The mathematical theory of black holes. Mathematical connections with some sectors of String Theory and Number Theory

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Abstract

In this paper we have described in the **Section 1**, some equations concerning the stellar evolution and their stability. In the **Section 2**, we have described some equations concerning the perturbations of Schwarschild black-hole, the Reissner-Nordstrom solution and the Schwarzschild geometry in D = d + 1 dimensions. Furthermore, in these sections, we have showed the mathematical connections with some sectors of Number Theory, principally with the Ramanujan's modular equations and the aurea ratio (or golden ratio)

1. On some equations concerning the stellar evolution and their stability. [1]

The success of the quantum theory may be traced to two basic facts: *first*, the Bohr radius of the ground state of the hydrogen atom, namely,

$$\frac{h^2}{4\pi^2 me^2} \approx 0.5\times 10^{-8}\,\text{cm}, \quad (1.1)$$

where h is Planck's constant, m is the mass of the electron and e is its charge, provides a correct measure of atomic dimensions; and *second*, the reciprocal of Sommerfeld's fine-structure constant,

$$\frac{hc}{2\pi e^2} \approx 137, \quad (1.2)$$

gives the maximum positive charge of the central nucleus that will allow a stable electron-orbit around it.
With regard the stellar structure and stellar evolution, the following combination of the dimensions of a mass provides a correct measure of stellar masses:

$$\left[\frac{hc}{G}\right]^{3/2}\frac{1}{H^2} \cong 29.2\,\odot, \quad (1.3)$$

where G is the constant of gravitation and H is the mass of the hydrogen atom.
The radiation pressure in the equilibrium of a star, is given from the following equation:

$$P = \left[\left(\frac{k}{\mu H} \right)^4 \frac{3}{a} \frac{1-\beta}{\beta^4} \right]^{1/3} \rho^{4/3} = C(\beta)\rho^{4/3} . \quad (1.4)$$

There is a general theorem (Chandrasekhar, 1936) which states that the pressure, P_c, at the centre of a star of a mass M in hydrostatic equilibrium in which the density, $\rho(r)$, at a point at a radial distance, r, from the centre does not exceed the mean density, $\bar{\rho}(r)$, interior to the same point r, must satisfy the inequality,

$$\frac{1}{2} G \left(\frac{4}{3}\pi \right)^{1/3} \bar{\rho}^{4/3} M^{2/3} \le P_c \le \frac{1}{2} G \left(\frac{4}{3}\pi \right)^{1/3} \rho_c^{4/3} M^{2/3} , \quad (1.5)$$

where $\bar{\rho}$ denotes the mean density of the star and ρ_c its density at the centre.
The right-hand side of inequality (1.5) together with P given by eq. (1.4), yields, for the stable existence of stars, the condition,

$$\left[\left(\frac{k}{\mu H} \right)^4 \frac{3}{a} \frac{1-\beta_c}{\beta_c^4} \right]^{1/3} \le \left(\frac{\pi}{6} \right)^{1/3} G M^{2/3} , \quad (1.6)$$

or, equivalently,

$$M \ge \left(\frac{6}{\pi} \right)^{1/2} \left[\left(\frac{k}{\mu H} \right)^4 \frac{3}{a} \frac{1-\beta_c}{\beta_c^4} \right]^{1/2} \frac{1}{G^{3/2}} , \quad (1.7)$$

where in the foregoing inequalities, β_c is a value of β at the centre of the star. Now Stefan's constant, a, by virtue of Planck's law, has the value

$$a = \frac{8\pi^5 k^4}{15 h^3 c^3} . \quad (1.8)$$

Inserting this value a in the equality (1.7) we obtain

$$\mu^2 M \left(\frac{\beta_c^4}{1-\beta_c} \right)^{1/2} \ge \frac{(135)^{1/2}}{2\pi^3} \left(\frac{hc}{G} \right)^{3/2} \frac{1}{H^2} = 0{,}1873 \left(\frac{hc}{G} \right)^{3/2} \frac{1}{H^2} . \quad (1.9)$$

We observe that the inequality (1.9) has isolated the combination (1.3) of natural constants of the dimensions of a mass; by inserting its numerical value given in eq. (1.3), we obtain the inequality,

$$\mu^2 M \left(\frac{\beta_c^4}{1-\beta_c} \right)^{1/2} \ge 5{,}48 \odot . \quad (1.10)$$

This inequality provides an upper limit to $(1-\beta_c)$ for a star of a given mass. Thus,

$$1-\beta_c \le 1-\beta_*, \quad (1.11)$$

where $(1-\beta_*)$ is uniquely determined by the mass M of the star and the mean molecular weight, μ, by the quartic equation,

$$\mu^2 M = 5{,}48\left(\frac{1-\beta_*}{\beta_*^4}\right)^{1/2} \odot. \quad (1.12)$$

We note that the following values:

$\left(\frac{\pi}{6}\right)^{1/3} = 0{,}80599 \approx 0{,}80901699$ **;** $\left(\frac{6}{\pi}\right)^{1/2} = 1{,}381976 \approx 1{,}39057647$ **;** $\frac{\sqrt{135}}{2\pi^3} = 0{,}18736 \approx 0{,}183689$

are well connected with various fractional powers of Phi $\Phi = \frac{\sqrt{5}+1}{2} = 1{,}61803399...$ **(see Appendix A)**

In a completely degenerate electron gas all the available parts of the phase space, with momenta less than a certain "threshold" value p_0 are occupied consistently with the Pauli exclusion-principle. If $n(p)dp$ denotes the number of electrons, per unit volume, between p and $p+dp$, then the assumption of complete degeneracy is equivalent to the assertion

$$n(p) = \frac{8\pi}{h^3} p^2 \quad (p \le p_0); \qquad n(p) = 0 \quad (p > p_0). \quad (1.13)$$

The value of the threshold momentum, p_0, is determined by the normalization condition

$$n = \int_0^{p_0} n(p)dp = \frac{8\pi}{3h^3} p_0^3, \quad (1.14)$$

where n denotes the total number of electrons per unit volume. For the distribution given by (1.13), the pressure p and the kinetic energy E_{kin} of the electrons (per unit volume), are given by

$$P = \frac{8\pi}{3h^3} \int_0^{p_0} p^3 v_p dp \quad (1.15)$$

and

$$E_{kin} = \frac{8\pi}{h^3} \int_0^{p_0} p^2 T_p dp, \quad (1.16)$$

where v_p and T_p are the velocity and the kinetic energy of an electron having a momentum p. If we set

$$v_p = p/m \quad \text{and} \quad T_p = p^2/2m, \quad (1.17)$$

appropriate for non-relativistic mechanics, in eqs. (1.15) and (1.16), we find

$$P = \frac{8\pi}{15h^3 m} p_0^5 = \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3} \frac{h^2}{m} n^{5/3} \quad (1.18)$$

and

$$E_{kin} = \frac{8\pi}{10h^3 m} p_0^5 = \frac{3}{40}\left(\frac{3}{\pi}\right)^{2/3} \frac{h^2}{m} n^{5/3}. \quad (1.19)$$

Thence, we obtain the following expressions:

$$P = \frac{8\pi}{3h^3}\int_0^{p_0} p^3 v_p dp = \frac{8\pi}{15h^3 m} p_0^5 = \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3} \frac{h^2}{m} n^{5/3}; \quad (1.18b)$$

$$E_{kin} = \frac{8\pi}{h^3}\int_0^{p_0} p^2 T_p dp = \frac{8\pi}{10h^3 m} p_0^5 = \frac{3}{40}\left(\frac{3}{\pi}\right)^{2/3} \frac{h^2}{m} n^{5/3}. \quad (1.19b)$$

We note that the following values:

$\frac{8\pi}{3} = 8{,}37758 \approx 8{,}34345$ **;** $\frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3} = 0{,}048486 \cong 0{,}04863$ **;** $\frac{3}{40}\left(\frac{3}{\pi}\right)^{2/3} = 0{,}072729 \cong 0{,}07294$ **;**

are well connected with various fractional powers of Phi $\Phi = \frac{\sqrt{5}+1}{2} = 1{,}61803399...$ **(see Appendix A)**

According to the expression for the pressure given by eq. (1.18), we have the relation

$$P = K_1 \rho^{5/3} \quad \text{where} \quad K_1 = \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3} \frac{h^2}{m(\mu_e H)^{5/3}}, \quad (1.20)$$

where μ_e is the mean molecular weight per electron. We can rewrite the expression also as follows:

$$P = \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3} \frac{h^2}{m(\mu_e H)^{5/3}} \rho^{5/3}. \quad (1.20b)$$

Thus, already for a degenerate star of solar mass (with $\mu_e = 2$) the central density is 4.19×10^6 g cm^3; and this density corresponds to a threshold momentum $p_0 = 1.29mc$ and a velocity which is $0.63c$. Consequently, the equation of state must be modified to take into account the effects of special relativity. And this is easily done by inserting in eqs. (1.15) and (1.16) the relations,

$$v_p = \frac{p}{m(1 + p^2/m^2c^2)^{1/2}} \quad \text{and} \quad T_p = mc^2\left[(1 + p^2/m^2c^2)^{1/2} - 1\right], \quad (1.21)$$

in place of the non-relativistic relations (1.17). We find that the resulting equation of state can be expressed, parametrically, in the form

$$P = Af(x) \quad \text{and} \quad \rho = Bx^3, \quad (1.22)$$

where

$$A = \frac{\pi m^4 c^5}{3h^3}, \quad B = \frac{8\pi m^3 c^3 \mu_e H}{3h^3} \quad (1.23)$$

and

$$f(x) = x(x^2+1)^{1/2}(2x^2-3) + 3\sinh^{-1} x. \quad (1.24)$$

Thence, P can be rewritten also as follows:

$$P = \frac{\pi m^4 c^5}{3h^3} \; x(x^2+1)^{1/2}(2x^2-3) + 3\sinh^{-1} x. \quad (1.24b)$$

And similarly

$$E_{kin} = Ag(x), \quad (1.25)$$

$$g(x) = 8x^3\left[(x^2+1)^{1/2} - 1\right] - f(x). \quad (1.26)$$

Thence, the eq. (1.25) can be rewritten also as follows:

$$E_{kin} = \frac{\pi m^4 c^5}{3h^3} \; 8x^3\left[(x^2+1)^{1/2} - 1\right] - x(x^2+1)^{1/2}(2x^2-3) + 3\sinh^{-1} x. \quad (1.26b)$$

