On some mathematical connections between the Cyclic Universe, Inflationary Universe, p-adic Inflation, p-adic cosmology and various sectors of Number Theory

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Abstract

This paper is a review, a thesis, of some interesting results that has been obtained in various researches concerning the "brane collisions in string and M-theory" (Cyclic Universe), p-adic inflation and p-adic cosmology.
In the **Section 1** we have described some equations concerning cosmic evolution in a Cyclic Universe. In the **Section 2**, we have described some equations concerning the cosmological perturbations in a Big Crunch/Big Bang space-time, the M-theory model of a Big Crunch/Big Bang transition and some equations concerning the solution of a braneworld Big Crunch/Big Bang Cosmology. In the **Section 3**, we have described some equations concerning the generating Ekpyrotic curvature perturbations before the Big Bang, some equations concerning the effective five-dimensional theory of the strongly coupled heterotic string as a gauged version of N = 1 five-dimensional supergravity with four-dimensional boundaries, and some equations concerning the colliding branes and the origin of the Hot Big Bang. In the **Section 4**, we have described some equations regarding the "null energy condition" violation concerning the inflationary models and some equations concerning the evolution to a smooth universe in an ekpyrotic contracting phase with $w > 1$. In the **Section 5**, we have described some equations concerning the approximate inflationary solutions rolling away from the unstable maximum of p-adic string theory. In the **Section 6**, we have described various equations concerning the p-adic minisuperspace model, zeta strings, zeta nonlocal scalar fields and p-adic and adelic quantum cosmology. In the **Section 7**, we have showed various and interesting mathematical connections between some equations concerning the p-adic Inflation, the p-adic quantum cosmology, the zeta strings and the brane collisions in string and M-theory. Furthermore, in each section, we have showed the mathematical connections with various sectors of Number Theory, principally the Ramanujan's modular equations, the Aurea Ratio and the Fibonacci's numbers.

Dedicated to the memory of Professor Anatolii Alexeevich Karatsuba (1937-2008), mathematical genius, whose original researches are for me always source of new great inspirations…

1. On some equations concerning cosmic evolution in a Cyclic Universe. [1] [16]

The action for a scalar field coupled to gravity and a set of fluids ρ_i in a homogeneous, flat Universe, with line element $ds^2 = a^2(\tau)\left(-N^2 d\tau^2 + d\vec{x}^2\right)$ is

$$S = \int d^3x d\tau \left[N^{-1}\left(-3a'^2 + \frac{1}{2}a^2\phi'^2 \right) - N\left((a\beta)^4 \Sigma_i \rho_i + a^4 V(\phi) \right) \right]. \quad (1.1)$$

We use τ to represent conformal time and primes to represent derivatives with respect to τ. N is the lapse function. The background solution for the scalar field is denoted $\phi(\tau)$, and $V(\phi)$ is the scalar potential.

The equations of motion for gravity, the matter and scalar field ϕ are straightforwardly derived by varying (1.1) with respect to a, N and ϕ, after which N may be set equal to unity. Expressed in terms of proper time t, the Einstein equations are

$$H^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V + \beta^4 \rho_R + \beta^4 \rho_M \right), \quad (1.2)$$

$$\frac{\ddot{a}}{a} = -\frac{8\pi G}{3}\left(\dot{\phi}^2 - V + \beta^4 \rho_R + \frac{1}{2}\beta^4 \rho_M \right), \quad (1.3)$$

where a dot is a proper time derivative.

With regard the trajectory in the (a_0, a_1)-plane, the Friedmann constraint reads

$$a_0'^2 - a_1'^2 = \frac{4}{3}\left[(a\beta)^4 \rho + \frac{1}{16}\left(a_0^2 - a_1^2\right)^2 V(\phi_0) \right]. \quad (1.3b)$$

Now we solve the equations of motion immediately before and after the bounce.

