{\sqrt{(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)}}\right)^2\right\} \approx -1.8495$$
at $x \approx 1.3675$

**Local minimum:**

$$\min\left\{\left(\frac{2(x-1)}{\sqrt{(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)}}\right)^2\right\} = 0 \text{ at } x = 1$$

**Limit:**

$$\lim_{x\to\pm\infty} \frac{4(-1+x)^2}{(-1-x+x^2)(-4+2x+x^2)(4-6x+5x^2-3x^3+x^4)} = 0$$

1/ (((((((2(x-1)) 1/ [(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)]^1/2)))^2)))

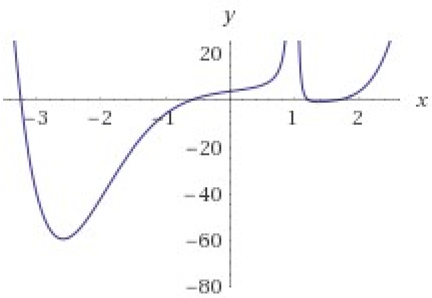
**Input:**

$$\frac{1}{\left((2(x-1)) \times \frac{1}{\sqrt{(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)}}\right)^2}$$

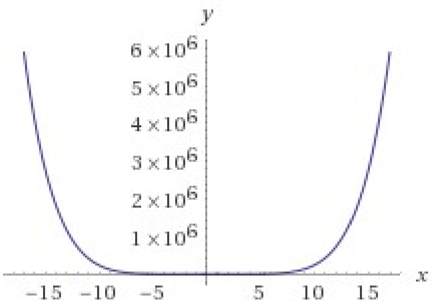
**Result:**

$$\frac{(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)}{4(x-1)^2}$$

**Plots:**




(x from -3.4 to 2.5)




(x from -18.2 to 17.3)

**Alternate forms:**

$$\frac{((x-1)x-1)(x(x+2)-4)((x-2)x((x-1)x+3)+4)}{4(x-1)^2}$$

$$\frac{x(x(x(x(x(x((x-2)x-5)+22)-39)+44)-20)-16)+16}{x(4x-8)+4}$$

$$\frac{x^6}{4}-\frac{3x^4}{2}+\frac{5x^3}{2}-\frac{13x^2}{4}+2x-\frac{3}{2(x-1)}+\frac{1}{4(x-1)^2}+\frac{9}{4}$$

**Expanded form:**

$$\frac{x^8}{4(x-1)^2}-\frac{x^7}{2(x-1)^2}-\frac{5x^6}{4(x-1)^2}+\frac{11x^5}{2(x-1)^2}-\frac{39x^4}{4(x-1)^2}+\frac{11x^3}{(x-1)^2}-\frac{5x^2}{(x-1)^2}-\frac{4x}{(x-1)^2}+\frac{4}{(x-1)^2}$$

**Alternate form assuming x>0:**

$$\frac{x^8-2x^7-5x^6+22x^5-39x^4+44x^3-20x^2-16x+16}{4(x-1)^2}$$

**Quotient and remainder:**

$$x^8-2x^7-5x^6+22x^5-39x^4+44x^3-20x^2-16x+16=\left(\frac{x^6}{4}-\frac{3x^4}{2}+\frac{5x^3}{2}-\frac{13x^2}{4}+2x+\frac{9}{4}\right)\times(4x^2-8x+4)+7-6x$$

**Real roots:**

$$x=-1-\sqrt{5}$$

$$x=\frac{1}{2}-\frac{\sqrt{5}}{2}$$

$$x=\sqrt{5}-1$$

$$x=\frac{1}{2}+\frac{\sqrt{5}}{2}$$

1.6180339887…

**Complex roots:**

$$x=\frac{1}{4}\left(3+\sqrt{5}-i\sqrt{18-6\sqrt{5}}\right)$$

$$x=\frac{1}{4}\left(3+\sqrt{5}+i\sqrt{18-6\sqrt{5}}\right)$$

$$x = \frac{1}{4}\left(3 - \sqrt{5} - i\sqrt{6\left(3 + \sqrt{5}\right)}\right)$$

$$x = \frac{1}{4}\left(3 - \sqrt{5} + i\sqrt{6\left(3 + \sqrt{5}\right)}\right)$$

**Properties as a real function:**

**Domain**

$$\left\{x \in \mathbb{R} : x + \sqrt{5} + 1 < 0 \text{ or } \frac{1}{2}\left(1 - \sqrt{5}\right) < x < 1 \text{ or } 1 < x < \sqrt{5} - 1 \text{ or } 2x > 1 + \sqrt{5}\right\}$$

**Range**

$\{y \in \mathbb{R} : y > 0\}$ (all positive real numbers)

$\mathbb{R}$ is the set of real numbers

**Series expansion at x = 0:**

$$4 + 4x - x^2 + 5x^3 + \frac{5x^4}{4} + O(x^5)$$

(Taylor series)

**Series expansion at x = ∞:**

$$\frac{x^6}{4} - \frac{3x^4}{2} + \frac{5x^3}{2} - \frac{13x^2}{4} + 2x + \frac{9}{4} + O\left(\left(\frac{1}{x}\right)^1\right)$$

(Taylor series)

**Derivative:**

$$\frac{d}{dx}\left(\frac{1}{\left(\frac{2(x-1)}{\sqrt{(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)}}\right)^2}\right) = \frac{3x^8 - 9x^7 - 3x^6 + 48x^5 - 94x^4 + 100x^3 - 66x^2 + 28x - 8}{2(x-1)^3}$$

**Indefinite integral:**

$$\int \frac{(-1 - x + x^2)(-4 + 2x + x^2)(4 - 6x + 5x^2 - 3x^3 + x^4)}{4(-1+x)^2}\,dx = \frac{1}{4}\left(\frac{x^7}{7} - \frac{6x^5}{5} + \frac{5x^4}{2} - \frac{13x^3}{3} + 4x^2 + 9x + \frac{1}{1-x} - 6\log(x-1) - \frac{2123}{210}\right) + \text{constant}$$

(assuming a complex-valued logarithm)

$\log(x)$ is the natural logarithm

**Local minima:**

$$\min\left\{\frac{1}{\left(\frac{2(x-1)}{\sqrt{(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)}}\right)^2}\right\} \approx -59.768 \text{ at } x \approx -2.5823$$

$$\min\left\{\frac{1}{\left(\frac{2(x-1)}{\sqrt{(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)}}\right)^2}\right\} \approx -0.54070 \text{ at } x \approx 1.3675$$

**Series representations:**

$$\frac{(-1-x+x^2)(-4+2x+x^2)(4-6x+5x^2-3x^3+x^4)}{4(-1+x)^2} = \sum_{n=-\infty}^{\infty} \left(\begin{cases} 4 & n=0 \\ -1 & n=2 \\ \frac{5}{4} & n=4 \\ \frac{7+n}{4} & (n=5 \text{ or } n>6) \\ \frac{7}{2} & n=6 \\ 4 & n=1 \\ 5 & n=3 \end{cases}\right) x^n$$

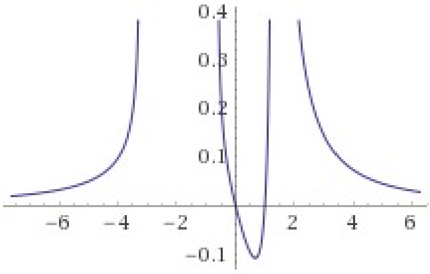
$$\frac{(-1-x+x^2)(-4+2x+x^2)(4-6x+5x^2-3x^3+x^4)}{4(-1+x)^2} = \sum_{n=-\infty}^{\infty} \left(\begin{cases} -\frac{3}{2} & (n=-1 \text{ or } n=1) \\ -1 & n=2 \\ \frac{1}{4} & (n=-2 \text{ or } n=6) \\ \frac{3}{2} & (n=3 \text{ or } n=5) \\ \frac{9}{4} & (n=0 \text{ or } n=4) \end{cases}\right) (-1+x)^n$$