According to eqs. (1.22) and (1.23), the pressure approximates the relation (1.20) for low enough electron concentrations $(x \ll 1)$; but for increasing electron concentrations $(x \gg 1)$, the pressure tends to

$$P = \frac{1}{8}\left(\frac{3}{\pi}\right)^{1/3} hcn^{4/3}. \quad (1.27)$$

This limiting form of relation can be obtained very simply by setting $v_p = c$ in eq. (1.15); then

$$P = \frac{8\pi c}{3h^3}\int_0^{p_0} p^3 dp = \frac{2\pi c}{3h^3} p_0^4; \quad (1.28)$$

and the elimination of p_0 with the aid of eq. (1.14) directly leads to eq. (1.27). The relation between P and ρ corresponding to the limiting form (1.28) is

$$P = \frac{1}{8}\left(\frac{3}{\pi}\right)^{1/3} \frac{hc}{(\mu_e H)^{4/3}} \rho^{4/3}. \quad (1.29)$$

Thence, the eq. (1.28) can be rewritten also as follows:

$$P = \frac{8\pi c}{3h^3}\int_0^{p_0} p^3 dp = \frac{2\pi c}{3h^3} p_0^4 = \frac{1}{8}\left(\frac{3}{\pi}\right)^{1/3} \frac{hc}{(\mu_e H)^{4/3}} \rho^{4/3}. \quad (1.29b)$$

We note that the following values:

$\frac{1}{8}\left(\frac{3}{\pi}\right)^{1/3} = 0{,}123093 \cong 0{,}12538820$ **and** $\frac{2\pi}{3} = 2{,}094395 \approx 2{,}12461180$**,**

are well connected with various fractional powers of Phi $\Phi = \frac{\sqrt{5}+1}{2} = 1{,}61803399...$ **(see Appendix A)**

In this limit, the configuration is an Emden polytrope of index 3. And it is well known that when the polytropic index is 3, the mass of the resulting equilibrium configuration is uniquely determined by the constant of proportionality, K_2, in the pressure-density relation. We have accordingly,

$$M_{\lim} = 4\pi\left(\frac{K_2}{\pi G}\right)^{3/2}(2.018) = 0.197\left(\frac{hc}{G}\right)^{3/2}\frac{1}{(\mu_e H)^2} = 5{,}76\mu_e^{-2}\odot. \quad (1.30)$$

We note that the value 5,76 is well connected with the mean of the following values: 5,562 and 5,890 that is equal to 5,726 that is a fractional powers of Phi $\Phi = \frac{\sqrt{5}+1}{2} = 1{,}61803399...$ **(see Appendix A)**

It is clear from general considerations that the exact mass-radius relation for the degenerate configurations must provide an upper limit to the mass of such configurations given by eq. (1.30); and further, that the mean density of the configuration must tend to infinity, while the radius tends to zero, and $M \to M_{\lim}$.

It is more convenient to express the electron pressure in terms of ρ and β_e defined in the manner

$$p_e = \frac{k}{\mu_e H}\rho T = \frac{\beta_e}{1-\beta_e}\frac{1}{3}aT^4, \quad (1.31)$$

where p_e now denotes the electron pressure. Then, analogous to eq. (1.4), we can write

$$p_e = \left[\left(\frac{k}{\mu_e H}\right)^4 \frac{3}{a}\frac{1-\beta_e}{\beta_e}\right]^{1/3}\rho^{4/3}. \quad (1.32)$$

Comparing this with eq. (1.29), we conclude that if

$$\left[\left(\frac{k}{\mu_e H}\right)^4 \frac{3}{a}\frac{1-\beta_e}{\beta_e}\right]^{1/3}\rho^{4/3} > K_2 = \frac{1}{8}\left(\frac{3}{\pi}\right)^{1/3}\frac{hc}{(\mu_e H)^{4/3}}, \quad (1.33)$$

the pressure p_e given by the classical perfect-gas equation of state will be greater than that given by the equation if degeneracy were to prevail, not only for the prescribed ρ and T, but for all ρ and T having the same β_e. Inserting for a its value given in eq. (1.8), we find that the inequality (1.33) reduces to

$$\frac{960}{\pi^4}\frac{1-\beta_e}{\beta_e} > 1, \quad (1.34)$$

or, equivalently

$$1-\beta_e > 0{,}0921 = 1-\beta_\omega . \quad (1.35)$$

We note that the value $\frac{960}{\pi^4} = 9{,}85534 \cong 9{,}88854382$ **and** $0{,}0921 \cong 0{,}09184494$**,**

are well connected with various fractional powers of Phi $\Phi = \frac{\sqrt{5}+1}{2} = 1{,}61803399...$ **(see Appendix A)**

On the standard model, the fraction β (=gas pressure/total pressure) is a constant through a star. On this assumption, the star is a polytrope of index 3 as is apparent from eq. (1.4); and, in consequence, we have the relation

$$M = 4\pi\left[\frac{C(\beta)}{\pi G}\right]^{3/2}(2{,}018) \quad (1.36)$$

where $C(\beta)$ is defined in eq. (1.4). Equation (1.36) provides a quartic equation for β analogous to eq. (1.12) for β_*. Equation (1.36) for $\beta = \beta_\omega$ gives

$$M = 0.197\beta_\omega^{-3/2}\left(\frac{hc}{G}\right)^{3/2}\frac{1}{(\mu H)^2} = 6{,}65\mu^{-2}\odot = \mathfrak{R} . \quad (1.37)$$

We note that the value 0,197 is well connected with 0,19513485 and that 6,65 is well connected with the mean of the following values: 6,47213 and 6,87538. It is equal to 6,6737 that is a fractional powers of Phi $\Phi = \frac{\sqrt{5}+1}{2} = 1{,}61803399...$ **(see Appendix A)**

By virtue of the inequality (1.5), the maximum central pressure attainable in a star must be less than that provided by the degenerate equation of state, so long as

$$\frac{1}{2}G\left(\frac{4}{3}\pi\right)^{1/3}M^{2/3} < K_2 = \frac{1}{8}\left(\frac{3}{\pi}\right)^{1/3}\frac{hc}{(\mu_e H)^{4/3}} \quad (1.38)$$

or, equivalently

$$M < \frac{3}{16\pi}\left(\frac{hc}{G}\right)^{3/2}\frac{1}{(\mu_e H)^2} = 1{,}74\mu_e^{-2}\odot . \quad (1.39)$$

Thence, we have the following connection between eq. (1.38) and eq. (1.39):

$$\frac{1}{2}G\left(\frac{4}{3}\pi\right)^{1/3}M^{2/3} < K_2 = \frac{1}{8}\left(\frac{3}{\pi}\right)^{1/3}\frac{hc}{(\mu_e H)^{4/3}} \Rightarrow M < \frac{3}{16\pi}\left(\frac{hc}{G}\right)^{3/2}\frac{1}{(\mu_e H)^2} = 1{,}74\mu_e^{-2}\odot. \quad (1.39b)$$

We conclude that there can be no surprises in the evolution of stars of mass less than $0{,}43\odot$ (if $\mu_e = 2$). The end stage in the evolution of such stars can only be that of the white dwarfs.

We note that the values $\left(\frac{4}{3}\pi\right)^{1/3} = 1{,}61199 \cong 1{,}618$ **and** $1{,}74 \cong 1{,}74535$ **are well connected with various fractional powers of Phi** $\Phi = \frac{\sqrt{5}+1}{2} = 1{,}61803399...$ **(see Appendix A)**

In the framework of the Newtonian theory of gravitation, the stability for radial perturbations depends only on an average value of the adiabatic exponent, Γ_1, which is the ratio of the fractional Lagrangian changes in the pressure and in the density experienced by a fluid element following the motion; thus,

$$\Delta P / P = \Gamma_1 \Delta\rho / \rho. \quad (1.40)$$

And the Newtonian criterion for stability is

$$\overline{\Gamma_1} = \int_0^M \Gamma_1(r)P(r)dM(r) \div \int_0^M P(r)dM(r) > \frac{4}{3}. \quad (1.41)$$

If $\overline{\Gamma_1} < \frac{4}{3}$, dynamical instability of a global character will ensue with an e-folding time measured by the time taken by a sound wave to travel from the centre to the surface. When one examines the same problem in the framework of the general theory of relativity, one finds that, again, the stability depends on an average value of Γ_1; but contrary to the Newtonian result, the stability now depends on the radius of the star as well. Thus, one finds that no matter how high $\overline{\Gamma_1}$ may be, instability will set in provided the radius is less than a certain determinate multiple of the Schwarzschild radius,

$$R_s = 2GM / c^2. \quad (1.42)$$

Thus, if for the sake of simplicity, we assume that Γ_1 is a constant through the star and equal to $5/3$, then the star will become dynamically unstable for radial perturbations, if $R_1 < 2{,}4R_S$. And further, if $\Gamma_1 \to \infty$, instability will set in for all $R < (9/8)R_S$. The radius $(9/8)R_S$ defines, in fact, the minimum radius which any gravitating mass, in hydrostatic equilibrium, can have in the framework of general relativity. It follows from the equations governing radial oscillations of a star, in a first post-Newtonian approximation to the general theory of relativity, that instability fro radial perturbations will set in for all

$$R < \frac{K}{\Gamma_1 - 4/3}\frac{2GM}{c^2}, \quad (1.43)$$

where K is a constant which depends on the entire march of density and pressure in the equilibrium configuration in the Newtonian framework. It is for this reason that we describe the instability as global. Thus, for a polytrope of index n, the value of the constant is given by

$$K = \frac{5-n}{18}\left[\frac{2(11-n)}{(n+1)\xi_1^4|\theta'_1|^3}\int_0^{\xi_1}\theta\left(\frac{d\theta}{d\xi}\right)^2\xi^2 d\xi + 1\right], \quad (1.44)$$

where θ is the Lane-Emden function in its standard normalization ($\theta = 1$ at $\xi = 0$), ξ is the dimensionless radial coordinate, ξ_1 defines the boundary of the polytrope (where $\theta = 0$) and θ'_1 is the derivative of θ at ξ_1. If we set $n = 2$ in the eq. (1.44), we obtain:

$$K = \frac{1}{6}\left[\frac{6}{\xi_1^4|\theta'_1|^3}\int_0^{\xi_1}\theta\left(\frac{d\theta}{d\xi}\right)^2\xi^2 d\xi + 1\right]. \quad (1.44b)$$

