Differential equations for conic section

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***Abstract**: The movement of a point along an ellipse under the action of a generalized force is studied. A well-known differential equation of second-order curves with respect to the focus is derived. Similar arguments are made for the differential equation of second-order curves with respect to the center. Received constant linear velocity for the motion along the ellipse. A comparison is made with the constant of Kepler's third law.*

keywords: *ellipse, angular acceleration, eccentricity, differential equation, constants of the linear speed of movement along the ellipse.*

Introduction

Philosophy considers motion and matter in a broad sense. The article examines the mechanical movement of material points.
F. Engels States: "... motion is unthinkable without matter" [1]. It is difficult to disagree with this statement. However, mathematics allows you to distinguish the mechanical movement of material points in a separate category with its own properties.
If simple equations of speed and acceleration are sufficient to describe rectilinear motion: V = S/t, a = S/t^2, then differential equations of motion are needed to solve problems on the curvilinear motion of material points and their systems. "The way we derive these equations doesn’t matter”: [2,§11,п.3]. In this paper the equation for conic section is derived by the investigation of movement of a body along an elipse under the influance of a summarized force.

Let us consider two variants of motion of a point along a second-order curve under the action of a generalized force. In the first variant, the ellipse, around the left focus, Fig. 1. In the second variant, the ellipse, around the center, Fig. 2.

The list of symbols

a, b - axes of the elipse
a - semi-major axis*, b* – semi-minor axis.

$p=\frac{b^2}{a}$ - focal parameter

$e=\sqrt{1-\frac{b^2}{a^2}}$ - eccentricity

x,y - coordinate of the mass point

m – the mass of the body

$\dot{x},\dot{y},\dot{\varphi}$ – time derivati\e

$\ddot{x},\ddot{y},\ddot{\varphi}$ - second time derivative

Option 1.

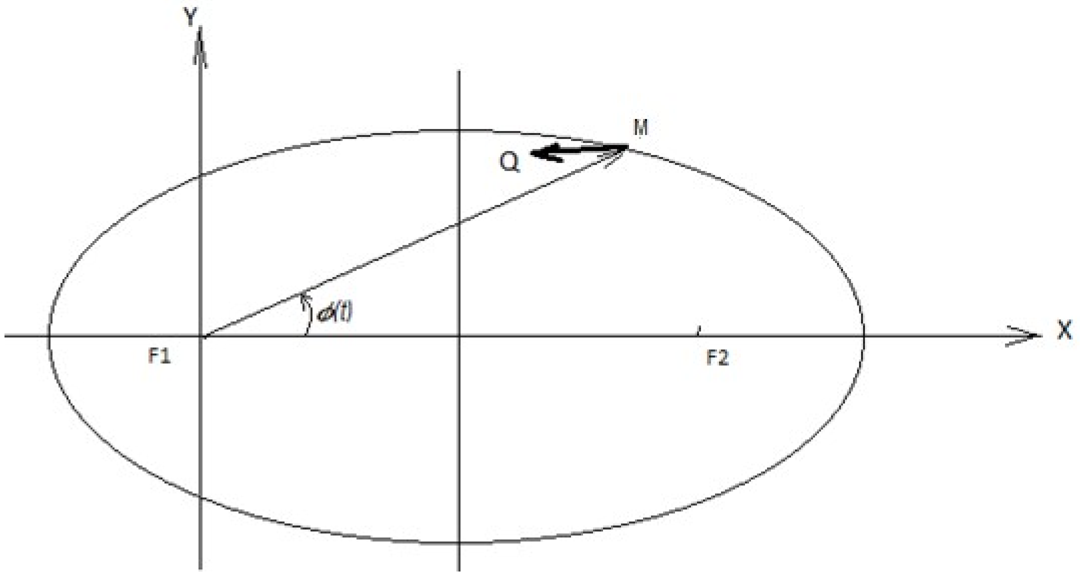

Figure 1.The mass point movement around the left focus.

M - material point.
Q – the force acting on the point.
F1- left focus.
F2 - right focus.
φ(t) - angle between X axis and the line connecting left focus and the point.