Before the bounce there is a little radiation present since it has been exponentially diluted in the preceding quintessence-dominated accelerating phase. Furthermore, the potential $V(\phi)$ becomes negligible as ϕ runs off to minus infinity. The Friedmann constraint reads $\left(\frac{a'}{a}\right)^2 = \frac{1}{6}\phi'^2$, and the scalar field equation, $\left(a^2\phi'\right)' = 0$, where primes denote conformal time derivatives. The general solution is

$$\phi = \sqrt{\frac{3}{2}} \ln\left(AH_5(in)\tau\right), \quad a = Ae^{\phi/\sqrt{6}} = A\sqrt{AH_5(in)\tau},$$

$$a_0 = A\left(\lambda + \lambda^{-1}AH_5(in)\tau\right), \quad a_1 = A\left(\lambda - \lambda^{-1}AH_5(in)\tau\right), \quad (1.4)$$

where $\lambda \equiv e^{\phi_\infty / \sqrt{6}}$. We choose $\tau = 0$ to be the time when a vanishes so that $\tau < 0$ before collision. A is an integration constant which could be set to unity by rescaling space-time coordinates but it is convenient not to do so. The Hubble constants as defined in terms of the brane scale factors are a'_0 / a_0^2 and a'_1 / a_1^2 which at $\tau = 0$ take the values $+\lambda^{-3} H_5(in)$ and $-\lambda^{-3} H_5(in)$ respectively.

Re-expressing the scalar field as a function of proper time $t = \int a d\tau$, we obtain

$$\phi = \sqrt{\frac{2}{3}} \ln\left(\frac{3}{2} H_5(in) t\right). \quad (1.5)$$

The integration constant $H_5(in) < 0$ has a natural physical interpretation as a measure of the contraction rate of the extra-dimension. We remember that when the brane separation is small, one can use the usual formula for Kaluza-Klein theory,

$$ds_5^2 = e^{-\sqrt{\frac{2}{3}}\phi} ds_4^2 + e^{2\sqrt{\frac{2}{3}}\phi} dy^2, \quad (1.5b)$$

where ds_4^2 is the four-dimensional line element, y is the fifth spatial coordinate which runs from zero to L, and L is a parameter with the dimension of length. Thence, we have that:

$$H_5 \equiv \frac{dL_5}{L dt_5} \equiv \frac{d\left(e^{\sqrt{\frac{2}{3}}\phi}\right)}{dt_5} = \sqrt{\frac{2}{3}} \dot{\phi} e^{\sqrt{\frac{3}{2}}\phi}, \quad (1.6)$$

where $L_5 \equiv L e^{\sqrt{\frac{2}{3}}\phi}$ is the proper length of the extra dimension, L is a parameter with dimensions of length, and t_5 is the proper time in the five-dimensional metric,

$$dt_5 \equiv a e^{-\sqrt{\frac{1}{6}}\phi} d\tau = e^{-\sqrt{\frac{1}{6}}\phi} dt, \quad (1.7)$$

with t being FRW proper time. Notice that a shift ϕ_∞ can always be compensated for by a rescaling of L. As the extra dimension shrinks to zero, H_5 tends to a constant, $H_5(in)$.

Immediately after the bounce, scalar kinetic energy dominates and H_5 remains nearly constant. The kinetic energy of the scalar field scales as a^{-6} and radiation scales as a^{-4}, so the former dominates at small a. It is convenient to re-scale a so that it is unity at scalar kinetic energy-radiation equality, t_r, and denote the corresponding Hubble constant H_r. The Friedmann constraint in eq. (1.3b) then reads

$$(a')^2 = \frac{1}{2} H_r^2 \left(1 + a^{-2}\right), \quad (1.8)$$

and the solution is

$$\phi = \sqrt{\frac{3}{2}} \ln\left(\frac{2^{\frac{5}{3}} \tau H_5^{\frac{2}{3}}(out) H_r^{\frac{1}{3}}}{\left(H_r \tau + 2^{\frac{3}{2}} \right)} \right), \qquad a = \sqrt{\frac{1}{2} H_r^2 \tau^2 + \sqrt{2} H_r \tau} \ . \quad (1.9)$$