We note that 6 is $\frac{1}{4}\times 24$ and 24 is related to the physical vibrations of the superstring by the following Ramanujan's modular equation:

$$24 = \frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}.$$

Thence, we have the following possible mathematical connection:

$$\frac{1}{4}\times\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} \Rightarrow \frac{1}{6}\left[\frac{6}{\xi_1^4|\theta'_1|^3}\int_0^{\xi_1}\theta\left(\frac{d\theta}{d\xi}\right)^2\xi^2 d\xi + 1\right]. \quad (1.44c)$$

In the following Table, we have the values of K for different polytropic indices. It should be particularly noted that K increases without limit for $n \to 5$ and the configuration becomes increasingly centrally condensed.

n	K
0	0.452381
1.0	0.565382
1.5	0.645063
2.0	0.751296
2.5	0.900302
3.0	1.12447
3.25	1.28503
3.5	1.49953
4.0	2.25338
4.5	4.5303
4.9	22.906
4.95	45.94

It has been possible to show that for $n \to 5$, the asymptotic behaviour of K is given by

$$K \to 2.3056/(5-n); \quad (1.45)$$

and, further, that along the polytropic sequence, the criterion for instability (1.43) can be expressed alternatively in the form

$$R < 0.2264\left(\frac{\rho_c}{\bar{\rho}}\right)^{1/3} \frac{2GM}{c^2}\frac{1}{\Gamma_1 - 4/3} \quad \left(\rho_c/\bar{\rho} \geq 10^6\right). \quad (1.46)$$

We note that the value 0,2264 is well connected with 0,229 that is a fractional powers of Phi $\Phi = \frac{\sqrt{5}+1}{2} = 1,61803399...$

Furthermore, the values of the precedent table, are well connected with the following fractional powers of Phi (see Appendix A):

0,445824; 0,572949; 0,629514; 0,752329; 0,891649; 1,128493; 1,259029; 1,500000; 2,250000; 4,500000; 22,180339; 41,1246 + 4,8541.

In this section we have considered only the restrictions on the last stages of stellar evolution that follow from the existence of an upper limit to the mass of completely degenerate configurations and from the instabilities of relativistic origin. From these and related considerations, the conclusion is inescapable that **black holes** will form as one of the natural end products of stellar evolution of massive stars; and further that they must exist in large numbers in the present astronomical universe.

2. On some equations concerning the perturbations of Schwarzschild black-hole and the Reissner-Nordstrom solution. [2]

With the reduction of the third-order system of the following equations

$$N_{,r} = aN + bL + cX\ , \quad (2.1) \qquad L_{,r} = \left(a - \frac{1}{r} + \nu_{,r}\right)N + \left(b - \frac{1}{r} - \nu_{,r}\right)L + cX\ , \quad (2.2)$$

$$X_{,r} = -\left(a - \frac{1}{r} + \nu_{,r}\right)N - \left(b + \frac{1}{r} - 2\nu_{,r}\right)L - \left(c + \frac{1}{r} - \nu_{,r}\right)X\ , \quad (2.3)$$

to the single second-order the following equation

$$Z^{(+)}_{,r_*,r} = (r-2M)V_{,r,r} + V_{,r} + \frac{3M(r-2M)}{r(nr+3M)}V_{,r} + 3M\frac{-nr^2+4Mnr+6M^2}{r^2(nr+3M)^2}V + \frac{3Mn}{(nr+3M)^3}[(2+n)r-M](L+X)+$$
$$-\frac{nr^2-3Mnr-3M^2}{r(r-2M)(nr+3M)^2}[(2r-5M)L+(r-3M)X], \quad (2.4)$$

for $Z^{(+)}$, it is clear that the solution for L, X, and N will require a further quadrature. Thus, rewriting the following equation

$$L_{,r} + \left(\frac{2}{r} - \nu_{,r}\right)L = -\left[X_{,r} + \left(\frac{1}{r} - \nu_{,r}\right)X\right], \quad (2.5)$$

in the form

$$\frac{d}{dr}\left(r^2 e^{-\nu}L\right) = -nr\frac{d}{dr}\left(re^{-\nu}V\right) \quad (2.6)$$

and replacing V by

$$V = \frac{nr+3M}{3Mr}Z^{(+)} + \frac{r}{3M}L\,, \quad (2.7)$$

we obtain the equation

$$\left(1+\frac{nr}{3M}\right)\frac{d}{dr}\left(r^2e^{-\nu}L\right) = -\frac{nr}{3M}\frac{d}{dr}\left[e^{-\nu}(nr+3M)Z^{(+)}\right]. \quad (2.8)$$

This equation yields the integral relation

$$r^2e^{-\nu}L = -\int\frac{nr}{(nr+3M)}\frac{d}{dr}\left[e^{-\nu}(nr+3M)Z^{(+)}\right]dr\,, \quad (2.9)$$

or, after an integration by parts

$$r^2 e^{-\nu} L = -nre^{-\nu} Z^{(+)} + 3Mn \int \frac{e^{-\nu}}{nr+3M} Z^{(+)} dr . \quad (2.10)$$

Thence, we have the following expression:

$$\left(1+\frac{nr}{3M}\right)\frac{d}{dr}\left(-nre^{-\nu}Z^{(+)} + 3Mn\int \frac{e^{-\nu}}{nr+3M} Z^{(+)} dr\right) = -\frac{nr}{3M}\frac{d}{dr}\left[e^{-\nu}(nr+3M)Z^{(+)}\right]. \quad (2.10b)$$

Defining

$$\Phi = ne^{\nu} \int \frac{e^{-\nu} Z^{(+)}}{nr+3M} dr , \quad (2.11)$$

we have the solution

$$L = -\frac{n}{r} Z^{(+)} + \frac{3M}{r^2}\Phi . \quad (2.12)$$

With this solution for L, equation (2.7) gives

$$X = \frac{n}{r}\left(Z^{(+)} + \Phi\right). \quad (2.13)$$

As a consequence of these solutions for L and X,

$$L + X = \frac{1}{r^2}(nr+3M)\Phi . \quad (2.14)$$

To obtain the solution for N, we take the following equation

$$Z^{(+)}_{,r_*} = \left(1-\frac{2M}{r}\right) Z_{,r} = (r-2M) V_{,r} + \frac{3M(r-2M)}{r(nr+3M)} V + \frac{nr^2 - 3nMr - 3M^2}{(nr+3M)^2}(L+X), \quad (2.15)$$

and substitute for $V_{,r} (= X_{,r}/n)$ on the right-hand side from the following equation

$$X_{,r} = -\frac{nr+3M}{r(r-2M)} N - \left[\frac{1}{r-2M} - \frac{M}{r(r-2M)} + \frac{M^2 + \sigma^2 r^4}{r(r-2M)^2}\right](L+X) + \frac{n+1}{r-2M} L . \quad (2.16)$$

In this manner, we obtain

$$Z_{,r*} = \frac{1}{n}(r-2M)\left\{-\frac{nr+3M}{r(r-2M)} N - \left[\frac{1}{r-2M} - \frac{M}{r(r-2M)} + \frac{M^2+\sigma^2 r^4}{r(r-2M)^2}\right] \times (L+X) + \frac{n+1}{r-2M} L\right\} + $$
$$+ \frac{3M(r-2M)}{nr(nr+3M)} X + \frac{nr^2 - 3Mnr - 3M^2}{(nr+3M)^2}(L+X). \quad (2.17)$$

Simplifying this last equation and substituting for L and $L+X$ their solutions (2.12) and (2.13), we find:

$$N=-\frac{nr}{nr+3M}Z^{(+)}_{,r*}-\frac{n}{(nr+3M)^2}\left[\frac{6M^2}{r}+3Mn+n(n+1)r\right]Z^{(+)}+\left(M-\frac{M^2+\sigma^2r^4}{r-2M}\right)\frac{\Phi}{r^2}. \quad (2.18)$$

For the eq. (2.11), we obtain the following expression:

$$N=-\frac{nr}{nr+3M}Z^{(+)}_{,r*}-\frac{n}{(nr+3M)^2}\left[\frac{6M^2}{r}+3Mn+n(n+1)r\right]Z^{(+)}+\left(M-\frac{M^2+\sigma^2r^4}{r-2M}\right)\frac{1}{r^2}\left(ne^{\nu}\int\frac{e^{-\nu}Z^{(+)}}{nr+3M}dr\right)$$

(2.18b)

This completes the formal solution of the basic equations.

Quasi-normal modes are defined as solutions of the perturbation equations, belonging to complex characteristics-frequencies and satisfying the boundary conditions appropriate for purely outgoing waves at infinity and purely ingoing waves at the horizon. The problem, then, is to seek solutions of the equations governing $Z^{(\pm)}$ which will satisfy the boundary conditions

$$Z^{(\pm)}\to A^{(\pm)}(\sigma)e^{-i\sigma r}\quad (r_*\to+\infty);\qquad Z^{(\pm)}\to A^{(\pm)}(\sigma)e^{+i\sigma r_*}\quad (r_*\to-\infty). \quad (2.19)$$

We observe that the characteristic frequencies σ are the same for $Z^{(-)}$ and $Z^{(+)}$; for, if σ is a characteristic frequency and $Z^{(-)}(\sigma)$ is the solution belonging to it, then the solution $Z^{(+)}(\sigma)$ derived from $Z^{(-)}(\sigma)$ in accordance with the following relations

$$\left[\mu^2(\mu^2+2)+12i\sigma M\right]Z^{(+)}=\left[\mu^2(\mu^2+2)+72M^2\frac{\Delta}{r^3(\mu^2r+6M)}\right]Z^{(-)}+12MZ^{(-)}_{,r_*}; \quad (2.20a)$$

$$\left[\mu^2(\mu^2+2)-12i\sigma M\right]Z^{(-)}=\left[\mu^2(\mu^2+2)+72M^2\frac{\Delta}{r^3(\mu^2r+6M)}\right]Z^{(+)}-12MZ^{(+)}_{,r_*}; \quad (2.20b)$$

will satisfy the boundary conditions (2.19) with

$$A^{(+)}(\sigma)=A^{(-)}(\sigma)\frac{\mu^2(\mu^2+2)-12i\sigma M}{\mu^2(\mu^2+2)+12i\sigma M}. \quad (2.21)$$