Let us place the left center into the origin of coordinates.

$$m\ddot{x} = -Q\cos(\varphi(t)) \qquad (1)$$
$$m\ddot{y} = -Q\sin(\varphi(t)) \qquad (2)$$

From equation (1) we can get

$$Q = \frac{-m\ddot{x}}{\cos(\varphi(t))} \qquad (3)$$

Let us substitute equation (3) into equation (2)

$$\ddot{y} = \frac{\ddot{x}}{\cos(\varphi(t))}\sin(\varphi(t)) \qquad (4)$$

The point coordinates can be represented as the function of angle of deflection φ(t) and radius r(t).

$$x = r(\varphi(t)) \cdot \cos(\varphi(t)) \qquad (5)$$
$$y = r(\varphi(t)) \cdot \sin(\varphi(t)) \qquad (6)$$
$$r(\varphi(t)) = \frac{p}{1 - e*\cos(\varphi(t))} \qquad (7)$$

Let us calculate the first and second time derivative From equations (5), (6), (7). Let the second time derivative be put in the equation (4) and move everything to the left side.

$$\ddot{\varphi} = \frac{2 * e * \sin(\varphi) * \dot{\varphi}^2}{1 - e * \cos(\varphi)} \qquad (8)$$

Equation (8) is differential equation of second-order **for conic section** with respect to the focus. Different values of the eccentricity will lead into a different shape of the curve.

Option 2.

Here the beginning of coordinates in the center of an ellipse, figure 2.

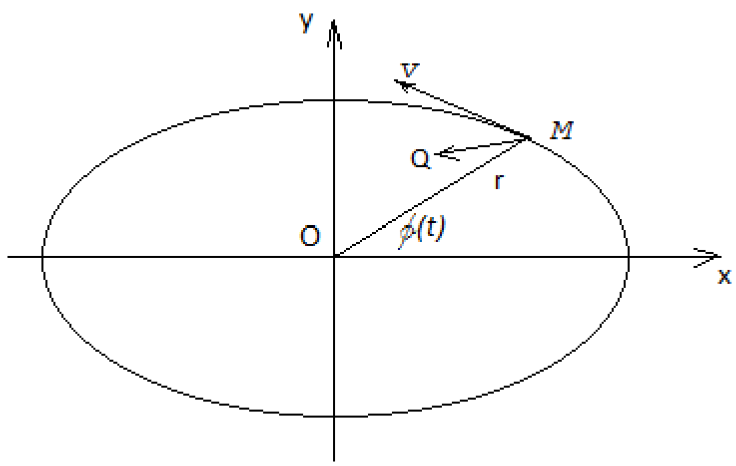

Figure 1.The mass point movement around the center.

M - material point.
Q – the force acting on the point.
O - center.
v - linear velocity of a point
φ(t) - angle between X axis and the line connecting center and the point.

Let us repeat the reasoning of option 1.

$$m\ddot{x} = -Q\cos(\varphi(t)) \qquad (9)$$

$$m\ddot{y} = -Q\sin(\varphi(t)) \qquad (10)$$

From equation (9) we can get

$$Q = \frac{-m\ddot{x}}{\cos(\varphi(t))} \qquad (11)$$

Let us substitute equation (11) into equation (10)

$$\ddot{y} = \frac{\ddot{x}}{\cos(\varphi(t))}\sin(\varphi(t)) \qquad (12)$$

The point coordinates can be represented as the function of angle of deflection *φ¿t)* and radius *r(t)*.

$$x = r(\varphi(t)) \cdot \cos(\varphi(t)) \qquad (13)$$

$$y = r(\varphi(t)) \cdot \sin(\varphi(t)) \quad (14)$$

$$r(\varphi(t)) = \frac{b\cos(\varphi(t))}{\sqrt{1 - e^2 \cos^2 \varphi(t)}} \quad (15)$$

Let us calculate the first and second time derivative from the equations (13), (14), (15). Let second time derivative be put in the equation (12) and move everything to the left side.

$$\ddot{\varphi} = \frac{2 * e^2 * \cos(\varphi) * \sin(\varphi) * \dot{\varphi}^2}{1 - e^2 * \cos(\varphi)^2} \quad (16)$$

Equation (16) differential equation of second-order **for conic section** with respect to the center. Different values of the eccentricity will lead into a different shape of the curve.
Kepler's second law is a consequence of the law of conservation of momentum.
The constant sectorial velocity is a property of equations (8), (16). Programs TygeBraheKepler2_focal.exe, TygeBraheKepler2_center.exe calculate the motion parameters and show the equality of areas of sectors with equal time intervals. Programs can be found [3].