(x(x-1)) / [(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)]^1/2

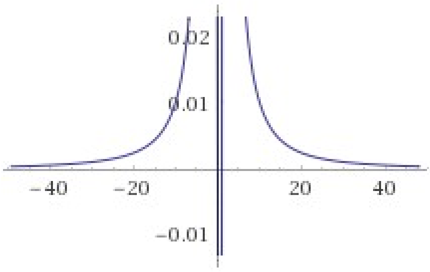
**Input:**

$$\frac{x(x-1)}{\sqrt{(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)}}$$

## Plots:




(x from −7.6 to 6.2)




(x from −49.2 to 47.8)

## Alternate forms:

$$\frac{(x-1)\,x}{\sqrt{((x-1)\,x-1)\,(x\,(x+2)-4)\,((x-2)\,x\,((x-1)\,x+3)+4)}}$$

$$\frac{(x-1)\,x}{\sqrt{x^8-2\,x^7-5\,x^6+22\,x^5-39\,x^4+44\,x^3-20\,x^2-16\,x+16}}$$

## Expanded form:

$$\frac{x^2\sqrt{(x^2-x-1)\,(x^2+2\,x-4)\,(x^4-3\,x^3+5\,x^2-6\,x+4)}}{(x^2-x-1)\,(x^2+2\,x-4)\,(x^4-3\,x^3+5\,x^2-6\,x+4)}-\frac{x\sqrt{(x^2-x-1)\,(x^2+2\,x-4)\,(x^4-3\,x^3+5\,x^2-6\,x+4)}}{(x^2-x-1)\,(x^2+2\,x-4)\,(x^4-3\,x^3+5\,x^2-6\,x+4)}$$

## Alternate forms assuming x>0:

$$\frac{(x-1)\,x\,(-1)^{\lfloor -(\arg(x^2-x-1)+\arg(x^2+2\,x-4)-\pi)/(2\,\pi)\rfloor}}{\sqrt{x^2-x-1}\,\sqrt{x^2+2\,x-4}\,\sqrt{x^4-3\,x^3+5\,x^2-6\,x+4}}$$

$$\frac{x^2\exp\left(i\,\pi\left\lfloor-\frac{\arg(x^2-x-1)}{2\,\pi}-\frac{\arg(x^2+2\,x-4)}{2\,\pi}+\frac{1}{2}\right\rfloor\right)}{\sqrt{x^2-x-1}\,\sqrt{x^2+2\,x-4}\,\sqrt{x^4-3\,x^3+5\,x^2-6\,x+4}}+\frac{x\exp\left(i\,\pi\left(\left\lfloor-\frac{\arg(x^2-x-1)}{2\,\pi}-\frac{\arg(x^2+2\,x-4)}{2\,\pi}+\frac{1}{2}\right\rfloor+1\right)\right)}{\sqrt{x^2-x-1}\,\sqrt{x^2+2\,x-4}\,\sqrt{x^4-3\,x^3+5\,x^2-6\,x+4}}$$

**Series expansion at x = 0:**

$$-\frac{x}{4}+\frac{x^2}{8}-\frac{x^3}{8}+\frac{9x^4}{32}-\frac{21x^5}{64}+O\left(x^6\right)$$

(Taylor series)

**Series expansion at x = -1 - sqrt(5):**

$$\frac{7+3\sqrt{5}}{4\sqrt[4]{5}\sqrt{6(38+17\sqrt{5})}\sqrt{-x-\sqrt{5}-1}}+$$
$$\frac{149(123+55\sqrt{5})(x+\sqrt{5}+1)}{192\times 5^{3/4}\sqrt{6}(38+17\sqrt{5})^{3/2}\sqrt{-x-\sqrt{5}-1}}+$$
$$\frac{7(35\,772\,855+15\,998\,107\sqrt{5})(x+\sqrt{5}+1)^2}{30\,720\times 5^{3/4}\sqrt{6}(9+4\sqrt{5})^2(38+17\sqrt{5})^{3/2}\sqrt{-x-\sqrt{5}-1}}+$$
$$\frac{(2\,805\,358\,112\,695+1\,254\,594\,288\,243\sqrt{5})(x+\sqrt{5}+1)^3}{4\,423\,680\times 5^{3/4}\sqrt{6}(9+4\sqrt{5})^4(38+17\sqrt{5})^{3/2}\sqrt{-x-\sqrt{5}-1}}+$$
$$\frac{(56\,013\,978\,923\,335+25\,050\,212\,912\,647\sqrt{5})(x+\sqrt{5}+1)^4}{849\,346\,560\sqrt[4]{5}(2+\sqrt{5})^3(9+4\sqrt{5})^4\sqrt{6(38+17\sqrt{5})}\sqrt{-x-\sqrt{5}-1}}+$$
$$O\left(\left(x+\sqrt{5}+1\right)^5\right)$$

(generalized Puiseux series)

**Series expansion at x = 1/2 (1 - sqrt(5)):**

$$\frac{1}{\sqrt[4]{5}\sqrt{6(2+\sqrt{5})}\sqrt{2x+\sqrt{5}-1}}-\frac{(267+113\sqrt{5})\left(x-\frac{1}{2}(1-\sqrt{5})\right)}{48\left(5^{3/4}\sqrt{6}(2+\sqrt{5})^{3/2}\sqrt{2x+\sqrt{5}-1}\right)}-$$
$$\frac{(30\,491+13\,632\sqrt{5})\left(x-\frac{1}{2}(1-\sqrt{5})\right)^2}{240\left(\sqrt[4]{5}\sqrt{6}(1+\sqrt{5})(2+\sqrt{5})^{3/2}(3+\sqrt{5})^2\sqrt{2x+\sqrt{5}-1}\right)}-$$
$$\frac{(14\,677\,235+6\,564\,009\sqrt{5})\left(x-\frac{1}{2}(1-\sqrt{5})\right)^3}{17\,280\left(5^{3/4}\sqrt{6}(2+\sqrt{5})^{3/2}(3+\sqrt{5})^4\sqrt{2x+\sqrt{5}-1}\right)}+$$
$$\frac{7(57\,709\,745+25\,745\,453\sqrt{5})\left(x-\frac{1}{2}(1-\sqrt{5})\right)^4}{13\,271\,040\times 5^{3/4}\sqrt{6}(2+\sqrt{5})^{3/2}(47+21\sqrt{5})\sqrt{2x+\sqrt{5}-1}}+$$
$$O\left(\left(x-\frac{1}{2}\left(1-\sqrt{5}\right)\right)^5\right)$$

(generalized Puiseux series)

**Derivative:**

$$\frac{d}{dx}\left(\frac{x(x-1)}{\sqrt{(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)}}\right) = \frac{-2x^9+6x^8-21x^6+33x^5-17x^4+2x^3-24x^2+40x-16}{(x^8-2x^7-5x^6+22x^5-39x^4+44x^3-20x^2-16x+16)^{3/2}}$$

((((x(x-1)) / [(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)]^1/2)))^2

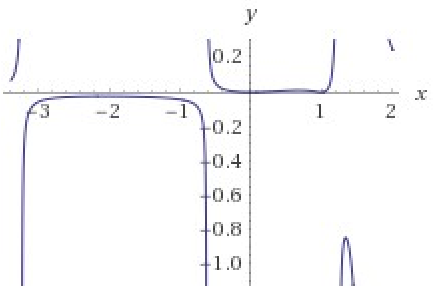
**Input:**

$$\left(\frac{x(x-1)}{\sqrt{(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)}}\right)^2$$