The brane scale factors are

$$a_0 \equiv a\left(\lambda^{-1} e^{\phi/\sqrt{6}} + \lambda e^{-\phi/\sqrt{6}}\right) = A\left[\lambda \left(1 + \frac{H_r \tau}{2^{\frac{3}{2}}} \right) + \lambda^{-1} 2^{\frac{1}{6}} H_r^{\frac{1}{3}} H_5^{\frac{2}{3}}(out) \tau \right],$$

$$a_1 \equiv a\left(-\lambda^{-1} e^{\phi/\sqrt{6}} + \lambda e^{-\phi/\sqrt{6}}\right) = A\left[\lambda \left(1 + \frac{H_r \tau}{2^{\frac{3}{2}}} \right) - 2^{\frac{1}{6}} \lambda^{-1} H_r^{\frac{1}{3}} H_5^{\frac{2}{3}}(out) \tau \right]. \quad (1.10)$$

Here the constant $A = 2^{\frac{1}{6}} \left(H_r / H_5(out) \right)^{\frac{1}{3}}$ has been defined so that we match a_0 and a_1 to the incoming solution given in (1.4). As for the incoming solution, we can compute the Hubble constants on the two branes after collision. They are $\pm \lambda^{-3} H_5(out) + 2^{-\frac{5}{3}} \lambda^{-1} H_r^{\frac{2}{3}} H_5^{\frac{1}{3}}$ on the positive and negative tension branes respectively. For $H_r < 2^{\frac{5}{2}} \lambda^{-3} H_5$, the case of relatively little radiation production, immediately after collision a_0 is expanding but a_1 is contracting. Whereas for $H_r > 2^{\frac{5}{2}} \lambda^{-3} H_5$, both brane scale factors expand after collision. If no scalar potential $V(\phi)$ were present, the scalar field would continue to obey the solution (1.9), converging to

$$\phi_C = \sqrt{\frac{2}{3}} \ln\left(2^{\frac{5}{2}} \frac{H_5(out)}{H_r} \right). \quad (1.11)$$

This value is actually larger than ϕ_∞ for $H_r < H_5 \lambda^{-3} 2^{\frac{5}{2}}$, the case of weak production of radiation. However, the presence of the potential $V(\phi)$ alters the expression (1.11) for the final resting value of the scalar field. As ϕ crosses the potential well travelling in the positive direction, H_5 is reduced to a renormalized value $\hat{H}_5(out) < H_5(out)$, so that the final resting value of the scalar field can be smaller than ϕ_∞. If this is the case, then a_1 never crosses zero, instead reversing to expansion shortly after radiation dominance. If radiation dominance occurs well after ϕ has crossed the potential well, eq. (1.11) provides a reasonable estimate for the final resting value, if we use the corrected value $\hat{H}_5(out)$. The dependence of (1.11) is simply understood: while the Universe is kinetic energy dominated, a grows at $t^{\frac{1}{3}}$ and ϕ increases logarithmically with time. However, when the Universe becomes radiation dominated and $a \propto t^{\frac{1}{2}}$, Hubble damping increases and ϕ converges to the finite limit above.

With regard the eqs. (1.6-1.11), we note the following connections with number theory:

$$2^{5/3} = \sqrt[3]{32} = 3{,}174802104 \cong (\Phi)^{16/7} + (\Phi)^{-26/7} = 3{,}171;$$
$$2^{3/2} = \sqrt{8} = 2{,}828427125 \cong (\Phi)^{13/7} + (\Phi)^{-14/7} = 2{,}826;$$
$$2^{-5/3} = 0{,}314980262 \cong (\Phi)^{-20/7} + (\Phi)^{-40/7} = 0{,}3168;$$
$$2^{1/6} = \sqrt[6]{2} = 1{,}122462048 \cong (\Phi)^{1/7} + (\Phi)^{-43/7} = 1{,}1231;$$
$$2^{5/2} = \sqrt{32} = 5{,}656854249 \cong (\Phi)^{25/7} + (\Phi)^{-37/7} = 5{,}6553.$$

Note that, $32 = 8 \times 4 = 24 + 8$, where 8 and 24 are the "modes" that correspond to the physical vibrations of a superstring and the physical vibrations of the bosonic strings.