It will suffice, then, to consider only the equation governing $Z^{(-)}$. Letting

$$Z^{(-)}=\exp\left(i\int^{r_*}\phi dr_*\right), \quad (2.22)$$

we find that the equation we have to solve is

$$i\phi_{,r_*}+\sigma^2-\phi^2-V^{(-)}=0; \quad (2.23)$$

and the appropriate boundary conditions are

$$\phi\to-\sigma \text{ as } r_*\to+\infty \quad\text{and}\quad \phi\to+\sigma \text{ as } r_*\to-\infty. \quad (2.24)$$

Furthermore, we can rewrite the eq. (2.20a) as follows:

$$\left[\mu^2\left(\mu^2+2\right)+12i\sigma M\right]Z^{(+)}=\left[\mu^2\left(\mu^2+2\right)+72M^2\frac{\Delta}{r^3\left(\mu^2 r+6M\right)}\right]\left(\exp\left(i\int^{r_*}\phi dr_*\right)\right)+12MZ^{(-)}_{,r_*}. \quad (2.24b)$$

Solutions of eq. (2.23), satisfying the boundary conditions (2.24), exist only when σ assumes one of a discrete set of values. A useful identity, which follows from integrating equation (2.23) and making use of the boundary conditions (2.24), is

$$-2i\sigma+\int_{-\infty}^{+\infty}\left(\sigma^2-\phi^2\right)dr_*=\int_{-\infty}^{+\infty}V^{(-)}dr_*=\frac{1}{2M}\left(\mu^2+\frac{1}{2}\right). \quad (2.25)$$

With regard the complex characteristic-frequency σ belonging to the quasi-normal modes of the Schwarzschild black-hole (σ is expressed in the unit $(2M)^{-1}$) for value of $l=4$, we have the following result:

$$l=4\,; \qquad 2M\sigma=1{,}61835+0{,}18832i\,. \quad \mathbf{(2.26)}$$

We note that the value 1,61835 is very near to the $\Phi=\frac{\sqrt{5}+1}{2}=1{,}61803398...$, thence to the value of the golden ratio.

The Reissner-Nordstrom solution

With regard the Reissner-Nordstrom solution, the reduction of the equations governing the polar perturbations of the Schwarzschild black-hole to Zerilli's equation, it is clear that the reducibility of a system of equations of order five to the pair of the following equations

$$\Lambda^2 H_2^{(+)}=\frac{\Delta}{r^5}\left[UH_2^{(+)}+W\left(-3MH_2^{(+)}+2Q_*\mu H_1^{(+)}\right)\right],\ \ \Lambda^2 H_1^{(+)}=\frac{\Delta}{r^5}\left[UH_1^{(+)}+W\left(+2Q_*\mu H_2^{(+)}+3MH_1^{(+)}\right)\right],$$

must be the result of the system allowing a special solution. Xanthopoulos has discovered that the present system of equations

$$B_{03}=\frac{1}{r^2}\left(r^2B_{23}\right)_{,r}=B_{23,r}+\frac{2}{r}B_{23}\,, \qquad r^4e^{2v}B_{02}=2Q_*^2\left[2T-l(l+1)V\right]-l(l+1)r^2B_{23}\,,$$

$$\left(r^2e^{2v}B_{03}\right)_{,r}+r^2e^{2v}B_{02}+\sigma^2r^2e^{-2v}B_{23}=2Q_*^2\frac{N+L}{r^2}\,,$$

and

$$N_{,r}=aN+bL+c\left(X-B_{23}\right), \qquad L_{,r}=\left(a-\frac{1}{r}+v_{,r}\right)N+\left(b-\frac{1}{r}-v_{,r}\right)L+c\left(X-B_{23}\right)-\frac{2}{r}B_{23}\,,$$

$$X_{,r}=-\left(a-\frac{1}{r}+v_{,r}\right)N-\left(b+\frac{1}{r}-2v_{,r}\right)L-\left(c+\frac{1}{r}-v_{,r}\right)\left(X-B_{23}\right)+B_{03}\,,$$

allows the special solution

$$N^{(0)} = r^{-2}e^{\nu}\left[M - \frac{r}{\Delta}\left(M^2 - Q_*^2 + \sigma^2 r^4\right) - 2\frac{Q_*^2}{r}\right], \quad L^{(0)} = r^{-3}e^{\nu}\left(3Mr - 4Q_*^2\right), \quad X^{(0)} = ne^{\nu}r^{-1},$$
$$B_{23}^{(0)} = -2Q_*^2 r^{-3}e^{\nu}, \quad \text{and} \quad B_{03}^{(0)} = 2Q_*^2 r^{-6}e^{-\nu}\left(2Q_*^2 + r^2 - 3Mr\right). \quad (2.27)$$

The completion of the solution for the remaining radial functions with the aid of the special solution (2.27) is relatively straightforward. Xanthopoulos finds

$$N = N^{(0)}\Phi + 2n\frac{e^{2\nu}}{\varpi}H_2^{(+)} - \frac{e^{2\nu}}{\varpi}\left(nrH_2^{(+)} + Q_*\mu H_1^{(+)}\right)_{,r} + \frac{1}{r\varpi^2}\left[e^{2\nu}(\varpi - 2nr - 3M) - (n+1)\varpi\right]\left(nrH_2^{(+)} + Q_*\mu H_1^{(+)}\right)$$
(2.28)

$$L = L^{(0)}\Phi - \frac{1}{r^2}\left(nrH_2^{(+)} + Q_*\mu H_1^{(+)}\right), \quad (2.29) \qquad X = X^{(0)}\Phi + \frac{n}{r}H_2^{(+)}, \quad (2.30)$$

$$B_{23} = B_{23}^{(0)}\Phi - \frac{Q_*\mu}{r^2}H_1^{(+)}, \quad (2.31) \qquad B_{03} = B_{03}^{(0)}\Phi - \frac{Q_*\mu}{r^2}H_{1,r}^{(+)} - 2\frac{Q_*^2}{r^4\varpi}\left(nrH_2^{(+)} + Q_*\mu H_1^{(+)}\right), \quad (2.32)$$

where

$$\Phi = \int\left(nrH_2^{(+)} + Q_*\mu H_1^{(+)}\right)\frac{e^{-\nu}}{\varpi r}dr\,. \quad (2.33)$$

Thence, the eqs. (2.31) and (2.32) can be written also as follows:

$$B_{23} = B_{23}^{(0)}\int\left(nrH_2^{(+)} + Q_*\mu H_1^{(+)}\right)\frac{e^{-\nu}}{\varpi r}dr - \frac{Q_*\mu}{r^2}H_1^{(+)}, \quad (2.34)$$

$$B_{03} = B_{03}^{(0)}\int\left(nrH_2^{(+)} + Q_*\mu H_1^{(+)}\right)\frac{e^{-\nu}}{\varpi r}dr - \frac{Q_*\mu}{r^2}H_{1,r}^{(+)} - 2\frac{Q_*^2}{r^4\varpi}\left(nrH_2^{(+)} + Q_*\mu H_1^{(+)}\right). \quad (2.35)$$

As in the case of the Schwarzschild perturbations, the potentials $V_i^{(\pm)}$ ($i = 1,2$), associated with the polar and the axial perturbations, are related in a manner which guarantees the equality of the reflexion and the transmission coefficients determined by the equations governing $Z_i^{(\pm)}$. Thus, it can be verified, the potentials are, in fact, given by

$$V_i^{(\pm)} = \pm\beta_i\frac{df_i}{dr_*} + \beta_i^2 f_i^2 + \kappa f_i\,, \quad (2.36)$$

where

$$\kappa = \mu^2\left(\mu^2 + 2\right), \quad \beta_i = q_j, \quad \text{and} \quad f_i = \frac{\Delta}{r^3\left(\mu^2 r + q_j\right)} \quad (i, j = 1,2; i \neq j), \quad (2.37)$$

thence, the eq. (2.36) can be rewritten also as follows

$$V_i^{(\pm)} = \pm q_j \frac{d}{dr_*}\frac{\Delta}{r^3\left(\mu^2 r + q_j\right)} + q_j^2\left[\frac{\Delta}{r^3\left(\mu^2 r + q_j\right)}\right]^2 + \mu^2\left(\mu^2+2\right)\frac{\Delta}{r^3\left(\mu^2 r + q_j\right)}. \quad (2.37b)$$

The solutions, $Z_i^{(+)}$ and $Z_i^{(-)}$ of the respective equations are, therefore, related in the manner

$$\left[\mu^2\left(\mu^2+2\right) \pm 2i\sigma q_j\right]Z_i^{(\pm)} = \left[\mu^2\left(\mu^2+2\right) + \frac{2q_j^2\Delta}{r^3\left(\mu^2 r + q_j\right)}\right]Z_i^{(\mp)} \pm 2q_j\frac{dZ_i^{(\mp)}}{dr_*} \qquad \left(i, j = 1,2; i \neq j\right). \quad (2.38)$$

It is the existence of this relation which guarantees the equality of the reflexion and the transmission coefficients determined by the wave equations governing $Z_i^{(+)}$ and $Z_i^{(-)}$.
In view of the relation (2.38) between the solutions belonging to axial and polar perturbations, the characteristic frequencies will be the same for $Z_i^{(+)}$ and $Z_i^{(-)}$. It should also be noticed that there is no quasi-normal mode which is purely electromagnetic or purely gravitational: any quasi-normal mode of oscillation will be accompanied by the emission of both electromagnetic and gravitational radiation in accordance with the following equation

$$H_1 = Z_1\cos\psi - Z_2\sin\psi; \qquad H_2 = Z_2\cos\psi + Z_1\sin\psi, \quad (2.39)$$

where the amplitudes H_1 and H_2 of the electromagnetic and gravitational (wave-like) disturbances (of some specified frequencies) are related to the function Z_1 and Z_2.
With regard the complex characteristic frequencies σ_1 and σ_2 (belonging to $Z_1^{(\pm)}$ and $Z_2^{(\pm)}$) of the quasi-normal modes for a range of values of Q_* and l, we obtain, for Z_1 and $Q_* = 0.9$ and $l = 2$, the value 0.61939 that is very near to the $\phi = \frac{\sqrt{5}-1}{2} = 0{,}61803398...$, thence to the value of the golden section.
The considerations, relative to the stability of the Schwarzschild black-hole to external perturbations apply, quite literally, to the Reissner-Nordstrom black-hole since the only fact relevant to those considerations was that the potential barriers, external to the event horizon, are real and positive; and stability follows from this fact.
While the equations governing $Z_i^{(\pm)}$ remain formally unaltered, the potential barriers, $V_i^{(\pm)}$, are negative in the interval, $r_- < r < r_+$, and in the associated range of r_*, namely $+\infty > r_* > -\infty$; they are in fact potential wells rather than potential barriers. Thus, the equation now governing $Z_i^{(-)}$ is, for example,

$$\frac{d^2 Z_i^{(-)}}{dr_*^2} + \sigma^2 Z_i^{(-)} = -\frac{|\Delta|}{r^5}\left[\left(\mu^2+2\right)r - q_j + \frac{4Q_*^2}{r}\right]Z_i^{(-)} \quad \left(i, j = 1,2; i \neq j\right) \text{ and } \left(r_- < r < r_+, +\infty > r_* > -\infty\right), \quad (2.40)$$

where

$$r_* = r + \frac{1}{2\kappa_+}\lg|r_+ - r| - \frac{1}{2\kappa_-}\lg|r - r_-|, \quad (2.41) \qquad \kappa_+ = \frac{r_+ - r_-}{2r_+^2}, \text{ and } \kappa_- = \frac{r_+ - r_-}{2r_-^2}. \quad (2.42)$$

