# On the Boltzmann Equation applied in various sectors of String Theory and the Black Hole Entropy in Canonical Quantum Gravity and Superstring Theory

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**Abstract**

In this paper we have showed the various applications of the Boltzmann equation in string theory and related topics. In the **Section 1**, we have described some equations concerning the time dependent multi-term solution of Boltzmann's equation for charged particles in gases under the influence of electric and magnetic fields, the Planck's blackbody radiation law, the Boltzmann's thermodynamic derivation and the connections with the superstring theory. In the **Section 2**, we have described some equations concerning the modifications to the Boltzmann equation governing the cosmic evolution of relic abundances induced by dilaton dissipative-source and non-critical-string terms in dilaton-driven non-equilibrium string cosmologies. In the **Section 3**, we have described some equations concerning the entropy of an eternal Schwarzschild black hole in the limit of infinite black hole mass, from the point of view of both canonical quantum gravity and superstring theory. We have described some equations regarding the quantum corrections to black hole entropy in string

theory. Furthermore, in this section, we have described some equations concerning the thesis "Can the Universe create itself?" and the adapted Rindler vacuum in Misner space. In the **Section 4**, we have described some equations concerning p-Adic models in Hartle-Hawking proposal and p-Adic and Adelic wave functions of the Universe. Furthermore, we have described in the various Sections the various possible mathematical connections that we've obtained with some sectors of Number Theory and, in the **Section 5**, we have showed some mathematical connections between some equations of arguments above described and p-adic and adelic cosmology.

## 1. On some equations concerning the time dependent multi-term solution of Boltzmann's equation for charged particles in gases under the influence of electric and magnetic fields. [1] [2] [3]

The behaviour of a uniform swarm of electrons in gases under the influence of electric and magnetic fields is described by the Boltzmann equation. This equation represents the time $t$ evolution of the distribution function $f(c,t)$ in velocity space $c$. The distribution function is defined such that $f(c,t)dc$ is the probability of finding a particle within $dc$ of $c$ at time $t$. The explicit form of Boltzmann's equation for charged particle of charge $q$ and mass $m$ under the influence of spatially homogeneous orthogonal electric $E$ and magnetic $B$ fields is

$$\frac{\partial f}{\partial t}+\frac{q}{m}[E+c\times B]\cdot\frac{\partial f}{\partial c}=-J(f,f_0) \quad (1.1)$$

Swarm conditions are assumed to apply and $J(f,f_0)$ denotes the rate of change of $f$ due to binary, particle conserving collisions with the neutral molecules only. The original Boltzmann collision operator and its semi-classical generalisation are used for elastic and inelastic process respectively:

$$J(f,f_0)=\sum_{jk}\int\left[f(c,t)f_{0j}(c_0)-f(c',t)f_{0k}(c'_0)\right]g\sigma(jk;g\chi)d\hat{g}'dc_0 . \quad (1.2)$$

Thence, the eq. (1.1) can be rewritten also

$$\frac{\partial f}{\partial t}+\frac{q}{m}[E+c\times B]\cdot\frac{\partial f}{\partial c}=-\sum_{jk}\int\left[f(c,t)f_{0j}(c_0)-f(c',t)f_{0k}(c'_0)\right]g\sigma(jk;g\chi)d\hat{g}'dc_0 . \quad (1.2b)$$

Dashed and undashed quantities refer to post- and pre-collision properties respectively. The quantity $\sigma(jk;g\chi)$ is the differential cross-section describing the scattering of a swarm particle of velocity $c$, from a neutral molecule in the $j$ th internal state of velocity $c_0$. The post-collision swarm particle and neutral velocities, and the final internal state of the neutral molecule are denoted by $c'$, $c'_0$ and $k$ respectively. The neutral molecules are assumed to remain in thermal equilibrium, characterized by a spatially homogeneous Maxwellian velocity distribution function $f_0(c_0)$. The quantity $d\hat{g}'=\sin\chi d\chi d\zeta$ represents the elements of angles of the post-collision relative velocity where $\chi$ and $\zeta$ are the scattering angles. In what follows, we employ a coordinate system in which $E$ in the $z$-direction, while $B$ is in the $y$-direction.

The angular dependence is represented in terms of a spherical harmonic expansion,

$$f(c,t)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_m^{(l)}(c,t)Y_m^{[l]}(\hat{c}), \quad (1.3)$$

where $Y_m^{[l]}(\hat{c})$ are spherical harmonics which are orthonormal on the angles of $c,\hat{c}$. The superscripts $[\ ]$ and $(\ )$ represent standard and contra-standard spherical tensor forms respectively. The speed distribution function is represented by an expansion about a Maxwellian at a temperature $T_b$ in terms of modified Sonine polynomials:

$$f_m^{(l)}(c,t)=w(\alpha,c)\sum_{\nu=0}^{\infty} F_m^{(\nu l)}(\alpha,t)R_{\nu l}(\alpha c), \quad (1.4)$$

where

$$w(\alpha,c)=\left(\frac{\alpha^2}{2\pi}\right)^{3/2}\exp\left\{\frac{-\alpha^2c^2}{2}\right\}, \quad (1.5) \qquad R_{\nu l}(\alpha c)=N_{\nu l}\left(\frac{\alpha c}{\sqrt{2}}\right)^l S_{l+1/2}^{(\nu)}\left(\frac{\alpha^2c^2}{2}\right), \quad (1.6)$$

$$N_{\nu l}^2=\frac{2\pi^{3/2}\nu!}{\Gamma(\nu+l+3/2)}, \quad (1.7)$$

$S_{l+1/2}^{(\nu)}\left(\frac{1}{2}\alpha^2c^2\right)$ are Sonine polynomials, and $\alpha^2 = m/(kT_b)$. The modified Sonine polynomials satisfy the orthonormality relation

$$\int_0^\infty w(\alpha,c)R_{\nu' l}(\alpha c)R_{\nu l}(\alpha c)c^2dc = \delta_{\nu'\nu}\,. \quad (1.8)$$

This equation can be rewritten also:

$$\int_0^\infty \left(\frac{\alpha^2}{2\pi}\right)^{3/2} \exp\left\{\frac{-\alpha^2c^2}{2}\right\}R_{\nu' l}(\alpha c)N_{\nu l}\left(\frac{\alpha c}{\sqrt{2}}\right)^l S_{l+1/2}^{(\nu)}\left(\frac{\alpha^2c^2}{2}\right)c^2dc = \delta_{\nu'\nu}\,. \quad (1.8b)$$

Making use of the orthogonality properties of the basis functions, the following complex doubly infinite coupled differential equations are generated under conservative conditions:

$$\sum_{\nu'=0}^{\infty}\sum_{l'=0}^{\infty}\sum_{m'=-l'}^{l'}\Bigg[\left(N\frac{d}{dt}\delta_{\nu\nu'} + NJ_{\nu\nu'}^{l}(\alpha)\right)\delta_{l'l}\delta_{m'm} + i\frac{qE}{m}\alpha(l'm10|lm)\langle \nu l\|K^{[1]}(\alpha)\|\nu' l'\rangle\delta_{m'm} +$$
$$+\frac{1}{2}\frac{qB}{m}\left\{\sqrt{(l-m)(l+m+1)}\delta_{m'm+1} - \sqrt{(l+m)(l-m+1)}\delta_{m'm-1}\right\}\delta_{l'l}\delta_{\nu'\nu}\Bigg]F_{m'}^{(\nu' l')} = 0\,, \quad (1.9)$$

where $N$ is the neutral number density and $(l'm10|lm)$ is a Clebsch-Gordan coefficient. For the crossed field configuration, symmetry requirements dictate that the drift velocity vector can only have components in the $E$ and $E\times B$ directions. The drift velocity components and the mean energy are expressed directly in terms of the calculated moment:

$$W_{E\times B} = \frac{1}{\alpha}\sqrt{2}\Im\{F_1^{(01)}\}, \qquad W_E = -\frac{1}{\alpha}\Im\{F_0^{(01)}\}, \qquad \varepsilon = \frac{3}{2}kT_b\left[1-\sqrt{\frac{2}{3}}\Re\{F_0^{(10)}\}\right], \quad (1.9b)$$

where $\Re\{\ \}$ and $\Im\{\ \}$ respectively represent the real and imaginary parts of the moments. For the crossed field configuration, the following symmetry property exists in the moments,

$$F_{-m}^{(\nu l)} = (-1)^m F_m^{(\nu l)}; \quad (1.10)$$

whereas the reality of the distribution function implies

$$\left(F_{-m}^{(\nu l)}\right)^* = (-1)^{l+m} F_m^{(\nu l)}. \quad (1.11)$$

On combining these relations we have

$$\left(F_m^{(\nu l)}\right)^* = (-1)^l F_m^{(\nu l)}, \quad (1.12)$$

and it follows that the system of complex equations can be recast into a form where the renormalized moments are real and only non-negative values of $m$ are required.

Now we take the current Boltzmann equation solution and note that are possible the following mathematical connections between these values and the Aurea ratio:

0.2689; 6.838; 0.1123; 2.318; 0.4154

We note that

$$\left[(\Phi)^{-21/7} + (\Phi)^{-35/7} + (\Phi)^{-70/7}\right] \times \frac{1}{3} = 0.1114 \approx 0.1123,$$

$$\left[(\Phi)^{-63/7} + (\Phi)^{-49/7} + (\Phi)^{-28/7} + (\Phi)^{-7/7}\right] \times \frac{1}{3} = 0.2705 \approx 0.2689,$$

$$\left[(\Phi)^{0} + (\Phi)^{-21/7}\right] \times \frac{1}{3} = 0.4120 \approx 0.4154,$$

$$\left[(\Phi)^{-56/7} + (\Phi)^{-35/7} + (\Phi)^{-21/7} + (\Phi)^{21/7}\right] \times \frac{1}{2} = 2.2917 \approx 2.318,$$

$$\left[(\Phi)^{14/7} + (\Phi)^{35/7}\right] \times \frac{1}{2} = 6.854 \approx 6.838.$$

Thence, mathematical connections with powers of aurea ratio, i.e. $\Phi = \frac{\sqrt{5}+1}{2} = 1{,}6180339$.

## 1.1 On some equations concerning the Planck's blackbody radiation law and the Boltzmann's thermodynamic derivation.

An example of a perfect blackbody radiation describes the radiation in a cavity bounded by any emitting and absorbing opaque substances of uniform temperature. According to Kirchhoff's findings, the state of the thermal radiation in such a cavity is entirely independent of the nature and properties of these substances and only depends on the absolute temperature, $T$, and the frequency, $\nu$ (or the radian frequency $\omega = 2\pi\nu$ or the wavelength $\lambda$). The radiation that ranges from $\nu$ to $\nu + d\nu$ contributes to the field of energy within a volume $dV$, on average, an amount of energy that is proportional to $d\nu$ and $dV$ expressed by

$$dE = U(\nu,T)d\nu dV = U(\omega,T)d\omega dV \,. \quad (1.13)$$

The quantity $U(\nu,T)$ (or $U(\omega,T)$) is called the monochromatic (or spectral) energy density of radiation. According to Planck, in the case of thermal equilibrium, it may be related to the average energy, $\overline{E}$, of a harmonic oscillator of the frequency $\nu$ located inside the cavity walls by

$$U(\nu,T) = A\overline{E}\,, \quad (1.14)$$

where $A$ is a constant. The quantities $A$ and $\overline{E}$ have to be determined. In the case of the thermal equilibrium, the probability, $P(E_j)$, to detect a stationary state with an energy $E_j$ is given by

$$P(E_j) = \alpha g_j \exp\left(-\frac{E_j}{kT}\right). \quad (1.15)$$

Here, $\alpha$ is a constant, $g_j$ is the number of stationary states, and $k = 1.3806 \cdot 10^{-23}\,\mathrm{JK^{-1}}$ is the Boltzmann constant. It reflects Boltzmann's connection between entropy and probability. Analogous to Boltzmann's formula, we express the probability that a harmonic oscillator occupies the $n^{th}$ level of energy, $E_n$, by

$$P_n = P(E_n) = C \exp\left(-\frac{E_n}{kT}\right), \quad (1.16)$$

where $C$ is another constant. Planck postulated that such an oscillator can only have the amount of energy

$$E_n = nh\nu = n\hbar\omega \quad (1.17a)$$

which, in principle, means that the energy is quantized.

Here, $n = 0,1,2,...,\infty$, is an integer, the so-called quantum number, $h = 6.626\times10^{-34}$ Js is the Planck constant, and $\hbar = h/(2\pi)$ is the Dirac constant. Planck assumed that the energy of an oscillator in the ground state (n = 0) equals zero. For n = 0 the zero energy is given by $E_0 = 1/2h\nu$ so that eq. (1.17a) becomes

$$E_n = \left(n + \frac{1}{2}\right) h\nu = \left(n + \frac{1}{2}\right) \hbar\omega. \quad (1.17b)$$

The quanta of energy are only emitted when an oscillator changes from one to another of its quantized energy states according to $\Delta E = E_{n+1} - E_n = h\nu = \hbar\omega$ for n = 0,1,2,... This value is called a quantum of energy. Obviously, the constant C occurring in eq. (1.16) can be determined from the condition that the sum over all probabilities must be equal to unity, i.e.,

$$\sum_{n=0}^{\infty} P_n = \sum_{n=0}^{\infty} Cexp\left(-\frac{E_n}{kT}\right) = C\sum_{n=0}^{\infty} exp\left(-\frac{E_n}{kT}\right) = 1. \quad (1.18)$$

Thus, we have

$$C = \frac{1}{\sum_{n=0}^{\infty} exp\left(-\frac{E_n}{kT}\right)}. \quad (1.19)$$

Now, we consider a lot of oscillators each being a vibrator of frequency $\nu$. Some of these oscillators, namely $N_0$, will be in the ground state (n = 0), $N_1$ will be in the next higher one (n = 1), and so forth. Thus, at the $n^{th}$ energy level we have an energy amount of $\varepsilon_n = E_n N_n$. The number of harmonic oscillators that occupies a level of energy is related to the corresponding probability by $N_n = NP_n(E_n)$ so that

$$\varepsilon_n = E_n N_n = E_n NCexp\left(-\frac{E_n}{kT}\right) = \frac{E_n Nexp\left(-\frac{E_n}{kT}\right)}{\sum_{n=0}^{\infty} exp\left(-\frac{E_n}{kT}\right)}. \quad (1.20)$$

According to

$$N = \sum_{n=0}^{\infty} N_n = \sum_{n=0}^{\infty} NCexp\left(-\frac{E_n}{kT}\right) = N\sum_{n=0}^{\infty} Cexp\left(-\frac{E_n}{kT}\right) = N, \quad (1.21)$$

we may state that N is the total number of harmonic oscillators. The total energy is then given by

$$E = \sum_{n=0}^{\infty} \varepsilon_n = \frac{N\sum_{n=0}^{\infty} E_n exp\left(-\frac{E_n}{kT}\right)}{\sum_{n=0}^{\infty} exp\left(-\frac{E_n}{kT}\right)}. \quad (1.22)$$

From this equation we can infer that the average energy per oscillator in thermal equilibrium as required in eq. (1.14) is given by

$$\bar{E} = \frac{E}{N} = \frac{\sum_{n=0}^{\infty} E_n exp\left(-\frac{E_n}{kT}\right)}{\sum_{n=0}^{\infty} exp\left(-\frac{E_n}{kT}\right)}. \quad (1.23)$$

For simplicity we set

$$Z = \sum_{n=0}^{\infty} exp\left(-\frac{E_n}{kT}\right). \quad (1.24)$$

The derivative of Z with respect to T amounts to

$$\frac{dZ}{dT} = \sum_{n=0}^{\infty} exp\left(-\frac{E_n}{kT}\right)\left(\frac{E_n}{kT^2}\right) = \frac{1}{kT^2}\sum_{n=0}^{\infty} E_n exp\left(-\frac{E_n}{kT}\right) \quad (1.25)$$

or

$$kT^2 \frac{dZ}{dT} = \sum_{n=0}^{\infty} E_n exp\left(-\frac{E_n}{kT}\right). \quad (1.26)$$

Combining eqs. (1.23) and (1.26) yields

$$\bar{E} = \frac{kT^2}{Z}\frac{dZ}{dT} = kT^2 \frac{d}{dT}(\ln Z). \quad (1.27)$$

As $E_n$ is quantized (see eq. (1.17a)), we obtain

$$Z = \sum_{n=0}^{\infty} exp\left(-\frac{nh\nu}{kT}\right) = \sum_{n=0}^{\infty} \left(exp\left(-\frac{h\nu}{kT}\right)\right)^n . \quad (1.28)$$

If we define

$$x = exp\left(-\frac{h\nu}{kT}\right), \quad (1.29)$$

we will easily recognize that

$$Z = \sum_{n=0}^{\infty} x^n \quad (1.30)$$

is a geometric series. As $0 \leq x < 1$, its sum is given by

$$Z = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} = \frac{1}{1-exp\left(-\frac{h\nu}{kT}\right)}. \quad (1.31)$$

Introducing this expression into eq. (1.27) yields

$$\bar{E} = kT^2 \frac{d}{dT}(\ln z) = -kT^2 \frac{d}{dT}\left\{\ln\left(1 - exp\left(-\frac{h\nu}{kT}\right)\right)\right\} =$$

$$= -kT^2 \frac{1}{1-exp\left(-\frac{h\nu}{kT}\right)}\left(-exp\left(-\frac{h\nu}{kT}\right)\left(\frac{h\nu}{kT^2}\right)\right) = \frac{h\nu exp\left(-\frac{h\nu}{kT}\right)}{1-exp\left(-\frac{h\nu}{kT}\right)} \quad (1.32)$$

or

$$\bar{E} = \frac{h\nu}{exp\left(\frac{h\nu}{kT}\right)-1}. \quad (1.33)$$

Introducing this equation into eq. (1.14) provides

$$U(\nu, T) = A\frac{h\nu}{exp\left(\frac{h\nu}{kT}\right)-1}. \quad (1.34)$$

The expression

$$\bar{n} = \frac{1}{exp\left(\frac{h\nu}{kT}\right)-1} = \frac{1}{exp\left(\frac{\hbar\omega}{kT}\right)-1} \quad (1.35)$$

is customarily called the Planck distribution. It may be regarded as a special case of the Bose-Einstein distribution when the chemical potential of a “gas” of photons is given by $\mu = 0$.

Now, we have to determine the constant A. It can be inferred from the classical blackbody radiation law,

$$U(\nu, T) = \frac{8\pi\nu^2}{c^3}kT, \quad (1.36)$$

where $c = 2.998\times10^8$ $ms^{-1}$ is the velocity of light in vacuum.