Equation (8) allows us to model orbits using Kepler's laws.

Equation (16) is applicable for modeling the streamlines of fluid and gas particles [4, 5].

The output of the constant linear velocity

Select the unit of time measurement. The point makes a complete revolution from 0 to 2π in the time T =1 year of the planet.

1. perihelion with radius vector $r_1 = a - c$, velocity V_1 2. the aphelion with the radius vector $r_2 = a + c$, velocity V_2

Option 1 — the left focus of the ellipse is the pole, the major axis is the polar axis

In perihelion and aphelion, sin(φ) = 0, so the acceleration at these points is zero (8), and the velocity difference modulo is a constant:

$$V_1 = V_2 + \delta, \delta = V_1 - V_2 \quad (17)$$

Let's Express the sector velocity, which is a constant value according to the law of conservation of the amount of motion, modulo the linear velocity:

$$V_s = 1/2 V_1 r_1 = 1/2 r_1 (V_2 + \delta) \quad (18)$$

$$V_s = 1/2 V_2 r_2 \quad (19)$$

$$1/2 r_1 (V_2 + \delta) = 1/2 r_2 V_2 \quad (20)$$

$$V_2 = \frac{r_1 \delta}{r_2 - r_1} \quad (21)$$

Substitute (21) in (19):

$$V_s = \frac{\delta r_1 r_2}{2(r_2 - r_1)} \tag{22}$$

Calculate the area of the ellipse along which the planet moves . On the one hand:

$$S_{ellipse} = \pi ab \tag{23}$$

where a is the length of the major half-axis and b is the length of the minor half-axis of the orbit.

On the other hand, taking advantage of the fact that to calculate the area of a sector, you can multiply the sector speed by the turnover period:

$$S_{ellipse} = V_s T = T \frac{\delta r_1 r_2}{2(r_2 - r_1)} \tag{24}$$

Therefore,

$$T \frac{\delta r_1 r_2}{2(r_2 - r_1)} = \pi ab \tag{25}$$

For further transformations, we will use the geometric properties of the ellipse. We have relations:

$$r_2 - r_1 = 2c, \ c = ae, \ r_1 r_2 = a^2 - c^2 = b^2$$

Substitute in (25):

$$T \frac{\delta b^2}{4ae} = \pi ab \tag{26}$$

$$T \frac{\delta b}{a^2 e} = 4\pi \tag{27}$$

In the formula (8) $T = 1$, therefore

$$\delta = \frac{4\pi a^2 e}{b} \tag{28}$$

$$\frac{T\delta}{\left(\frac{4\pi a^2 e}{b}\right)} = const \tag{29}$$

The program _Planet_left_focus_LM.exe [6] calculates V_1, V_2 using the formula (8),

where $T = 1$ [*year of the planet*] and $\delta = V_1 - V_2$ [*a. e. / year of the planet*]

In tables 1-6, the distance unit is the distance from the Earth to the Sun [*a. e.*], $T_{Earth} = 1$[*a.e.*]

Name	*e*	*a* [*a.e.*]	*b* [*a.e.*]	V_1[*a. e. / year planet*]	V_2[*a. e. / year planet*]	$const_1$
Mercury	0.2056359	0.3870992	0.3788264	2.99636006	1.97423744	1.02214694
Venus	0.0068	0.7282313	0.7282144	4.60678244	4.5445137	1.00062621
Earth	0.0167112	1.0000026	0.999863	6.38903284	6.17898464	1.00008965
Mars	0.0549	1.523662	1.5170023	10.5133467	8.71732235	0.99998128
X	0.1	5	4.9749369	34.7310257	28.4163113	0.99998021

Table 1

Kepler's third law

$$\frac{T^2}{a^3} = const_2 \tag{30}$$

Name	*T*	*a* [a.e.]	$const_2$
Mercury	0.2408	0.38709926	0,99965
Venus	0,615	0.72823125	0.97994
Earth	1	1.00000262	1
Mars	1,88081	1.52366197	1.00007
X	11,1803399	5	0.99999

Table 2

From tables 1 and 2, we see that $const_1 = const_2 = 1$.

