# Simplest electromagnetic felds and their sources

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**Abstract**

The problem of generation of plane and evanescent waves by electric charge and current densities on a plane is considered. It is shown that, first, both ordinary and evanescent waves can be emitted by such a source, second, that source of evanescent wave is perfectly static in an appropriate frame. The source is found as as explicit form of surface charge and current densities on a plane, which satisfy the continuity condition. One of components of retarded potential of the source is calculated. It is shown that the expression derived provides an erroneous representation of the field.

# 1 Introduction

Electromagnetic fields of the simplest form possess planar symmetry, so, the simplest electromagnetic fields are plane waves in form. There also exist evanescent waves whose mathematical representation is actually equivalent to that of plane ones. At the same time, these two forms of electromagnetic field seem to be very different in sense of opportunity to be generated by charge and current oscillations. Indeed, oscillations of this sort are believed to generate only ordinary waves which propagate in space without any specific exponential extinction which evanescent waves expose. Consequently, according to classical concepts, charge and current oscillations generate only ordinary waves, whehereas evanescent ones cannot be produced this way. This fact is interesting from mathematical point of view because both ordinary and evanescent waves satisfy Maxwell equations and the source can be introduced as a certain boundary condition on a certain surface, so, no reason is seen, why it is possible for one solution and impossible for another. Our primary goal was to investigate this phenomenon, but the study itself has produced a number of interesting facts to study.

To study all these facts, we first present both solutions in same form and second, find out a pair of surface charge and current densities on a plane, which privide boundary conditions for Maxwell equations as the source of the field at both sides of the plane. Note that the fact alone of existence of such a source indicates opportunity of emitting both plane and evanescent waves by putting high frequency voltage onto a conducting plane. This fact actually allows us to formulate two paradoxes. Both of them are analysed. Besides, one of them makes it possible to calculate the retarded potential for the source constructed that removes both discrepancies and shows that in this case retarded poptential provides a wrong representation of the field.

# 2 Electromagnetic fields possessing planar symmetry

Cartesian coordinates $\{x, y, z\}$ provide the simplest and the most natural realization of planar symmetry, so, it is natural to compose solutions of field equations possessing this symmetry in this coordinate system. It turns out that plane wave solutions of all (scalar, vector, spinor, etc.) field equations can easily be obtained in Cartesian coordinates from the plane wave solution of the D'Alembert equation. For simplicity, below we represent a plane wave solution of the D'Alembert equation as the real or imaginary part of the function

$$\phi = ae^{\imath(\omega t - px - qz)},$$

where the constants $\omega, p$ and $q$ satisfy the condition

$$\omega^2 = p^2 + q^2, \tag{1}$$

where we used special choice of the coordinate system. To obtain a vector potential we multiply it by a unit vector orthogonal to the wave vector. If, for instance, the vector lies in the $(xz)$-plane, we have a vector potential with components

$$\begin{aligned} A_x = &\quad a\frac{q}{\omega}e^{\imath(\omega t - px - qz)}, \\ A_z = &\quad -a\frac{p}{\omega}e^{\imath(\omega t - px - qz)} \end{aligned} \tag{2}$$

and take only real or imaginary parts. This way we obtain the corresponding field strengths

$$
\begin{aligned}
E_x = - &\quad aq \sin(\omega t - px - qz), \ E_z = ap \sin(\omega t - px - qz), \\
H_y = &\quad a\omega \sin(\omega t - px - qz).
\end{aligned} \tag{3}
$$

Besides, there exists another solution which does not possess planar symmetry, but is mathematically is acually identical to plane waves. Suppose that $\omega^2$ is less than $p^2$. Then the component $q$ of the wave vector is imaginary and the strengths become

$$
\begin{aligned}
E_x = &\quad aqe^{-qz} \sin(\omega t - px), \ E_z = -ape^{-qz} \cos(\omega t - px), \\
H_y = &\quad a\omega e^{-qz} \cos(\omega t - px).
\end{aligned} \tag{4}
$$

Note the difference in phases of the strength components between plane and evanescent waves. The field represented by these strengths has no planar symmetry, but mathematically is related to the field of plane wave. Now we consider the problem of existence of sources of the field in both cases, i.e. such a pair of chagre and current densities which can produce fields under consideration.