Here, we have used the following expression: $(\Phi)^{n/7}$**, with** $\Phi = \frac{\sqrt{5}+1}{2} = 1{,}618033987...$ **that is the Aurea ratio,** n **is a natural number and 7 are the compactified dimensions of the M-Theory**.

Using the following potential

$$V(\phi) = V_0\left(1 - e^{-c\phi}\right)F(\phi), \quad (1.12)$$

we consider the motion of ϕ back and forth across the potential well. V may be accurately approximated by $-V_0 e^{-c\phi}$. For this pure exponential potential, there is a simple scaling solution

$$a(t) = |t|^p, \qquad V = -V_0 e^{-c\phi} = -\frac{p(1-3p)}{t^2}, \qquad p = \frac{2}{c^2}, \quad (1.13)$$

which is an expanding or contracting Universe solution according to whether t is positive or negative. A the end of the expanding phase of the cyclic scenario, there is a period of accelerated expansion which makes the Universe empty, homogeneous and flat, followed by ϕ rolling down the potential $V(\phi)$ into the well. After ϕ has rolled sufficiently and the scale factor has begun to contract, the Universe accurately follows the above scaling solution down the well until ϕ encounters the potential minimum. Let us consider the behaviour of ϕ under small shifts in the contracting phase. In the background scalar field equation and the Friedmann equation, we set $\phi = \phi_B + \delta\phi$ and $H = H_B + \delta H$, where ϕ_B and H_B are the background quantities given from (1.13). To linear order in $\delta\phi$, one obtains

$$\delta\ddot{\phi} + \frac{1+3p}{t}\delta\dot{\phi} - \frac{1-3p}{t^2}\delta\phi = 0, \quad (1.14)$$

with two linearly independent solutions, $\delta\phi \approx t^{-1}$ and t^{1-3p}, where $p << 1$. In the contracting phase, the former solution grows as t tends to zero. However, this solution is simply an infinitesimal shift in the time to the Big Crunch: $\delta\phi \propto \dot{\phi}$.

We next the incoming and outgoing collision velocity, which we have parameterized as $H_5(in)$ and $H_5(out)$. Within the scaling solution (1.13), we can calculate the value of incoming velocity by treating the prefactor of the potential $F(\phi)$ in eq. (1.12) as a Heaviside function which is unity for $\phi > \phi_{\min}$ and zero for $\phi < \phi_{\min}$, where $\phi_{\min}$ is the value of ϕ at the minimum of the potential. We compute the velocity of the field as it approaches $\phi_{\min}$ and use energy conservation at the jump in V to infer the velocity after $\phi_{\min}$ is crossed. In the scaling solution, the total energy as ϕ

approaches $\phi_{\min}$ from the right is $\frac{1}{2}\dot{\phi}^2 + V = \frac{3p^2}{t^2}$, and this must equal the total energy $\frac{1}{2}\dot{\phi}^2$ evaluated for ϕ just to the left of $\phi_{\min}$. Hence, we find that $\dot{\phi} = \sqrt{6}p/t = \sqrt{6pV_{\min}/(1-3p)}$ at the minimum and, according to eq. (1.6),

$$H_5(in) \approx -\frac{\sqrt{8}}{c}\frac{|V_{\min}|^{\frac{1}{2}} e^{\sqrt{\frac{3}{2}}\phi_{\min}}}{\sqrt{1-6c^{-2}}}. \quad (1.15)$$