The classical radiation law fulfils both (a) Kirchhoff’s findings regarding the state of the thermal radiation in a cavity, and (b) the requirements of Wien’s displacement law that reads

$$U(\nu, T) \propto \nu^3 f\left(\frac{\nu}{T}\right). \quad (1.37)$$

For $\nu \to 0$ eq. (1.34) provides $U(\nu, T) \to 0/0$. Thus, we have to use the de l'Hospital's rule. For $f(\nu) = Ah\nu$ and $g(\nu) = exp\big(h\nu/(kT)\big) - 1$ we obtain

$$\lim_{\nu \to 0} \frac{f'(\nu)}{g'(\nu)} = \lim_{\nu \to 0} \frac{Ah}{\frac{h}{kT} exp\left(\frac{h\nu}{kT}\right)} = AkT. \quad (1.38)$$

Comparing eqs. (1.36) and (1.38) yields

$$A = \frac{8\pi\nu^2}{c^3}, \quad (1.39)$$

as already mentioned by Planck. Inserting this expression in eq. (1.34) leads to

$$U(\nu, T) = \frac{8\pi h}{c^3} \frac{\nu^3}{exp\left(\frac{h\nu}{kT}\right) - 1}. \quad (1.40)$$

Consequently, eq. (1.13) may be written as

$$dE = \frac{8\pi h}{c^3} \frac{\nu^3}{exp\left(\frac{h\nu}{kT}\right) - 1} d\nu dV \quad (1.41)$$

or

$$dE = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{exp\left(\frac{\hbar\omega}{kT}\right) - 1} d\omega dV. \quad (1.42)$$

The monochromatic intensity, $B(\nu, T)$, is generally related to the differential amount of radiant energy, dE, that crosses an area element dA in directions confined to a differential solid angle $d\Omega$ being oriented at an angle $\theta$ to the normal of dA,

$$dE = B(\nu, T) \cos\theta dA d\Omega d\nu dt, \quad (1.43)$$

in the time interval between t and t + dt and the frequency interval between $\nu$ and $\nu + d\nu$ . Thus, we obtain

$$dE = \frac{8\pi h}{c^3}\frac{\nu^3}{exp\left(\frac{h\nu}{kT}\right)-1}d\nu dV = B(\nu,T)\cos\theta dAd\Omega dtd\nu =$$

$$= \frac{4\pi}{c}B(\nu,T)\cos\theta\, dA\frac{d\Omega}{4\pi}cdtd\nu = \frac{4\pi}{c}B(\nu,T)d\nu dV \quad (1.44)$$

and, hence

$$B(\nu,T) = \frac{2h}{c^2}\frac{\nu^3}{exp\left(\frac{h\nu}{kT}\right)-1}. \quad (1.45)$$

The quantity $\frac{d\Omega}{4\pi}$ in eq. (1.44) expresses the probability of radiation propagation in a certain direction. Using the relationship

$$B(\vartheta,T)d\vartheta = B(\nu(\vartheta),T)d\nu, \quad (1.46)$$

where $\vartheta$ stands for any variable like radian frequency, $\omega$, wavelength, $\lambda$, wave number as defined in spectroscopy, $n_s = \frac{1}{\lambda} = \nu/c$ or the wave number as defined in physics $n_p = \frac{2\pi}{\lambda} = \frac{2\pi\nu}{c} = \omega/c$, that can be related to the frequency $\nu$ via the transformation $\nu(\vartheta)$, yields then

$$B(\omega,T) = B(\nu(\omega),T)\frac{d\nu}{d\omega} = \frac{\hbar}{4\pi^3c^2}\frac{\omega^3}{exp\left(\frac{\hbar\omega}{kT}\right)-1}. \quad (1.47)$$

Equations (1.45) and (1.47) are customarily called the Planck functions for these two frequency domains. The frequency domain is given by $[0,\infty]$. As

$$B(\lambda,T) = B(\nu(\lambda),T)\frac{d\nu}{d\lambda} = -B(\nu(\lambda),T)\frac{c}{\lambda^2}, \quad (1.48)$$

we obtain for the Planck function in the wavelength domain $[\infty,0]$

$$B(\lambda, T) = -\frac{2hc^2}{\lambda^5\left\{exp\left(\frac{hc}{\lambda kT}\right)-1\right\}}. \quad (1.49)$$

Since the wave numbers defined in physics is

$$n_p = \frac{2\pi}{\lambda} = \frac{2\pi\nu}{c} = \omega/c, \quad (1.50)$$

we obtain the following result

$$B(n_p, T) = B(\nu(n_p), T)\frac{d\nu}{dn_p} = \frac{2\hbar c^2}{(2\pi)^3}\frac{n_p^3}{exp\left(\frac{\hbar c n_p}{kT}\right)-1}. \quad (1.51)$$

Integrating eq. (1.47) over all frequencies yields for the total intensity

$$B(T) = \int_0^\infty B(\nu, T)d\nu = \frac{2h}{c^2}\int_0^\infty \frac{\nu^3}{exp\left(\frac{h\nu}{kT}\right)-1}d\nu. \quad (1.52)$$

Defining $X = \frac{h\nu}{kT}$ leads to

$$B(T) = \int_0^\infty B(\nu, T)d\nu = \frac{2k^4}{c^2h^3}T^4\int_0^\infty \frac{X^3}{exp(X)-1}dX, \quad (1.53)$$

where the integral amounts to

$$\int_0^\infty \frac{X^3}{exp(X)-1}dX = \frac{\pi^4}{15}. \quad (1.54)$$

Thus, eqs. (1.53) and (1.54) provides for the total intensity

$$B(T) = \beta T^4 \quad (1.55)$$

with

$$\beta = \frac{2\pi^4 k^4}{15c^2h^3}. \quad (1.56)$$

Since blackbody radiance may be considered as an example of isotropic radiance, we obtain for the radiative flux density also called the irradiance

$$F(T) = \pi B(T) = \pi\beta T^4 \quad (1.57)$$

or

$$F(T) = \sigma T^4, \quad (1.58)$$

where $\sigma = \pi\beta = 5.67 \times 10^{-8} Jm^{-2}s^{-1}K^{-4}$ is the Stefan constant.

According to Stefan's empirical findings and Boltzmann's thermodynamic derivation, eq. (1.58) is customarily called the power law of Stefan and Boltzmann. On the other hand, the integration of eq. (1.49) over all wavelengths yields

$$B(T) = \int_{\infty}^{0} B(\lambda, T) d\lambda = -\int_{\infty}^{0} \frac{2hc^2}{\lambda^5\left\{exp\left(\frac{hc}{\lambda kT}\right)-1\right\}} d\lambda = \int_{0}^{\infty} \frac{2hc^2}{\lambda^5\left\{exp\left(\frac{hc}{\lambda kT}\right)-1\right\}} d\lambda. \quad (1.59)$$

By using the definition $X = \frac{hc}{\lambda kT}$ the total intensity becomes

$$B(T) = \frac{2k^4}{c^2h^3} T^4 \int_0^{\infty} \frac{X^3}{exp(X)-1} dx = \frac{2\pi^4 k^4}{15c^2h^3} T^4 = \beta T^4, \quad (1.60)$$

i.e., eq. (1.55) deduced in the frequency domain and eq. (1.60) deduced in the wavelength domain provide identical results, and, hence, $\frac{k^4}{(c^2h^3)} = \frac{15\sigma}{(2\pi^5)} = const.$ The same is true for the two wave number domains. If we consider the transformation from the frequency domain to any $\vartheta$ domain, where $\vartheta$ stands for $\omega$, $\lambda$, $n_s$, or $n_p$, we have to consider that $B(\vartheta, T)d\vartheta = B(\nu, T)d\nu$. This relationship was used in eqs. (1.47), (1.48) and (1.51). On the other hand, according to the theorem for the substitution of variables in an integral, we have

$$\int_{\nu_1}^{\nu_2} B(\nu, T) d\nu = \int_{\vartheta_1}^{\vartheta_2} B(\nu(\vartheta), T) \frac{d\nu(\vartheta)}{d\vartheta} d\vartheta, \quad (1.61)$$

where $\nu(\vartheta)$ is the transformation from $\nu$ to $\vartheta$, $\nu_1 = \nu(\vartheta_1)$, and $\nu_2 = \nu(\vartheta_2)$. If we consider, for instance, Rayleigh's radiation formula

$$U(\nu, T) = \frac{8\pi\nu^2}{c^3} kTexp\left(-\frac{h\nu}{kT}\right) \quad (1.62)$$

which satisfies , like the Planck function, Ehrenfest's requirements in the red and violet ranges, the monochromatic intensity will read

$$B(\nu, T) = \frac{2\nu^2}{c^2} kTexp\left(-\frac{h\nu}{kT}\right). \quad (1.63)$$

Thus, the total intensity reads

$$B(T) = \int_0^\infty B(\nu, T)d\nu = \frac{2k^4}{c^2h^3}T^4 \int_0^\infty X^2 exp(-X)dX = \frac{2k^4}{c^2h^3}T^4\Gamma(3) = \beta_R T^4 \quad (1.64)$$

where $X = \frac{h\nu}{kT}$ and $\beta_R = \frac{4k^4}{c^2h^3}$. Here, the subscript R characterizes the value deduced from Rayleigh's radiation formula. This value differs from that obtained with the Planck function by a factor, expressed by $\beta_R = 30\beta/\pi^4 \cong 0.308\beta$. Thus, in the case of Rayleigh's radiation formula the value of the Stefan constant would be given by $\sigma_R = 0.308\sigma = 1.75 \times 10^{-8} Jm^{-2}s^{-1}K^{-4}$. If we consider any finite or filtered spectrum ranging, for instance, from $\nu_1$ to $\nu_2$ , eq. (1.53) will become

$$B(T)_{X_1}^{X_2} = \frac{2k^4}{c^2h^3}T^4 \int_{X_1}^{X_2} \frac{X^3}{exp(X)-1} dX = \beta_F T^4, \quad (1.65)$$

where the value of the filtered spectrum, $\beta_F$, is defined by

$$\beta_F = \frac{2k^4}{c^2h^3} \int_{X_1}^{X_2} \frac{X^3}{exp(X)-1} dX. \quad (1.66)$$

This quantity reflects the real world situations. In such a case the Stefan's constant would become

$$\sigma_F = \sigma \frac{15}{\pi^4} \int_{X_1}^{X_2} \frac{X^3}{exp(X)-1} dX = e_F(X_1, X_2)\sigma, \quad (1.67)$$

where the characteristic value of the filtered spectrum, $e_F(X_1, X_2)$, is defined by

$$e_F(X_1, X_2) = \frac{15}{\pi^4} \int_{X_1}^{X_2} \frac{X^3}{exp(X)-1} dX. \quad (1.68)$$

This means that in the instance of a filtered spectrum the power law of Stefan and Boltzmann must read

$$F_F(T) = e_F(X_1, X_2)\sigma T^4. \quad (1.69)$$

This formula describes the fractional emission of a blackbody due to a finite or filtered spectrum. Equation (1.69) has the form commonly used in the case of gray bodies that are characterized by imperfect absorption and emission.

Even though Wien's displacement relationship was well-known before Planck published his famous radiation law, it can simply be derived using eq. (1.45) by determining its maximum, i.e., $\frac{\partial B(\nu,T)}{\partial \nu} = 0$ and $\frac{\partial^2 B(\nu,T)}{\partial \nu^2} < 0$ . The first derivative reads

$$\frac{\partial}{\partial \nu} B(\nu, T) = \frac{2h\nu^2}{c^2 \left\{exp\left(\frac{h\nu}{kT}\right)-1\right\}^2} \left(3\left\{exp\left(\frac{h\nu}{kT}\right) - 1\right\} - \frac{h\nu}{kT} exp\left(\frac{h\nu}{kT}\right)\right). \quad (1.70)$$

This derivative is only equal to zero when the numerator of the term on the right-hand side of eq. (1.70) is equal to zero (the corresponding denominator is larger than zero for $0 < \nu < \infty$ ), i.e.,

$$3\{exp(x_\nu) - 1\} = x_\nu exp(x_\nu) \quad (1.71)$$

with $x_\nu = h\nu/(kT)$. This transcendental function can only be solved numerically. One obtains $x_\nu \cong 2.8214$ , and in a further step

$$\frac{\nu_{ext}}{T} = x_\nu \frac{k}{h}, \quad (1.72)$$

where $\nu_{ext}$ is the frequency at which the extreme (either a minimum or a maximum) of the Planck function in the frequency domain occurs. We note that the value 2.8214 is related to the Aurea ratio as follow:

$$\left[(\Phi)^{28/7} + (\Phi)^{7/7}\right] \times \frac{1}{3} = 2.8240 \approx 2.8214,$$

with $\Phi = \frac{\sqrt{5}+1}{2} = 1{,}6180339.$

It can simply be proofed that for this extreme the second derivative fulfils the condition $\frac{\partial^2 B(\nu,T)}{\partial \nu^2} < 0$ so that the extreme corresponds to the maximum, $\nu_{max}$. If we use that $c = \lambda\nu$ we will obtain

$$\lambda_{max}^{(\nu)} T = \frac{c}{x_\nu}\frac{h}{k} \cong 5.098 \times 10^{-3} mK. \quad (1.73)$$

This formula should be called Wien's displacement relationship, rather than Wien's displacement law because the latter is customarily used for eq. (1.37).

For the two different wave number domains, we obtain the same result. Thus, we have

$$\lambda_{max}^{(n_p)} = \lambda_{max}^{(n_s)} = \lambda_{max}^{(\nu)}. \quad (1.74)$$

On the other hand, if we consider the Planck function (1.49) formulated for the wavelength domain, the first derivative will read

$$\frac{\partial}{\partial\lambda} B(\lambda, T) = \frac{2hc^2}{\lambda^6 \left\{exp\left(\frac{hc}{\lambda kT}\right) - 1\right\}^2} \left(5\left\{exp\left(\frac{hc}{\lambda kT}\right) - 1\right\} - \frac{hc}{\lambda kT} exp\left(\frac{hc}{\lambda kT}\right)\right). \quad (1.75)$$

Again, this derivative is only equal to zero when the numerator of the term on the right-hand side of eq. (1.75) equals zero. The corresponding denominator is also larger than zero for $0 < \lambda < \infty$. Thus, defining

$$x_\lambda = \frac{hc}{\lambda kT} \quad (1.76)$$

leads to

$$5\{exp(x_\lambda) - 1\} = x_\lambda exp(x_\lambda). \quad (1.77)$$

The numerical solution of this transcendental function reads $x_\lambda \cong 4.9651$. Using this result yields

$$\lambda_{max}^{(\lambda)} T = \frac{c}{x_\lambda} \frac{h}{k} \cong 2.897 \times 10^{-3} mK. \quad (1.78)$$

We note that the value 4.9651 is related with the Aurea ratio as follow:

$$\left[(\Phi)^{28/7} + (\Phi)^{-14/7} + (\Phi)^{-28/7} + (\Phi)^{-49/7}\right] \times \frac{2}{3} = 4.9442 \approx 4.9651,$$

with $\Phi = \frac{\sqrt{5}+1}{2} = 1{,}6180339.$

### 1.2 On some equations concerning the adelic harmonic oscillator.

With regard the ordinary quantum oscillator, the evolution operator $U_\infty(t)$ is defined by

$$U_\infty(t)\psi_n^{(\infty)}(x) = \int_R K_t^{(\infty)}(x,y)\psi_n^{(\infty)}(y)dy, \quad (1.79)$$

where the kernel $K_t^{(\infty)}(x,y)$ for the harmonic oscillator is

$$K_t^{(\infty)}(x,y) = \lambda_\infty(2\sin t)|\sin t|_\infty^{-1/2} exp 2\pi i \left(\frac{x^2+y^2}{2\tan t} - \frac{xy}{\sin t}\right). \quad (1.80)$$

With regard the p-Adic quantum oscillator, a p-adic evolution operator is given by

$$U_p(t)\psi^{(p)}(x) = \int_{Q_p} K_t^{(p)}(x,y)\psi^{(p)}(y)dy, \quad (1.81)$$

where the kernel for the harmonic oscillator is

$$K_t^{(p)}(x,y) = \lambda_p(2t)|t|_p^{-1/2}\chi_p\left(\frac{xy}{\sin t} - \frac{x^2+y^2}{2\tan t}\right), \quad t \in G_p\backslash\{0\}. \quad (1.82)$$

Thence, the eq. (1.81) can be written also as follow

$$U_p(t)\psi^{(p)}(x) = \int_{Q_p} \lambda_p\,(2t)|t|_p^{1/2}\chi_p\left(\frac{xy}{\sin t} - \frac{x^2+y^2}{2\tan t}\right)\psi^p(y)dy. \quad (1.82b)$$

Furthermore, the operator $U_p(t)$ and its kernel $K_t^{(p)}(x,y)$ satisfy the group relations

$$U_p(t+t') = U_p(t)U_p(t'), \quad (1.83)$$

$$K_{t+t'}^{(p)}(x,y) = \int_{Q_p} K_t^{(p)}(x,z)K_{t'}^{(p)}(z,y)dz. \quad (1.84)$$

With regard the harmonic oscillator over adeles, let the evolution operator $U(t)$ be defined by

$$U(t)\psi(x) = \int_A K_t(x,y)\psi(y)dy, \quad (1.85)$$

where $t \in G \subset A$ , $x,y \in A$, and $\psi(x) \in L_2(A)$. Also $U(t) = U_\infty(t_\infty)\prod_p U_p(t_p)$ and

$$K_t(x,y) = K_{t_\infty}^{(\infty)}(x_\infty,y_\infty)\prod_p K_{t_p}^{(p)}(x_p,y_p), \quad (1.86)$$

where $K_{t_\infty}^{(\infty)}(x_\infty,y_\infty)$ and $K_{t_p}^{(p)}(x_p,y_p)$ for the harmonic oscillator are given by (1.80) and (1.82). Denoting $\lambda(a) = \lambda_\infty(a_\infty)\prod_p \lambda_p(a_p)$ one can write

$$K_t(x,y) = \lambda(2\sin t)|\sin t|^{-1/2}\chi\left(\frac{xy}{\sin t} - \frac{x^2+y^2}{2\tan t}\right), \quad (1.87)$$

which resembles the form of real and p-adic kernels. Thence, the eq. (1.85) can be written also as follow:

$$U(t)\psi(x) = \int_A \lambda(2\sin t)|\sin t|^{-1/2}\chi\left(\frac{xy}{\sin t} - \frac{x^2+y^2}{2\tan t}\right)\psi(y)dy. \quad (1.87b)$$

We define an orthonormal basis for the corresponding adelic evolution operator,

$$U(t) = U_\infty(t_\infty)\prod_p U_p(t_p), \quad t \in A, \quad (1.88)$$

as

$$\psi_{\alpha\beta}(x) = \psi_{nm}^{(\infty)}(x_\infty)\prod_p \psi_{\alpha_p\beta_p}^{(p)}(x_p), \quad x \in A. \quad (1.89)$$

According to the definition (1.89), it follows that any eigenstate of the adelic harmonic oscillator is

$$\psi_{\alpha\beta}(x) = \frac{2^{1/4}}{(2^n n!)^{1/2}} e^{-\pi x^2} H_n\left(x\sqrt{2\pi}\right)\prod_{p\in\Gamma_{\alpha\beta}}\psi_{\alpha_p\beta_p}(x_p)\prod_{p\notin\Gamma_{\alpha\beta}}\Omega\left(|x_p|_p\right). \quad (1.90)$$

## 2. On some equations concerning the modifications to the Boltzmann equation governing the cosmic evolution of relic abundances induced by dilaton dissipative-source and non-critical-string terms in dilaton-driven non-equilibrium string cosmologies. [4]

The main relationship between the Einstein and $\sigma$-model frames is

$$g_{\mu\nu} = e^{-2\Phi}g_{\mu\nu}^{\sigma}, \quad \frac{\partial t}{\partial t^{\sigma}} = e^{-\Phi} \quad (2.1)$$

where $\Phi$ is the dilaton field, and the superscript $\sigma$ denotes quantities evaluated in the $\sigma$-model frame. To discuss the non-critical string (off-shell) corrections to Boltzmann equation, it will be necessary to consider time derivatives in the $\sigma$-model frame. This is due to the fact that it is in this frame that the target time $X^0 \equiv t_\sigma$ is related simply to the Liouville mode $\varphi$ in non-critical string theories,

$$\varphi + t_\sigma = 0. \quad (2.2)$$

The solution of the generalized conformal invariance conditions, after Liouville dressing, in the $\sigma$-model frame is:

$$-\tilde{\beta}^i = g^{i''} + Qg^{i'} \quad (2.3)$$

where the prime denotes differentiation with respect to the Liouville zero mode $\rho$, and the overall minus sign on the left-hand side of the above equation pertains to supercritical strings, with a time like signature of the Liouville mode, for which the central charge deficit $Q^2 > 0$ by convention. Furthermore, we have that upon the inclusion of matter backgrounds, including dark matter species, the associated equations, after compactification to four target-space dimensions, read in the Einstein frame:

$$3H^2 - \widetilde{\varrho_m} - \varrho_\Phi = \frac{e^{2\Phi}}{2}\widetilde{G_\Phi},$$

$$2\dot{H} + \widetilde{\varrho_m} + \varrho_\Phi + \widetilde{p_m} + p_\Phi = \frac{\widetilde{G_{ii}}}{a^2},$$

$$\ddot{\Phi} + 3H\dot{\Phi} + \frac{1}{4}\frac{\partial \hat{V}_{all}}{\partial \Phi} + \frac{1}{2}(\widetilde{\varrho_m} - 3\widetilde{p_m}) = -\frac{3}{2}\frac{\widetilde{G_{ii}}}{a^2} - \frac{e^{2\Phi}}{2}\widetilde{G_\Phi}, \quad (2.4)$$

where $\widetilde{\varrho_m}(\widetilde{p_m})$ denotes the matter energy density (pressure), including dark matter contributions, and $\varrho_\Phi(p_\Phi)$ the corresponding quantities for the dilaton dark-energy fluid. All derivatives in (2.4) are with respect the Einstein time $t$ which is related to the Robertson-Walker cosmic time. The modified Boltzmann equation for a four-dimensional effective field theory after string compactification (or restriction on three-brane worlds), in the presence of non-critical (off-shell) string backgrounds and dilaton source terms is:

$$\frac{dn}{dt} = \frac{\dot{a}}{a}\int d^3p\frac{|\vec{p}|^2}{E}\frac{\partial f}{\partial E} - \left(\dot{\Phi} + \eta e^{\Phi}\Phi'\right)\int d^3p\frac{\partial f}{\partial \Phi} - \eta e^{\Phi}\left(\frac{a'}{a} + \Phi'\right)$$

$$\int d^3p|\vec{p}|\frac{\partial f}{\partial|\vec{p}|} + \int d^3p\frac{C[f]}{E} \rightarrow \frac{dn}{dt} + 3\frac{\dot{a}}{a}n = 3\eta e^{\Phi}\left(\frac{a'}{a} + \Phi'\right)n +$$

$$+\int \frac{d^3p}{E}C[f] - \left(\dot{\Phi} + \eta e^{\Phi}\Phi'\right)\int d^3p\frac{\partial f}{\partial \Phi}. \quad (2.5)$$