# 3 Charge and current oscillations producing plane and evanescent waves

Suppose, the plane $z = 0$ is conducting and carries some surface charge and current densities which produce the fields (3) and (4). In other words, the strengths of this form in the semispaces $z < 0$ and $z > 0$ have jumps on the plane which can be matched so that the field in the whole space and some densities specified by the jumps satisfy Maxwell equations. According to Maxwell equations, tangential component of electric strength is continuous on the plane and hence, in the whole space. Unlike it, normal component of electric and tangential comopnent of magnetic strengths have jumps. These jumps simply expose mirror symmetry of the field with respect to the plane. This symmetry signifies that in the formulas (2) components of the potential $A_z$ and of the wave vector $q$ change sign, so that under $z < 0$ instead of these formulas we have

$$A_x = \quad a\frac{q}{\omega}e^{\imath(\omega t-px+qz)}, \tag{5}$$
$$A_z = \quad a\frac{p}{\omega}e^{\imath(\omega t-px+qz)}$$

Indeed, $z < 0$ the corresponding strengths are

$$E_x = \quad -aq\sin(\omega t - px + qz),\ \ E_z = -ap\sin(\omega t - px + qz), \tag{6}$$
$$H_y = \quad -a\omega\sin(\omega t - px - qz).$$

By construction, these strengths also satisfy source-free Maxwell equations under $z \neq 0$ and satisfy the desired boundary conditions which will be discussed below. Similarly, the second solution in the $z < 0$ semispace has the form

$$E_x = \quad aqe^{-qz}\sin(\omega t - px),\ \ E_z = ape^{-qz}\cos(\omega t - px), \tag{7}$$
$$H_y = - \quad a\omega e^{-qz}\cos(\omega t - px).$$

Now, according to the theory of simple and double layers, the jump of $E_z$ is equal to $4\pi\sigma$ and that of $H_y$ equals $4\pi j_x$ where $\sigma$ and $j_x$ are surface densities of charge and current correspondingly, speed of light is put equal to unity. Hence, the densities are

$$\sigma = \frac{ap}{2\pi}\sin(\omega t - px),\ j_x = \frac{a\omega}{2\pi}\sin(\omega t - px) \tag{8}$$

and it is easy to verify that they satisfy the continuity condition:

$$\frac{\partial\sigma}{\partial t} + \frac{\partial j_x}{\partial x} = 0. \tag{9}$$

Thus, according to Maxwell equations and theory of single and double layers, surface charge and current densities just obtained, produce the field with strengths (3)under $z < 0$ and (6) in the another semispace of (4) and (7) above and below the plane $z = 0$ correspondingly depending on the relation between $\omega^2$ and $p^2$. Taking into account the fact that the vector $\vec{H}$ is actually an antisymmetric tensor so that $H_y$ is actually its $zx$-component, the field as a whole possesses exact mirror symmetry with respect to the plane. And so is the field obtained from the strengths (3,6) and (4,7) by the duality transformation $\vec{E} \to \vec{H}$, $\vec{H} \to -\vec{E}$. The source of the field composed this way is presented by current density alone with current flowing in the $y$-direction, so that there is no surface charge density on the plane. Hence, a conclusion can be made that in general, the field produced by a non-stationary charge and current densities on a plane, can be obtaines as a source-fraa solution of the Maxwell equations in one semi-space and its mirror reflection in another.

# 4 Other frames of reference

The field of static charge distribution is a subject of electrostatics. In fact, so is electromagnetic field of such a carge distribution moving uniformly because there exists a frame of reference in which the field is purely electrostatic. In all the rest cases the field is believed to possess some radiation part. So is the field of any oscillation of a charge distributions so that, according to classical concepts, whenever such an oscillation occurs, electromagnetic waves appear which reaches infinity. Now, let us look at the resuits obtained.