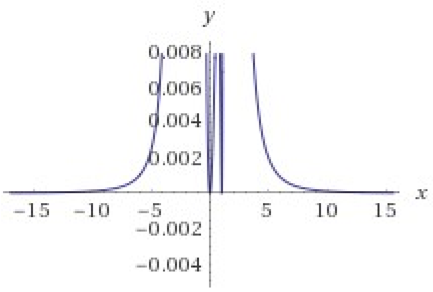
**Result:**

$$\frac{(x-1)^2 x^2}{(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)}$$

**Plots:**




(x from −3.4 to 2)




(x from −16.9 to 15.5)

**Alternate forms:**

$$\frac{(x-1)^2 x^2}{((x-1)x-1)(x(x+2)-4)((x-2)x((x-1)x+3)+4)}$$

$$\frac{x^2((x-2)x+1)}{x(x(x(x(x(x((x-2)x-5)+22)-39)+44)-20)-16)+16}$$

**Expanded form:**

$$\frac{x^4 - 2x^3 + x^2}{x^8 - 2x^7 - 5x^6 + 22x^5 - 39x^4 + 44x^3 - 20x^2 - 16x + 16}$$

**Roots:**

$x = 0$

$x = 1$

**Properties as a real function:**

**Domain**

$\{x \in \mathbb{R} : x + \sqrt{5} + 1 < 0 \text{ or } \frac{1}{2}\left(1 - \sqrt{5}\right) < x < \sqrt{5} - 1 \text{ or } 2x > 1 + \sqrt{5}\}$

**Range**

$\{y \in \mathbb{R} : y \geq 0\}$ (all non-negative real numbers)

$\mathbb{R}$ is the set of real numbers

**Series expansion at x = 0:**

$$\frac{x^2}{16} - \frac{x^3}{16} + \frac{5x^4}{64} - \frac{11x^5}{64} + \frac{x^6}{4} + O(x^7)$$

(Taylor series)

**Series expansion at x = ∞:**

$$\left(\frac{1}{x}\right)^4 + \frac{6}{x^6} - \frac{10}{x^7} + O\left(\left(\frac{1}{x}\right)^8\right)$$

(Laurent series)

**Derivative:**

$$\frac{d}{dx}\left(\left(\frac{x(x-1)}{\sqrt{(x^2 - x - 1)(x^2 + 2x - 4)(x^4 - 3x^3 + 5x^2 - 6x + 4)}}\right)^2\right) =$$
$$-\frac{2x(2x^{10} - 8x^9 + 6x^8 + 21x^7 - 54x^6 + 50x^5 - 19x^4 + 26x^3 - 64x^2 + 56x - 16)}{(x^2 - x - 1)^2 (x^2 + 2x - 4)^2 (x^4 - 3x^3 + 5x^2 - 6x + 4)^2}$$

## Indefinite integral:

$$\int \left( \frac{x(x-1)}{\sqrt{(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)}} \right)^2 dx =$$

$$\frac{1}{240} \left( -30 \sum_{\{\omega:\, \omega^4 - 3\omega^3 + 5\omega^2 - 6\omega + 4 = 0\}} \frac{\omega^3 \log(x-\omega) - \omega^2 \log(x-\omega) + \omega \log(x-\omega) - 2\log(x-\omega)}{4\omega^3 - 9\omega^2 + 10\omega - 6} + 4(5+2\sqrt{5}) \log(-2x+\sqrt{5}+1) - (5+\sqrt{5}) \log(-x+\sqrt{5}-1) + (\sqrt{5}-5) \log(x+\sqrt{5}+1) + 4(5-2\sqrt{5}) \log(2x+\sqrt{5}-1) \right) + \text{constant}$$

(assuming a complex-valued logarithm)

$\log(x)$ is the natural logarithm

## Local maxima:

$$\max\left\{ \left( \frac{x(x-1)}{\sqrt{(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)}} \right)^2 \right\} \approx -0.023622$$

at $x \approx -2.0737$

$$\max\left\{ \left( \frac{x(x-1)}{\sqrt{(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)}} \right)^2 \right\} \approx 0.011887$$

at $x \approx 0.67630$

## Local minima:

$$\min\left\{ \left( \frac{x(x-1)}{\sqrt{(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)}} \right)^2 \right\} = 0 \text{ at } x = 0$$

$$\min\left\{ \left( \frac{x(x-1)}{\sqrt{(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)}} \right)^2 \right\} = 0 \text{ at } x = 1$$

## Limit:

$$\lim_{x \to \pm\infty} \frac{(-1+x)^2 x^2}{(-1-x+x^2)(-4+2x+x^2)(4-6x+5x^2-3x^3+x^4)} = 0$$

1/[((((x(x-1)) / [(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)]^1/2)))^2]

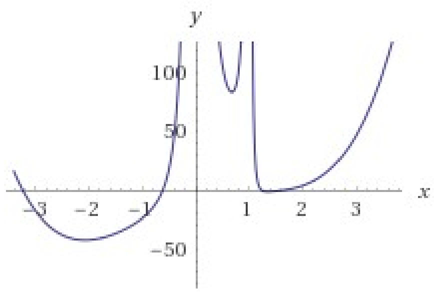
**Input:**

$$\frac{1}{\left(\frac{x(x-1)}{\sqrt{(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)}}\right)^2}$$

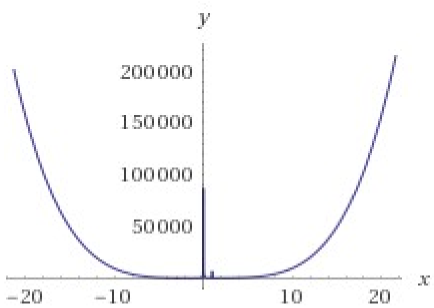
**Result:**

$$\frac{(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)}{(x-1)^2 x^2}$$

**Plots:**




(x from −3.4 to 3.7)




(x from −21.3 to 21.6)

**Alternate forms:**

$$x^4-6x^2+\frac{16}{x^2}+10x-\frac{8}{x-1}+\frac{1}{(x-1)^2}+\frac{16}{x}-13$$

$$\frac{((x-1)x-1)(x(x+2)-4)((x-2)x((x-1)x+3)+4)}{(x-1)^2 x^2}$$

$$\frac{x(x(x(x(x(x((x-2)x-5)+22)-39)+44)-20)-16)+16}{x^2((x-2)x+1)}$$

**Expanded form:**

$$\frac{x^6}{(x-1)^2} - \frac{2x^5}{(x-1)^2} - \frac{5x^4}{(x-1)^2} + \frac{22x^3}{(x-1)^2} - \frac{39x^2}{(x-1)^2} + \frac{16}{(x-1)^2 x^2} + \frac{44x}{(x-1)^2} - \frac{20}{(x-1)^2} - \frac{16}{(x-1)^2 x}$$

**Alternate form assuming x>0:**

$$\frac{x^8 - 2x^7 - 5x^6 + 22x^5 - 39x^4 + 44x^3 - 20x^2 - 16x + 16}{(x-1)^2 x^2}$$

**Quotient and remainder:**

$$x^8 - 2x^7 - 5x^6 + 22x^5 - 39x^4 + 44x^3 - 20x^2 - 16x + 16 = \boxed{(x^4 - 6x^2 + 10x - 13)} \times (x^4 - 2x^3 + x^2) + \boxed{8x^3 - 7x^2 - 16x + 16}$$

**Real roots:**

$$x = -1 - \sqrt{5}$$

$$x = \frac{1}{2} - \frac{\sqrt{5}}{2}$$

$$x = \sqrt{5} - 1$$

$$x = \frac{1}{2} + \frac{\sqrt{5}}{2}$$

1.6180339887…

**Complex roots:**

$$x = \frac{1}{4}\left(3 + \sqrt{5} - i\sqrt{18 - 6\sqrt{5}}\right)$$

$$x = \frac{1}{4}\left(3 + \sqrt{5} + i\sqrt{18 - 6\sqrt{5}}\right)$$

$$x = \frac{1}{4}\left(3 - \sqrt{5} - i\sqrt{6\left(3 + \sqrt{5}\right)}\right)$$

$$x = \frac{1}{4}\left(3 - \sqrt{5} + i\sqrt{6\left(3+\sqrt{5}\right)}\right)$$

**Properties as a real function:**

**Domain**

$$\{x \in \mathbb{R} : x + \sqrt{5} + 1 < 0 \text{ or } \frac{1}{2}\left(1-\sqrt{5}\right) < x < 0 \text{ or } 0 < x < 1 \text{ or } 1 < x < \sqrt{5} - 1 \text{ or } 2x > 1 + \sqrt{5}\}$$