With regard the form of the dependence of $f$ on the dilaton source terms, which would survive a dilaton-driven critical-string cosmology case, we constrain this form by requiring that in the Einstein frame these are two types of dependence on $\Phi$: (i) *explicit*, of the form $e^{-4\Phi}$, arising from the fact that the phase space density is constructed as a quantity in the $\sigma$-model frame of the string, which is then expressed in terms of quantities in the Einstein frame. As such, it is by definition (as a density) *inversely* proportional to the proper $\sigma$-model frame volume $V^\sigma = \int d^4x\sqrt{-g^\sigma} \propto e^{4\Phi}$ on account of (2.1); (ii) *implicit*, corresponding to a dependence on $\Phi$ through the Einstein-frame metric $g_{ii}$ (2.1). Hence the general structure of $f$ is of the form:

$$f\left(\Phi, \vec{p}, \vec{x}, g^\sigma_{\mu\nu} = e^{2\Phi} g_{\mu\nu}; t\right) \propto e^{-4\Phi} F(|\vec{p}|, \vec{x}, t). \quad (2.6)$$

This implies that:

$$\int d^3p \frac{\partial f}{\partial \Phi} = -4\int d^3p f + \sum_{i=1}^{3}\int d^3p \frac{\partial g_{ii}}{\partial \Phi}\frac{\partial f}{\partial g_{ii}} = -4n - 2\int d^3p \sum_{i=1}^{3} g_{ii}\frac{\partial |\vec{p}|}{\partial g_{ii}}\frac{\partial f}{\partial |\vec{p}|} =$$

$$= -4n - \int d^3p |\vec{p}| \frac{\partial f}{\partial |\vec{p}|} = -4n + 3\int d^3p f(|\vec{p}|, t) = -n, \quad (2.7)$$

where in the last step we have performed appropriate partial (momentum-space) integrations. The final form of the Liouville operation (2.5), then, reads:

$$\frac{dn}{dt} + 3\left(\frac{\dot{a}}{a}\right)n - \dot{\Phi}n = 3\eta e^{\Phi}\frac{a'}{a}n + 4\eta e^{\Phi}\Phi' n + \int \frac{d^3p}{E} C[F]. \quad (2.8)$$

We now notice that non-critical terms can be expressed in terms of the Weyl anomaly coefficients for the ($\sigma$-model) graviton and dilaton backgrounds as:

$$3\eta e^{\Phi}\frac{a'}{a} = 4\eta e^{-\Phi}\left(\frac{1}{8} g^{\mu\nu}\tilde{\beta}^{Grav}_{\mu\nu} - \frac{3}{4}\tilde{\beta}^{\Phi} e^{2\Phi}\right)$$

$$4\eta e^{\Phi}\Phi' = 4\eta e^{\Phi}\tilde{\beta}^{\Phi}. \quad (2.9)$$

where we used the Einstein frame metric to contract indices, with $\tilde{\beta}^{Grav}$ denoting the graviton Weyl anomaly coefficient. We are concentrating in this chapter to the $\tilde{\beta}^{Grav}_{00} = 0$. Thus the Boltzmann equation finally becomes

$$\frac{dn}{dt} + 3\left(\frac{\dot{a}}{a}\right)n - \dot{\Phi}n = \frac{1}{2}\eta\left(e^{-\Phi} g^{\mu\nu}\tilde{\beta}^{Grav}_{\mu\nu} + 2e^{\Phi}\tilde{\beta}^{\Phi}\right)n + \int \frac{d^3p}{E} C[f]. \quad (2.10)$$

Now we consider solutions of the modified Boltzmann equation (2.8), or equivalently (2.10), for a particle species density $n$ in the physically interesting case of supersymmetric dark matter species, viewed as the lightest supersymmetric particles. It is convenient to write the Boltzmann equation for the density of species $n$ in a compact form that represents collectively the dilaton-dissipative-source and non-critical-string contributions as external-source $\Gamma(t)n$ terms:

$$\frac{dn}{dt} + 3\frac{\dot{a}}{a}n = \Gamma(t)n + \int \frac{d^3p}{E}C[f], \quad \Gamma(t) \equiv \dot{\Phi} + \frac{1}{2}\eta\left(e^{-\Phi}g^{\mu\nu}\tilde{\beta}^{Grav}_{\mu\nu} + 2e^{\Phi}\tilde{\beta}^{\Phi}\right) \quad (2.11)$$

where we work in the physical scheme (2.2) from now on, for which $\eta = -1$. Depending on the sign of $\Gamma(\mathrm{t})$ one has different effects on the relic abundance of the species $X$ with density $n$, which we now proceed to analyze. To find an explicit expression for $\Gamma(t)$ in our case we should substitute the solution of (2.3), more specifically (2.4). Regarding the form of (2.11) it is nice to see that the extra terms can be cast in a simple-looking form of a source term $\Gamma(t)n$ including both the dilaton dissipation and the non-critical-string terms. In a more familiar form, the interaction term $C[f]$ of the above modified Boltzmann equation can be expressed in terms of the thermal average of the cross section $\sigma$ times the Moeller velocity $v$ of the annihilated particles

$$\frac{dn}{dt} = -3\frac{\dot{a}}{a}n - \langle v\sigma\rangle\left(n^2 - n_{eq}^2\right) + \Gamma n. \quad (2.12)$$

Let us assume that $n = n_{eq}^{(0)}$ at a very early epoch $t_0$. Then the solution of the modified Boltzmann equations at all times $t < t_f$ is given by

$$n_{eq}a^3 = n_{eq}^{(0)}a^3(t_0)exp\left(\int_{t_0}^{t}\Gamma dt\right). \quad (2.13)$$

The time $t_0$ characterizes a very early time, which is not unreasonable to assume that it signals the exit from the inflationary period. Soon after the exit from inflation, all particles are in thermal equilibrium, for all times $t < t_f$, with the source term modifying the usual Boltzmann distributions in the way indicated in eq. (2.13) above. It has been tacitly assumed that the entropy is conserved despite the presence of the source and the non-critical-string contributions. In our approach this is an *approximation*, since we know that non-critical strings lead to entropy production. However, the entropy increase is most significant during the inflationary era, and hence it is not inconsistent to assume that there is no significant entropy production after the exit from inflation. We assume that above the freeze-out point the density is the equilibrium density as provided by eq. (2.13), while below this the interaction terms starts becoming unimportant. It is customary to define $x \equiv T/m_{\tilde{\chi}}$ and restrict the discussion on a particular species $\tilde{\chi}$ of mass $m_{\tilde{\chi}}$, which eventually may play the role of the dominant Dark Matter candidate. It also proves convenient to trade the number density $n$ for the quantity $Y \equiv n/s$, that is the number per entropy density. The equation for $Y$ is derived from (2.12) and is given by

$$\frac{dY}{dx} = m_{\tilde{\chi}}\langle v\sigma\rangle \left(\frac{45}{\pi} G_N \tilde{g}_{eff}\right)^{-1/2} \left(h + \frac{x}{3}\frac{dh}{dx}\right)\left(Y^2 - Y_{eq}^2\right) - \frac{\Gamma}{Hx}\left(1 + \frac{x}{3h}\frac{dh}{dx}\right) Y. \quad (2.14)$$

where $G_N = 1/M_{Planck}^2$ is the four-dimensional gravitational constant, the quantity $H$ is the Hubble expansion rate, $h$ denote the entropy degrees of freedom, and $\langle v\sigma\rangle$ is the thermal average of the relative velocity times the annihilation cross section and $\tilde{g}_{eff}$ is simply defined by the relation

$$\varrho + \Delta\varrho \equiv \frac{\pi^2}{30} T^4 \tilde{g}_{eff}. \quad (2.15)$$

We next remark that $\rho$, as well as $\Delta\rho$, as functions of time are known, once one solves the cosmological equations. However, only the degrees of freedom involved in $\rho$ are thermal, the rest, like the cosmological-constant term if present in a model, are included in $\Delta\rho$. Therefore, the relation between temperature and time is provided by

$$\rho = \frac{\pi^2}{30} T^4 g_{eff}(T) \quad (2.16)$$

while $\rho + \Delta\rho$ are involved in the evolution through

$$H^2 = \frac{8\pi G_N}{3}(\rho + \Delta\rho). \quad (2.17)$$

We note that $\frac{45}{\pi} = 14.323944$ and $\frac{\pi^2}{30} = 0.328986813$ can be connected with the Aurea ratio as follow:

$$\left[(\Phi)^{28/7} + (\Phi)^{14/7} + (\Phi)^{-21/7}\right] \times \frac{3}{2} = 14.5623 \approx 14.3239,$$

$$\left[(\Phi)^{-14/7} + (\Phi)^{-42/7}\right] \times \frac{3}{4} = 0.32827 \approx 0.3289.$$

with $\Phi = \frac{\sqrt{5}+1}{2} = 1,6180339.$

Thus, it is important to bear in mind that $\Delta\rho$ contributes to the dynamical expansion, through eq. (2.17), but not to the thermal evolution of the Universe. The quantity $\tilde{g}_{eff}$ defined in (2.15), is therefore given by

$$\tilde{g}_{eff} = g_{eff} + \frac{30}{\pi^2} T^{-4} \Delta\rho . \quad (2.18)$$

The meaning of the above expression is that time has been replaced by temperature, through eq. (2.16), after solving the dynamical equations. In terms of $\tilde{g}_{eff}$ the expansion rate $H$ is written as

$$H^2 = \frac{4\pi^3 G_N}{45} T^4 \tilde{g}_{eff} . \quad (2.19)$$

This is used in the Boltzmann equation for $Y$ and the conversion from the time variable $t$ to temperature or, equivalently, the variable $x$. For $x$ above the freezing point $x_f$, $Y \approx Y_{eq}$ and, upon omitting the contributions of the derivative terms $dh/dx$, an approximation which is also adopted in the standard cosmological treatments, we obtain for the solution of (2.14)

$$Y_{eq} = Y_{eq}^{(0)} \exp\left( - \int_x^\infty \frac{\Gamma H^{-1}}{x} dx \right). \quad (2.20)$$

Here, $Y_{eq}^{(0)}$ corresponds to $n_{eq}^{(0)}$ and in the non-relativistic limit is given by

$$Y_{eq}^{(0)} = \frac{45}{2\pi^2} \frac{g_s}{h} (2\pi x)^{-3/2} \exp(-1/x) \quad (2.21)$$

where $g_s$ counts the particle's spin degrees of freedom. In the regime $x < x_f$, $Y >> Y_{eq}^{(0)}$ the eq. (2.14) can be written as

$$\frac{d}{dx}\frac{1}{Y} = -m_{\widetilde{\chi}}\langle v\sigma\rangle\left(\frac{45}{\pi}G_N\widetilde{g}_{eff}\right)^{-1/2} h + \frac{\Gamma H^{-1}}{xY}. \quad (2.22)$$

We note that $\frac{45}{2\pi^2} = 2.279726$, $\frac{30}{\pi^2} = 3.03963$ and $\frac{4\pi^3}{45} = 2.75611$ can be connected with the Aurea ratio as follow:

$$\left[(\Phi)^{21/7} + (\Phi)^0 + (\Phi)^{-21/7} + (\Phi)^{-35/7}\right] \times \frac{1}{2} = 2.7811 \approx 2.75611,$$

$$\left[(\Phi)^{28/7}\right] \times \frac{4}{9} = 3.046 \approx 3.03963,$$

$$\left[(\Phi)^{28/7}\right] \times \frac{1}{3} = 2.2847 \approx 2.279726,$$

with $\Phi = \frac{\sqrt{5}+1}{2} = 1{,}6180339$.

Applying (2.22) at the freezing point $x_f$ and using (2.20) and (2.21), leads, after a straightforward calculation, to the determination of $x_f = T_f / m_{\widetilde{\chi}}$ through

$$x_f^{-1} = \ln\left[0.03824 g_s \frac{M_{Planck} m_{\widetilde{\chi}}}{\sqrt{g*}} x_f^{1/2}\langle v\sigma\rangle_f\right] + \frac{1}{2}\ln\left(\frac{g*}{\widetilde{g}*}\right) + \int_{x_f}^{x_{in}} \frac{\Gamma H^{-1}}{x} dx. \quad (2.23)$$

As usual, all quantities are expressed in terms of the dimensionless $x \equiv T / m_{\tilde{\chi}}$ and $x_{in}$ corresponds to the time $t_0$, taken to represent the exit from the inflationary period of the Universe.

Now, we note that, in order to calculate the relic abundance, we must solve (2.22) from $x_f$ to today's value $x_0$, corresponding to a temperature $T_0 \approx 2.7°K$. Then, we arrive at the result:

$$Y^{-1}(x_0) = Y^{-1}(x_f) + \left(\frac{\pi}{45}\right)^{1/2} m_{\tilde{\chi}} M_{Planck} \tilde{g}_*^{-1/2} h(x_0) J - \int_{x_0}^{x_f} \frac{\Gamma H^{-1}}{xY} dx . \quad (2.24)$$

We note that $\sqrt{\frac{\pi}{45}} = 0.26422$ is related to the Aurea ratio as follow:

$$(\Phi)^{-35/7} + (\Phi)^{-56/7} + (\Phi)^{-77/7} + (\Phi)^{-91/7} \times \frac{9}{4} = 0.2664 \cong 0.2642 \ldots$$

with $\Phi = \frac{\sqrt{5}+1}{2} = 1{,}6180339$.

In conventional Cosmology $\tilde{g}_*$ is replaced by $g_*$ and the last term in (2.24) is absent. The quantity $J$ is $J \equiv \int_{x_0}^{x_f} \langle v\sigma \rangle dx$. By replacing $Y(x_f)$ by its equilibrium value (2.20) the ratio of the first term on the right hand-side of (2.24) to the second is found to be exactly the same as in the no-dilaton case. Therefore, by the same token as in conventional Cosmology, the first term can be safely omitted, as long as $x_f$ is of order of 1/10 or less. Furthermore, the integral on the right hand-side of (2.24) can be simplified if one uses the fact that $\langle v\sigma \rangle n$ is small as compared with the expansion rate $\dot{a}/a$ after decoupling. For the purposes of the evaluation of this integral, therefore, this term can be omitted in (2.22), as long as we stay within the decoupling regime, and one obtains:

$$\frac{d}{dx}\frac{1}{Y} = \frac{\Gamma H^{-1}}{xY} . \quad (2.25)$$

By integration this yields $Y(x) = Y(x_0)\exp\left(-\int_{x_0}^{x} \Gamma H^{-1} dx / x\right)$. Using this inside the integral in (2.24) we get

$$(h(x_0)Y(x_0))^{-1} = \left(1 + \int_{x_0}^{x_f} \frac{\Gamma H^{-1}}{\psi(x)} dx\right)^{-1} \left(\frac{\pi}{45}\right)^{1/2} m_{\tilde{\chi}} M_{Planck} \tilde{g}_*^{-1/2} J \quad (2.26)$$

where the function $\psi(x)$ is given by $\psi(x) \equiv x \exp\left(-\int_{x_0}^{x} \Gamma H^{-1} dx / x\right)$. With the exception of the prefactor on the right hand-side of (2.26), this is identical in form to the result derived in standard treatments, if $\tilde{g}_*$ is replaced by $g_*$ and the value of $x_f$, implicitly involved in the integral $J$, is replaced by its value found in ordinary treatments in which the dilaton-dynamics and non-critical-string effects are absent.

The matter density of species $\tilde{\chi}$ is then given by

$$\rho_{\tilde{\chi}} = f \left(\frac{4\pi^3}{45}\right)^{1/2} \left(\frac{T_{\tilde{\chi}}}{T_\gamma}\right)^3 \frac{T_\gamma^3}{M_{Planck}} \frac{\sqrt{\tilde{g}_*}}{J} \quad (2.27)$$

where the prefactor $f$ is:

$$f = 1 + \int_{x_0}^{x_f} \frac{\Gamma H^{-1}}{\psi(x)}$$

thence, the eq. (2.27) can be rewritten also:

$$\rho_{\tilde{\chi}} = \left(1 + \int_{x_0}^{x_f} \frac{\Gamma H^{-1}}{\psi(x)}\right) \left(\frac{4\pi^3}{45}\right)^{1/2} \left(\frac{T_{\tilde{\chi}}}{T_\gamma}\right)^3 \frac{T_\gamma^3}{M_{Planck}} \frac{\sqrt{\tilde{g}_*}}{J} . \quad (2.27b)$$

We note that the value $\sqrt{\frac{4\pi^3}{45}} = 1.6601546$ is related with the Aurea ratio as follow:

$$\Phi + \frac{\frac{1}{2}(\phi)}{10} = 1.648.. \approx 1.660,$$

$$(\Phi)^{14/7} + (\Phi)^0 + (\Phi)^{-35/7} \times \frac{4}{9} = 1.648 \ldots$$

where $\Phi = \frac{\sqrt{5}+1}{2}$ and $\phi = \frac{\sqrt{5}-1}{2}$.

It is important to recall that the thermal degrees of freedom are counted by $g_{eff}$, and not $\tilde{g}_{eff}$, the latter being merely a convenient device connecting the total energy, thermal and non-thermal, to the temperature $T$. Hence,

$$\left(\frac{T_{\tilde{\chi}}}{T_\gamma}\right)^3 = \frac{g_{eff}(1MeV)}{g_{eff}(T_{\tilde{\chi}})} \frac{4}{11} = \frac{43}{11}\frac{1}{g_*}. \quad (2.28)$$

In deriving (2.28) only the thermal content of the Universe is used, while the dilaton and the non-critical terms do not participate. Therefore the $\tilde{\chi}$'s matter density is given by

$$\rho_{\tilde{\chi}} = f\left(\frac{4\pi^3}{45}\right)^{1/2} \frac{43}{11} \frac{T_\gamma^3}{M_{Planck}} \frac{\sqrt{\tilde{g}_*}}{g_* J}. \quad (2.29)$$

This formula tacitly assumes that the $\tilde{\chi}$ is decoupled before neutrinos. For the relic abundance, then, we derive the following approximate result

$$\Omega_{\tilde{\chi}} h_0^2 = \left(\Omega_{\tilde{\chi}} h_0^2\right)_{no-source} \times \left(\frac{\tilde{g}_*}{g_*}\right)^{1/2} \exp\left(\int_{x_0}^{x_f} \frac{\Gamma H^{-1}}{x} dx\right), \quad (2.30)$$

where we used that fact that: $1+\int_{x_0}^{x_f}\frac{\Gamma H^{-1}}{\psi(x)}dx=\exp\left(\int_{x_0}^{x_f}\frac{\Gamma H^{-1}}{x}dx\right)$. In (2.30), the quantity referred to as no-source is the well known no-source expression

$$\left(\Omega_{\tilde{\chi}}h_0^2\right)_{no-source}=\frac{1.066\times10^9\,GeV^{-1}}{M_{Planck}\sqrt{g_*}J} \qquad (2.31)$$

where $J\equiv\int_{x_0}^{x_f}\langle v\sigma\rangle dx$. Thence, the eq. (2.30) can be rewritten also:

$$\Omega_{\tilde{\chi}}h_0^2=\frac{1.066\times10^9\,GeV^{-1}}{M_{Planck}\sqrt{g_*}\int_{x_0}^{x_f}\langle v\sigma\rangle dx}\times\left(\frac{\tilde{g}_*}{g_*}\right)^{1/2}\exp\left(\int_{x_0}^{x_f}\frac{\Gamma H^{-1}}{x}dx\right). \qquad (2.31b)$$

We note that the value $\frac{43}{11}=3.9090$ is related to the Aurea ratio as follow

$$(2\times\Phi)+\phi=3.8541\cong3.9090,$$

$$(\Phi)^{28/7}+(\Phi)^0\times\frac{1}{2}=3.9270,$$

where $\Phi=\frac{\sqrt{5}+1}{2}$ and $\phi=\frac{\sqrt{5}-1}{2}$.

However, as already remarked, the end point $x_f$ in the integration is the shifted freeze-out point as determined by eq. (2.23). The merit of casting the relic density in such a form is that it clearly exhibits the effect of the presence of the source.