Therefore,

$$\frac{\delta}{\left(\frac{4\pi a^2 e}{b}\right)} = \frac{T^2}{a^3} \tag{31}$$

$$T = \sqrt{\frac{\delta b a}{4\pi e}} \tag{32}$$

Using the formula (32), we get table 3

Name	e	a [$a.e.$]	b [$a.e.$]	V_1 - V_2 [$a.$ $e.$ / $year\ planet$]	T
Mercury	0.2056359	0.3870992	0.3788264	1.02212262	0.240839854
Venus	0.0068	0.7282313	0.7282144	0.06226873	0.621641338
Earth	0.01671123	1.0000026	0.999863	0.210048199	1.00008965
Mars	0.0549	1.523662	1.5170023	1.79602432	1.88074052
X	0.1	5	4.9749369	6.31471443	11.1802301

Table 3

Option 2 - the center of the ellipse is the pole, the major axis is the polar axis

At the intersection of the ellipse and the major axis cos(φ) = 1, it follows that (15) takes the form

$$r=\frac{b}{\sqrt{1-e^2}} \qquad (33)$$

Let's Express the sector velocity (which is a constant value) modulo the linear velocity

$$V_s=1/2Vr \qquad (34)$$

Calculate the area of the ellipse. On the one hand:

$$S_{ellipse}=\pi ab \qquad (35)$$

where a is the length of the major half-axis and b is the length of the minor half-axis of the orbit.

On the other hand, taking advantage of the fact that to calculate the area of a sector, you can multiply the sector speed by the turnover period:

$$S_{ellipse}=V_s T=1/2TVr \qquad (36)$$

Since T = 1, then

$$Vr=2\pi ab \qquad (37)$$

With the pole in the center of the ellipse

$$r_1=r_2=a \qquad (38)$$

Get

$$V_a=2\pi b \qquad (39)$$

where V_a is the speed at the poles of the major axis.

$$\frac{v_a}{b}=2\pi=const_3 \qquad (40)$$

$$\frac{v_a}{2\pi b} = const_3 = 1 \tag{41}$$

The program _Planet_left_focus_LM.exe [6] calculates *Va* using the formula (16), where *T = 1* [*year of the planet*]

Name	*const₃*	*a* [*a.e.*]	*b* [*a.e.*]	V_a [*a.e./ year planet*]
Mercury	1.0218372	0.387099	0.378826	2.4322
Venus	1.0000166	0.728231	0.728214	4.57558
Earth	1.0001384	1.000002	0.999863	6.2832
Mars	1.0043862	1.523662	1.517002	9.5734
X	1.0050234	5	4.974937	31.41547

Table 4

From tables 2 and 4, we see that $const_2 = const_3 = 1$.
Formally, you can equate (41) and (30),

$$\frac{v_a}{2\pi b} = \frac{T^2}{a^3} \tag{42}$$

$$T = \sqrt{\frac{v_a a^3}{2\pi b}} \tag{43}$$

Using the formula (43), we get table 5, where the period of the planet is calculated relative to one year of the Earth:

Name	*a* [*a.e.*]	*b* [*a.e.*]	V_a [*a.e./год планеты*]	*T*
Mercury	0.3870992	0.3788264	1.02212262	0.243458182
Venus	0.7282313	0.7282144	0.06226873	0.621452034
Earth	1.0000026	0.999863	0.210048199	1.00007319
Mars	1.523662	1.5170023	1.79602432	1.88487852
X	5	4.9749369	6.31471443	11.2083855

Table 5

Here is a summary table of the periods obtained in various ways:

Name	*T (справочное)*	*T =sqrt(a^3)*	*T (32)*	*T (43)*
Mercury	0.240841791	0.240842715	0.240839854	0.243458182
Venus	0.615178823	0.621446848	0.621641338	0.621452034
Earth	1	1.00000393	1.00008965	1.00007319
Mars	1.88081586	1.88075805	1.88074052	1.88487852
X		11.1803398	11.1802301	11.2083855

Table 6

External links

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