On one hand, it looks quite natural if an alternate current produces on conducting plane produces a plane wave. After all, antennas work this way. Nevertheless, the result obtained rises some questions. The point is that charge and current densities found as the source of the field, are formed by a static charge density $\sigma$ which moves uniformly in the $x$-direction with constant velocity $\omega/p$, thus, uniformly. In a frame moving so, this charge distribution is static, so that its field cannot contain radiation. So, one of results obtained look somewhat surprising because it describes electromagnetic wave emitted by a source which seems to be a static charge distribution. In fact, there is no paradox because charge density moves with phase velocity $\omega/p$ which is greater than speed of light, hence, no such an inertial frame exists.

Now, consider the case of slow wave $\omega < p$. As the speed is less that that of light, a frame exists in which the source is static and then the field cannot contain radiation. Our result reads that in this case the field has the form of evanescent wave which cannot be assiciated with radiation. Indeed, passage to the frame in which the charge density rests, exposes a static state in which current density is zero so that both source and the field are static. In other words, in this frame one obsrvees electrostatic field of a static charge density on the plane. However, combination of two running charge distributions with opposite velocities form a source whose time dependense cannot be removed by changing frame of reference, thus, is genuinely non-stationary with charges oscillating within wavelength, and nevertheless, its field is not radiation. According to classical concepts, this is impossible, therefore the concept of non-static source of electromagnetic field needs to be revised.

# 5 Retarded potential

The main task of classical electrodynamics as a branch of the field theory is to find electromagnetic field of a given source. One particular example of the goal reached is presented above as charge and current densities on plane (8) along with strengths (3,6) or (4,7). In this particular case the goal was reached by solving Maxwell euqations as they stand that is quite unusual. The main apprjach to the task is to calculate retarded potential for a given source. Note that this approach can well be applied to the densities (8) that allows us to compare results the two approach give.

The reason to do so consists in the followig. Till now Maxwell equations have been solved for a given non-stationary source only two times, first, by G. Mie [1] more than a century ago, in spherical coordinates, thus, in particular, for a pont-like oscillating dipole and for the second time, for uniformly accelerated charge about three decades ago in our work [2]. In both cases retarded potentials were known and are in complete contradiction with analytical solution of the Maxwell equations for the same source. Now we have the third example of exact colution for the field of a source which can well be integrated over. That is what we are going to do now.

Surface charge and current densities (8) constitute a source whose retarded potential can well be found. This will be done below. In fact, for our aims it suffices to find only one component of the potential, say, $A_x$. The value of this component in a point $P$ with coordinates $(t_0, x_0, y_0, z_0)$ does not depend on the variable $y_0$ due to translational invariance in along the $y$-axis. The component in question is equal to the integral

$$A_x(t_0, x_0, y_0, z_0) = \frac{a\omega}{2\pi} \int \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}y \, \frac{\sin[\omega(t - t_0) - p(x - x_0)]}{\sqrt{(t - t_0)^2 - (x - x_0)^2 - (y - y_0)^2 - z_0^2}} \tag{10}$$

where the domain of integration is interior of the light cone of the past with the top in the point $P$.

Integration over the variable $y$ in the formula (11) can be completed independently. To do so, introduce a new variable $\eta = y - y_0$ and its limits

$$Y = \pm\sqrt{(t-t_0)^2 - (x-x_0)^2 - z_0^2}.$$

The integral takes the form

$$A_x(t_0, x_0, y_0, z_0) =$$
$$\frac{a\omega}{2\pi} \int \mathrm{d}t\, \mathrm{d}x\, \sin[\omega(t-t_0) - p(x-x_0)] \int\limits_{-Y}^{Y} \frac{\mathrm{d}y}{\sqrt{Y^2-\eta^2}}$$

where the second integral is just a number:

$$\int\limits_{-Y}^{Y} \frac{\mathrm{d}y}{\sqrt{Y^2-\eta^2}} = \pi.$$

Therefore, the form of $A_x$ simplifies:

$$A_x(t_0, x_0, y_0, z_0) = \frac{a\omega}{2} \int \mathrm{d}t\, \mathrm{d}x\, \sin[\omega(t-t_0) - p(x-x_0)]$$

with domain of integration bordered with hyperbolas

$$(t-t_0)^2 - (x-x_0)^2 - z_0^2 < 0.$$

To integrate it over the variable $x$, substitute $\xi = x - x_0$ with new variable ranging as

$$-X \leq \xi \leq X,\ X = \sqrt{(t-t_0)^2 - z_0^2}$$

so that

$$A_x = \frac{a\omega}{2} \int\limits_{-\infty}^{z_0} \mathrm{d}t \int\limits_{-X}^{X} \mathrm{d}\xi\, \sin[\omega(t-t_0) - p(x-x_0)].$$