**3. On some equations concerning the entropy of an eternal Schwarzschild black hole in the limit of infinite black hole mass, from the point of view of both canonical quantum gravity and superstring theory. [5] [6] [7]**

The counting of string states near a horizon of a black hole, gives a finite entropy which agrees with the usual Bekenstein-Hawking result

$$\sigma_{BH} = \frac{4\pi M^2}{G} = \frac{A}{4G} . \quad (3.1)$$

The partition function for a single mode, labelled by the quantum numbers $n$ and $\vec{k}$, is given by

$$Z\left(\beta; n, \vec{k}\right) = \sum_{m=0}^{\infty} e^{-m\beta\omega_n\left(\vec{k}\right)} = \left(1 - e^{-\beta\omega_n\left(\vec{k}\right)}\right)^{-1} . \quad (3.2)$$

Since the modes are independent, the total partition function is

$$Z(\beta) = \prod_{n,\vec{k}} Z\left(\beta; n, \vec{k}\right) = \exp(-\beta F(\beta)), \quad (3.3)$$

and the Helmholtz free energy is

$$F(\beta) = -\frac{1}{\beta} \sum_{n,\vec{k}} \log\left(Z\left(\beta; n, \vec{k}\right)\right). \quad (3.4)$$

Approximating the sums by integrals, equation (3.4) becomes

$$F(\beta)=\frac{L^2}{\beta}\int_{R^2}\frac{d^2k}{(2\pi)^2}\int_{\xi\varepsilon}^{\infty}d\omega\frac{dn}{d\omega}\log\left(1-e^{-\beta\omega}\right). \quad (3.5)$$

Differentiating the following equation

$$n=\frac{\omega}{2\pi}\left[\log\left(\frac{1+\sqrt{1-(\xi\varepsilon/\omega)^2}}{1-\sqrt{1-(\xi\varepsilon/\omega)^2}}\right)-2\sqrt{1-(\xi\varepsilon/\omega)^2}\right], \quad (3.5b)$$

which is an implicit equation for the frequencies $\omega$, gives the density of levels

$$\frac{dn}{d\omega}=\frac{1}{2\pi}\log\left(\frac{1+\sqrt{1-(\xi\varepsilon/\omega)^2}}{1-\sqrt{1-(\xi\varepsilon/\omega)^2}}\right), \quad (3.6)$$

and changing the orders of integration, one obtains the expression

$$F(\beta)=\frac{A}{(2\pi)^2\beta}\int_{\varepsilon m}^{\infty}d\omega\log\left(1-e^{-\beta\omega}\right)\int_0^{\sqrt{(\omega/\varepsilon)^2-m^2}}dkk\log\left(\frac{1+\sqrt{1-(\varepsilon/\omega)^2(k^2+m^2)}}{1-\sqrt{1-(\varepsilon/\omega)^2(k^2+m^2)}}\right), \quad (3.7)$$

where $L^2=A$ is the area of the horizon. After performing the integral over $k$, equation (3.7) becomes

$$F(\beta)=\frac{A}{(2\pi)^2\beta}\int_{\varepsilon m}^{\infty}d\omega\log\left(1-e^{-\beta\omega}\right)\left[(\omega/\varepsilon)^2\sqrt{1-(\varepsilon m/\omega)^2}+\frac{m^2}{2}\log\left(\frac{1-\sqrt{1-(\varepsilon m/\omega)^2}}{1+\sqrt{1-(\varepsilon m/\omega)^2}}\right)\right]. \quad (3.8)$$

Expanding in powers of the field mass $m$, the leading term is

$$F(\beta) = \frac{A}{(2\pi\varepsilon)^2 \beta} \int_0^\infty d\omega\omega^2 \log\left(1 - e^{-\beta\omega}\right), \quad (3.9)$$

which is integrated to yield

$$F(\beta) = -\frac{\pi^2 A}{180\varepsilon^2 \beta^4}. \quad (3.10)$$

We note that the value $\frac{\pi^2}{180} = 18.2378$ is related to the Aurea ratio as follow

$$(\Phi)^{35/7} + (\Phi)^{14/7} \times \frac{4}{3} = 18.2776 \cong 18.2378,$$

where $\Phi = \frac{\sqrt{5}+1}{2}$.

We note that the eq. (3.3) can be rewritten also as follow:

$$Z(\beta) = \prod_{n,\vec{k}} Z\left(\beta; n, \vec{k}\right) = \exp\left(-\beta \frac{A}{(2\pi\varepsilon)^2 \beta} \int_0^\infty d\omega\omega^2 \log\left(1 - e^{-\beta\omega}\right)\right). \quad (3.10b)$$

The resulting entropy, evaluated at the Rindler temperature $\tau_R = 1/2\pi$, is

$$\sigma_\phi = \frac{A}{360\pi\varepsilon^2}. \quad (3.11)$$

The result (3.11) agrees with the entropy of a scalar field propagating outside a finite mass black hole as calculated by ‘t Hooft. In the limit of a large sphere, the entropy per unit area should agree with that in Rindler space, i.e. in qualitative agreement with the result (3.11).

We will calculate the entropy of an infinitely massive, eternal Schwarzschild black hole by (formally) evaluating the functional integral of Euclidean canonical quantum gravity. We begin by formulating the functional integral representation of the partition function. If a spacetime manifold $\mathcal{M}$ is static, there exist coordinates $\{x^{\mu}\}$ such that the metric may be written

$$g = g_{00} dx^0 \otimes dx^0 + g_{ij} dx^i \otimes dx^j \,, \quad (3.12)$$

where $g_{\mu\nu}$ is independent of $x^0$, $i, j \in \{1,2,3\}$, and $\mathcal{M}$ has the topology $R \times \Sigma$. Next, define the Euclidean manifold $\overline{\mathcal{M}}$, with topology $S^1 \times \Sigma$, by defining the Euclidean “time” coordinate $\theta = ix^0$ and periodically identifying $\theta$ with period $\beta$. The metric on $\overline{\mathcal{M}}$ then has signature +4, and is written

$$\overline{g} = -g_{00} d\theta \otimes d\theta + g_{ij} dx^i \otimes dx^j \,. \quad (3.13)$$

The partition function for fields $\phi$ propagating on $\mathcal{M}$ can be expressed as a functional integral

$$Z(\beta) = \mathcal{N} \int \mathcal{D}[\phi] e^{-I[\phi]} \,, \quad (3.14)$$

where $\mathcal{N}$ is a normalization factor and $I$ is the action of the theory on the Euclidean manifold. For the theory of canonical quantum gravity, in which the metric is one of the fields to be integrated over, the generalization of equation (3.14) is taken to *define* the partition function. The partition function is formally written as

$$Z(\beta) = \mathcal{N} \int_{\mathcal{F}} \mathcal{D}[g] \int \mathcal{D}[\phi] \exp(-I[g,\phi]) \,, \quad (3.15)$$

where the space $\mathcal{F}$ of Euclidean metrics is restricted by boundary conditions, such as the total energy contained in spacetime and behaviour at infinity. The Euclidean action functional $I$ appearing in the integral is

$$I[g,\phi] = I_{EH}[g] + I_{\phi}[\phi, g], \quad (3.16)$$

where $I_{EH}$ is the Euclidean Einstein-Hilbert action

$$I_{EH}[g] = \frac{1}{16\pi G_0}\left(-\int_{\mathcal{M}} \varepsilon_g R + 2\int_{\partial\mathcal{M}} \varepsilon_h K\right), \quad (3.17)$$

and $I_{\phi}$ is the action of the "matter" (non-gravitational) fields $\phi$. The bare gravitational coupling is explicitly denoted by $G_0$. Thence, the eq. (3.15), can be rewritten also as follow:

$$Z(\beta) = \mathcal{N}\int_{\mathcal{F}} \mathcal{D}[g] \int \mathcal{D}[\phi] \exp\left[-\left(\frac{1}{16\pi G_0}\left(-\int_{\mathcal{M}} \varepsilon_g R + 2\int_{\partial\mathcal{M}} \varepsilon_h K\right)\right) - I_{\phi}[\phi, g]\right]. \quad (3.17b)$$

The usual method of calculation of the partition function (3.15) is to find a manifold $\overline{\mathcal{M}}$ with metric $\hat{g}$ which is a stationary point of the classical action and satisfies the boundary conditions. Then, writing an arbitrary metric $g$ as $g = \hat{g} + f$, one quantizes the fluctuations $f$ and $\phi$ in the background metric $\hat{g}$. The action (3.17) can be expanded in powers of $f$ as

$$I[g,\phi] = I[\hat{g} + f, \phi] = I_{EH}[\hat{g}] + I_{\phi}[\phi, \hat{g}] + \int_{\overline{\mathcal{M}}} \varepsilon_{\hat{g}} \left.\frac{\delta I}{\delta g}\right|_{\hat{g}} f + \frac{1}{2}\int_{\overline{\mathcal{M}}} \varepsilon_{\hat{g}} \left.\frac{\delta^2 I}{\delta g^2}\right|_{\hat{g}} f^2 + \ldots, \quad (3.18)$$

and the partition function (3.15) can be written

$$Z(\beta) = e^{-\beta F} = \exp(-I_{EH}[g])Z' \quad (3.19)$$

where

$$Z' = \mathcal{N}\int \mathcal{D}[f]\int \mathcal{D}[\phi]\exp(-(I[\hat{g}+f,\phi]-I_{EH}[\hat{g}])). \quad (3.20)$$

Thence, the eq. (3.19) can be rewritten also as follow:

$$Z(\beta) = e^{-\beta F} = \exp(-I_{EH}[g])\,\mathcal{N}\int \mathcal{D}[f]\int \mathcal{D}[\phi]\exp(-(I[\hat{g}+f,\phi]-I_{EH}[\hat{g}])). \quad (3.20b)$$

To study the partition function for gravitational and matter fields propagating outside an infinitely massive, eternal black hole, the stationary point to expand around is a Euclidean continuation $\overline{\mathcal{R}}$ of Rindler space, with metric

$$g = s^2 d\theta \otimes d\theta + ds \otimes ds + dx^2 \otimes dx^2 + dx^3 \otimes dx^3. \quad (3.21)$$

The Euclidean "time" coordinate $\theta$ is periodic with period $\beta$, and we again restrict $x^2, x^3 \in \left[-\frac{L}{2}, \frac{L}{2}\right]$ to regulate divergences due to the horizon area $A$.

Consider now the factor $\exp(-I_{EH}[\hat{g}])$ in equation (3.19), which gives the contribution to the entropy from the classical geometry. The effect of the curvature singularity is that

$$\int_{\overline{\mathcal{R}}_\beta} \varepsilon_{\hat{g}} R = 2A(2\pi - \beta). \quad (3.22)$$

Thus the Einstein-Hilbert action is

$$S_{EH}[\hat{g}] = -\frac{(2\pi - \beta)A}{8\pi G_0} = \beta F \; . \quad (3.23)$$

From equation (3.23) one obtains the Bekenstein-Hawking formula for the entropy per unit area,

$$\frac{\sigma}{A} = \frac{1}{4G_0} . \quad (3.24)$$

Integrating out all the matter fields and tree level gravitons gives a contribution proportional to $\frac{1}{\varepsilon^2}$. Including this term, the entropy per unit area is

$$\frac{\sigma}{A} = \frac{1}{4}\left(\frac{1}{G_0} + \frac{C}{90\pi\varepsilon^2}\right), \quad (3.25)$$

where $C$ is a constant which depends on the matter content of the theory. After integrating out the matter fields and fluctuations of the metric, $Z'$ has the form $Z' = \exp(-W')$, where on general grounds $W'$ must be a diffeomorphism-invariant functional of the background metric $g$. $W'$ will contain all possible covariant terms, and may be expanded in powers of the Riemann tensor and its derivatives as

$$W'[g] = \int_{\mathcal{M}} \varepsilon_g \left[ -\frac{1}{16\pi} aR + Q(R) \right]. \quad (3.26)$$

Here $a$ is a constant and $Q$ contains all other induced covariant terms. We neglect a possible renormalization of the cosmological constant. The effect of $a$ is to renormalize the value of the gravitational coupling in the effective action from $G_0$ to $G_R$, given by

$$\frac{1}{G_R} = \frac{1}{G_0} + a \,. \quad (3.27)$$

The next step is to evaluate equation (3.26) for Euclidean Rindler space $\overline{\mathcal{R}}$. To regulate the curvature singularity at the origin, define $R = (2\pi - \beta)f$, where $f$ is a smooth function supported only on an $\varepsilon$-neighbourhood of the origin. The condition that (3.22) be satisfied means that $f$ must satisfy

$$\int_{\overline{\mathcal{R}}_\beta} \varepsilon_g f = 2A \,. \quad (3.28)$$

We also require that the scale of variation of $f$ is independent of the conical angle, so that derivatives of $f$ do not introduce additional dependence on $\beta$. Now consider the possible types of terms that can appear in $Q$.

1. Any local or non-local term with $n \geq 2$ powers of the Riemann tensor $R_{\alpha\beta\mu\nu}$ will be proportional to $(2\pi - \beta)^n$. This includes terms with arbitrary numbers of derivative operators acting on $R_{\alpha\beta\mu\nu}$. Their contribution to $W'$ may be represented as

$$-\sum_{n=2}^{\infty} b_n (2\pi - \beta)^n \,, \quad (3.29)$$

   where the $b_n$ are constants.

2. Now consider terms linear in $R_{\alpha\beta\mu\nu}$, with arbitrary derivatives acting on them. For example, consider

$$I = \int_{\overline{\mathcal{R}}} \varepsilon_g \left(\nabla^2\right) R . \quad (3.30)$$

Since $R$ is now a smooth function, by use of Stokes' theorem equation (3.30) can be rewritten as an integral over the boundary of $\overline{\mathcal{R}}$ ,

$$I = \int_{\partial\overline{\mathcal{R}}} \varepsilon_h n^{\mu} \nabla_{\mu} R , \quad (3.31)$$

where $n$ is a unit vector normal to $\partial\overline{\mathcal{R}}$ . But $R$ vanishes outside a small neighbourhood of

the origin, so the integral $I$ vanishes. It is obvious that all such terms will vanish after integration by parts.

Due to the rapid falloff of the Green functions in four dimensions, non-local terms proportional to one power of $R_{\mu\nu\alpha\beta}$ will not appear, and the above list covers all possible terms. Thus, using the condition (3.22), the full Helmholtz free energy $\beta F = S_{EH}[g] + W'[g]$ can be written

$$\beta F = -\sum_{n=1}^{\infty} b_n (2\pi - \beta)^n , \quad (3.32)$$

where $b_1 = \dfrac{A}{8\pi G_R}$ comes only from the Einstein-Hilbert term. The entropy is therefore

$$\sigma(\beta) = \frac{\beta A}{8\pi G_R} + \sum_{n=1}^{\infty} \left(b_n + \beta(n+1) b_{n+1}\right)(2\pi - \beta)^n . \quad (3.33)$$

Setting $\beta = 2\pi$, equation (3.33) reduces to the Bekenstein-Hawking entropy (3.1), but with the renormalized gravitational coupling $G_R$ given by equation (3.27).

Thus we arrive at the conclusion that for the case of canonical quantum gravity coupled to matter fields, *the expression* (3.1) *for the Bekenstein-Hawking entropy of the fields propagating outside a black hole is a general result, but the gravitational coupling appearing in equation* (3.1) *is the renormalized gravitational coupling* $G_R$ *given by equation* (3.27). Comparing equations (3.27) and (3.25), we see that the divergences in the entropy are the same divergences which renormalize the gravitational coupling.

Now we examine how quantum corrections affect the entropy of a two dimensional black hole. In the two dimensional model proposed by Callan, Giddings, Harvey and Strominger (CGHS), the divergence in the entropy of scalar fields moving in a black hole background is not the same as the divergence which renormalizes the gravitational coupling. The CGHS model is defined by the action functional

$$S_{CGHS} = \frac{1}{2\pi}\int \varepsilon_g \left( e^{-2\phi}\left[R + 4(\nabla\phi)^2 + 4\lambda^2\right] - \frac{1}{2}(\nabla f)^2 \right), \quad (3.34)$$

where $g$, $\phi$, and $f$ are the metric, dilaton, and matter fields, respectively, and $\lambda^2$ is a cosmological constant which defines a length scale for the theory. The classical theory defined by the action (3.34) has eternal black hole solutions. Defining light cone coordinates $x^{\pm}$ and choosing the line element to have the form $ds^2 = -e^{2\rho}dx^+dx^-$, these solutions are given by

$$e^{-2\rho} = e^{-2\phi} = \frac{M}{\lambda} - \lambda^2 x^+ x^-, \quad (3.35)$$

where $M$ is the black hole mass. The future horizon is the curve $x^- = 0$.

Due to the bad infrared behaviour of scalar fields in two dimensions, in addition to the horizon cutoff $\varepsilon$ one must also introduce an infrared cutoff $\ell$. The entropy is found to be

$$\sigma = \frac{1}{6}\log\left(\frac{\ell}{\varepsilon}\right). \quad (3.36)$$

Note that this entropy is not proportional to the horizon area. Instead, it represents an infinite additive constant to the entropy. The origin of this entropy can also be understood by examining the effective action obtained after integrating out the $f$ field. This action is given by the original action (3.34) plus a Liouville action, which can be written using the above metric as

$$S_L = -\frac{1}{12\pi}\int d^2x(\rho - \log(\ell/\varepsilon))(\nabla)^2(\rho - \log(\ell/\varepsilon)) = -\frac{1}{96\pi}\int \varepsilon_g R\frac{1}{\nabla^2}R - \frac{\log(\ell/\varepsilon)}{12\pi}\int \varepsilon_g R.$$

(3.37)

Note that the $\rho$ field only appears in the combination $\rho - \log(\ell/\varepsilon)$. The first term in equation (3.37) is the familiar correction to the classical action, and is responsible for the Hawking radiation from the two dimensional black hole. The second term is proportional to the Euler class, and is the term which gives rise to the divergent entropy of the scalar field.

The starting point for our discussion is the two-dimensional supersymmetric sigma model describing the propagation of superstrings in a background spacetime metric $g$. The generating functional for the two-dimensional superconformal field theory on a world sheet of genus $n$ is

$$Z^{(n)} = \frac{\kappa_0^{2(n-1)}}{Vol(\mathcal{G})}\int \mathcal{D}[e]\int \mathcal{D}[\chi]\int \mathcal{D}[X]\int \mathcal{D}[\Psi]\exp(-I[X,\Psi;e,\chi]), \quad (3.38)$$

where $X^\mu$ and $\Psi$ are the bosonic and fermionic coordinates of the superstring, respectively, $e$ is the world sheet zweibein, and $\chi$ is the gravitino. $\kappa_0$ is the bare string coupling, and $\mathcal{G}$ denotes the symmetry group of the two dimensional action $I$, which includes diffeomorphisms, superconformal transformations, and an on shell local supersymmetry. The first step is to gauge fix the world sheet zweibein to $e = e^\Lambda \hat{e}$ and the gravitino to $\chi = \rho\lambda$, where $\hat{e}$ is a fiducial zweibein, $\rho$ are the two dimensional Dirac matrices, and $\lambda$ is a Grassmann variable. This introduces reparametrization ghosts $b, c, \beta$, and $\gamma$, and equation (3.38) becomes

$$Z^{(n)} = \kappa_0^{2(n-1)} \frac{\int \mathcal{D}[\Lambda,\lambda]}{Vol(SC)} \int_{F_n} d^{2m_n}\tau \frac{\int \mathcal{D}[X]\mathcal{D}[\Psi]\mathcal{D}[b,c,\beta,\gamma]}{Vol(\Omega)} \exp\left(-I\left[X,b,c,\beta,\gamma;e^{\Lambda}\hat{e},\rho\lambda\right]\right).$$

(3.39)

Here SC denotes the group of superconformal transformations, $F_n$ is a fundamental region for the integration over the $2m_n$ supermoduli $\tau$, and $\Omega$ denotes the additional subgroup of symmetries which remains after the gauge fixing. $\Omega$ is generated by the conformal Killing vectors and spinors. We imagine regulating the two dimensional field theory by replacing the world sheet by a finite lattice. The volume of the group $\Omega$ is then also naturally regulated.

Consider first the case of genus zero. After integrating over the world sheet fields and dividing out the volume of $\Omega$, equation (3.39) for $Z^{(0)}$ takes the form

$$Z^{(0)} = \kappa_0^{-2} \frac{\int \mathcal{D}[\Lambda,\lambda]}{Vol(SC)} F(g;\varepsilon,\Lambda,\lambda) \quad (3.40)$$

where $F$ is a generally covariant functional of the background metric $g$, and also depends on the world sheet regulator parameter $\varepsilon$ and the superconformal parameters $\Lambda$ and $\lambda$. The basic structure of $F$ can be determined by quite general arguments. To begin with, for a fixed value of the regulator, $F$ may be expanded as a sum of integrals of powers of the Riemann tensor and its derivatives.