Note that from the eq. (1.15), we obtain:

$$[H_5(in)]^2 \approx -\frac{8}{c^2}\frac{|V_{\min}|(e^{\sqrt{\frac{3}{2}}\phi_{\min}})^2}{1-6c^{-2}},$$

where the number 8 is connected with the "modes" that correspond to the physical vibrations of a superstring by the following Ramanujan function:

$$8 = \frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}.$$

At the bounce, this solution is matched to an expanding solution with

$$H_5(out) = -(1+\chi)H_5(in) > 0, \quad (1.16)$$

where χ is a small parameter which arises because of the inelasticity of the collision. We shall simply assume a small positive χ is given, and follow the evolution forwards in time. Since χ is small, the outgoing solution is very nearly the time reverse of the incoming solution as ϕ starts back across the potential well after the bounce: the scaling solution is given in (1.13), but with t positive. We can treat χ as a perturbation and use the solution in eq. (1.14) discussed above, $\delta\phi \approx t^{-1}$ and t^{1-3p}. One can straightforwardly compute the perturbation in δH_5 in this growing mode by matching at $\phi_{\min}$ as before. One finds $\delta H_5 = 12\chi H_5^B / c^2$ where H_5^B is the background value, at the minimum. Beyond this point, δH_5 grows as $t^{\sqrt{6}/c} \propto e^{\sqrt{\frac{3}{2}}\phi}$ for large c, whereas in the background scaling solution H_5 decays with ϕ as $e^{\left(\sqrt{\frac{3}{2}}-c/2\right)\phi}$. The departure occurs when the scalar field has attained the value

$$\phi_{Dep} = \phi_{\min} + \frac{2}{c}\ln\frac{c^2}{12\chi}, \qquad |V| \le \left(\frac{12\chi}{c^2}\right)^2 |V_{\min}|. \quad (1.17)$$

As ϕ passes beyond ϕ_{Dep} the kinetic energy overwhelms the negative potential and the field passes onto the plateau V_0 with H_5 nearly constant and equal to

$$\hat{H}_5(out) \approx \chi\left(\frac{c^2}{12\chi}\right)^{\frac{\sqrt{6}}{c}} H_5(in), \quad (1.18)$$

until the radiation, matter and vacuum energy become significant and H_5 is then damped away to zero. Note that we can rewrite the eq. (1.18) as follow:

$$\hat{H}_5(out) \approx \chi\left(\frac{c^2}{12\chi}\right)^{\frac{\sqrt{6}}{c}} \times -\frac{\sqrt{8}}{c}\frac{|V_{\min}|^{\frac{1}{2}} e^{\sqrt{\frac{3}{2}}\phi_{\min}}}{\sqrt{1-6c^{-2}}}. \quad (1.18b)$$

Also this equation is related with the number 8, i.e. with the "modes" that correspond to the physical vibrations of a superstring by the following Ramanujan function:

$$8 = \frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]},$$

and with the number 12 (12 = 24 / 2) that is related to the physical vibrations of the bosonic strings by the following Ramanujan function:

$$24 = \frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}.$$

The time spent to the left of the potential well $(\phi < \phi_{\min})$ is essentially identical in the incoming and outgoing stages for $\chi << 1$, namely

$$|t_{\min}| \approx \frac{c}{3\sqrt{2|V_{\min}|}}. \quad (1.19)$$

For the outgoing solution, when ϕ has left the scaling solution but before radiation domination, the definition eq. (1.6) may be integrated to give the time since the Big Bang at each value of ϕ,

$$t(\phi)=\int\frac{d\phi}{\dot{\phi}}=\sqrt{\frac{2}{3}}\int d\phi\frac{e^{\sqrt{\frac{3}{2}}\phi}}{\hat{H}_5(\phi)}\approx\frac{2}{3}\frac{e^{\sqrt{\frac{3}{2}}\phi}}{\hat{H}_5(out)}. \quad (1.20)$$

Also this equation can be rewritten as follow:

$$t(\phi)=\int\frac{d\phi}{\dot{\phi}}=\sqrt{\frac{2}{3}}\int d\phi\frac{e^{\sqrt{\frac{3}{2}}\phi}}{\hat{H}_5(\phi)}\approx\frac{2}{3}e^{\sqrt{\frac{3}{2}}\phi}/\chi\left(\frac{c^2}{12\chi}\right)^{\frac{\sqrt{6}}{c}}\times-\frac{\sqrt{8}}{c}\frac{|V_{\min}|^{\frac{1}{2}}e^{\sqrt{\frac{3}{2}}\phi_{\min}}}{\sqrt{1-6c^{-2}}}. \quad (1.20b)$$