**Range**

$\{y \in \mathbb{R} : y > 0\}$ (all positive real numbers)

$\mathbb{R}$ is the set of real numbers

**Series expansion at x = 0:**

$$\frac{16}{x^2} + \frac{16}{x} - 4 + 20x + 5x^2 + O(x^3)$$

(Laurent series)

**Series expansion at x = ∞:**

$$x^4 - 6x^2 + 10x - 13 + O\left(\left(\frac{1}{x}\right)^1\right)$$

(Taylor series)

**Derivative:**

$$\frac{d}{dx}\left(\frac{1}{\left(\frac{x(x-1)}{\sqrt{(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)}}\right)^2}\right) = \frac{2(2x^9 - 6x^8 + 21x^6 - 33x^5 + 17x^4 - 2x^3 + 24x^2 - 40x + 16)}{(x-1)^3 x^3}$$

**Indefinite integral:**

$$\int \frac{(-1-x+x^2)(-4+2x+x^2)(4-6x+5x^2-3x^3+x^4)}{(-1+x)^2 x^2}\,dx = \frac{x^5}{5} - 2x^3 + 5x^2 - 13x + \frac{1}{1-x} - \frac{16}{x} - 8\log(1-x) + 16\log(x) + \text{constant}$$

(assuming a complex-valued logarithm)

log(x) is the natural logarithm

**Local minima:**

$$\min\left\{\frac{1}{\left(\frac{x(x-1)}{\sqrt{(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)}}\right)^2}\right\} \approx -42.333 \text{ at } x \approx -2.0737$$

$$\min\left\{\frac{1}{\left(\frac{x(x-1)}{\sqrt{(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)}}\right)^2}\right\} \approx 84.126 \text{ at } x \approx 0.67630$$

**Series representations:**

$$\frac{(-1-x+x^2)(-4+2x+x^2)(4-6x+5x^2-3x^3+x^4)}{(-1+x)^2 x^2} = \sum_{n=-\infty}^{\infty} \left(\begin{cases} 16 & -3 < n < 0 \\ -4 & n = 0 \\ 5 & n = 2 \\ 9+n & (n = 3 \text{ or } n > 4) \\ 14 & n = 4 \\ 20 & n = 1 \end{cases}\right) x^n$$

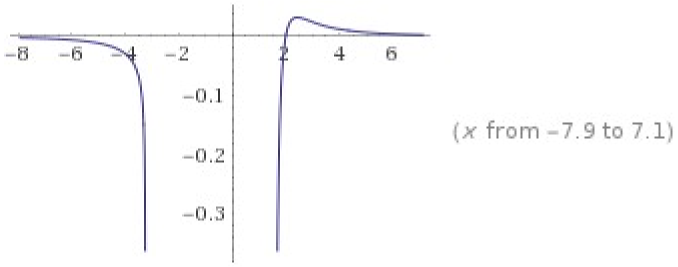
$$\frac{(-1-x+x^2)(-4+2x+x^2)(4-6x+5x^2-3x^3+x^4)}{(-1+x)^2 x^2} = \sum_{n=-\infty}^{\infty} \left(\begin{cases} -8 & n = -1 \\ 1 & n = -2 \\ 16(-1)^n(2+n) & (n = 2 \text{ or } n > 4) \\ 24 & n = 0 \\ -76 & n = 3 \\ 97 & n = 4 \\ -46 & n = 1 \end{cases}\right) (-1+x)^n$$

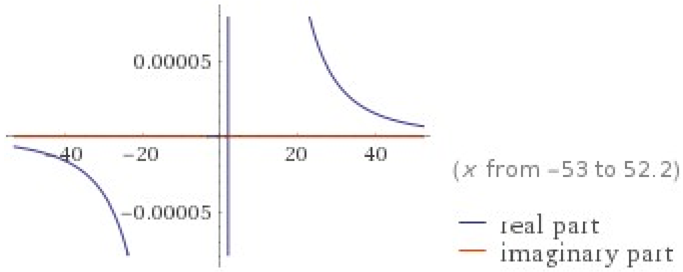
(x-2) / [(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)]^1/2

**Input:**

$$\frac{x-2}{\sqrt{(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)}}$$

**Plots:**




**Alternate forms:**

$$\frac{x-2}{\sqrt{((x-1)\,x-1)\,(x\,(x+2)-4)\,((x-2)\,x\,((x-1)\,x+3)+4)}}$$

$$\frac{x-2}{\sqrt{x^8-2\,x^7-5\,x^6+22\,x^5-39\,x^4+44\,x^3-20\,x^2-16\,x+16}}$$

**Expanded form:**

$$\frac{x\sqrt{(x^2-x-1)\,(x^2+2\,x-4)\,(x^4-3\,x^3+5\,x^2-6\,x+4)}}{(x^2-x-1)\,(x^2+2\,x-4)\,(x^4-3\,x^3+5\,x^2-6\,x+4)}-\frac{2\sqrt{(x^2-x-1)\,(x^2+2\,x-4)\,(x^4-3\,x^3+5\,x^2-6\,x+4)}}{(x^2-x-1)\,(x^2+2\,x-4)\,(x^4-3\,x^3+5\,x^2-6\,x+4)}$$

**Alternate forms assuming x>0:**

$$\frac{(x-2)\,(-1)^{\left\lfloor -(\arg(x^2-x-1)+\arg(x^2+2\,x-4)-\pi)/(2\,\pi)\right\rfloor}}{\sqrt{x^2-x-1}\,\sqrt{x^2+2\,x-4}\,\sqrt{x^4-3\,x^3+5\,x^2-6\,x+4}}$$

$$\frac{x \exp\left(i\pi \left\lfloor -\frac{\arg(x^2-x-1)}{2\pi} - \frac{\arg(x^2+2x-4)}{2\pi} + \frac{1}{2} \right\rfloor\right)}{\sqrt{x^2-x-1}\,\sqrt{x^2+2x-4}\,\sqrt{x^4-3x^3+5x^2-6x+4}} - \frac{2 \exp\left(i\pi \left\lfloor -\frac{\arg(x^2-x-1)}{2\pi} - \frac{\arg(x^2+2x-4)}{2\pi} + \frac{1}{2} \right\rfloor\right)}{\sqrt{x^2-x-1}\,\sqrt{x^2+2x-4}\,\sqrt{x^4-3x^3+5x^2-6x+4}}$$

$\arg(z)$ is the complex argument

$\lfloor x \rfloor$ is the floor function

**Series expansion at x = 0:**

$$-\frac{1}{2} - \frac{3x^2}{8} + \frac{5x^3}{16} - \frac{5x^4}{8} + O(x^5)$$

(Taylor series)