The coefficients of the terms in the expansion of $F$ will depend on $\varepsilon,\Lambda$, and $\lambda$, and will in general diverge as $\varepsilon$ goes to zero. This is one of the difficulties involved in defining string theory off shell. The coefficient of the term $\int \varepsilon_g R$ is independent of $\varepsilon,\Lambda,$ and $\lambda$. For this term the integral over the superconformal parameters simply cancels $Vol(SC)$, and so $Z^{(0)}$ can be written

$$Z^{(0)} = -\kappa_0^{-2} \int \varepsilon_g R + \kappa_0^{-2} \frac{\int \mathcal{D}[\Lambda,\lambda]}{Vol(SC)} Q(g;\varepsilon,\Lambda,\lambda), \quad (3.41)$$

where $Q$ contains all the other terms in $F$.

Although it is apparent that a unique definition of the off shell amplitude does not exist, it is obvious that the first term in equation (3.41) governs the low energy scattering of gravitons, and that its coefficient can be related in the usual way to the bare gravitational coupling. The genus zero generating functional $Z^{(0)}$ has been written down for a ten dimensional background metric, but we want to study four dimensional physics, so we must introduce a compactification scheme. For simplicity, we will consider a target space $\overline{\mathcal{T}}$ which is a product manifold $\overline{\mathcal{M}} \times K$, where $\overline{\mathcal{M}}$ is a four dimensional manifold coordinatized by $\{x^i\}_{i=1}^{4}$ (which will eventually be identified with four dimensional Euclidean Rindler space), and $K$ is a $D-4$ dimensional compact manifold coordinatized by $\{x^i\}_{i=5}^{D}$ and having no intrinsic curvature. The metric on $\overline{\mathcal{T}}$ is block diagonal, decomposing into a metric $g^{(4)}$ on $\overline{\mathcal{M}}$ and a metric $g^{(D-4)}$ on $K$. A simple choice for $K$ is the product manifold $(S^1)^{D-4}$, with $x^i \in [0, L_i]$ for $i \in \{5,...,D\}$, and $g_{ij}^{(D-4)} = \delta_{ij}$. The $L_i$ are taken to be on the order of $\sqrt{\alpha'}$. The genus zero generating functional can now be written

$$Z^{(0)} = -\frac{1}{16\pi G_0}\left( \int_{\overline{\mathcal{M}}} \varepsilon_{g^{(4)}} R + \frac{\int \mathcal{D}[\Lambda, \lambda]}{Vol(SC)} \int_{\overline{\mathcal{M}}} \varepsilon_{g^{(4)}} Q\left(g^{(4)}; \varepsilon, \Lambda, \lambda\right) \right), \quad (3.42)$$

where the bare four dimensional gravitational coupling is

$$G_0 = \frac{\kappa_0^2}{16\pi \int_K \varepsilon_{g^{(D-4)}}}. \quad (3.43)$$

All that is left now is to specify the four dimensional manifold $\overline{\mathcal{M}}$ and the background metric. The manifold $\overline{\mathcal{M}}$ is taken to be Euclidean Rindler space $\overline{\mathcal{R}}_\beta$, which has an angle deficit $(2\pi - \beta)$, and metric $g^{(4)}$ given by equation (3.21). From here the argument proceeds exactly as before, when we found that the entropy per unit area at $\beta = 2\pi$ depend only on the coefficient of the integral of $R$, and we obtain the result that the entropy per unit area obtained from genus zero string graphs is given by the Bekenstein-Hawking formula,

$$\frac{\sigma}{A}=\frac{1}{4G_0}. \quad (3.44)$$

It should by now be apparent that this result does not depend on the exact definition of off shell superstring generating functionals, because changes in the prescription for off shell functionals can only influence the result through the terms which depend on the regulator or the superconformal parameters. These terms all give contributions to the entropy which vanish when one sets $\beta = 2\pi$.

Now we define the quantity $\widetilde{Z}^{(0)}$ using equation (3.39) by

$$Z^{(0)}=\kappa_0^{-2}\frac{\int \mathcal{D}[\Lambda,\lambda]}{Vol(SC)}\frac{1}{Vol(\Omega)}\widetilde{Z}^{(0)}, \quad (3.45)$$

so that $\widetilde{Z}^{(0)}$ is the regulated generating functional for the two dimensional field theory, computed in a particular conformal gauge, and without the volume of $\Omega$ removed from it. We have that $Vol(\Omega)\propto \log(\varepsilon)$ and there exists a field redefinition such that

$$\frac{\partial \widetilde{Z}^{(0)}}{\partial(\log(\varepsilon))}=I_{eff}, \quad (3.46)$$

where $I_{eff}$ is the spacetime action which generates equations of motion equivalent to the superconformal invariance conditions. $I_{eff}$ has an expansion of the form

$$I_{eff}=c\int \varepsilon_g\left(-R+\frac{4}{(D-2)}(\nabla\Phi)^2+\frac{1}{3}\exp\left(\frac{8\Phi}{D-2}\right)H^2+\alpha' Q(g;\alpha',\varepsilon,\Lambda,\lambda)\right), \quad (3.47)$$

where $c$ is a constant and $Q$ contains terms which are higher order in $R_{\alpha\beta\mu\nu}$, with coefficients that depend on $\alpha',\varepsilon,\Lambda,$ and $\lambda$. We can now integrate equation (3.47) with respect to $\log(\varepsilon)$ to obtain $\widetilde{Z}^{(0)}$, and insert this quantity in equation (3.45). Dividing out the volume Vol $(\Omega) \propto \log(\varepsilon)$, we see that the coefficient of the integral of $R$ is independent of $\varepsilon, \Lambda,$ and $\lambda$. For this term, the integral over the superconformal parameters cancels Vol(SC), and we arrive at equation (3.41).

A major puzzle in the physics of black holes concerns the interpretation of the entropy associated with a black hole. In the semi-classical approximation, the entropy is given by the Bekenstein-Hawking formula:

$$S_{BH} = \frac{A}{4G\hbar}, \quad (3.48)$$

where $A$ is the area of the event horizon of the black hole and $G$ is Newton's constant. Using Planck's formula for a single massless boson we get the entropy density:

$$s(z) = \frac{4}{3}\frac{\pi^2}{30}\left(\frac{1}{2\pi z}\right)^3. \quad (3.49)$$

We note that the value $\frac{4\pi^2}{90} = 0.438649$ is related with the Aurea ratio as follow

$(\Phi)^0 + (\Phi)^{-21/7} + (\Phi)^{-42/7} + (\Phi)^{-56/7} \times \frac{1}{3} = 0.437694 \cong 0.438649,$

where $\Phi = \frac{\sqrt{5}+1}{2}$.

Note that we have been able to define the entropy density because entropy is an *extensive* quantity as it should be. However, the dominant contribution comes from the region near the horizon $z = 0$ and is not extensive but proportional to the area. If we put a cutoff on the proper distance at $z = \varepsilon$ the total entropy is:

$$S = \int_{\varepsilon}^{\infty} s(z) A dz = \frac{A}{360\pi\varepsilon^2}, \quad (3.50)$$

where $A$ is the area in the transverse dimensions.

Now, we derive the expression for the entropy in the bosonic string theory at one loop. This derivation is essentially an application of equations (3.49) and (3.50). These considerations are also relevant to Rindler strings. As our starting point we take the expression for the entropy in field theory in the proper time formalism. The reasoning leading to the corresponding expression in string theory is similar to the one employed in the derivation of the cosmological constant. Let us consider the free-energy density for a single boson of mass $m$ at finite temperature $\beta^{-1}$:

$$f(\beta, m^2) = \frac{1}{\beta} \int \frac{1}{(2\pi)^{d-1}} d^{d-1}k \log\left(1 - e^{-\beta\omega_k}\right). \quad (3.51)$$

To write it in the proper time formalism we first introduce $1 = \int_0^{\infty} 2\omega \int_{-i\infty}^{i\infty} \frac{1}{2\pi i} ds e^{s\left(\omega^2 - \omega_k^2\right)}$, expand the logarithm and then perform the gaussian momentum integrals to obtain

$$f(\beta, m^2) = -\int_0^{\infty} \frac{1}{(2\pi s)^{d/2}} \frac{ds}{s} \sum_{r=1}^{\infty} e^{-m^2 s/2 - r^2\beta^2/2s}. \quad (3.52)$$

The entropy density is as usual $s(\beta, m^2) = \beta^2 \frac{\partial f}{\partial \beta}$. For Rindler observers we simply put $\beta = 2\pi z$ to obtain the local entropy density and then integrate as in (3.50) to get the total entropy:

$$S(m^2) = A \int_0^{\infty} dz (2\pi z)^3 \int_0^{\infty} \frac{1}{(2\pi s)^{d/2}} \frac{ds}{s^2} \sum_{r=1}^{\infty} r^2 e^{-m^2 s/2 - 2\pi^2 r^2 z^2/s}. \quad (3.53)$$

There is an ultraviolet divergence as in (3.50) because the $s$ integral diverges near $s = 0$ for small $z$. Notice that we have to be careful while interchanging the order of integration because the integral over $s$ is not uniformly convergent as a function of $z$. We can interchange the order by putting appropriate cutoffs for both the integrals.

It is straightforward to generalize these formulae to the spectrum of the bosonic string in $d = 26$ by summing the expression (3.52) over $m^2$. In the light cone gauge the spectrum is given in terms of the occupation numbers of the right-moving and the left-moving oscillators $N_{ni}$ and $\widetilde{N}_{ni}$:

$$m^2 = \frac{2}{\alpha'}\left[-2 + \sum_{i=1}^{24}\sum_{n=1}^{\infty} n\left(N_{ni} + \widetilde{N}_{ni}\right)\right], \quad (3.54)$$

subject to the constraint

$$\sum_{i=1}^{24}\sum_{n=1}^{\infty} n\left(N_{ni} - \widetilde{N}_{ni}\right) = 0 . \quad (3.55)$$

The constraint can be enforced by introducing

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} d\theta e^{2\pi i n\theta\left(N_{ni} - \widetilde{N}_{ni}\right)} \quad (3.56)$$

into the sum. It is convenient to introduce a complex variable $\tau = \theta + i\frac{s}{2\pi\alpha'}$. The sum can then be easily performed to obtain

$$f(\beta) = -\frac{1}{2}\left(\frac{1}{4\pi^2\alpha'}\right)^{-13} \int_S \frac{1}{\operatorname{Im}\tau^2} d^2\tau (\operatorname{Im}\tau)^{-12} \left|\eta\left(e^{2\pi i\tau}\right)\right|^{-48} \sum_{r=1}^{\infty} \exp - \frac{r^2\beta^2}{4\pi\alpha' \operatorname{Im}\tau}. \quad (3.57)$$

Here $\eta$ is the Dedekind eta function,

$$\eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \quad (3.58)$$

and the region of integration is the strip $S$

$$-\frac{1}{2} \leq \operatorname{Re}\tau \leq \frac{1}{2}, \qquad 0 \leq \operatorname{Im}\tau \leq \infty. \quad (3.59)$$

Thence, the equation (3.57) can be written also as follow:

$$f(\beta) = -\frac{1}{2}\left(\frac{1}{4\pi^2\alpha'}\right)^{-13} \int_S \frac{1}{\operatorname{Im}\tau^2} d^2\tau (\operatorname{Im}\tau)^{-12} \left| q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)\left(e^{2\pi i\tau}\right) \right|^{-48} \sum_{r=1}^{\infty} \exp - \frac{r^2\beta^2}{4\pi\alpha' \operatorname{Im}\tau}.$$

(3.59b)

The total entropy computed from this formula has a divergence for each mode coming from the region near $\operatorname{Im}\tau = 0$. This would be the end of the story if we were dealing with a field theory of the string modes. But string theory is not merely a sum of field theories because of duality. The sum of field theories overcounts the correct string answer. The correct generalization of the above formulae to string theory is more subtle and requires a proper treatment of this overcounting. We do this by noting that using a modular transformation of $\tau$, every point in $S$ can be mapped onto the fundamental domain $\mathcal{F}$ of a torus:

$$|\tau| > 1, \ -\frac{1}{2} < \operatorname{Re}\tau < \frac{1}{2}, \ \operatorname{Im}\tau > 0. \quad (3.60)$$

Recall that the modular group $\Gamma$ at one-loop is the group of disconnected diffeomorphisms of a torus up to conformal equivalences. It is isomorphic to the group $SL(2,Z)/Z_2$ under which $\tau$ transforms as

$$\tau' = \frac{a\tau + b}{c\tau + d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,Z). \quad (3.61)$$

We have to divide the $SL(2,Z)$ by $Z_2$ because the elements $\{I, -I\}$ leave $\tau$ unchanged. The strip $S$ consists of an infinite number of domains $\mathcal{F}_\gamma$ each of which can be obtained from $\mathcal{F}$ by the action of an element of $\Gamma$, $\gamma(\rho_1\rho_2) = \begin{pmatrix} a & b \\ \rho_1 & \rho_2 \end{pmatrix}$ where $\rho_1$ and $\rho_2$ are relatively prime. The integers $a$ and $b$ can be chosen in such a way that $\operatorname{Im}\tau' = \operatorname{Im}\tau / E(\tau, \rho_1, \rho_2)$ where we have defined

$$E(\tau, \rho_1, \rho_2) \equiv |\rho_1\tau + \rho_2|^2. \quad (3.62)$$

Note that in expression (3.57), if we replace the summation by 1 then what we have is the cosmological constant at one loop for the bosonic string at zero temperature which is invariant under modular transformations. Using these facts we see that

$$f(\beta) = -\frac{1}{2}\left(\frac{1}{4\pi^2\alpha'}\right)^{-13} \int_S \frac{1}{\operatorname{Im}\tau'^2} d^2\tau' (\operatorname{Im}\tau')^{-12} \left|\eta\left(e^{2\pi i\tau'}\right)\right|^{-48} \sum_{r=1}^{\infty} \exp - \frac{\beta^2 r^2}{4\pi\alpha' \operatorname{Im}\tau'} =$$

$$= -\frac{1}{2}\left(\frac{1}{4\pi^2\alpha'}\right)^{-13} \int_{\mathcal{F}} \frac{1}{\operatorname{Im}\tau^2} d^2\tau (\operatorname{Im}\tau)^{-12} \left|\eta\left(e^{2\pi i\tau}\right)\right|^{-48} \sum_{r_1, r_2} \exp - \frac{\beta^2 E(\tau, r_1, r_2)}{4\pi\alpha' \operatorname{Im}\tau}. \quad (3.63)$$

It follows now that the total entropy in the bosonic string at one loop is given by

$$S = \left(\frac{1}{4\pi^2\alpha'}\right)^{-13} 2\pi A \int_0^\infty dz (2\pi z)^3 E(\tau, r_1, r_2) \times \int_{\mathcal{F}} \frac{1}{\operatorname{Im}\tau^2} d^2\tau (\operatorname{Im}\tau)^{-13} \left|\eta\left(e^{2\pi i\tau}\right)\right|^{-48} \sum_{r_1 r_2} \exp - \frac{\pi z^2 E(\tau, r_1, r_2)}{\alpha' \operatorname{Im}\tau}.$$

(3.64)

We note that the eq. (3.64) can be written also as follow:

$$\left(\frac{1}{4\pi^2\alpha'}\right)^{-13} 2\pi A \int_0^\infty dz (2\pi z)^3 E(\tau, r_1, r_2) \int_{\mathcal{F}} \frac{1}{\operatorname{Im}\tau^2} d^2\tau (\operatorname{Im}\tau)^{-13} \left| q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \left(e^{2\pi i\tau}\right) \right|^{-48} \sum_{r_1 r_2} \exp - \frac{\pi z^2 E(\tau, r_1, r_2)}{\alpha' \operatorname{Im}\tau}$$

(3.64b)

It is easy to check that this expression is modular invariant using the Poisson resummation formula. This means that the restriction of the modular integration to the fundamental domain $\mathcal{F}$ is a consistent procedure.

Thence, we conclude that in string theory the modular integration is over the fundamental domain $\mathcal{F}$ and not over the strip $S$. The entropy is ultraviolet finite because the region of short proper time near $\operatorname{Im}\tau = 0$ is excluded by modular invariance. Infrared divergences coming from very large $\operatorname{Im}\tau$ may still be present.

## 3.1 On some equations concerning the thesis "Can the Universe create itself?". The adapted Rindler vacuum in Misner space.

We find that the Universe does *not* seem to be created from nothing. If the Universe is created from *something*, that something could have been *itself*. Thus it is possible that the Universe is its own mother. In such a case, if we trace the history of the Universe backward, inevitably we will enter in a region of **closed timelike curves** (CTCs). Therefore CTCs may play an important role in the creation of the Universe.

A simple spacetime with CTCs is obtained from Minkowski spacetime by indentifying points that are related by time translation. Minkowski spacetime is $(R^4, \eta_{ab})$. In Cartesian coordinates $(t, x, y, z)$ the Lorentzian metric $\eta_{ab}$ is given by

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2. \quad (3.65)$$

Now we identify points $(t, x, y, z)$ with points $(t + nt_0, x, y, z)$ where $t_0$ is a positive constant and $n$ is any integer. Then we obtain a spacetime with topology $S^1 \times R^3$ and the Lorentzian metric. Such a spacetime is closed in the time direction and has no Cauchy horizon. All events in this spacetime are threaded by CTCs. Minkowski spacetime $(R^4, \eta_{ab})$ is the covering space of this spacetime.