In view of the relation (2.38) between the solutions, belonging to axial and polar perturbations, it will, again, suffice to restrict our consideration to equation (2.40); and for convenience, we shall suppress the distinguishing superscript. An important consequence of the fact that we are now concerned with a short-range one-dimensional potential-well, is that equation (2.40) will allow a finite number of discrete, non-degenerate, bound states:

$$\sigma = \pm i\sigma_j \quad [j=1,2; n=1,2...,m]. \quad (2.43)$$

The boundary conditions we must now impose are

$$\begin{aligned} Z(r_*) &\to A(\sigma)e^{-i\sigma r_*} + B(\sigma)e^{+i\sigma r_*} \qquad (r \to r_- + 0; r_* \to +\infty) \\ &\to e^{-i\sigma r_*} \qquad (r \to r_+ - 0; r_* \to -\infty). \quad (2.44) \end{aligned}$$

The coefficients $A(\sigma)$ and $B(\sigma)$ in equation (2.44) are related to the reflexion and the transmission amplitudes,

$$A(\sigma) = \frac{1}{T^*(\sigma)} = \frac{1}{T_1(-\sigma)}, \quad B(\sigma) = \frac{R^*(\sigma)}{T^*(\sigma)} = \frac{R_1(-\sigma)}{T_1(-\sigma)}, \quad (2.45)$$

so that

$$|A(\sigma)|^2 - |B(\sigma)|^2 = 1. \quad (2.46)$$

With regard the *amplification factors*, $|A(\sigma)|^2$, appropriate for the potential $V_1^{(\pm)}$ and $V_2^{(\pm)}$ for $Q_*^2 = 0.75M^2$, we observe that $|A(\sigma)|^2$, and, therefore, also $|B(\sigma)|^2$, tend to finite limits as $\sigma \to 0$. This fact has its origin in the existence of bound states of zero energy in the potential wells, V_1 and V_2. **Furthermore, we note that for $\sigma = 0.30$, for V_1 we have the value 1.6168, while for V_2 we have the value 1.6286. It is easy note that these values are very near to the $\Phi = \frac{\sqrt{5}+1}{2} = 1{,}61803398...$, thence to the value of the golden ratio.**

In analyzing the radiation arriving at the Cauchy horizon at r_-, we must distinguish the edges EC' and EF in the Penrose diagram. For this reason, we restore the time-dependence, $e^{i\sigma t}$, of the solutions; and remembering that in the interval, $r_- < r < r_+$,

$$u = r_* + t \quad \text{and} \quad v = r_* - t, \quad (2.47)$$

we write, in place of equation (2.44),

$$Z(r_*, t) \to e^{-i\sigma v} + [A(\sigma) - 1]e^{-i\sigma v} + B(\sigma)e^{+i\sigma u}. \quad (2.48)$$

If we now suppose that the flux of radiation emerging from $D'C'$ is $\hat{Z}(v)$, then

$$Z(\sigma) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{Z}(v)e^{i\sigma v}dv. \quad (2.49)$$

This flux disperses in the domain between the two horizons and at the Cauchy horizon it is determined by

$$Z(r_*, t) \to X(v) + Y(u) \quad (v \to \infty; u \to \infty), \quad (2.50)$$

where

$$X(v) = \int_{-\infty}^{+\infty} Z(\sigma)[A(\sigma)-1]e^{-i\sigma v} d\sigma, \quad (2.51)$$

and

$$Y(u) = \int_{-\infty}^{+\infty} Z(\sigma) B(\sigma) e^{i\sigma u} d\sigma. \quad (2.52)$$

Thence, the eq. (2.50) can be rewritten also as follows:

$$Z(r_*, t) \to \int_{-\infty}^{+\infty} Z(\sigma)[A(\sigma)-1]e^{-i\sigma v} d\sigma + \int_{-\infty}^{+\infty} Z(\sigma) B(\sigma) e^{i\sigma u} d\sigma. \qquad (2.52b)$$

However, our interest is not in $X(v)$ or $Y(u)$, per se, but rather in quantities related to them. We are primarily interested in the radiation an observer receives at the instant of his (or her) crossing the Cauchy horizon. To evaluate this quantity, we consider a freely falling observer following a radial geodesic. The four-velocity, **U**, of the observer is given by the following equations

$$\left(\frac{dr}{d\tau}\right)^2 + \frac{\Delta}{r^2}\left(1+\frac{L^2}{r^2}\right) = E^2; \qquad \frac{dt}{d\tau} = E\frac{r^2}{\Delta}; \qquad \text{and} \qquad \frac{d\varphi}{d\tau} = \frac{L}{r^2}, \quad \text{where} \qquad \Delta = r^2 - 2Mr + Q_*^2,$$

for $L = 0$; thus,

$$U^t = \frac{r^2}{\Delta} E, \qquad U^{r_*} = \frac{r^2}{\Delta}\left(E^2 - \frac{\Delta}{r^2}\right)^{1/2}, \qquad \text{and} \qquad U^\theta = U^\varphi = 0, \quad (2.53)$$

where, consistently with the time-like character of the coordinate r in the interval $r_- < r < r_+$, we have chosen the positive square-root in the expression for U^{r_*}. Also, it should be noted that we are allowed to assign negative values for E since the coordinate t is space-like in the same interval. With the prevalent radiation-field expressed in terms of $Z(r_*, t)$, a measure of the flux of radiation, $\mathcal{F}$, received by the freely falling observer is given by

$$\mathcal{F} = U^j Z_{,j} = \frac{r^2}{\Delta}\left[E Z_{,t} + \sqrt{\left(E^2 - \frac{\Delta}{r^2}\right)} Z_{,r_*}\right]. \quad (2.54)$$

We have seen that as we approach the Cauchy horizon (cf.equations (2.50) – (2.52)),

$$Z(r_*, t) \to X(t - r_*) + Y(t + r_*). \quad (2.55)$$

Accordingly,

$$Z_{,t} \to X_{,-v} + Y_{,u} \quad \text{and} \quad Z_{,r_*} \to -X_{,-v} + Y_{,u}; \quad (2.56)$$

and the expression (2.54) for $\mathcal{F}$ becomes

$$\mathcal{F} \to \frac{r^2}{\Delta}\left\{X_{,-v}\left[E - \sqrt{\left(E^2 - \frac{\Delta}{r^2}\right)}\right] + Y_{,u}\left[E + \sqrt{\left(E^2 - \frac{\Delta}{r^2}\right)}\right]\right\}. \quad (2.57)$$

On EF, v remains finite while $u \to \infty$; therefore,

$$r_* \to +t \ , \ u \to 2r_* \to -\frac{1}{\kappa_-}\lg|r - r_-| \quad \text{as} \quad r \to r_- \quad \text{on} \quad EF . \quad (2.58)$$

Also for $E > 0$, the term in $X_{,-v}$ remains finite while the term in $Y_{,u}$ has a divergent factor (namely, $1/\Delta$). Hence,

$$\mathcal{F}_{EF} \to -\frac{2r_-^2}{r_+ - r_-} E Y_{,u} e^{\kappa_- u} \quad (u \to \infty \ \text{on} \ EF). \quad (2.59)$$

On EC', u remains finite while $v \to \infty$; therefore,

$$r_* \to -t \ , \ v \to 2r_* \to -\frac{1}{\kappa_-}\lg|r - r_-| \quad \text{as} \quad r \to r_- \quad \text{on} \quad EC'. \quad (2.60)$$

And for $E < 0$, the term in $Y_{,u}$ remains finite while the term in $X_{,-v}$ has the divergent factor. Hence,

$$\mathcal{F}_{EC'} \to +\frac{2r_-^2}{r_+ - r_-}|E| X_{,-v} e^{\kappa_- v} \quad (v \to \infty \ \text{on} \ EC'). \quad (2.61)$$

We conclude from the equations (2.59) and (2.61) that the divergence, or otherwise, of the received fluxes on the Cauchy horizon, at EF and EC', depend on

$$Y_{,u} = \int_{-\infty}^{+\infty} i\sigma \frac{R_1(-\sigma)}{T_1(-\sigma)} Z(\sigma) e^{i\sigma u} d\sigma; \quad (2.62) \quad \text{and} \quad X_{,-v} = \int_{-\infty}^{+\infty} i\sigma \left[\frac{1}{T_1(-\sigma)} - 1\right] Z(\sigma) e^{-i\sigma v} d\sigma, \quad (2.63)$$

where we have substituted for $A(\sigma)$ and $B(\sigma)$ from equation (2.45). Furthermore, we can rewrite the eq. (2.57) as follows:

$$\mathcal{F} \to \frac{r^2}{\Delta}\left\{\int_{-\infty}^{+\infty} i\sigma \left[\frac{1}{T_1(-\sigma)} - 1\right] Z(\sigma) e^{-i\sigma v} d\sigma \left[E - \sqrt{\left(E^2 - \frac{\Delta}{r^2}\right)}\right] + \int_{-\infty}^{+\infty} i\sigma \frac{R_1(-\sigma)}{T_1(-\sigma)} Z(\sigma) e^{i\sigma u} d\sigma \left[E + \sqrt{\left(E^2 - \frac{\Delta}{r^2}\right)}\right]\right\}$$

(2.63b)