**Series expansion at x = -1 - sqrt(5):**

$$-\frac{3+\sqrt{5}}{4\left(\sqrt[4]{5}\,\sqrt{6(38+17\sqrt{5})}\,\sqrt{-x-\sqrt{5}-1}\right)} - \frac{(14563+6513\sqrt{5})(x+\sqrt{5}+1)}{192\left(5^{3/4}\,\sqrt{6}\,(38+17\sqrt{5})^{3/2}\,\sqrt{-x-\sqrt{5}-1}\right)} - \frac{(6378683805+2852634119\sqrt{5})(x+\sqrt{5}+1)^2}{30720\left(5^{3/4}\,\sqrt{6}\,(38+17\sqrt{5})^{7/2}\,\sqrt{-x-\sqrt{5}-1}\right)} - \frac{(166257002805+74352392003\sqrt{5})(x+\sqrt{5}+1)^3}{4423680\left(\sqrt[4]{5}\,\sqrt{6}\,(38+17\sqrt{5})^{7/2}\,\sqrt{-x-\sqrt{5}-1}\right)} - \frac{(901474208694741+403151522120839\sqrt{5})(x+\sqrt{5}+1)^4}{849346560\left(\sqrt[4]{5}\,\sqrt{6}\,(38+17\sqrt{5})^{9/2}\,\sqrt{-x-\sqrt{5}-1}\right)} + O\left(\left(x+\sqrt{5}+1\right)^5\right)$$

(generalized Puiseux series)

**Series expansion at x = 1/2 (1 - sqrt(5)):**

$$-\frac{3+\sqrt{5}}{2\left(\sqrt[4]{5}\,\sqrt{6\,(2+\sqrt{5})}\,\sqrt{2\,x+\sqrt{5}-1}\right)}-\frac{(397+201\sqrt{5})\left(x-\frac{1}{2}(1-\sqrt{5})\right)}{48\left(5^{3/4}\,\sqrt{6}\,(2+\sqrt{5})^{3/2}\,\sqrt{2\,x+\sqrt{5}-1}\right)}-$$

$$\frac{(115\,145+50\,869\sqrt{5})\left(x-\frac{1}{2}(1-\sqrt{5})\right)^2}{960\left(5^{3/4}\,\sqrt{6}\,(2+\sqrt{5})^{3/2}\,(3+\sqrt{5})^2\,\sqrt{2\,x+\sqrt{5}-1}\right)}-$$

$$\frac{(506\,486+208\,953\sqrt{5})\left(x-\frac{1}{2}(1-\sqrt{5})\right)^3}{69\,120\left(\sqrt[4]{5}\,\sqrt{6}\,(2+\sqrt{5})^{3/2}\,(7+3\sqrt{5})\,\sqrt{2\,x+\sqrt{5}-1}\right)}-$$

$$\frac{(69\,849\,065+29\,802\,963\sqrt{5})\left(x-\frac{1}{2}(1-\sqrt{5})\right)^4}{207\,360\left(\sqrt[4]{5}\,(1+\sqrt{5})^4\,\sqrt{6\,(2+\sqrt{5})}\,(3+\sqrt{5})^3\,\sqrt{2\,x+\sqrt{5}-1}\right)}+$$

$$O\left(\left(x-\frac{1}{2}\left(1-\sqrt{5}\right)\right)^5\right)$$

(generalized Puiseux series)

**Series expansion at x = sqrt(5) - 1:**

$$\frac{\sqrt{5}-3}{4\sqrt[4]{5}\sqrt{6(17\sqrt{5}-38)}\sqrt{-x+\sqrt{5}-1}}+$$

$$\left(\frac{(\sqrt{5}-3)(1592\sqrt{5}-3561)}{192\times 5^{3/4}\sqrt{6(9-4\sqrt{5})}(\sqrt{5}-2)^{3/2}(4\sqrt{5}-9)\sqrt{-x+\sqrt{5}-1}}+\frac{1}{4\sqrt[4]{5}\sqrt{6(17\sqrt{5}-38)}\sqrt{-x+\sqrt{5}-1}}\right)\left(x-\sqrt{5}+1\right)+$$

$$\left(\frac{1592\sqrt{5}-3561}{192\times 5^{3/4}\sqrt{6(9-4\sqrt{5})}(\sqrt{5}-2)^{3/2}(4\sqrt{5}-9)\sqrt{-x+\sqrt{5}-1}}-\frac{(\sqrt{5}-3)(4859280\sqrt{5}-10865681)}{30720\sqrt[4]{5}(\sqrt{5}-2)^2(4\sqrt{5}-9)^2\sqrt{6(17\sqrt{5}-38)}\sqrt{-x+\sqrt{5}-1}}\right)$$

$$\left(x-\sqrt{5}+1\right)^2+$$

$$\frac{(74352392003\sqrt{5}-166257002805)(x-\sqrt{5}+1)^3}{4423680\sqrt[4]{5}(\sqrt{5}-2)^3(4\sqrt{5}-9)^3\sqrt{6(17\sqrt{5}-38)}\sqrt{-x+\sqrt{5}-1}}+$$

$$\frac{(403151522120839\sqrt{5}-901474208694741)(x-\sqrt{5}+1)^4}{849346560\sqrt[4]{5}(\sqrt{5}-2)^4(4\sqrt{5}-9)^4\sqrt{6(17\sqrt{5}-38)}\sqrt{-x+\sqrt{5}-1}}+$$

$$O\left(\left(x-\sqrt{5}+1\right)^5\right)$$

(generalized Puiseux series)

**Derivative:**

$$\frac{d}{dx}\left(\frac{x-2}{\sqrt{(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)}}\right)=$$

$$-\frac{x(3x^7-13x^6+4x^5+63x^4-149x^3+178x^2-132x+48)}{(x^8-2x^7-5x^6+22x^5-39x^4+44x^3-20x^2-16x+16)^{3/2}}$$

((((x-2) / [(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)]^1/2)))^2

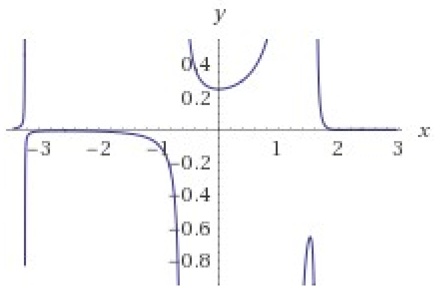
**Input:**

$$\left(\frac{x-2}{\sqrt{(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)}}\right)^2$$

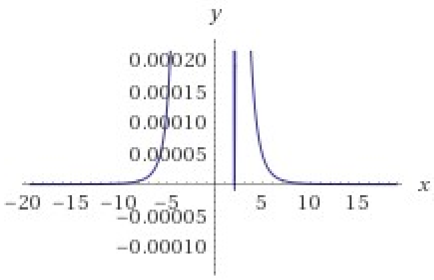
**Result:**

$$\frac{(x-2)^2}{(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)}$$

**Plots:**



(x from -3.4 to 2.9)



(x from -19.3 to 18.8)

**Alternate forms:**

$$\frac{(x-2)^2}{((x-1)\,x-1)\,(x\,(x+2)-4)\,((x-2)\,x\,((x-1)\,x+3)+4)}$$

$$\frac{(x-4)\,x+4}{x\,(x\,(x\,(x\,(x\,(x\,((x-2)\,x-5)+22)-39)+44)-20)-16)+16}$$

**Partial fraction expansion:**

$$\frac{-5\,x-16}{24\,(x^2+2\,x-4)}+\frac{x-1}{12\,(x^2-x-1)}+\frac{x^3-x^2+x}{8\,(x^4-3\,x^3+5\,x^2-6\,x+4)}$$

**Expanded form:**

$$\frac{x^2-4\,x+4}{x^8-2\,x^7-5\,x^6+22\,x^5-39\,x^4+44\,x^3-20\,x^2-16\,x+16}$$

**Root:**

$x = 2$

**Properties as a real function:**

**Domain**

$\{x \in \mathbb{R} : x + \sqrt{5} + 1 < 0 \text{ or } \frac{1}{2}\left(1 - \sqrt{5}\right) < x < \sqrt{5} - 1 \text{ or } 2x > 1 + \sqrt{5}\}$

**Range**

$\{y \in \mathbb{R} : y \geq 0\}$ (all non-negative real numbers)

$\mathbb{R}$ is the set of real numbers

**Series expansion at x = 0:**

$$\frac{1}{4} + \frac{3x^2}{8} - \frac{5x^3}{16} + \frac{49x^4}{64} + O(x^5)$$

(Taylor series)