Now let us consider a particle detector moving in this spacetime. The particle detector is coupled to the field $\phi$ by the interaction Lagrangian $cm(\tau)\phi[X(\tau)]$, where $c$ is a small coupling constant, $m$ is the detector's monopole moment, $\tau$ is the proper time of the detector's worldline, and $X(\tau)$ is the trajectory of the particle detector. Suppose initially the detector is in its ground state with energy $E_0$ and the field $\phi$ is in some quantum state $|\ \rangle$. Then the transition probability for the detector to all possible excited states with energy $E > E_0$ and the field $\phi$ to all possible quantum states is given by

$$P = c^2 \sum_{E > E_0} \left| \langle E | m(0) | E_0 \rangle \right|^2 \mathcal{F}(\Delta E), \quad (3.66)$$

where $\Delta E = E - E_0 > 0$ and $\mathcal{F}(\Delta E)$ is the response function

$$\mathcal{F}(\Delta E) = \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' e^{-i\Delta E(\tau - \tau')} G^+(X(\tau), X(\tau')), \quad (3.67)$$

which is independent of the details of the particle detector and is determined by the positive frequency Wightman function $G^{+}(X,X') \equiv \langle|\phi(X)\phi(X')|\rangle$. Thence, the eq. (3.66) can be written also as follow:

$$P = c^2 \sum_{E>E_0} \left|\langle E|m(0)|E_0\rangle\right|^2 \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' e^{-i\Delta E(\tau-\tau')} G^{+}(X(\tau), X(\tau')). \quad (3.67b)$$

The response function represents the bath of particles that the detector effectively experiences. The remaining factor in eq. (3.66) represents the selectivity of the detector to the field and depends on the internal structure of the detector. The Wightman function for the Minkowski vacuum is

$$G_M^{+}(X,X') = \frac{1}{4\pi^2} \frac{1}{-(t-t'-i\varepsilon)^2 + (x-x')^2 + (y-y')^2 + (z-z')^2}, \quad (3.68)$$

where $\varepsilon$ is an infinitesimal positive real number which is introduced to indicate that $G^{+}$ is the boundary value of a function which is analytic in the lower-half of the complex $\Delta t \equiv t - t'$ plane. For the adapted Minkowski vacuum in the spacetime $(S^1 \times R^3, \eta_{ab})$, the Wightman function is

$$G^{+}(X,X') = \frac{1}{4\pi^2} \sum_{n=-\infty}^{\infty} \frac{1}{-(t-t'+nt_0-i\varepsilon)^2 + (x-x')^2 + (y-y')^2 + (z-z')^2}. \quad (3.69)$$

Assume that the detector moves along the geodesic $x = \beta t \ (\beta < 1)$, $y = z = 0$, then the proper time is $\tau = t/\zeta$ with $\zeta = 1/\sqrt{1-\beta^2}$. On the geodesic, the Wightman function is reduced to

$$G^{+}(\tau,\tau') = \frac{1}{4\pi^2} \sum_{n=-\infty}^{\infty} \frac{1}{-(t-t'+nt_0-i\varepsilon)^2 + \beta^2(t-t')^2} = -\frac{1}{4\pi^2\zeta^2} \sum_{n=-\infty}^{\infty} \frac{1}{\left(\tau-\tau'+\frac{nt_0}{\zeta}-i\varepsilon/\zeta\right)^2 - \beta^2(\tau-\tau')^2}. \quad (3.70)$$

Inserting eq. (3.70) into eq. (3.67), we obtain

$$\mathcal{F}(\Delta E) = -\frac{1}{4\pi^2\zeta^2}\sum_{n=-\infty}^{\infty} dT \int_{-\infty}^{\infty} d\Delta\tau e^{-i\Delta E\Delta\tau} \frac{1}{\left(\Delta\tau+\frac{nt_0}{\zeta}-i\varepsilon/\zeta\right)^2-\beta^2(\Delta\tau)^2}, \quad (3.71)$$

where $\Delta\tau = \tau - \tau'$ and $T = (\tau + \tau')/2$. The integration over $\Delta\tau$ is taken along a contour closed in the lower-half plane of complex $\Delta\tau$. Inspecting the poles of the integrand, we find that all poles are in the upper-half plane of complex $\Delta\tau$ (we note that $\beta < 1$). Therefore according to the residue theorem we have

$$\boldsymbol{\mathcal{F}}(\Delta E\ ) = 0. \quad (3.72)$$

Such a particle detector perceives no particles, though the renormalized energy-momentum tensor of the field has the form of radiation. Let us consider a particle detector moving in Misner space with the adapted Rindler vacuum. Suppose the detector moves along a geodesic with $x = a$, $y = \beta t$, and $z = 0$ ($a$ and $\beta$ are constants and $a$ is positive), which goes through the P, R, and F regions. The proper time of the detector is $\tau = t/\zeta$ with $\zeta = 1/\sqrt{1-\beta^2}$. We note that the Hadamard function for the adapted Rindler vacuum in Misner space is

$$G^{(1)}(X, X') = \frac{1}{2\pi^2}\sum_{n=-\infty}^{\infty} \frac{\gamma}{\xi\xi\prime \sin h\gamma[-(\eta-\eta'+nb)^2+\gamma^2]}, \quad (3.73)$$

thence, on this geodesic, the Hadamard function in (3.73) is reduced to

$$G^{(1)}(t, t') = \frac{1}{2\pi^2}\frac{\gamma}{\sin h\gamma\sqrt{(a^2-t^2)(a^2-t\prime^2)}}\sum_{n=-\infty}^{\infty} \frac{1}{-(\eta-\eta'+nb)^2+\gamma^2}, \quad (3.73b)$$

where $\gamma$ is given by

$$\cos h\,\gamma = \frac{2a^2-t^2-t\prime^2+\beta^2(t-t\prime)^2}{2\sqrt{(a^2-t^2)(a^2-t\prime^2)}}, \quad (3.74)$$

and $\eta - \eta'$ is given by

$$\sin h\,(\eta - \eta') = \frac{a(t-t')}{\sqrt{(a^2-t^2)(a^2-t'^2)}}. \quad (3.75)$$

Though this Hadamard function is originally defined only in R, it can be analytically extended to F, P, and L. The Wightman function is equal to 1/2 of the Hadamard function with $t$ replaced by $t - i\varepsilon/2$ and $t'$ replaced by $t' + i\varepsilon/2$, where $\varepsilon$ is an infinitesimal positive real number. Then the response function is

$$\boldsymbol{\mathcal{F}}(E) = \frac{1}{4\pi^2}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} dT \int_{-\infty}^{\infty} d\Delta\tau \frac{\gamma^+ e^{-iE\Delta\tau}}{\sin h\gamma^+ \sqrt{\left[a^2-\zeta^2\left(T+\frac{\Delta\tau}{2}-\frac{i\varepsilon}{2\zeta}\right)^2\right]\left[a^2-\zeta^2\left(T-\frac{\Delta\tau}{2}+\frac{i\varepsilon}{2\zeta}\right)^2\right]}\{-[(\eta-\eta')^+ nb]^2+\gamma^{+2}\}},$$

(3.76)

where $T \equiv (\tau + \tau')/2$, $\Delta\tau \equiv \tau - \tau'$; $\gamma^+$ and $(\eta - \eta')^+$ are given by (3.74) and (3.75) with $t$ replaced by $t - i\varepsilon/2$ and $t'$ replaced by $t' + i\varepsilon/2$. The integral over $\Delta\tau$ can be worked out by the residue theorem where we choose the integration contour to close in the lower-half complex-$\Delta\tau$ plane. The result is zero since there are no poles in the lower-half plane. Therefore such a detector cannot be excited and so it detects nothing. We have also calculated the response functions for detectors on worldlines with constants $\xi, y$ and $z$ and worldlines with constants $\widetilde{\xi, y}$, and $z$ - both are zero. Now we consider a particle detector moving along a geodesic with $\chi, \theta, \phi$=constants. The response function is given by eq. (3.67) but with the integration over $\tau$ and $\tau'$ ranging from 0 to $\infty$. The Wightman function is obtained from the corresponding Hadamard function by the relation

$$G^+(\tau, \chi, \theta, \phi; \tau', \chi', \theta', \phi') = \frac{1}{2}G^{(1)}\left(\tau - \frac{i\varepsilon}{2}, \chi, \theta, \phi; \tau' + \frac{i\varepsilon}{2}, \chi', \theta', \phi'\right), \quad (3.77)$$

where $\varepsilon$ is an infinitesimal positive real number. Along the worldline of the detector, we have

$$Z(\tau, \tau') = -\sin h\frac{\tau}{r_0}\sin h\frac{\tau'}{r'_0} + \cos h\frac{\tau}{r_0}\cos h\frac{\tau'}{r'_0} = \cos h\frac{\tau-\tau'}{r_0}, \quad (3.78)$$

$$Z(-\tau,\tau') = +\sin h\frac{\tau}{r_0}\sin h\frac{\tau\prime}{r\prime_0} + \cos h\frac{\tau}{r_0}\cos h\frac{\tau\prime}{r\prime_0} = \cos h\frac{\tau+\tau\prime}{r_0}, \quad (3.79)$$

and

$$G^+(X,X') = \frac{1}{8\pi^2 r_0^2}\left(\frac{1}{1-\cos h\frac{\tau-\tau'-i\varepsilon}{r_0}} + \frac{1}{1-\cos h\frac{\tau+\tau\prime}{r_0}}\right). \quad (3.80)$$

Then the response function is

$$\boldsymbol{\mathcal{F}}(\Delta E) = \frac{1}{8\pi^2}\int_0^\infty dT\int_{-\infty}^\infty d\Delta\tau e^{-i\Delta E r_0\Delta\tau}\left[\frac{1}{1-\cos h(\Delta\tau-i\varepsilon)} + \frac{1}{1-\cos h2T}\right], \quad (3.81)$$

where $\Delta\tau = (\tau-\tau')/r_0$ and $T = (\tau+\tau')/2r_0$. It is easy to calculate the contour integral over $\Delta\tau$. We find that the integration of the second term is zero and therefore, the result is the same as that for an inertial particle detector in an eternal de Sitter space. Thus we have

$$\frac{d\mathcal{F}}{dT} = \frac{r_0}{2\pi}\frac{\Delta E}{e^{2\pi r_0\Delta E}-1}, \quad (3.82)$$

which is just the response function for a detector in a thermal radiation with the Gibbons-Hawking temperature

$$T_{G-H} = \frac{1}{2\pi r_0}. \quad (3.83)$$

For a conformally coupled scalar field in a conformally flat spacetime, the Green function $G(X,X')$ of the conformal vacuum is related to the corresponding Green function $\bar{G}(X,X')$ in the flat spacetime by

$$G(X,X') = \Omega^{-1}(X)\bar{G}(X,X')\Omega^{-1}(X'). \quad (3.84)$$

It is well known that in the simply connected de Sitter space, an inertial particle detector perceives thermal radiation with the Gibbons-Hawking temperature if the conformally coupled scalar field is in the conformal Minkowski vacuum. Now we want to find what a particle detector perceives in the adapted conformal Rindler vacuum in our multiply connected de Sitter space. The response function of the particle detector is still given by eq. (3.67). The Wightman function is obtained from the corresponding Hadamard function by eq. (3.77). The Hadamard function for the conformally coupled scalar field in multiply connected de Sitter space is related to that in Misner space via eq. (3.84). The Hadamard function for the adapted Rindler vacuum in Misner space is given by eq. (3.73). Now we consider particle detectors moving along two kinds of worldlines in our multiply connected de Sitter space.

1. *A particle detector moving along a geodesic with $l, \theta, \phi$=constant in region $\mathcal{F}$.*

In this region the Hadamard function is

$$G_{CR}^{(1)}(X, X') = -\frac{1}{2\pi^2}\sum_{n=-\infty}^{\infty}\frac{\tilde{\gamma}}{\sin h\tilde{\gamma}\sqrt{\left(\frac{\tilde{t}^2}{r_0^2}-1\right)\left(\frac{\tilde{t'}^2}{r_0^2}-1\right)}\left[-(l-l'+n\beta)^2+r_0^2\tilde{\gamma}^2\right]}, \quad (3.85)$$

where $\tilde{\gamma}$ is given by

$$\cos h\,\tilde{\gamma} = \frac{1}{\sqrt{\left(\frac{\tilde{t}^2}{r_0^2}-1\right)\left(\frac{\tilde{t'}^2}{r_0^2}-1\right)}}\left\{-1+\frac{\tilde{t}\tilde{t'}}{r_0^2}\left[\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi-\phi')\right]\right\}. \quad (3.86)$$

On the worldline of the particle detector, the Hadamard function is reduced to

$$G_{CR}^{(1)}(\tilde{t}, \tilde{t'}) = -\frac{1}{2\pi^2}\sum_{n=-\infty}^{\infty}\frac{\tilde{\gamma}}{\sin h\tilde{\gamma}\sqrt{\left(\frac{\tilde{t}^2}{r_0^2}-1\right)\left(\frac{\tilde{t'}^2}{r_0^2}-1\right)}\left(-n^2\beta^2+r_0^2\tilde{\gamma}^2\right)}, \quad (3.87)$$

and $\cos h\,\tilde{\gamma}$ is reduced to

$$\cos h\,\tilde{\gamma} = \frac{\frac{\tilde{t}\tilde{t}'}{r_0^2}-1}{\sqrt{\left(\frac{\tilde{t}^2}{r_0^2}-1\right)\left(\frac{\tilde{t}'^2}{r_0^2}-1\right)}}. \quad (3.88)$$

Using the proper time $\tau$ defined by the following equation

$$\tilde{t} = r_0 \cos h \frac{\tau}{r_0}, \quad (3.89)$$

on the worldline of the particle detector $\cos h\,\tilde{\gamma}$ and $G_{CR}^{(1)}$ can be written as

$$\cos h\,\tilde{\gamma} = \frac{\cos h2T+\cos h\Delta\tau-2}{\cos h2T-\cos h\Delta\tau}, \quad (3.89b)$$

and

$$G_{CR}^{(1)}(T,\Delta\tau) = \frac{1}{\pi^2 r_0^2}\sum_{n=-\infty}^{\infty}\frac{\tilde{\gamma}}{\sin h\tilde{\gamma}(\cos h2T-\cos h\Delta\tau)(n^2b^2-\tilde{\gamma}^2)}, \quad (3.90)$$

where $\tau > 0, \tau' > 0, \Delta\tau = \frac{(\tau-\tau')}{r_0}, T = \frac{(\tau+\tau')}{2r_0}$, and $b = \beta/r_0$. The Wightman function is equal to one half of the Hadamard function with $\Delta\tau$ replaced by $\Delta\tau - i\varepsilon$. Thus the response function is

$$\mathcal{F}(\Delta E) = \sum_{n=-\infty}^{\infty}\mathcal{F}_n(\Delta E), \quad (3.91)$$

where

$$\mathcal{F}_n(\Delta E) = \frac{1}{2\pi^2 r_0^2}\int_0^{\infty} dT\int_{-\infty}^{\infty} d\Delta\tau e^{-i\Delta E r_0\Delta\tau} \times \left[\frac{\tilde{\gamma}}{\sin h\tilde{\gamma}(\cos h2T-\cos h\Delta\tau)(n^2b^2-\gamma^2)}\right]_{\Delta\tau\to\Delta\tau-i\varepsilon}. \quad (3.92)$$

Thence, the eq. (3.91) can be written also as follow

$$\mathcal{F}(\Delta E) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi^2 r_0^2} \int_0^\infty dT \int_{-\infty}^{\infty} d\Delta\tau e^{-i\Delta E r_0 \Delta\tau} \times \left[\frac{\tilde{\gamma}}{\sin h\tilde{\gamma}(\cos h2T - \cos h\Delta\tau)(n^2b^2-\gamma^2)}\right]_{\Delta\tau\to\Delta\tau - i\varepsilon}. \quad (3.92b)$$

2. *A particle detector moving along a co-moving worldline in the steady-state coordinate system.* Suppose the detector moves along the geodesic $\rho, \theta, \phi$ = constants (such a worldline is a timelike geodesic passing through $\mathcal{R}$ and into $\mathcal{F}$) where $\rho \equiv (x^2 + y^2 + z^2)^{1/2}$ and the proper time $\tau$ are related to the static radius $r$ by

$$r = -r_0 \rho / \overline{\eta} = \rho e^{\tau / r_0} . \quad (3.93)$$

The Cauchy horizon is at $r = r_0$, or $\rho = -\overline{\eta} = r_0 e^{-\tau / r_0}$. On the worldline of the detector the Hadamard function is

$$G_{CR}^{(1)}(T, \Delta\tau) = \frac{1}{2\pi^2 r_0^2} \frac{\gamma}{2L \sinh \frac{\Delta\tau}{2}} \sum_{n=-\infty}^{\infty} \frac{1}{\gamma^2 - \left(\frac{t-t'}{r_0} + nb\right)^2}, \quad (3.94)$$

where $\Delta\tau = (\tau - \tau')/r_0$, $T = (\tau + \tau')/2r_0$, $L = \rho e^T / r_0 \equiv r(T)/r_0$, $\gamma$ is given by

$$\cosh \gamma = \frac{1 - L^2}{\sqrt{1 + L^4 - 2L^2 \cosh \Delta\tau}}, \quad (3.95)$$

and $t - t'$ is related to $T$ and $\Delta\tau$ by

$$\cosh \frac{t-t'}{r_0} = \frac{\cosh \Delta\tau - L^2}{\sqrt{1 + L^4 - 2L^2 \cosh \Delta\tau}} . \quad (3.96)$$

By analytical continuation, eqs. (3.94-3.96) hold in the whole region covered by the steady-state coordinates in de Sitter space. The Wightman function $G^+$ is equal to one half of $G^{(1)}$ with $\Delta\tau$ replaced by $\Delta\tau - i\varepsilon$. The response function is $\mathcal{F}(\Delta E) = \sum_{n=-\infty}^{\infty} \mathcal{F}_n(\Delta E)$ where

$$\mathcal{F}_n(\Delta E) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} dT \int_{-\infty}^{\infty} d\Delta\tau e^{-i\Delta E r_0 \Delta\tau} \times \left\{ \frac{\gamma}{2L\sinh\frac{\Delta\tau}{2}\left[\gamma^2 - \left(\frac{t-t'}{r_0} + nb\right)^2\right]} \right\}_{\Delta\tau\to\Delta\tau - i\varepsilon} . \quad (3.97)$$

The contribution of all $n \neq 0$ terms is

$$\frac{d}{dT}\sum_{n\neq 0}\mathcal{F}_n = \frac{1}{4\pi^2\left(e^{2\pi r_0 \Delta E} - 1\right)}\sum_{n=1}^{\infty}\frac{\sin\left(\Delta E r_0 \Delta\tau_n^+\right)}{L\sinh\frac{\Delta\tau_n^+}{2}} \times \frac{\alpha_1\left(1 + L^4 - 2L^2\cosh\Delta\tau_n^+\right)}{\alpha_1 L\left(L^2 - 1\right)\cosh\frac{\Delta\tau_n^+}{2} - \left(\alpha_2 + nb\right)\left(L^2\cosh\Delta\tau_n^+ - 1\right)},$$

(3.98)

which represents a "grey-body" Hawking radiation. As $T \to \infty$ (or $L \to \infty$), $\frac{d}{dT}\sum_{n\neq 0}\mathcal{F}_n$ exponentially drops to zero; therefore, at events far from the Cauchy horizon in $\mathcal{F}$, the particle detector only perceives pure Hawking radiation. As $L \to 1$ (approaching the Cauchy horizon), we also have $\frac{d}{dT}\sum_{n\neq 0}\mathcal{F}_n \to 0$. Thus as the Cauchy horizon is approached from the side of region $\mathcal{F}$, the particle detector co-moving in the steady-state coordinate system perceives pure Hawking radiation with Gibbons-Hawking temperature. We find that in our multiply connected de Sitter space with the adapted Rindler vacuum, region $\mathcal{R}$ is cold (where the temperature is zero) but region $\mathcal{F}$ is hot (where the temperature is $T_{G-H}$). Similarly, region $\mathcal{L}$ is cold but $\mathcal{P}$ is hot, the above results can be easily extended to these regions. This gives rise to an arrow of increasing entropy, from a cold region to a hot region.

## 4. p-Adic Models in Hartle-Hawking proposal and p-Adic and Adelic wave function of the Universe. [8] [9]

Ordinary and p-adic quantum mechanics can be unified in the form of adelic quantum mechanics

$$\left(L_2(A), W(z), U(t)\right). \quad (4.1)$$

$L_2(A)$ is the Hilbert space on $A$, $W(z)$ is a unitary representation of the Heisenberg-Weyl group on $L_2(A)$ and $U(t)$ is a unitary representation of the evolution operator on $L_2(A)$. The evolution operator $U(t)$ is defined by

$$U(t)\psi(x) = \int_A K_t(x,y)\psi(y)dy = \prod_v \int_{Q_v} K_t^{(v)}(x_v,y_v)\psi^{(v)}(y_v)dy_v. \quad (4.2)$$

The eigenvalue problem for $U(t)$ reads

$$U(t)\psi_{\alpha\beta}(x) = \chi(E_\alpha t)\psi_{\alpha\beta}(x), \quad (4.3)$$

where $\psi_{\alpha\beta}$ are adelic eigenfunctions, $E_\alpha = \left(E_\infty, E_2, \dots, E_p, \dots\right)$ is the corresponding adelic energy, indices $\alpha$ and $\beta$ denote energy levels and their degeneration. Any adelic eigenfunction has the form

$$\Psi_S(x) = \Psi_\infty(x_\infty)\prod_{p\in S}\Psi_p\left(x_p\right)\prod_{p\notin S}\Omega\left(\left|x_p\right|_p\right), \quad x \in A, \quad (4.4)$$

where $\Psi_\infty \in L_2(R)$, $\Psi_p \in L_2\left(Q_p\right)$ are ordinary and p-adic eigenfunctions, respectively. The $\Omega$-function defined from the following formula

$$\Omega\left(|x|_p\right) = 1, \ |x|_p \le 1; \qquad \Omega\left(|x|_p\right) = 0, \ |x|_p > 1 \quad (4.5),$$

is an element of the Hilbert space $L_2\left(Q_p\right)$, and provides convergence of the infinite product (4.4). A suitable way to calculate p-adic propagator $K_p(x'',t'';x',t')$ is to use Feynman's path integral method, i.e.

$$K_p(x'',t'';x',t') = \int_{x',t'}^{x'',t''} \chi_p\left(-\frac{1}{h}\int_{t'}^{t''} L(\dot{q},q,t)dt\right)Dq. \quad (4.6)$$

For quadratic Lagrangians it has been evaluated in the same way for real and p-adic cases, and the following exact general expression is obtained:

$$K_v(x'',t'';x',t') = \lambda_v\left(-\frac{1}{2h}\frac{\partial^2\bar{S}}{\partial x''\partial x'}\right)\left|\frac{1}{h}\frac{\partial^2\bar{S}}{\partial x''\partial x'}\right|_v^{1/2}\chi_v\left(-\frac{1}{h}\bar{S}(x'',t'';x',t')\right). \quad (4.7)$$

With regard the Hartle-Hawking proposal for the wave function of the Universe, the p-adic wave function is given by the integral

$$\Psi_p(q^\alpha) = \int_{G_p} dN K_p(q^\alpha, N; 0{,}0), \quad (4.8)$$

where, according to the adelic structure of $N$, $G_p = Z_p$ (i.e. $|N|_p \leq 1$) for every or almost every $p$.