In particular, if we wish to evaluate the infinite integrals, as is naturally suggested, by contour integration, closing the contour appropriately in the upper half-plane and in the lower half-plane, then we need to specify the domains of analyticity of $A(\sigma)$ and $B(\sigma)$, as defined in equations (2.45). Returning then to the definitions of $A(\sigma)$ and $B(\sigma)$, we can write

$$B(\sigma) = \frac{R_1(-\sigma)}{T_1(-\sigma)} = \frac{1}{2i\sigma}[f_2(x,-\sigma), f_1(x,+\sigma)] \quad (2.64)$$

and

$$A(\sigma) = \frac{1}{T_1(-\sigma)} = \frac{1}{2i\sigma}[f_1(x,-\sigma), f_2(x,-\sigma)], \quad (2.65)$$

where for convenience, we have written x in place of r_* and $f_1(x,\pm\sigma)$, and $f_2(x,+\sigma)$ are solutions of the one-dimensional wave equations which satisfy the boundary conditions

$$f_1(x,\pm\sigma) \to e^{\mp i\sigma x} \quad x \to +\infty; \qquad \text{and} \qquad f_2(x,\pm\sigma) \to e^{\pm i\sigma x} \quad x \to -\infty. \quad (2.66)$$

Also $f_2(x,-\sigma)$ satisfies the integral equation

$$f_2(x,-\sigma) = e^{-i\sigma x} + \int_{-\infty}^{x} \frac{\sin\sigma(x-x')}{\sigma} V(x') f_2(x',-\sigma) dx'. \quad (2.67)$$

The corresponding integral equation satisfied by $f_1(x,\pm\sigma)$ is

$$f_1(x,\mp\sigma) = e^{\pm i\sigma x} - \int_{x}^{\infty} \frac{\sin\sigma(x-x')}{\sigma} V(x') f_1(x',\mp\sigma) dx'. \quad (2.68)$$

Adapting a more general investigation of Hartle and Wilkins to the simpler circumstances of our present problem, we can determine the domains of analyticity of the functions, $f_2(x,-\sigma)$ and $f_1(x,\pm\sigma)$, in the complex σ-plane, by solving the Volterra integral-equations (2.67) and (2.68) by successive iterations. Thus, considering equation (2.67), we may express its solution as a series in the form

$$f_2(x,-\sigma) = e^{-i\sigma x} + \sum_{n=1}^{\infty} f_2^{(n)}(x,-\sigma), \quad (2.69)$$

where

$$f_2^{(n)}(x,-\sigma) = \int_{-\infty}^{x} dx_1 \frac{\sin\sigma(x-x_1)}{\sigma} V(x_1) f_2^{(n-1)}(x_1,-\sigma). \quad (2.70)$$

By this last recurrence relation,

$$f_2^{(n)}(x,-\sigma) = \int_{-\infty}^{x_0} dx_1 \int_{-\infty}^{x_1} dx_2 \ldots \int_{-\infty}^{x_{n-1}} dx_n \prod_{i=1}^{n} \frac{\sin\sigma(x_{i-1}-x_i)}{\sigma} V(x_i) e^{-i\sigma x_n} \quad (2.71)$$

where $x_0 = x$; or, after some rearrangement,

$$f_2^{(n)}(x,-\sigma) = \frac{e^{-i\sigma x}}{(2i\sigma)^n} \int_{-\infty}^{x_0} dx_1 \ldots \int_{-\infty}^{x_{n-1}} dx_n \prod_{i=1}^{n} \left\{\left[e^{2i\sigma(x_{i-1}-x_i)} - 1\right] V(x_i)\right\}. \quad (2.72)$$

Thence, the eq. (2.69) can be rewritten also as follows:

$$f_2(x,-\sigma) = e^{-i\sigma x} + \sum_{n=1}^{\infty} \frac{e^{-i\sigma x}}{(2i\sigma)^n} \int_{-\infty}^{x_0} dx_1 \ldots \int_{-\infty}^{x_{n-1}} dx_n \prod_{i=1}^{n} \left\{\left[e^{2i\sigma(x_{i-1}-x_i)} - 1\right] V(x_i)\right\}. \quad (2.72b)$$

Since

$$V(x) \to \text{ constant } e^{2\kappa_+ x} \quad (x \to -\infty), \quad (2.73)$$

it is manifest that each of the multiplicands in (2.72) tends to zero, exponentially, for $x \to -\infty$ for all

$$\operatorname{Im}\sigma > -\kappa_+ . \quad (2.74)$$

In view of the asymptotic behaviour (2.73) for $V(x)$, we may, compatible with this behaviour, expect a representation of $V(x)$ for $x<0$, in the manner of a Laplace transform, by

$$V(x) = \int_{2\kappa_+}^{\infty} d\mu \mathcal{V}(\mu) e^{\mu x} , \quad (2.75)$$

where $\mathcal{V}(\mu)$ includes δ-functions at various locations, i.e., $\mathcal{V}(\mu)$ is a distribution in the technical sense. With the foregoing representation for $V(x)$, the first iterate, $f_2^{(1)}(x,-\sigma)$, of the solution for $f_2(x,-\sigma)$, becomes

$$f_2^{(1)}(x,-\sigma) = \frac{e^{-i\sigma x}}{2i\sigma}\int_{-\infty}^{x} dx_1 \left[e^{2i\sigma(x-x_1)} - 1\right]\int_{2\kappa_+}^{\infty} d\mu \mathcal{V}(\mu) e^{\mu x_1} ; \quad (2.76)$$

or, inverting the order of the integrations, we have

$$f_2^{(1)}(x,-\sigma) = \frac{e^{-i\sigma x}}{2i\sigma}\int_{2\kappa_+}^{\infty} d\mu \mathcal{V}(\mu) e^{\mu x}\int_{-\infty}^{x} dx_1 \left[e^{2i\sigma(x-x_1)} - 1\right] e^{\mu(x_1 - x)} . \quad (2.77)$$

After effecting the integration over x_1, we are left with

$$f_2^{(1)}(x,-\sigma) = e^{-i\sigma x}\int_{2\kappa_+}^{\infty} d\mu \frac{\mathcal{V}(\mu)}{\mu(\mu - 2i\sigma)} e^{\mu x} . \quad (2.78)$$

From this last expression, it is evident that $f_2^{(1)}(x,-\sigma)$ has singularities along the negative imaginary axis beginning at $\operatorname{Im}\sigma = -\kappa_+$. Thence, the eq. (2.76) can be rewritten also as follows:

$$f_2^{(1)}(x,-\sigma) = \frac{e^{-i\sigma x}}{2i\sigma}\int_{-\infty}^{x} dx_1 \left[e^{2i\sigma(x-x_1)} - 1\right]\int_{2\kappa_+}^{\infty} d\mu \mathcal{V}(\mu) e^{\mu x_1} = e^{-i\sigma x}\int_{2\kappa_+}^{\infty} d\mu \frac{\mathcal{V}(\mu)}{\mu(\mu - 2i\sigma)} e^{\mu x} . \quad (2.78b)$$

Entropy of strings and black holes: Schwarzschild geometry in D = d + 1 dimensions **[3]**

The black hole metric found by solving Einstein's equation in D dimensions is given by

$$d\tau^2 = \left(1 - \frac{R_S^{D-3}}{r^{D-3}}\right)dt^2 - \left(1 - \frac{R_S^{D-3}}{r^{D-3}}\right)^{-1} dr^2 - r^2 d\omega_{D-2} . \quad (2.79)$$

The horizon is defined by

$$R_S = \left[\frac{16\pi(D-3)GM}{\Omega_{D-2}(D-2)}\right]^{\frac{1}{D-3}} \quad (2.80)$$

and its D-2 dimensional “area” is given by

$$A = R_S^{D-2} \int d\Omega_{D-2} = R_S^{D-2} \Omega_{D-2} . \quad (2.81)$$

Furthermore, the entropy is given by

$$S = \frac{A}{4G} = \frac{(2GM)^{\frac{D-2}{D-3}} \Omega_{D-2}}{4G} . \quad (2.82)$$

The entropy in equation (2.82) is what is required by black hole thermodynamics.
To extend a static spherically symmetric geometry to D = d + 1 dimensions, the metric can be assumed to be of the form

$$ds^2 = -e^{2\Phi} dt^2 + e^{2\Delta} dr^2 + r^2 \left(d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + ... + \sin^2 \theta_1 ... \sin^2 \theta_{d-2} d\theta_{d-1}^2 \right). \quad (2.83)$$

Using orthonormal coordinates, the $G_{\hat{t}}^{\hat{t}}$ component of the Einstein tensor can be directly calculated to be of the form

$$G_{\hat{t}}^{\hat{t}} = -\left[(D-2)\Delta' \frac{e^{-2\Delta}}{r} + \frac{(D-2)(D-3)}{2r^2} \left(1 - e^{-2\Delta} \right) \right]. \quad (2.84)$$

Form Einstein’s equation for ideal pressureless matter, $G_{\hat{t}\hat{t}} = \kappa\rho$. This means

$$G_{\hat{t}}^{\hat{t}} = -\frac{(D-2)}{2r^{D-2}} \frac{d}{dr} \left[r^{D-3} \left(1 - e^{-2\Delta} \right) \right] = -\kappa\rho \quad (2.85)$$

which can be solved to give

$$\left(1 - e^{-2\Delta} \right) r^{D-3} = \frac{2\kappa}{D-2} \int_0^r \rho(r') r'^{D-2} dr' = \frac{2\kappa}{(D-2)} \frac{M}{\Omega_{D-2}} \quad (2.86)$$

where the solid angle is given by $\Omega_{D-2} = \frac{2\pi^{(D-1)/2}}{\Gamma((D-1)/2)}$. The $G_{\hat{r}}^{\hat{r}}$ component of the Einstein tensor satisfies

$$G_{\hat{r}}^{\hat{r}} = -\left[-(D-2)\Phi' \frac{e^{-2\Delta}}{r} + \frac{(D-2)(D-3)}{2r^2} \left(1 - e^{-2\Delta} \right) \right] = -\left[-(D-2)(\Phi' + \Delta') \frac{e^{-2\Delta}}{r} + \kappa\rho \right]. \quad (2.87)$$