**Series expansion at x = ∞:**

$$\left(\frac{1}{x}\right)^6 - \frac{2}{x^7} + \frac{5}{x^8} - \frac{22}{x^9} + O\left(\left(\frac{1}{x}\right)^{10}\right)$$

(Laurent series)

**Derivative:**

$$\frac{d}{dx}\left(\left(\frac{x-2}{\sqrt{(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)}}\right)^2\right) = -\frac{2(x-2)x(3x^7-13x^6+4x^5+63x^4-149x^3+178x^2-132x+48)}{(x^2-x-1)^2(x^2+2x-4)^2(x^4-3x^3+5x^2-6x+4)^2}$$

**Indefinite integral:**

$$\int\left(\frac{x-2}{\sqrt{(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)}}\right)^2 dx =$$

$$\frac{1}{240}\left(30\sum_{\{\omega:\,\omega^4+5\omega^3+11\omega^2+10\omega+4=0\}}\frac{\omega^3\log(x-\omega-2)+5\omega^2\log(x-\omega-2)+9\omega\log(x-\omega-2)+6\log(x-\omega-2)}{4\omega^3+15\omega^2+22\omega+10} - 2(\sqrt{5}-5)\log(-2x+\sqrt{5}+1) - (25+11\sqrt{5})\log(-x+\sqrt{5}-1) + (11\sqrt{5}-25)\log(x+\sqrt{5}+1) + 2(5+\sqrt{5})\log(2x+\sqrt{5}-1)\right) + \text{constant}$$

(assuming a complex-valued logarithm)

$\log(x)$ is the natural logarithm

**Local maxima:**

$$\max\left\{\left(\frac{x-2}{\sqrt{(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)}}\right)^2\right\} \approx -0.0068270$$
at $x \approx -2.6226$

$$\max\left\{\left(\frac{x-2}{\sqrt{(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)}}\right)^2\right\} \approx -0.65930$$
at $x \approx 1.5215$

**Local minima:**

$$\min\left\{\left(\frac{x-2}{\sqrt{(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)}}\right)^2\right\} = \frac{1}{4} \text{ at } x = 0$$

$$\min\left\{\left(\frac{x-2}{\sqrt{(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)}}\right)^2\right\} = 0 \text{ at } x = 2$$

**Limit:**

$$\lim_{x\to\pm\infty} \frac{(-2+x)^2}{(-1-x+x^2)(-4+2x+x^2)(4-6x+5x^2-3x^3+x^4)} = 0$$

1/(((((((x-2) / [(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)]^1/2)))^2)))

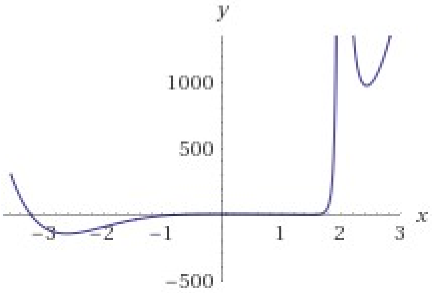
**Input:**

$$\frac{1}{\left(\frac{x-2}{\sqrt{(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)}}\right)^2}$$

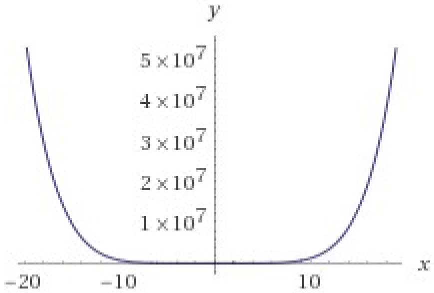
**Result:**

$$\frac{(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)}{(x-2)^2}$$

## Plots:




(x from −3.6 to 2.9)




(x from −19.7 to 19)

## Alternate forms:

$$\frac{((x-1)x-1)(x(x+2)-4)((x-2)x((x-1)x+3)+4)}{(x-2)^2}$$

$$x^6+2x^5-x^4+10x^3+5x^2+24x+\frac{112}{x-2}+\frac{16}{(x-2)^2}+56$$

$$\frac{x(x(x(x(x(x((x-2)x-5)+22)-39)+44)-20)-16)+16}{(x-4)x+4}$$

## Expanded form:

$$\frac{x^8}{(x-2)^2}-\frac{2x^7}{(x-2)^2}-\frac{5x^6}{(x-2)^2}+\frac{22x^5}{(x-2)^2}-\frac{39x^4}{(x-2)^2}+\frac{44x^3}{(x-2)^2}-\frac{20x^2}{(x-2)^2}-\frac{16x}{(x-2)^2}+\frac{16}{(x-2)^2}$$

## Alternate form assuming x>0:

$$\frac{x^8-2x^7-5x^6+22x^5-39x^4+44x^3-20x^2-16x+16}{(x-2)^2}$$

## Quotient and remainder:

$$x^8-2x^7-5x^6+22x^5-39x^4+44x^3-20x^2-16x+16 = \left(x^6+2x^5-x^4+10x^3+5x^2+24x+56\right)\times(x^2-4x+4)+112x-208$$

**Real roots:**

$x = -1 - \sqrt{5}$

$x = \frac{1}{2} - \frac{\sqrt{5}}{2}$

$x = \sqrt{5} - 1$

$x = \frac{1}{2} + \frac{\sqrt{5}}{2}$

1.6180339887…

**Complex roots:**

$x = \frac{1}{4}\left(3 + \sqrt{5} - i\sqrt{18 - 6\sqrt{5}}\right)$

$x = \frac{1}{4}\left(3 + \sqrt{5} + i\sqrt{18 - 6\sqrt{5}}\right)$

$x = \frac{1}{4}\left(3 - \sqrt{5} - i\sqrt{6\left(3 + \sqrt{5}\right)}\right)$

$x = \frac{1}{4}\left(3 - \sqrt{5} + i\sqrt{6\left(3 + \sqrt{5}\right)}\right)$

**Properties as a real function:**

**Domain**

$\{x \in \mathbb{R} :$
$x + \sqrt{5} + 1 < 0$ or $\frac{1}{2}\left(1 - \sqrt{5}\right) < x < \sqrt{5} - 1$ or $\frac{1}{2}\left(1 + \sqrt{5}\right) < x < 2$ or $x > 2\}$

**Range**

$\{y \in \mathbb{R} : y > 0\}$ (all positive real numbers)

$\mathbb{R}$ is the set of real numbers

**Series expansion at x = 0:**

$$4 - 6x^2 + 5x^3 - \frac{13x^4}{4} + O(x^5)$$

(Taylor series)

**Series expansion at x = ∞:**

$$x^6 + 2x^5 - x^4 + 10x^3 + 5x^2 + 24x + 56 + O\left(\left(\frac{1}{x}\right)^1\right)$$

(Taylor series)

**Derivative:**

$$\frac{d}{dx}\left(\frac{1}{\left(\frac{x-2}{\sqrt{(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)}}\right)^2}\right) =$$

$$\frac{2x(3x^7 - 13x^6 + 4x^5 + 63x^4 - 149x^3 + 178x^2 - 132x + 48)}{(x-2)^3}$$

**Indefinite integral:**

$$\int \frac{(-1 - x + x^2)(-4 + 2x + x^2)(4 - 6x + 5x^2 - 3x^3 + x^4)}{(-2 + x)^2}\, dx =$$

$$\frac{x^7}{7} + \frac{x^6}{3} - \frac{x^5}{5} + \frac{5x^4}{2} + \frac{5x^3}{3} + 12x^2 + 56x - \frac{16}{x-2} + 112\log(x-2) + \text{constant}$$

(assuming a complex-valued logarithm)

$\log(x)$ is the natural logarithm

**Local maximum:**

$$\max\left\{\frac{1}{\left(\frac{x-2}{\sqrt{(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)}}\right)^2}\right\} = 4 \text{ at } x = 0$$