Note that

$$\int\limits_{-X}^{X} \mathrm{d}\xi\, \sin[\omega(t-t_0) - p(x-x_0)] =$$
$$\frac{1}{p}\{\cos[\omega(t-t_0+pX)] - \cos[\omega(t-t_0-pX)]\} = \frac{2}{p} sin\omega(t-t_0)\sin X.$$

Therefore

$$A_x = \frac{a\omega}{\pi} \int\limits_{-\infty}^{z_0} \mathrm{d}\tau \sin\omega\tau \, \sin\sqrt{\tau^2 - z_0^2},$$

where $\tau = t - t_0$. It looks somewhat unexpected, but the result reads that retarded potential is not function of $\omega t_0$. This fact signifies that retarded potential of the source under consideration cannot represent the field of the source.

# 6 Discussion

The new results derived in this work fall into three categories. First, we show that the field produced by a non-stationary charge and current densities on a plane, possesses mirror symmetry with respect to the plane. By mirror symmetry of electromagnetic field we mean symmetry with respect to transformation which includes ordinary reflection of points of the space and vectors in a given plane. The corresponding transformation of tensor components particularly, of the component $H_{zx}$ usually regarded as the $y$-component of magnetic strength, follows from that of vector components. This result is equally valid for plane and evanescent waves.

Second, we have shown that the field of evanescent wave is static, thus, has nothing to do with electromagnetic radiation. In other words, there always exists a frame of reference in which both the field and its source are perfectly static. At the same time, theoretically, there also exist standing evanescent waves whose fields cannot be turned into electro- or magnetostatic ones by chosing a relevant frame. Formally they are presented by non-stationary fields which have non-stationary sources, but, at the same time cannot be identified with electromagnetic radiation.

The first result provides an explicit form of both field and its source obtained by solving Maxwell equations as they stand that gives an opportunity to compare analytical solution with expression of the field obtained by the method of retarded potential. Unlike other possible solutions with extended sources, plane is a domain integration over which is quite possible. So, our first result gives the third opportunity to verify the theory of retarded potential after H. Hertz' dipole and uniformly accelerated charge. Verification in two other cases shows that theory of retarded pontentials is wrong. First, in case of point-like oscillating dipole, the exact solution of Maxwell equations in spherical coordinates obtained by G. Mie [1], provides an expression of the field strengths in terms of Whittaker functions of the radial coordinate whereas retarded potential contains only elementary functions of the same coordinate. Second, the field of uniformly accelerated charge obtained in our work [2] does not contain any radiation part, whereas retarded potential describes radiation from it [3].

Third, we have calculated retarded potential for the source presented by the surface densities (8). The result of integration is quite strange because its time dependence apparently differs form harmonic oscillation. Thus, result confirms that theory of retarded potential is wrong. Hence, straightforward calculation of the retarded potential in this case shows that theory of retarded potentials is erroneous. Moreover, in case $\omega < p$ when the field is static, subsitution of the D'Alembert equation for that of vector potential is correct, nevertheless the method gives a wrong result. In this case the result demonstrates non-existence of Green function for the D'Alembert equation claimed in our work [4].

# References


[1] Zafar Turakulov *The dipole radiation. Retarded Potential and Maxwell Equations.* Lambert, 2011

[2] Z. Ya. Turakulov, *Electromagnetic field of a charge moving with constant asseleration* J. Geom. Phys. 14(1994)305

[3] J. D. Jackson, *Classical electrodynamics*, Wiley, New York, 1975

[4] Zafar Turakulov. *CFT vs QFT. Green's functions* https://www.researchgate.net/publication/325545752