For pressureless matter in the exterior region $(\rho = 0 = P)$, we can immediately conclude that $\Phi = -\Delta$. Defining the Schwarzschild radius $R_S^{D-3} = \frac{2\kappa M}{(D-2)\Omega_{D-2}}$ we obtain the form of the metric

$$e^{-2\Delta} = 1 - \left(\frac{R_S}{r} \right)^{D-3} = e^{2\Phi} . \quad (2.88)$$

If we write $F(r) \equiv e^{2\Phi}$, a useful shortcut for calculating the solution to Einstein's equation (2.85) is to note its equivalence to the Newtonian Poisson equation in the exterior region

$$\nabla^2 F(r) = -\kappa\rho, \qquad F(r) = 1 + 2\phi_{Newton}. \quad (2.89)$$

The Hawking temperature can be calculated by determining the dimensional factor between the Rindler time and Schwarzschild time. Near the horizon, the proper distance to the horizon is given by

$$\rho = \frac{2R_S}{(D-3)}\sqrt{\left(\frac{r}{R_S}\right)^{D-3} - 1} \quad (2.90)$$

which gives the relation between Rindler time/temperature units and Schwarzschild time/temperature units

$$d\omega = \frac{(D-3)}{2R_S}dt. \quad (2.91)$$

Thus, the Hawking temperature of the black hole is given by

$$T_H = \frac{1}{2\pi}\frac{(D-3)}{2R_S}. \quad (2.92)$$

Using the first law of thermodynamics, the entropy can be directly calculated to be of the form

$$S = \frac{2\pi(D-3)A}{\kappa}. \quad (2.93)$$

Substituting the form $\kappa = 8\pi(D-3)G$ for the gravitational coupling gives the previous results in D-dimensions (see eq. (2.82)). Furthermore, if we substitute $\kappa = 8\pi(D-3)G$ in the eq. (2.86), we obtain the following expression:

$$\left(1 - e^{-2\Delta}\right)r^{D-3} = \frac{2\kappa}{D-2}\int_0^r \rho(r')r'^{D-2}\,dr' = \frac{2\cdot 8\pi(D-3)G}{(D-2)}\frac{M}{\Omega_{D-2}}. \quad (2.93b)$$

We note that this expression can be related by the number 8, with the "modes" that correspond to the physical vibrations of a superstring by the following Ramanujan function:

$$8 = \frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}.$$

Thence, we obtain the following mathematical connection:

$$\left(1-e^{-2\Delta}\right)r^{D-3}=\frac{2\kappa}{D-2}\int_0^r \rho(r')r'^{D-2}\,dr'=\frac{2\cdot 8\pi(D-3)G}{(D-2)}\frac{M}{\Omega_{D-2}}\Rightarrow$$

$$\Rightarrow\frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (2.93c)$$

The quantization of the string defines a 1 + 1 dimensional quantum field theory in which the (D – 2) transverse coordinates $X^i(\sigma)$ play the role of free scalar fields. The spatial coordinate of this field theory is σ_1, and it runs from 0 to 2π.
The entropy and energy of such a quantum field theory can be calculated by standard means. The leading contribution for large energy is (setting the string length $\ell_S=1$)

$$E=\pi T^2(D-2),\qquad S=2\pi T(D-2). \quad (2.94)$$

Using $E=\frac{m^2}{2}$ and eliminating the temperature yields $S=\sqrt{2(D-2)\pi}m$ or, restoring the units, i.e. the string length ℓ_S:

$$S=\sqrt{2(D-2)\pi}m\ell_S. \quad (2.95)$$

Subleading corrections can also be calculated to give

$$S=\sqrt{2(D-2)\pi}m\ell_S-c\log(m\ell_S) \quad (2.96)$$

where c is a positive constant. The entropy is the log of the density of states. Therefore the number of states with mass m is

$$N_m=\left(\frac{1}{m\ell_S}\right)^c\exp\left(\sqrt{2\pi(D-2)}m\ell_S\right). \quad (2.97)$$

The formula (2.97) is correct for the simplest bosonic string, but similar formulae exist for the various versions of superstring theory.
Now let us compare the entropy of the single string with that of n strings, each carrying mass $\frac{m}{n}$. Call this entropy $S_n(m)$. Then

$$S_n(m)=nS(m/n) \quad (2.98)$$

or

$$S_n(m) = \sqrt{2(D-2)}\pi m\ell_S - nc\log\left(\frac{m\ell_S}{n}\right). \quad (2.99)$$

Obviously for large n the single string is favored. For a given total mass, the statistically most likely state in free string theory is a single excited string. Thus it is expected that when the string coupled goes to zero, most of the black hole states will evolve into a single excited string.

These observations allow us to estimate the entropy of a black hole. The assumptions are the following:

- A black hole evolves into a single string in the limit $g \to 0$
- Adiabatically sending g to zero is an isentropic process; the entropy of the final string is the same as that of the black hole
- The entropy of a highly excited string of mass m is of order $S \approx m\ell_S$ (2.100)
- At some points as $g \to 0$ the black hole will male a transition to a string. The point at which this happens is when the horizon radius is of the order of the string scale.

The string and the Planck length scales are related by

$$g^2\ell_S^{D-2} = \ell_p^{D-2}. \quad (2.101)$$

At some value of the coupling that depends on the mass of the black hole, the string length will exceed the Schwarzschild radius of the black hole. This is the point at which the transition from black hole to string occurs.

Let us begin with a black hole of mass M_0 in a string theory with coupling constant g_0. The Schwarzschild radius is of order

$$R_S \approx (M_0 G)^{\frac{1}{D-3}}, \quad (2.102)$$

and using

$$G \approx g^2\ell_S^{D-2} \quad (2.103)$$

we find

$$\frac{R_S}{\ell_S} \approx \left(\ell_S M_0 g_0^2\right)^{\frac{1}{D-3}}. \quad (2.104)$$

Thus for fixed g_0 if the mass is large enough, the horizon radius will be much bigger than ℓ_S. Now start to decrease g. In general the mass will vary during an adiabatic process. Let us call the g-dependent mass $M(g)$. Note

$$M(g_0) = M_0. \quad (2.105)$$

The entropy of a Schwarzschild black hole (in any dimension) is a function of the dimensionless variable $M\ell_P$. Thus, as long as the system remains a black hole,

$$M(g)\ell_P = constant. \quad (2.106)$$

Since $\ell_P \approx \ell_S g^{\frac{2}{D-2}}$ we can write equation (2.106) as

$$M(g)=M_0\left(\frac{g_0^2}{g^2}\right)^{\frac{1}{D-2}}. \quad (2.107)$$

Now as $g\to 0$ the ratio of the g-dependent horizon radius to the string scale decreases. From equation (2.80) it becomes of order unity at

$$M(g)\ell_P^{D-2}\approx \ell_S^{D-3} \quad (2.108)$$

which can be written

$$M(g)\ell_S\approx\frac{1}{g^2}. \quad (2.109)$$

Combining equations (2.107) and (2.109) we find

$$M(g)\ell_S\approx M_0^{\frac{D-2}{D-3}}G_0^{\frac{1}{D-3}}. \quad (2.110)$$

As we continue to decrease the coupling, the weakly coupled string mass will not change significantly. Thus we see that a black hole of mass M_0 will evolve into a free string satisfying equation (2.110). But now we can compute the entropy of the free string. From equation (2.100) we find

$$S\approx M_0^{\frac{D-2}{D-3}}G_0^{\frac{1}{D-3}}. \quad (2.111)$$

This is a very pleasing result in that it agrees with the Bekenstein-Hawking entropy in equation (2.82). In this calculation the entropy is calculated as the microscopic entropy of fundamental strings.

Appendix A

FINAL TABLES

In this tables we have the various fractional powers of Phi $\Phi=\frac{\sqrt{5}+1}{2}=1{,}61803399...$ that we have obtained by the following expression: $\sum\left(\Phi^{n/7}\right)\times$ *fractions or numbers of the first line*; for n included in the following numerical interval: [– 113; + 38] . For example:

$$2{,}6666667=\left[(\Phi)^{7/7}+(\Phi)^{-14/7}\right]\times\frac{4}{3}=(1{,}61803399+0{,}38196601)\times\frac{4}{3}=2\times\frac{4}{3}=2{,}6666667$$

	*9/4	*4/9	*1/3	*1/2	*2/3	*3/4
a0	0,20288237	0,04007553	0,03005665	0,04508497	0,06011330	0,06762746
b0	0,21952671	0,04336330	0,03252248	0,04878371	0,06504495	0,07317557
h0	0,23248251	0,04592247	0,03444185	0,05166278	0,06888371	0,07749417
c1	0,25077641	0,04953608	0,03715206	0,05572809	0,07430412	0,08359214
d1	0,26640125	0,05262247	0,03946685	0,05920028	0,07893370	0,08880042
es1	0,28736419	0,05676330	0,04257247	0,06385871	0,08514495	0,09578806
e1	0,30997668	0,06122996	0,04592247	0,06888371	0,09184494	0,10332556
f1	0,32827058	0,06484357	0,04863268	0,07294902	0,09726536	0,10942353
fis1	0,35520167	0,07016329	0,05262247	0,07893370	0,10524494	0,11840056
gis1	0,37616461	0,07430412	0,05572809	0,08359214	0,11145618	0,12538820
a1	0,40576475	0,08015106	0,06011330	0,09016994	0,12022659	0,13525492
b1	0,43104628	0,08514495	0,06385871	0,09578806	0,12771742	0,14368209
h1	0,46496503	0,09184494	0,06888371	0,10332556	0,13776741	0,15498834
c2	0,50155281	0,09907216	0,07430412	0,11145618	0,14860824	0,16718427
d2	0,53115295	0,10491910	0,07868933	0,11803399	0,15737865	0,17705098
es2	0,57472838	0,11352659	0,08514495	0,12771742	0,17028989	0,19157613
e2	0,60864712	0,12022659	0,09016994	0,13525492	0,18033989	0,20288237
f2	0,65654115	0,12968714	0,09726536	0,14589803	0,19453071	0,21884705
fis2	0,69744754	0,13776741	0,10332556	0,15498834	0,20665112	0,23248251
gis2	0,75232922	0,14860824	0,11145618	0,16718427	0,22291236	0,25077641
a2	0,81152949	0,16030212	0,12022659	0,18033989	0,24045318	0,27050983
b2	0,85942353	0,16976267	0,12732200	0,19098301	0,25464401	0,28647451
h2	0,92993005	0,18368989	0,13776741	0,20665112	0,27553483	0,30997668
c3	0,98481173	0,19453071	0,14589803	0,21884705	0,29179607	0,32827058
d3	1,06230590	0,20983820	0,15737865	0,23606798	0,31475730	0,35410197
es3	1,12849382	0,22291236	0,16718427	0,25077641	0,33436854	0,37616461
e3	1,21729424	0,24045318	0,18033989	0,27050983	0,36067977	0,40576475
f3	1,31308230	0,25937428	0,19453071	0,29179607	0,38906142	0,43769410
fis3	1,39057647	0,27468177	0,20601133	0,30901699	0,41202266	0,46352549
gis3	1,50465843	0,29721648	0,22291236	0,33436854	0,44582472	0,50155281
a3	1,59345885	0,31475730	0,23606798	0,35410197	0,47213595	0,53115295
b3	1,71884705	0,33952534	0,25464401	0,38196601	0,50928802	0,57294902
h3	1,82594136	0,36067977	0,27050983	0,40576475	0,54101966	0,60864712
c4	1,96962346	0,38906142	0,29179607	0,43769410	0,58359214	0,65654115
d4	2,12461180	0,41967640	0,31475730	0,47213595	0,62951461	0,70820393
es4	2,25000000	0,44444444	0,33333333	0,50000000	0,66666667	0,75000000
e4	2,43458848	0,48090637	0,36067977	0,54101966	0,72135955	0,81152949
f4	2,57827058	0,50928802	0,38196601	0,57294902	0,76393202	0,85942353
fis4	2,78115295	0,54936355	0,41202266	0,61803399	0,82404532	0,92705098
gis4	2,95443518	0,58359214	0,43769410	0,65654115	0,87538820	0,98481173
a4	3,18691770	0,62951461	0,47213595	0,70820393	0,94427191	1,06230590
b4	3,43769410	0,67905069	0,50928802	0,76393202	1,01857603	1,14589803
h4	3,64057647	0,71912622	0,53934466	0,80901699	1,07868933	1,21352549
c5	3,93924691	0,77812285	0,58359214	0,87538820	1,16718427	1,31308230
d5	4,17172942	0,82404532	0,61803399	0,92705098	1,23606798	1,39057647