**Global minimum:**

$$\min\left\{\frac{1}{\left(\frac{x-2}{\sqrt{(x^2-x-1)(x^2+2x-4)(x^4-3x^3+5x^2-6x+4)}}\right)^2}\right\} \approx -146.48 \text{ at } x \approx -2.6226$$

**Series representations:**

$$\frac{(-1-x+x^2)(-4+2x+x^2)(4-6x+5x^2-3x^3+x^4)}{(-2+x)^2} = \sum_{n=-\infty}^{\infty} \left( \begin{cases} 16(-6+n) & n>6 \\ 0 & n=2 \\ -8 & n=4 \\ -4 & n=1 \\ 1 & n=6 \\ -8 & n=5 \\ -2 & n=3 \\ 1 & n=0 \end{cases} \right) (-1+x)^n$$

$$\frac{(-1-x+x^2)(-4+2x+x^2)(4-6x+5x^2-3x^3+x^4)}{(-2+x)^2} = \sum_{n=-\infty}^{\infty} \left( \begin{cases} -\frac{13}{4} & n=4 \\ 2^{2-n}(-13+n) & n>6 \\ 0 & n=1 \\ 4 & n=0 \\ 1 & n=5 \\ 5 & n=3 \\ \frac{9}{16} & n=6 \\ -6 & n=2 \end{cases} \right) x^n$$

We note that:

1/((1/sqrt(1/2 + sqrt(5)/2)))^2

**Input:**

$$\frac{1}{\left(\frac{1}{\sqrt{\frac{1}{2}+\frac{\sqrt{5}}{2}}}\right)^2}$$

**Result:**

$$\frac{1}{2} + \frac{\sqrt{5}}{2}$$

**Decimal approximation:**

1.6180339887498948482045868343656381177203091798057628621354...

1.6180339887…

**Alternate form:**

$$\frac{1}{2}\left(1+\sqrt{5}\right)$$

**Minimal polynomial:**

$$x^2 - x - 1$$

**Observations**

Figs.

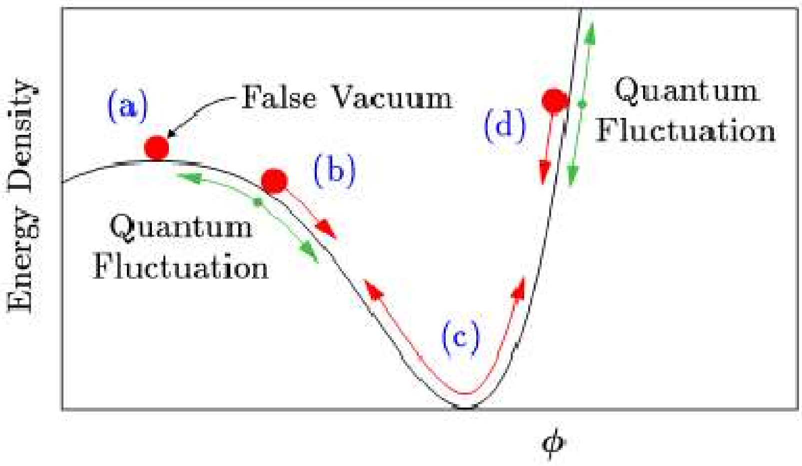
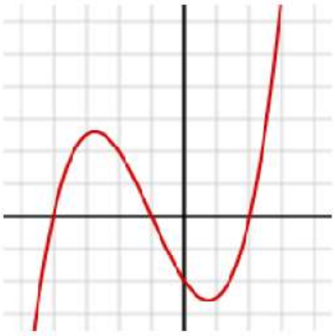


FIG. 1: In simple inflationary models, the universe at early times is dominated by the potential energy density of a scalar field, $\phi$. Red arrows show the classical motion of $\phi$. When $\phi$ is near region (a), the energy density will remain nearly constant, $\rho \cong \rho_f$, even as the universe expands. Moreover, cosmic expansion acts like a frictional drag, slowing the motion of $\phi$: Even near regions (b) and (d), $\phi$ behaves more like a marble moving in a bowl of molasses, slowly creeping down the side of its potential, rather than like a marble sliding down the inside of a polished bowl. During this period of "slow roll," $\rho$ remains nearly constant. Only after $\phi$ has slid most of the way down its potential will it begin to oscillate around its minimum, as in region (c), ending inflation.



Graph of a cubic function with 3 real roots (where the curve crosses the horizontal axis at $y = 0$). The case shown has two critical points. Here the function is $f(x) = (x^3 + 3x^2 - 6x - 8)/4$.

The ratio between $M_0$ and q

$$M_0 = \sqrt{3q^2 - \Sigma^2},$$

$$q = \frac{(3\sqrt{3})\, M_s}{2}.$$

i.e. the gravitating mass $M_0$ and the Wheelerian mass q of the wormhole, is equal to:

$$\frac{\sqrt{3\,(2.17049 \times 10^{37})^2 - 0.001^2}}{\frac{1}{2}\,((3\sqrt{3})\,(4.2 \times 10^6 \times 1.9891 \times 10^{30}))}$$

1.732050787905194420703947625671018160083566548802082460520...

1.7320507879

$1.7320507879 \approx \sqrt{3}$ that is the ratio between the gravitating mass $M_0$ and the Wheelerian mass q of the wormhole

We note that:

$$\left(-\frac{1}{2}+\frac{i}{2}\sqrt{3}\right)-\left(-\frac{1}{2}-\frac{i}{2}\sqrt{3}\right)$$

$i$ is the imaginary unit

$$i\sqrt{3}$$

1.732050807568877293527446341505872366942805253810380628055... $i$

$r \approx 1.73205$ (radius), $\theta = 90°$ (angle)

1.73205

This result is very near to the ratio between $M_0$ and q, that is equal to 1.7320507879 $\approx \sqrt{3}$

With regard $\sqrt{3}$ , we note that is a fundamental value of the formula structure that we need to calculate a Cubic Equation

We have that the previous result

$$\left(-\frac{1}{2}+\frac{i}{2}\sqrt{3}\right)-\left(-\frac{1}{2}-\frac{i}{2}\sqrt{3}\right) = i\sqrt{3} =$$

= 1.732050807568877293527446341505872366942805253810380628055... $i$

$r \approx 1.73205$ (radius), $\theta = 90°$ (angle)

can be related with:

$$u^2(-u)\left(\frac{1}{2}\pm\frac{i\sqrt{3}}{2}\right)+v^2(-v)\left(\frac{1}{2}\pm\frac{i\sqrt{3}}{2}\right)=q$$

Considering:

$$(-1)\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)-(-1)\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)=q$$

$= i\sqrt{3} = 1.7320508075688772935274463415058723669428052538103806280 55... i$

$r \approx 1.73205$ (radius), $\theta = 90°$ (angle)

Thence:

$$\left(-\frac{1}{2}+\frac{i}{2}\sqrt{3}\right)-\left(-\frac{1}{2}-\frac{i}{2}\sqrt{3}\right) \Rightarrow$$

$$\Rightarrow(-1)\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)-(-1)\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)=q=1.73205\approx\sqrt{3}$$

**We observe how the graph above, concerning the cubic function, is very similar to the graph that represent the scalar field (in red). It is possible to hypothesize that cubic functions and cubic equations, with their roots, are connected to the scalar field.**

*From:*
*https://www.scientificamerican.com/article/mathematics-ramanujan/?fbclid=IwAR2caRXrn_RpOSvJ1QxWsVLBcJ6KVgd_Af_hrmDYBNyU8mpSjRs1BDeremA*

*Ramanujan's statement concerned the deceptively simple concept of partitions—the different ways in which a whole number can be subdivided into smaller numbers. Ramanujan's original statement, in fact, stemmed from the observation of patterns, such as the fact that p(9) = 30, p(9 + 5) = 135, p(9 + 10) = 490, p(9 + 15) = 1,575 and so on are all divisible by 5. Note that here the n's come at intervals of five units.*