Models of the de Sitter type are models with cosmological constant Λ and without matter fields. We consider two minisuperspace models of this type, with D = 4 and D = 3 space-time dimensions. The corresponding real Einstein-Hilbert action is

$$S = \frac{1}{16\pi G}\int_M d^Dx\sqrt{-g}(R-2\Lambda) + \frac{1}{8\pi G}\int_{\partial M} d^{D-1}x\sqrt{h}K, \quad (4.9)$$

where $R$ is the scalar curvature of D-dimensional manifold $M$, Λ is the cosmological constant, and $K$ is the trace of the extrinsic curvature $K_{ij}$ on the boundary $\partial M$. The metric for this model is of the Robertson-Walker type

$$ds^2 = \sigma^{D-2}[-N^2dt^2 + a^2(t)d\Omega_{D-1}^2]. \quad (4.10)$$

In this expression $d\Omega_{D-1}^2$ denotes the metric on the unit (D – 1)-sphere, $\sigma^{D-2} = 8\pi G/[V^{D-1}(D-1)(D-2)]$, where $V^{D-1}$ is the volume of the unit (D – 1)-sphere. In the real D = 3 case, the model is related to the multiple-sphere configuration and wormhole solutions. $v$-adic classical action for this model is

$$\bar{S}_v(a'',N;a',0)=\frac{1}{2\sqrt{\lambda}}\left[N\sqrt{\lambda}+\lambda\left(\frac{2a''a'}{\sin h(N\sqrt{\lambda})}-\frac{a'^2+a''^2}{\tan h(N\sqrt{\lambda})}\right)\right]. \quad (4.11)$$

Let us note that $\lambda$, ($\lambda=\Lambda G^2$), denotes the rescaled cosmological constant $\Lambda$. Using (4.7) for the propagator of this model we have

$$K_v(a'',N;a',0)=\lambda_v\left(-\frac{2\sqrt{\lambda}}{\sin h(N\sqrt{\lambda})}\right)\left|\frac{\sqrt{\lambda}}{\sin h(N\sqrt{\lambda})}\right|_v^{1/2}\chi_v\big(-\bar{S}_v(a'',N;a',0)\big). \quad (4.12)$$

The p-adic Hartle-Hawking wave function is

$$\Psi_p(a)=\int_{|N|_{p\leq 1}}dN\frac{\lambda_p(-2N)}{|N|_p^{1/2}}\chi_p\left(-\frac{N}{2}+\frac{\sqrt{\lambda}\cot h(N\sqrt{\lambda})}{2}a^2\right), \quad (4.13)$$

which after p-adic integration becomes

$$\Psi_p(a)=\Omega\big(|a|_p\big),\ \ |\lambda|_p\leq p^{-2},\ p\neq 2, \qquad \Psi_p(a)=\frac{1}{2}\Omega(|a|_2),\ \ |\lambda|_2\leq 2^{-4},\ p=2, \quad (4.14)$$

The de Sitter model in D = 4 space-time dimensions may be described by the metric

$$ds^2=\sigma^2\left[-\frac{N^2}{q(t)}dt^2+q(t)d\Omega_3^2\right],\quad \sigma^2=\frac{2G}{3\pi}, \quad (4.15)$$

and the corresponding action $S_v[q]=\frac{1}{2}\int_{t'}^{t''}dtN\left(-\frac{\dot{q}^2}{4N^2}-\lambda q+1\right)$ where $\lambda=2\Lambda G/(9\pi)$, thence the action can be written also as follow

$$S_v[q]=\frac{1}{2}\int_{t'}^{t''}dtN\left(-\frac{\dot{q}^2}{4N^2}-[\frac{2\Lambda G}{9\pi}]q+1\right).$$

For $N = 1$, the equation of motion $\ddot{q} = 2\lambda$ has solution $q(t) = \lambda t^2 + \left(\frac{q''-q'}{T} - \lambda T\right) t + q'$, where $q'' = q(t'')$, $q' = q(t')$ and $T = t'' - t'$. Note that this classical solution resembles motion of a particle in a constant field and defines an algebraic manifold. The choice of metric in the form (4.15) yields quadratic $v$-adic classical action

$$\bar{S}_v(q'', T; q', 0) = \frac{\lambda^2 T^3}{24} - [\lambda(q' + q'') - 2]\frac{T}{4} - \frac{(q''-q')^2}{8T}. \quad (4.16)$$

According to (4.7), the corresponding propagator is

$$K_v(q'', T|q', 0) = \frac{\lambda_v(-8T)}{|4T|_v^{1/2}} \chi_v\left(-\bar{S}_v(q'', T|q', 0)\right). \quad (4.17)$$

We obtain the p-adic Hartle-Hawking wave function by the integral

$$\Psi_p(q) = \int_{|T|_p \le 1} dT \frac{\lambda_p(-8T)}{|4T|_p^{1/2}} \chi_p\left(-\frac{\lambda^2 T^3}{24} + (\lambda q - 2)\frac{T}{4} + \frac{q^2}{8T}\right), \quad (4.18)$$

and as result we get also $\Omega\left(|q|_p\right)$ function with the condition $\lambda = 4 \cdot 3 \cdot l$, $l \in Z_p$. The above Ω-functions allow adelic wave functions of the form (4.4) for both D = 3 and D = 4 cases. Since $|\lambda|_p \le p^{-2}$ in (4.14) for all $p \ne 2$, it means that $\lambda$ cannot be a rational number and consequently the above de Sitter minisuperspace model in D = 3 space-time dimensions is not adelic one. However D = 4 case is adelic, because $\lambda = 4 \cdot 3 \cdot l$ is a rational number when $l \in Z \subset Z_p$.

In the Vladimirov-Volovich formulation, p-adic quantum mechanics is a triple

$$(L_2(Q_p), W_p(z), U_p(t)), \quad (4.19)$$

where $W_p(z)$ corresponds to $W_p(\alpha\hat{x}, \beta\hat{k})$ defined in the following equation

$$W_v(\alpha\hat{x}, \beta\hat{k}) = \chi_v\left(\frac{1}{2}\alpha\beta\right)\chi_v(-\beta\hat{k})\chi_v(-\alpha\hat{x}). \quad (4.20)$$

Adelic quantum mechanics is a natural generalization of the above formulation of ordinary and p-adic quantum mechanics: $(L_2(A), W_A(z), U_A(t))$. In complex-valued adelic analysis it is worth mentioning an additive character

$$\chi_A(x) = \chi_\infty(x_\infty)\prod_p \chi_p(x_p), \quad (4.21)$$

a multiplicative character

$$|x|_A^s = |x_\infty|_\infty^s \prod_p |x_p|_p^s, \quad s \in C, \quad (4.22)$$

and elementary functions of the form

$$\varphi_\wp(x) = \varphi_\infty(x_\infty)\prod_{p\in\wp}\varphi_p(x_p)\prod_{p\notin\wp}\Omega\left(|x_p|_p\right), \quad (4.23)$$

where $\varphi_\infty(x_\infty)$ is an infinitely differentiable function on R and $|x_\infty|_\infty^n\varphi_\infty(x_\infty) \to 0$ as $|x_\infty|_\infty \to \infty$ for any $n \in \{0,1,2,\dots\}$, $\varphi_p(x_p)$ are some locally constant functions with compact support, and

$$\Omega\left(|x_p|_p\right) = 1,\ |x_p|_p \leq 1, \qquad \Omega\left(|x_p|_p\right) = 0,\ |x_p|_p > 1. \quad (4.24)$$

All finite linear combinations of elementary functions (4.23) make the set $\mathcal{L}(A)$ of the Schwartz-Bruhat adelic functions. The Fourier transform of $\varphi(x) \in \mathcal{L}(A)$, which maps $\mathcal{L}(A)$ onto $\mathcal{L}(A)$, is

$$\tilde{\varphi}(y) = \int_A \varphi(x)\chi_A(xy)dx, \quad (4.25)$$

where $\chi_A(xy)$ is defined by (4.21) and $dx = dx_\infty dx_2 dx_3$ ... is the Haar measure on $A$. A basis of $L_2(A(\wp))$ may be given by the corresponding orthonormal eigenfunctions in a spectral problem of the evolution operator $U_A(t)$, where $t \in A$. Such eigenfunctions have the form

$$\Psi_\wp(x,t) = \Psi_\infty(x_\infty, t_\infty) \prod_{p\in\wp} \Psi_p(x_p, t_p) \prod_{p\notin\wp} \Omega\left(|x_p|_p\right), \quad (4.26)$$

where $\Psi_\infty \in L_2(R)$ and $\Psi_p \in L_2(Q_p)$ are eigenfunctions in ordinary and p-adic cases, respectively. $\Omega\left(|x_p|_p\right)$ is an element of $L_2(Q_p)$, defined by (4.24), which is invariant under transformation of an evolution operator $U_p(t_p)$ and provides convergence of the infinite product (4.26).

p-Adic and adelic minisuperspace quantum cosmology is an application of p-adic and adelic quantum mechanics to the cosmological models, respectively. In the path integral approach to standard quantum cosmology, the starting point is Feynman's path integral method. The amplitude to go from one state with intrinsic metric $h'_{ij}$ and matter configuration $\phi'$ on an initial hypersurface $\Sigma'$ to another state with metric $h''_{ij}$ and matter configuration $\phi''$ on a final hypersurface $\Sigma''$ is given by the path integral

$$K_\infty(h''_{ij}, \phi'', \Sigma''; h'_{ij}, \phi', \Sigma') = \int \chi_\infty(-S_\infty[g_{\mu\nu}, \Phi]) D_\infty g_{\mu\nu} D_\infty \Phi \quad (4.27)$$

over all four-geometries $g_{\mu\nu}$ and matter configurations $\Phi$, which interpolate between the initial and final configurations. In (4.27) $S_\infty[g_{\mu\nu}, \Phi]$ is an Einstein-Hilbert action for the gravitational and matter fields. To perform p-adic and adelic generalization we make first p-adic counterpart of the action using form-invariance under change of real to the p-adic number fields. Then we generalize (4.27) and introduce p-adic complex-valued cosmological amplitude

$$K_p(h''_{ij}, \phi'', \Sigma''; h'_{ij}, \phi', \Sigma') = \int \chi_p(-S_p[g_{\mu\nu}, \Phi]) D_p g_{\mu\nu} D_p \Phi. \quad (4.28)$$

The standard minisuperspace ground-state wave function in the Hartle-Hawking (no-boundary) proposal is defined by functional integration in the Euclidean version of

$$\psi_\infty[h_{ij}] = \int \chi_\infty(-S_\infty[g_{\mu\nu},\Phi])D_\infty g_{\mu\nu}D_\infty\Phi, \quad (4.29)$$

over all compact four-geometries $g_{\mu\nu}$ which induce $h_{ij}$ at the compact three-manifold. This three-manifold is the only boundary of the all four-manifolds. Extending Hartle-Hawking proposal to the p-adic minisuperspace, an adelic Hartle-Hawking wave function is the infinite product

$$\psi_A(q) = \prod_v \int \chi_v(-S_v[g_{\mu\nu},\Phi])D_v g_{\mu\nu}D_v\Phi, \quad (4.30)$$

where path integration must be performed over both, Archimedean and non-Archimedean geometries. If an evaluation of the corresponding functional integrals for a minisuperspace model yields $\psi(q_\alpha)$ in the form (4.26), then such cosmological model is a Hartle-Hawking adelic one. Now we consider the approach consists in the following p-adic proposal for the Hartle-Hawking type of the wave function:

$$\psi_\infty(q) = \sum_{a.m.} \prod_p \int \chi_p(-S_p[g_{\mu\nu},\Phi])D_p g_{\mu\nu}D_p\Phi, \quad (4.31)$$

where summation is over algebraic manifolds. The de Sitter minisuperspace model in D = 4 space-time dimensions is the Hartle-Hawking adelic one. Namely, according to the Hartle-Hawking proposal one has

$$\psi_v(q) = \int K_v(q,T;0,0)dT, \quad v = \infty, 2, 3, \dots, p, \dots, \quad (4.32)$$

where

$$K_v(q'',T;q',0) = \lambda_v(-8T)|4T|_v^{-1/2}\chi_v\left[-\frac{\lambda^2T^3}{24} + (\lambda q - 2)\frac{T}{4} + \frac{q^2}{8T}\right] \quad (4.33)$$

is the kernel of the $\nu$-adic evolution operator. The functions $\lambda_\nu(a)$ have the properties

$$|\lambda_\nu(a)|_\nu = 1,\ \lambda_\nu(b^2 a) = \lambda_\nu(a),\ \lambda_\nu(a)\lambda_\nu(b) = \lambda_\nu(a+b)\lambda_\nu\big(ab(a+b)\big). \quad (4.34)$$

Employing the p-adic Gauss integral

$$\int_{Q_p} \chi_p(\alpha x^2 + \beta x)dx = \lambda_p(\alpha)|2\alpha|_p^{-1/2}\chi_p\left(-\frac{\beta^2}{4\alpha}\right), \quad \alpha \neq 0, \quad (4.35)$$

one can rewrite p-adic version of (4.32) in the form

$$\psi_p(q) = \int_{Q_p} dx \chi_p(qx) \int DT \chi_p\left[-\frac{\lambda^2 T^3}{24} + \left(\frac{\lambda q}{4} - \frac{1}{2} - 2x^2\right)T\right]. \quad (4.36)$$

Taking the region of integration to be $|T|_p \leq 1$ one obtains

$$\psi_p(q) = \int_{Q_p} dx \chi_p(qx)\Omega\left(\left|\frac{\lambda q}{4} - \frac{1}{2} - 2x^2\right|_p\right), \quad \left|\frac{\lambda^2}{24}\right|_p \leq 1. \quad (4.37)$$

An evaluation of the integral (4.37) yields

$$\psi_p(q) = exp\big(i\pi\delta^1_{|q|_2}\delta^2_p\big)\Omega\big(|q|_p\big), \quad \left|\frac{\lambda^2}{24}\right|_p \leq 1, \quad (4.38)$$

where $\delta^b_a$ is the Kronecker symbol. With regard $\psi_\infty(q_\infty)$ the result depends on the contour of integration and has an exact solution

$$\psi_\infty(q_\infty) = exp\left(\frac{1}{3\lambda}\right) Ai\left(\frac{1-\lambda q_\infty}{(2\lambda)^{2/3}}\right), \quad (4.39)$$

where $Ai(x)$ is the Airy function. Thence, we obtain an adelic wave function for the de Sitter cosmological model in the form

$$\psi_A(q) = \psi_\infty(q_\infty) \prod_p exp\left(i\pi\delta^1_{|q|_2}\delta^2_p\right)\Omega\left(\left|q_p\right|_p\right), \quad \left|\frac{\lambda^2}{24}\right|_p \leq 1. \quad (4.40)$$

The necessary condition that a system can be regarded as the adelic one is the existence of p-adic ground state $\Omega\left(|q_\alpha|_p\right)$ $(\alpha = 1,2,\dots,n)$ in the way

$$\int_{|q'_\alpha|_p \leq 1} K_p(q''_\alpha, T; q'_\alpha, 0) dq'_\alpha = \Omega\left(|q''_\alpha|_p\right) \quad (4.41)$$

for all $p$ but a finite set $\wp$. For the case of de Sitter model one obtains

$$\psi_p(q) = \Omega\left(|q|_p\right), \quad |T|_p \leq 1, \quad \left|\frac{\lambda^2}{24}\right|_p \leq 1, \quad p \neq 2,$$

$$\psi_p(q) = \Omega(|q|_2), \quad |T|_2 \leq 1/2, \quad \left|\frac{\lambda^2}{24}\right|_2 \leq 1, \quad p = 2, \quad (4.42)$$

what is in a good agreement with the result (4.40) obtained by the Hartle-Hawking proposal.

## 5. Mathematical connections.

Now we describe some possible mathematical connections between some equations of arguments above described, some sectors of string theory and p-adic and adelic cosmology.

With regard the **Section 1**, we have the following connection between the eq. (1.2) and the fundamental equation concerning the Palumbo-Nardelli model:

$$J(f,f_0)=\sum_{jk}\int\left[f(c,t)f_{0j}(c_0)-f(c',t)f_{0k}(c'_0)\right]g\sigma(jk;g\chi)d\hat{g}'dc_0 \rightarrow$$

$$\rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]=$$

$$=\int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\widetilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(\left|F_2\right|^2\right)\right]. \quad (5.1)$$

Furthermore, we have the following mathematical connection between the eqs. (1.20), (1.22) and the eq. (1.87b):

$$\varepsilon_n = E_n N_n = E_n NCexp\left(-\frac{E_n}{kT}\right) = \frac{E_n Nexp\left(-\frac{E_n}{kT}\right)}{\sum_{n=0}^{\infty} exp\left(-\frac{E_n}{kT}\right)} \rightarrow$$

$$\rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]=$$

$$=\int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\widetilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(\left|F_2\right|^2\right)\right]. \quad (5.2)$$

$$E = \sum_{n=0}^{\infty} \varepsilon_n = \frac{N\sum_{n=0}^{\infty} E_n exp\left(-\frac{E_n}{kT}\right)}{\sum_{n=0}^{\infty} exp\left(-\frac{E_n}{kT}\right)} \rightarrow$$

$$\rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]=$$

$$=\int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\widetilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(\left|F_2\right|^2\right)\right]. \quad (5.3)$$

With regard the **Section 3**, we have the following mathematical connections. We have the following mathematical connection between eq. (3.9) and eq. (3.3), i.e.

$$Z(\beta)=\prod_{n,\vec{k}} Z(\beta;n,\vec{k})=\exp(-\beta F(\beta)) \to F(\beta)=\frac{A}{(2\pi\varepsilon)^2\beta}\int_0^\infty d\omega\omega^2\log\left(1-e^{-\beta\omega}\right)$$

This relationship can be connected with the eqs. (3.92b) and (4.18) of **Section 4** as follow:

$$Z(\beta)=\prod_{n,\vec{k}} Z(\beta;n,\vec{k})=\exp(-\beta F(\beta)) \to F(\beta)=\frac{A}{(2\pi\varepsilon)^2\beta}\int_0^\infty d\omega\omega^2\log\left(1-e^{-\beta\omega}\right) \to$$

$$\to \mathcal{F}(\Delta E) = \sum_{n=-\infty}^{\infty}\frac{1}{2\pi^2 r_0^2}\int_0^\infty dT\int_{-\infty}^{\infty} d\Delta\tau e^{-i\Delta E r_0\Delta\tau}\times\left[\frac{\tilde{\gamma}}{\sin h\tilde{\gamma}(\cos h2T-\cos h\Delta\tau)(n^2b^2-\gamma^2)}\right]_{\Delta\tau\to\Delta\tau-i\varepsilon} \to$$

$$\to \int_{|T|_p\leq 1} dT\frac{\lambda_p(-8T)}{|4T|_p^{1/2}}\chi_p\left(-\frac{\lambda^2T^3}{24}+(\lambda q-2)\frac{T}{4}+\frac{q^2}{8T}\right). \quad (5.4)$$

Also equation (3.20b) can be related with the eq. (3.92b) as follow:

$$Z(\beta)=e^{-\beta F}=\exp(-I_{EH}[g])\,\mathcal{N}\int\mathcal{D}[f]\int\mathcal{D}[\phi]\exp\left(-\left(I[\hat{g}+f,\phi]-I_{EH}[\hat{g}]\right)\right) \to$$

$$\to \mathcal{F}(\Delta E) = \sum_{n=-\infty}^{\infty}\frac{1}{2\pi^2 r_0^2}\int_0^\infty dT\int_{-\infty}^{\infty} d\Delta\tau e^{-i\Delta E r_0\Delta\tau}\times\left[\frac{\tilde{\gamma}}{\sin h\tilde{\gamma}(\cos h2T-\cos h\Delta\tau)(n^2b^2-\gamma^2)}\right]_{\Delta\tau\to\Delta\tau-i\varepsilon}$$

(5.5)

The eq. (3.37) can be related with the eqs. (2.10) and (4.18) as follow:

$$S_L=-\frac{1}{12\pi}\int d^2x(\rho-\log(\ell/\varepsilon))(\nabla)^2(\rho-\log(\ell/\varepsilon))=-\frac{1}{96\pi}\int\varepsilon_g R\frac{1}{\nabla^2}R-\frac{\log(\ell/\varepsilon)}{12\pi}\int\varepsilon_g R \to$$

$$\to \frac{dn}{dt}+3\left(\frac{\dot{a}}{a}\right)n-\dot{\Phi}n=\frac{1}{2}\eta\left(e^{-\Phi}g^{\mu\nu}\tilde{\beta}_{\mu\nu}^{Grav}+2e^{\Phi}\tilde{\beta}^{\Phi}\right)n+\int\frac{d^3p}{E}C[f] \to$$

$$\to \int_{|T|_p\leq 1} dT\frac{\lambda_p(-8T)}{|4T|_p^{1/2}}\chi_p\left(-\frac{\lambda^2T^3}{24}+(\lambda q-2)\frac{T}{4}+\frac{q^2}{8T}\right). \quad (5.6)$$