es5	4,50000000	0,88888889	0,66666667	1,00000000	1,33333333	1,50000000
e5	4,78037654	0,94427191	0,70820393	1,06230590	1,41640786	1,59345885
f5	5,15654115	1,01857603	0,76393202	1,14589803	1,52786405	1,71884705
fis5	5,56230590	1,09872709	0,82404532	1,23606798	1,64809064	1,85410197
gis5	5,89057647	1,16357066	0,87267800	1,30901699	1,74535599	1,96352549
a5	6,37383539	1,25902921	0,94427191	1,41640786	1,88854382	2,12461180
b5	6,75000000	1,33333333	1,00000000	1,50000000	2,00000000	2,25000000
h5	7,28115295	1,43825243	1,07868933	1,61803399	2,15737865	2,42705098
c6	7,73481173	1,52786405	1,14589803	1,71884705	2,29179607	2,57827058
d6	8,34345885	1,64809064	1,23606798	1,85410197	2,47213595	2,78115295
es6	9,00000000	1,77777778	1,33333333	2,00000000	2,66666667	3,00000000
e6	9,53115295	1,88269688	1,41202266	2,11803399	2,82404532	3,17705098
f6	10,31308230	2,03715206	1,52786405	2,29179607	3,05572809	3,43769410
fis6	10,92172942	2,15737865	1,61803399	2,42705098	3,23606798	3,64057647
gis 6	11,78115295	2,32714132	1,74535599	2,61803399	3,49071198	3,92705098
a6	12,51518827	2,47213595	1,85410197	2,78115295	3,70820393	4,17172942
b6	13,50000000	2,66666667	2,00000000	3,00000000	4,00000000	4,50000000
h6	14,56230590	2,87650487	2,15737865	3,23606798	4,31475730	4,85410197
c7	15,42172942	3,04626754	2,28470066	3,42705098	4,56940131	5,14057647
d7	16,68691770	3,29618127	2,47213595	3,70820393	4,94427191	5,56230590
es7	17,67172942	3,49071198	2,61803399	3,92705098	5,23606798	5,89057647
e7	19,06230590	3,76539376	2,82404532	4,23606798	5,64809064	6,35410197
f7	20,25000000	4,00000000	3,00000000	4,50000000	6,00000000	6,75000000
fis7	21,84345885	4,31475730	3,23606798	4,85410197	6,47213595	7,28115295
gis7	23,56230590	4,65428265	3,49071198	5,23606798	6,98142397	7,85410197
a7	24,95288237	4,92896442	3,69672331	5,54508497	7,39344663	8,31762746
b7	27,00000000	5,33333333	4,00000000	6,00000000	8,00000000	9,00000000
h7	28,59345885	5,64809064	4,23606798	6,35410197	8,47213595	9,53115295
c8	30,84345885	6,09253508	4,56940131	6,85410197	9,13880262	10,28115295

*4/3	*3/2	*2	*3	Phi
0,12022659	0,13525492	0,18033989	0,27050983	0,14589803
0,13008990	0,14635114	0,19513485	0,29270228	0,15786741
0,13776741	0,15498834	0,20665112	0,30997668	0,16718427
0,14860824	0,16718427	0,22291236	0,33436854	0,18033989
0,15786741	0,17760084	0,23680111	0,35520167	0,19157613
0,17028989	0,19157613	0,25543484	0,38315225	0,20665112
0,18368989	0,20665112	0,27553483	0,41330224	0,22291236
0,19453071	0,21884705	0,29179607	0,43769410	0,23606798
0,21048988	0,23680111	0,31573482	0,47360223	0,25543484

0,22291236	0,25077641	0,33436854	0,50155281	0,27050983
0,24045318	0,27050983	0,36067977	0,54101966	0,29179607
0,25543484	0,28736419	0,38315225	0,57472838	0,30997668
0,27553483	0,30997668	0,41330224	0,61995337	0,33436854
0,29721648	0,33436854	0,44582472	0,66873708	0,36067977
0,31475730	0,35410197	0,47213595	0,70820393	0,38196601
0,34057978	0,38315225	0,51086967	0,76630451	0,41330224
0,36067977	0,40576475	0,54101966	0,81152949	0,43769410
0,38906142	0,43769410	0,58359214	0,87538820	0,47213595
0,41330224	0,46496503	0,61995337	0,92993005	0,50155281
0,44582472	0,50155281	0,66873708	1,00310562	0,54101966
0,48090637	0,54101966	0,72135955	1,08203932	0,58359214
0,50928802	0,57294902	0,76393202	1,14589803	0,61803399
0,55106966	0,61995337	0,82660449	1,23990673	0,66873708
0,58359214	0,65654115	0,87538820	1,31308230	0,70820393
0,62951461	0,70820393	0,94427191	1,41640786	0,76393202
0,66873708	0,75232922	1,00310562	1,50465843	0,81152949
0,72135955	0,81152949	1,08203932	1,62305899	0,87538820
0,77812285	0,87538820	1,16718427	1,75077641	0,94427191
0,82404532	0,92705098	1,23606798	1,85410197	1,00000000
0,89164944	1,00310562	1,33747416	2,00621124	1,08203932
0,94427191	1,06230590	1,41640786	2,12461180	1,14589803
1,01857603	1,14589803	1,52786405	2,29179607	1,23606798
1,08203932	1,21729424	1,62305899	2,43458848	1,31308230
1,16718427	1,31308230	1,75077641	2,62616461	1,41640786
1,25902921	1,41640786	1,88854382	2,83281573	1,52786405
1,33333333	1,50000000	2,00000000	3,00000000	1,61803399
1,44271910	1,62305899	2,16407865	3,24611797	1,75077641
1,52786405	1,71884705	2,29179607	3,43769410	1,85410197
1,64809064	1,85410197	2,47213595	3,70820393	2,00000000
1,75077641	1,96962346	2,62616461	3,93924691	2,12461180
1,88854382	2,12461180	2,83281573	4,24922359	2,29179607
2,03715206	2,29179607	3,05572809	4,58359214	2,47213595
2,15737865	2,42705098	3,23606798	4,85410197	2,61803399
2,33436854	2,62616461	3,50155281	5,25232922	2,83281573
2,47213595	2,78115295	3,70820393	5,56230590	3,00000000
2,66666667	3,00000000	4,00000000	6,00000000	3,23606798
2,83281573	3,18691770	4,24922359	6,37383539	3,43769410
3,05572809	3,43769410	4,58359214	6,87538820	3,70820393
3,29618127	3,70820393	4,94427191	7,41640786	4,00000000
3,49071198	3,92705098	5,23606798	7,85410197	4,23606798
3,77708764	4,24922359	5,66563146	8,49844719	4,58359214
4,00000000	4,50000000	6,00000000	9,00000000	4,85410197
4,31475730	4,85410197	6,47213595	9,70820393	5,23606798
4,58359214	5,15654115	6,87538820	10,31308230	5,56230590
4,94427191	5,56230590	7,41640786	11,12461180	6,00000000
5,33333333	6,00000000	8,00000000	12,00000000	6,47213595
5,64809064	6,35410197	8,47213595	12,70820393	6,85410197
6,11145618	6,87538820	9,16718427	13,75077641	7,41640786
6,47213595	7,28115295	9,70820393	14,56230590	7,85410197
6,98142397	7,85410197	10,47213595	15,70820393	8,47213595
7,41640786	8,34345885	11,12461180	16,68691770	9,00000000
8,00000000	9,00000000	12,00000000	18,00000000	9,70820393

8,62951461	9,70820393	12,94427191	19,41640786	10,47213595
9,13880262	10,28115295	13,70820393	20,56230590	11,09016994
9,88854382	11,12461180	14,83281573	22,24922359	12,00000000
10,47213595	11,78115295	15,70820393	23,56230590	12,70820393
11,29618127	12,70820393	16,94427191	25,41640786	13,70820393
12,00000000	13,50000000	18,00000000	27,00000000	14,56230590
12,94427191	14,56230590	19,41640786	29,12461180	15,70820393
13,96284794	15,70820393	20,94427191	31,41640786	16,94427191
14,78689326	16,63525492	22,18033989	33,27050983	17,94427191
16,00000000	18,00000000	24,00000000	36,00000000	19,41640786
16,94427191	19,06230590	25,41640786	38,12461180	20,56230590
18,27760524	20,56230590	27,41640786	41,12461180	22,18033989

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