*Ramanujan posited that this pattern should go on forever, and that similar patterns exist when 5 is replaced by 7 or 11—there are infinite sequences of p(n) that are all divisible by 7 or 11, or, as mathematicians say, in which the "moduli" are 7 or 11.*

*Then, in nearly oracular tone Ramanujan went on: "There appear to be corresponding properties," he wrote in his 1919 paper, "in which the moduli are powers of 5, 7 or 11...and no simple properties for any moduli involving primes other than these three." (Primes are whole numbers that are only divisible by themselves or by 1.) Thus, for instance, there should be formulas for an infinity of n's separated by 5^3 = 125 units, saying that the corresponding p(n)'s should all be divisible by 125.*

*In the past methods developed to understand partitions have later been applied to physics problems such as the theory of the strong nuclear force or the entropy of black holes.*

*From Wikipedia*

*In particle physics, Yukawa's interaction or Yukawa coupling, named after Hideki Yukawa, is an interaction between a scalar field $\phi$ and a Dirac field $\psi$. The Yukawa interaction can be used to describe the nuclear force between nucleons (which are fermions), mediated by pions (which are pseudoscalar mesons). The Yukawa interaction is also used in the Standard Model to describe the coupling between the Higgs field and massless quark and lepton fields (i.e., the fundamental fermion particles). Through spontaneous symmetry breaking, these fermions acquire a mass proportional to the vacuum expectation value of the Higgs field.*

*Can be this the motivation that from the development of the Ramanujan's equations we obtain results very near to the dilaton mass calculated as a type of Higgs boson: 125 GeV for T = 0 and to the Higgs boson mass 125.18 GeV and practically equal to the rest mass of Pion meson 139.57 MeV*

*Note that:*

$$g_{22} = \sqrt{(1+\sqrt{2})}.$$

Hence

$$\begin{aligned} 64g_{22}^{24} &= e^{\pi\sqrt{22}} - 24 + 276e^{-\pi\sqrt{22}} - \cdots, \\ 64g_{22}^{-24} &= \qquad\qquad\qquad 4096e^{-\pi\sqrt{22}} + \cdots, \end{aligned}$$

so that

$$64(g_{22}^{24} + g_{22}^{-24}) = e^{\pi\sqrt{22}} - 24 + 4372e^{-\pi\sqrt{22}} + \cdots = 64\{(1+\sqrt{2})^{12} + (1-\sqrt{2})^{12}\}.$$

Hence

$$e^{\pi\sqrt{22}} = 2508951.9982\ldots.$$

*Thence:*

$$64g_{22}^{-24} \quad = \qquad\qquad 4096e^{-\pi\sqrt{22}} + \cdots$$

*And*

$$64(g_{22}^{24} + g_{22}^{-24}) = e^{\pi\sqrt{22}} - 24 + 4372e^{-\pi\sqrt{22}} + \cdots = 64\{(1+\sqrt{2})^{12} + (1-\sqrt{2})^{12}\}$$

*That are connected with 64, 128, 256, 512, 1024 and 4096 = $64^2$*

*(Modular equations and approximations to π - S. Ramanujan - Quarterly Journal of Mathematics, XLV, 1914, 350 – 372)*

*All the results of the most important connections are signed in blue throughout the drafting of the paper. We highlight as in the development of the various equations we use always the constants π, ϕ, 1/ϕ, the Fibonacci and Lucas numbers, linked to the golden ratio, that play a fundamental role in the development, and therefore, in the final results of the analyzed expressions.*

*In mathematics, the Fibonacci numbers, commonly denoted $F_n$, form a sequence, called the Fibonacci sequence, such that each number is the sum of the two preceding ones, starting from 0 and 1. Fibonacci numbers are strongly related to the golden ratio: Binet's formula expresses the nth Fibonacci number in terms of n and the golden ratio, and implies that the ratio of two consecutive Fibonacci numbers tends to the golden ratio as n increases.*
*Fibonacci numbers are also closely related to Lucas numbers ,in that the Fibonacci and Lucas numbers form a complementary pair of Lucas sequences*

*The beginning of the sequence is thus:*

*0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, 10946, 17711, 28657, 46368, 75025, 121393, 196418, 317811, 514229, 832040, 1346269, 2178309, 3524578, 5702887, 9227465, 14930352, 24157817, 39088169, 63245986, 102334155...*

*The Lucas numbers or Lucas series are an integer sequence named after the mathematician François Édouard Anatole Lucas (1842–91), who studied both that sequence and the closely related Fibonacci numbers. Lucas numbers and Fibonacci numbers form complementary instances of Lucas sequences.*

*The Lucas sequence has the same recursive relationship as the Fibonacci sequence, where each term is the sum of the two previous terms, but with different starting values. This produces a sequence where the ratios of successive terms approach the golden ratio, and in fact the terms themselves are roundings of integer powers of the golden ratio.[1] The sequence also has a variety of relationships with the Fibonacci numbers, like the fact that adding any two Fibonacci numbers two terms apart in the Fibonacci sequence results in the Lucas number in between.*

*The sequence of Lucas numbers is:*

*2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, 2207, 3571, 5778, 9349, 15127, 24476, 39603, 64079, 103682, 167761, 271443, 439204, 710647, 1149851, 1860498, 3010349, 4870847, 7881196, 12752043, 20633239, 33385282, 54018521, 87403803……*

*All Fibonacci-like integer sequences appear in shifted form as a row of the Wythoff array; the Fibonacci sequence itself is the first row and the Lucas sequence is the second row. Also like all Fibonacci-like integer sequences, the ratio between two consecutive Lucas numbers converges to the golden ratio.*

*A Lucas prime is a Lucas number that is prime. The first few Lucas primes are:*

*2, 3, 7, 11, 29, 47, 199, 521, 2207, 3571, 9349, 3010349, 54018521, 370248451, 6643838879, ... (sequence A005479 in the OEIS).*

*In geometry, a golden spiral is a logarithmic spiral whose growth factor is φ, the golden ratio.[1] That is, a golden spiral gets wider (or further from its origin) by a factor of φ for every quarter turn it makes. Approximate logarithmic spirals can occur in nature, for example the arms of spiral galaxies[3] - golden spirals are one special case of these logarithmic spirals*

We observe that 1728 and 1729 are results very near to the mass of candidate glueball **$f_0$(1710) scalar meson.** Furthermore, 1728 occurs in the algebraic formula for the j-invariant of an elliptic curve. As a consequence, it is sometimes called a Zagier as a pun on the Gross–Zagier theorem. The number 1728 is one less than the Hardy–Ramanujan number 1729 (taxicab number).

Furthermore, we obtain as results of our computations, always values very near to the Higgs boson mass 125.18 GeV and practically equals to the rest mass of Pion meson 139.57 MeV. In conclusion we obtain also many results that are very good approximations to the value of the golden ratio 1.618033988749... and to $\zeta(2) = \frac{\pi^2}{6} = 1.644934 \ldots$

**We note how the following three values: 137.508 (golden angle), 139.57 (mass of the Pion - meson Pi) and 125.18 (mass of the Higgs boson), are connected to each other. In fact, just add 2 to 137.508 to obtain a result very close to the mass of the Pion and subtract 12 to 137.508 to obtain a result that is also very close to the mass of the Higgs boson. We can therefore hypothesize that it is the golden angle (and the related golden ratio inherent in it) to be a fundamental ingredient both in the structures of the microcosm and in those of the macrocosm.**

## References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*Salvatore Capozziello, Rocco D'Agostino and Daniele Gregoris* -
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**INTEGRALS ASSOCIATED WITH RAMANUJAN AND ELLIPTIC FUNCTIONS**
*BRUCE C. BERNDT*

**Incomplete Elliptic Integrals in Ramanujan's Lost Notebook**
*Dan Schultz* - March 17, 2015