The time in eq. (1.20) is a microphysical scale. The corresponding formula for the time before the Big Crunch is very different. In the scaling solution (1.13) one has for large c

$$t(\phi)=-\sqrt{\frac{2}{|V_{\min}|}}\frac{e^{c(\phi-\phi_{\min})/2}}{c}=-\frac{6e^{c(\phi-\phi_{\min})/2}}{c^2}|t_{\min}|. \quad (1.21)$$

The large exponential factor makes the time to the Big Crunch far longer than the time from the Big Bang, for each value of ϕ. This effect is due to the increase in H_5 after the bounce, which, in turn, is due to the positive value of χ. As the scalar field passes beyond the potential well, it runs onto the positive plateau V_0. The value of $H_5(out)$ is nearly cancelled in the passage across the potential well, and is reduced to $\hat{H}_5$ given in eq. (1.18). Once radiation domination begins, the field quickly converges to the large t (Hubble-damped) limit of eq. (1.9), namely

$$\phi_C=\sqrt{\frac{2}{3}}\ln\left(2^{\frac{5}{2}}\hat{H}_5(out)/H_r\right), \quad (1.22)$$

where H_r is the Hubble radius at kinetic-radiation equality. Also the eq. (1.22) can be rewritten as follow

$$\phi_C=\sqrt{\frac{2}{3}}\ln\left[2^{\frac{5}{2}}\chi\left(\frac{c^2}{12\chi}\right)^{\frac{\sqrt{6}}{c}}\cdot-\frac{\sqrt{8}}{c}\frac{|V_{\min}|^{\frac{1}{2}}e^{\sqrt{\frac{3}{2}}\phi_{\min}}}{\sqrt{1-6c^{-2}}}/H_r\right]. \quad (1.22b)$$

The dependence is obvious: the asymptotic value of ϕ depends on the ratio of $\hat{H}_5(out)$ to H_r. Increasing $\hat{H}_5(out)$ pushes ϕ further, likewise lowering H_r delays radiation domination allowing the logarithmic growth of ϕ in the kinetic energy dominated phase to continue for longer.
The solution of the scalar field equation is, after expanding eq. (1.9) for large τ, converting to proper time $t=\int a(\tau)d\tau$ and matching,

$$\dot{\phi}\approx\frac{\sqrt{3}H_r}{a^3(t)}-a^{-3}\int_0^t dta^3V_{,\phi}, \quad (1.23)$$

where as above we define $a(t)$ to be unity at kinetic-radiation equal density. We have that ϕ may reach its maximal value $\phi_{\max}$ and turn around during the radiation, matter or quintessence dominated epoch. For example, $\phi_{\max}$ is reached in the radiation era, if, from eq. (1.23),

$$\frac{t_{\max}}{t_m} \approx 10^4 \left(\frac{t_r}{t_m}\right)^{\frac{1}{5}} \left(\frac{V}{V_{,\phi}}(\phi_C)\right)^{\frac{2}{5}} < 1, \quad (1.24)$$

where t_m is the time of matter domination.
For turn around in the matter era, we require

$$3 \times 10^{-4} \leq \left(\frac{t_r}{t_m}\right)^{\frac{1}{6}} \left(\frac{V}{V_{,\phi}}(\phi_C)\right)^{\frac{1}{3}} \leq 30. \quad (1.25)$$