The eq. (3.42), can be connected with the eq. (3.92b) as follow:

$$Z^{(0)} = -\frac{1}{16\pi G_0}\left(\int_{\mathcal{M}} \varepsilon_{g^{(4)}} R + \frac{\int \mathcal{D}[\Lambda, \lambda]}{Vol(SC)} \int_{\mathcal{M}} \varepsilon_{g^{(4)}} Q\left(g^{(4)}; \varepsilon, \Lambda, \lambda\right)\right) \rightarrow$$

$$\rightarrow \mathcal{F}(\Delta E) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi^2 r_0^2} \int_0^{\infty} dT \int_{-\infty}^{\infty} d\Delta\tau e^{-i\Delta E r_0 \Delta\tau} \times \left[\frac{\tilde{\gamma}}{\sin h\tilde{\gamma}(\cos h2T - \cos h\Delta\tau)(n^2b^2 - \gamma^2)}\right]_{\Delta\tau \rightarrow \Delta\tau - i\varepsilon}$$

(5.7)

The eq. (3.47) can be related with the fundamental equation concerning the Palumbo-Nardelli model and the Ramanujan's modular equation as follow:

$$I_{eff} = c\int \varepsilon_g \left( -R + \frac{4}{(D-2)}(\nabla\Phi)^2 + \frac{1}{3}\exp\left(\frac{8\Phi}{D-2}\right)H^2 + \alpha' Q(g; \alpha', \varepsilon, \Lambda, \lambda)\right) \rightarrow$$

$$\rightarrow 8 = \frac{1}{3}\frac{4\left[anti\log\frac{\int_0^{\infty}\frac{\cos \pi t x w'}{\cosh \pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} \rightarrow$$

$$\rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] =$$

$$= \int_0^{\infty}\frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(\left|F_2\right|^2\right)\right]. \quad (5.8)$$

With regard the eq. (3.53), we have the following mathematical connections with the eqs. (2.10) and (3.92b):

$$S(m^2)=A\int_0^\infty dz(2\pi z)^3\int_0^\infty \frac{1}{(2\pi s)^{d/2}}\frac{ds}{s^2}\sum_{r=1}^\infty r^2 e^{-m^2 s/2-2\pi^2 r^2 z^2/s} \rightarrow$$

$$\rightarrow \frac{dn}{dt}+3\left(\frac{\dot{a}}{a}\right)n-\dot{\Phi}n=\frac{1}{2}\eta\left(e^{-\Phi}g^{\mu\nu}\tilde{\beta}_{\mu\nu}^{Grav}+2e^{\Phi}\tilde{\beta}^{\Phi}\right)n+\int\frac{d^3p}{E}C[f]\rightarrow$$

$$\rightarrow \mathcal{F}(\Delta E)=\sum_{n=-\infty}^{\infty}\frac{1}{2\pi^2 r_0^2}\int_0^\infty dT\int_{-\infty}^{\infty} d\Delta\tau e^{-i\Delta E r_0\Delta\tau}\times\left[\frac{\tilde{\gamma}}{\sin h\tilde{\gamma}(\cos h2T-\cos h\Delta\tau)(n^2b^2-\gamma^2)}\right]_{\Delta\tau\to\Delta\tau-i\varepsilon}$$

(5.9)

The eq. (3.59b) can be connected with the fundamental equation concerning the Palumbo-Nardelli model and the Ramanujan's modular equation as follow:

$$f(\beta)=-\frac{1}{2}\left(\frac{1}{4\pi^2\alpha'}\right)^{-13}\int_S\frac{1}{\operatorname{Im}\tau^2}d^2\tau(\operatorname{Im}\tau)^{-12}\left|q^{\frac{1}{24}}\prod_{n=1}^\infty(1-q^n)(e^{2\pi i\tau})\right|^{-48}\sum_{r=1}^\infty \exp-\frac{r^2\beta^2}{4\pi\alpha'\operatorname{Im}\tau}\rightarrow$$

$$\rightarrow 8=\frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}\rightarrow$$

$$\rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr(G_{\mu\nu}G_{\rho\sigma})f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]=$$

$$=\int_0^\infty\frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(\left|F_2\right|^2\right)\right].\quad(5.10)$$

While, the eq. (3.64b) can be connected with the fundamental equation concerning the Palumbo-Nardelli model, the Ramanujan's modular equation and the Boltzmann equation as follow:

$$\left(\frac{1}{4\pi^2\alpha'}\right)^{-13} 2\pi A\int_0^\infty dz(2\pi z)^3 E(\tau,r_1,r_2)\int_{\mathcal{F}}\frac{1}{\operatorname{Im}\tau^2}d^2\tau(\operatorname{Im}\tau)^{-13}\left|q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)(e^{2\pi i\tau})\right|^{-48}\sum_{r_1 r_2}\exp-\frac{\pi z^2 E(\tau,r_1,r_2)}{\alpha'\operatorname{Im}\tau}$$

→

$$\rightarrow \frac{dn}{dt}+3\left(\frac{\dot{a}}{a}\right)n-\dot{\Phi}n=\frac{1}{2}\eta\left(e^{-\Phi}g^{\mu\nu}\tilde{\beta}_{\mu\nu}^{Grav}+2e^{\Phi}\tilde{\beta}^{\Phi}\right)n+\int\frac{d^3p}{E}C[f]\rightarrow$$

$$\rightarrow 8=\frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}\rightarrow$$

$$\rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]=$$

$$=\int_0^\infty\frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(\left|F_2\right|^2\right)\right]. \quad (5.11)$$

## Appendix A

***Notions of thermodynamics*.**

The laws of thermodynamics, in principle, describe the specifics for the transport of heat and work in thermodynamic processes. Since their inception, however, these laws have become some of the most important in all of physics and other types of science associated with thermodynamics. In this paper we consider principally the first and the second law of thermodynamics:

1. The first law mandates conservation of energy, and states in particular that heat is a form of energy.
2. The second law which states that the entropy of the Universe always increases, or (equivalently) that perpetual motion machines are impossible.

***First Law***

Energy can neither be created nor destroyed. It can only change forms.

In any process, the total energy of the Universe remains the same.

For a thermodynamic cycle the net heat supplied to the system equals the net work done by the system.

The First Law states that energy cannot be created or destroyed; rather, the amount of energy lost in a steady state process cannot be greater than the amount of energy gained. This is the statement of conservation of energy for a thermodynamic system. It refers to the two ways that a closed system transfers energy to and from its surroundings – by the process of heating (or cooling) and the process of mechanical work. The rate of gain or loss in the stored energy of a system is determined by the rates of these two processes. In open systems, the flow of matter is another energy transfer mechanism, and extra terms must be included in the expression of the first law.

The first law can be expressed as the Fundamental Thermodynamic Relation:

$$dE = TdS - pdV.$$

Here, E is internal energy, T is temperature, S is entropy, p is pressure and V is volume. This is a statement of conservation of energy: the net change in internal energy (dE) equals the heat energy that flows in (TdS), minus the energy that flows out via the system performing work (pdV).

***Second Law***

The entropy of an isolated system not in equilibrium will tend to increase over time, approaching a maximum value at equilibrium.

In a simple manner, the second law states "energy systems have a tendency to increase their entropy rather than decrease it". A way of thinking about the second law is also to consider entropy as a measure of disorder.

***The Clausius Theorem***

The Clausius Theorem (1854) states that in a cyclic process

$$\oint \frac{dQ}{T} \leq 0 \quad \text{(a)}$$

The equality holds in the reversible case and the "<" is in the irreversible case. The reversible case is used to introduce the function state entropy. This is because in cyclic process the variation of a state function is zero.

Now we consider a reversible process *a-b*. A series of isothermal and adiabatic processes can replace this process if the heat and work interaction in those processes is the same as that in the process *a-b*. Let this process be replaced by the process *a-c-d-b*, where *a-c* and *d-b* are reversible adiabatic processes, while *c-d* is a reversible isothermal process. The isothermal line is chosen such that the area *a-e-c* is the same as the area *b-e-d*. Now, since the area under the *p-V* diagram is the work done for a reversible process, we have that the total work done in the cycle *a-c-d-b-a* is zero. Applying the first law (dU = dQ + dW), we have that the total heat transferred is also zero as the process is a cycle (and hence dU = 0). Since *a-c* and *d-b* are adiabatic processes, the heat transferred in process *c-d* is the same as that in the process *a-b*. Now applying first law between the states *a* and *b* along *a-b* and *a-c-d-b*, we have, the work done is the same. Thus the heat and work in the process *a-b* and *a-c-d-b* are the same and any reversible process *a-b* can be replaced with a combination of isothermal and adiabatic processes, which is the Clausius Theorem.

A corollary of this theorem is that any reversible cycle can be replaced by a series of Carnot cycles. Suppose each of these Carnot cycles absorbs heat $dQ_1^i$ at temperature $T_1^i$ and rejects heat $dQ_2^i$ at $T_2^i$ . Then, for each of these engines, we have

$$\frac{dQ_1^i}{dQ_2^i} = -\frac{T_1^i}{T_2^i}$$

or, equivalently

$$\frac{dQ_1^i}{T_1^i} + \frac{dQ_2^i}{T_2^i} = 0$$

The negative sign is included as the heat lost from the body has a negative value. Summing over a large number of these cycles, we have, in the limit,

$$\oint_R \frac{dQ}{T} = 0.$$

This means that the quantity $dQ_{rev}/T$ is a property. It is given the name *entropy*. Further, using Carnot's principle, for an irreversible cycle, the efficiency is less than that for the Carnot cycle, so that

$$\eta_{irr} = 1 - \frac{dQ_2}{dQ_1} < \eta_{Carnot} \qquad \frac{dQ_1}{T_1} - \frac{dQ_2}{T_2} < 0.$$

As the heat is transferred out of the system in the second process, we have, assuming the normal conventions for heat transfer,

$$\frac{dQ_1}{T_1} + \frac{dQ_2}{T_2} < 0.$$

So that, in the limit we have,

$$\oint_I \frac{dQ}{T} < 0 \qquad \oint \frac{dQ}{T_r} \leq 0.$$

The above inequality is called *inequality of Clausius*. Here the equality holds in the reversible case.

Now we consider a whatever real transformation I of a thermodynamic system that lead the system from the state A to the state B. We imagine then to again bring the system to the initial state through a reversing transformation II. Applying the eq. (a) to the whole cycle so built:

$$0 \geq \oint \frac{\partial Q}{T} = \int_{A\,(I)}^{B} \frac{\partial Q}{T} + \int_{B\,(II)}^{A} \frac{\partial Q}{T} \quad \text{(b)}$$

But $\int_{B\,(II)}^{A} \frac{\partial Q}{T}$ , being effected on a reversing transformation, it is equal to $\Delta S = S(A) - S(B)$, thence the eq. (b) becomes:

$$S(B) - S(A) \geq \int_A^B \frac{\partial Q}{T} \quad \text{(c).}$$

The quantity $\int_A^B \frac{\partial Q}{T}$ is defined the Clausius's integral: it coincides with the variation of entropy only if the transformation along which is calculated is reversing. Particularly, if the transformation I completed by the system is an adiabatic transformation, the right hand side of (c) becomes zero, thence $S(B) - S(A) \geq 0$.

If any system completes an adiabatic transformation that lead it from the state A to the state B, the entropy of the final state B is great or equal to the entropy of the initial state A (the sign equal is valid if the transformation is reversing). Particularly, this is valid for the transformations of the isolated systems: in whatever spontaneous transformation of whatever thermodynamic system the entropy cannot decrease. Whatever system, together with his sources (if are sources only of this system), constitutes an isolated system. Therefore: in whatever transformation, the sum of the entropy of the system and its sources cannot decrease. This is a further way to enunciate the second law of thermodynamics.

Thence, the entropy is an indicator of the state of disorder of a determinate set of bodies. Great is the disorder, great is the entropy.

The transformations toward increasing entropy "produce" *positive entropy* (the difference of entropy between the final state and that initial $\geq 0$), while those to decreasing entropy produce *negative entropy*. The entropy results a non conservative entity: in the realizable transformations, in which there exists an interaction among the system under observation and the environment, we have an increase of entropy after the transformation. We take as example a "cold" body with temperature $T_1$ that is put in contact with a "heat" body with temperature $T_2$: the variables are the quantity of heat exchanged Q and the temperatures T. We know that the quantity of heat Q will go from the body with the great temperature to that with smaller temperature up to the attainment of an intermediary temperature $T_e$ of equilibrium among the two.

Clausius says that the relationship among the quantity of heat "transformed" Q and the temperatures, initial and final, is:

$$\frac{Q}{T_e} - \frac{Q}{T_2} > 0,$$

from the moment that

$$T_e < T_2.$$

Clausius defined "*Entropy*" the relationship S = Q/T. We can say that the exchanged heat has realized a transformation in which

$$\Delta S > 0.$$

Zeuner proposed an interesting analogy among the gravitational potential energy of a body P and the entropy of a mass with a heat Q having temperature T. We know that the potential energy (that is the mechanical work) of the mass of water of the basin is

$$L = P \times \Delta H.$$

Zeuner consider the obtainable work from a thermal motor able to turn the heat into work with a Carnot's Cycle from the moment that it allows to express only the output of transformation heat/work only in terms of temperatures (rather than quantity of heat). In fact, can be possible to show that the output of the Carnot's Cycle is equal to:

$$\eta = \frac{T - T_0}{T} = 1 - \frac{T_0}{T},$$

where $T - T_0$ is the difference of temperature among the "source" and the "refrigerant". Thence, introducing in the motor the quantity of heat Q, the obtainable mechanical work L is:

$$L = \frac{Q}{T}\Delta T,$$

where **the entropy Q/T is a factor of proportionality analogous to the weight P**, being the jump of level ΔH corresponding to the jump of temperature ΔT that the motor realize. We observe that between Q and T exists a functional link such that increasing Q increases in direct proportion T and, therefore, gives a certain initial entropy, the obtainable work depends exclusively on the realizable ΔT. The motor that expels heat to smaller temperature produces more mechanic work to parity of "consumption": this is the meaning of the compare between the two thermal motors, working with "jumps" of different temperature and same initial temperature. Thence, is necessary compare two cases at identical initial temperature and keep in mind that is the factor ΔT to determine the result of transformation.

## Entropy and Life

An organism is alive when, to his own inside, produces some transformations at negative entropy (or with $\Delta S < 0$) that contradict the second principle (law). We observe a vegetable seed: if is alive, under the anticipated conditions from the Nature, it *spontaneously* germinates and grows capturing the Carbon from the atmosphere, giving origin to the plant, freeing the Oxygen through the photosynthesis. The general equation for photosynthesis is:

$$nCO_2 + nH_2A + photons \rightarrow (CH_2O)n + nO_2 + nA$$

carbon dioxide + electron donor + light energy → carbohydrate + oxygen + oxidized electron donor

Since water is used as the electron donor in oxygenic photosynthesis, the equation for this process is:

$$2nCO_2 + 2nH_2O + photons \rightarrow 2(CH_2O)n + 2nO_2$$

carbon dioxide + water + light energy → carbohydrate + oxygen

With regard the equation for the type of photosynthesis that occurs in plants, we have that

$$6CO_2 + 6H_2O \rightleftarrows C_6H_{12}O_6 + 6O_2 \quad \text{(where } \xrightarrow{light}\text{, while } \xleftarrow[cellulose]{}\text{)}$$

carbon dioxide + water ⇄ sugar + oxygen

Naturally, if we consider also the interaction of the plant with the quanta of the solar energy and the minerals, we have that all the transformations gives positive entropy (thence, is correct the affirmation for which the entropy of Universe always increases).

The only certainty is that with the death begins an irreversible trial, with production of positive entropy as affirms the second principle. In conclusion, can be said that the property of the entropy is that to increase in every practically realizable transformation (i.e., in every irreversible transformation) **locally excluded in the case of living organisms**.

The messy velocity of the molecules of a gas (but also those of the liquids and the solids), to a date temperature, assumes values continually and casually variables *following a particular distribution*. **Through this distribution, discovered by Boltzmann, the living nature, vegetable and animal can effect local transformations to decreasing entropy**.

The irreversibility seems to be a "*defect*" of Cosmos having the function to force it to a progressive entropic enrichment (and, therefore, to an energetic decadence) so that the final form of all the available energy is the thermal and unusable from the point of view entropic. Thence, all of a sudden of the evolution of the Universe, *in an ended time*, it won't be more possible to realize in practice some thermodynamic cycle.

## Thermodynamics and Life

Boltzmann got the graph of the probability in function of the temperature postulating that, gives a certain number *m* of indistinguishable particles among them, (A,B,C,...,M) and existing a number *n* of possible states (*a,b,c,...,n*) in which one or more particles (also everybody *m*) can be found, the presence of the particles in every state can happen with different possibilities. If the identical particles are free to occupy the various states (as in the case of a gas), they can also exchange continually the state among them also maintaining "approximately" a certain distribution. According to the boundary conditions (for example the temperature) a certain distribution of the possible configurations will be typical of such conditions. If for state of the particle we intend the possession of a certain quantity of kinetic energy E associated to every molecule of a gas, in a certain interval of values of energy ΔE, there will be some molecules in constant quantity even if between them continuous exchanges of energy happen and, therefore, within the same interval, some particles enter and others go out.

## Conclusion

1. The *correct* application of the Boltzmann's statistic, to phenomena for which are asked the probability, show that combinations of numerous particles, such to produce complex organisms, asks very long times to the comparison of which the age of Universe is less than a pulsation of eyelashes.
2. Probabilities assume the greatest values in correspondence to the messiest configurations.
3. The most orderly combinations are those that characterize the organic structures and is necessary *the action of an intelligent entity* to select, order and preserve in the time the favourable combinations.
4. The Gaussian point of view implicates that the "casual phenomena" are necessarily associated to a program which implicates the existence of an objective around of which we have a great concentration of events.
5. It is necessary to postulate the existence of an intelligent project (**Intelligent Design**) without which the configurations and the favourable events constitute episodes without a functional connection between them.
6. The existence of the life what phenomenon producing positive entropy in continuous way (and not cyclical) is a real fact.

The everything induces to affirm the existence of an Supreme Being Coordinator that possesses the life "previously" and that is able to transmit to the animals and vegetables. We note that the two living systems, animal and vegetable, are complementary among them in the sense that *the project* foresees that the issues of the first ones ($CO_2$) are the food for the seconds, and that the discard of the seconds ($O_2$) is essential for the first ones. The cycle of the Carbon and the Oxygen is really a great idea! There is only one explanation: we are in presence of the greatest Physicist Planner of every time: Creative God! In other terms: **The Creation is a thermodynamic necessity!**

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**References**


[1] R.D. White, S.Dujko, K.F. Ness, R. E. Robson, Z. Raspopovic, Z. Lj. Petrovic – "*Time dependent multi-term solution of Boltzmann's equation for magnetised low temperature plasmas*" – Austral. Mathematical Soc. May 1, 2007.

[2] Gerhard Kramm and Nicole Molders – "*Planck's blackbody radiation law: Presentation in different domains and determination of the related dimensional constants*" - http://arxiv.org/ftp/arxiv/papers/0901/0901.1863.pdf - Jan 2009.

[3] Branko Dragovich – "*Adelic Harmonic Oscillator*" – arXiv:hep-th/0404160v1 – 21.04.2004.

[4] AB Lahanas, NE Mavromatos and DV Nanopoulos – "*Dilaton and off-shell (non-critical string) effects in Boltzmann equation for species abundances*" – PMC Physics A 2007 – Oct 2007.

[5] Leonard Susskind and John Uglum – "*Black Hole Entropy in Canonical Quantum Gravity and Superstring Theory*" – arXiv:hep-th/9401070v3 – 14.07.1994.

[6] Atish Dabholkar – "*Quantum corrections to Black Hole Entropy in String theory*" – arXiv:hep-th/9409158v3 – 9.04.1995.

[7] J. Richard Gott, III and Li-Xin Li – "*Can the Universe Create Itself?*" – arXiv:astro-ph/9712344 v1 – 30.12.1997.

[8] G. Djordjevic, B. Dragovich, L.D. Nesic, I.V. Volovich – "*p-Adic and Adelic Minisuperspace Quantum Cosmology*" – arXiv:gr-qc/0105050v2 – 02.04.2002.

[9] Branko Dragovich – "*p-Adic and Adelic Cosmology: p-Adic Origin of Dark Energy and Dark Matter*" – arXiv:hep-th/0602044v1 – 04.02.2006.