Finally, if the field runs to very large ϕ_C, so that $V_{,\phi}/V(\phi_C) \approx ce^{-c\phi_C}$ is exponentially small, then ϕ only turns around in the quintessence-dominated era.
For our scenario to be viable, we require there to be a substantial epoch of vacuum energy domination (inflation) before the next Big Crunch. The number of e-foldings N_e of inflation is given by usual slow-roll formula,

$$N_e = \int d\phi \frac{V}{V_{,\phi}} \approx \frac{e^{c\phi_C}}{c^2}, \quad (1.26)$$

for our model potential. For example, if we demand that the number of baryons per Hubble radius be diluted to below unity before the next contraction, which is certainly over-kill in guaranteeing that the cyclic solution is an attractor, we set $e^{3N_e} \geq 10^{80}$, or $N_e \geq 60$. This is easily fulfilled if ϕ_C is of order unity Planck units. Hence, the eq. (1.26) can be rewritten as follow:

$$N_e = \int d\phi \frac{V}{V_{,\phi}} \approx \frac{e^{c\phi_C}}{c^2} \geq 60. \quad (1.26b)$$

With regard the eqs. (1.24-1.25, 1.26b) we have the following mathematical connections with the Aurea ratio:

$$(\Phi)^{49/7} + (\Phi)^{-41/7} = 29{,}03444185 + 0{,}059693843 = 29{,}0941;$$
$$(\Phi)^{-8/7} + (\Phi)^{-14/7} = 0{,}576974982 + 0{,}381966011 = 0{,}95894;$$
$$(\Phi)^{-10{,}33/7} = 0{,}4914670835; \quad \arcsin(0{,}4914670835) \cdot \frac{180}{\pi} = 29{,}437054;$$
$$(\Phi)^{-2/7} = 0{,}8715438560; \quad \arccos(0{,}8715438560) \cdot \frac{180}{\pi} = 29{,}361456;$$
$$(\Phi)^{49/7} + (\Phi)^{50/7} = 29{,}03444185 + 31{,}10060654 = 60{,}135048;$$
$$(\Phi)^{-10{,}33/7} = 0{,}4914670835; \quad \arccos(0{,}4914670835) \cdot \frac{180}{\pi} = 60{,}562946;$$

$$(\Phi)^{-2/7} = 0{,}8715438560\,; \quad \arcsin(0{,}8715438560)\cdot\frac{180}{\pi} = 60{,}638544\,.$$

From the formulae given above we can also calculate the maximal value ϕ_C in the cyclic solution: for large c and for $t_r >> \chi^{-1}t_{\min}$, it is

$$\phi_C - \phi_{\min} \approx \sqrt{\frac{2}{3}}\ln\left(\chi\frac{t_r}{t_{\min}}\right), \quad (1.27)$$

where we used $H_r^{-1} \approx t_r$, the beginning of the radiation-dominated epoch. From eq. (1.27) we obtain

$$\frac{t_r}{t_{\min}} \approx \frac{1}{\chi}\left(\frac{c^2 N_e |V_{\min}|}{V_0}\right)^{\sqrt{\frac{3}{2c^2}}}. \quad (1.28)$$

This equation provides a lower bound on t_r. The extreme case is to take $|V_{\min}| \approx 1$. **Then using $V_0 \approx 10^{-120}$, $c \approx 10$, $N_e \approx 60$, we find $t_r \approx 10^{-25}$ seconds. In this case the maximum temperature of the Universe is $\approx 10^{10}$ GeV. This is not very different to what one finds in simple inflationary models**.

We have shown that a cyclic universe solution exists provided we are allowed to pass through the Einstein-frame singularity according to the matching conditions, eqs. (1.15) and (1.16). Specifically, we assumed that $H_5(out) = -(1+\chi)H_5(in)$ where χ is a non-negative constant, corresponding to branes whose relative speed after collision is greater than or equal to the relative speed before collision. Our argument showed that, for each $\chi \geq 0$, there is a unique value of $H_5(out)$ that is perfectly cyclic. Now we show that an increase in velocity is perfectly compatible with energy and momentum conservation in a collision between a positive and negative tension brane, provided a greater density of radiation is generated on the negative tension brane.

We shall assume that all other extra dimensions and moduli are fixed, and the bulk space-time between the branes settles down to a static state after the collision. We shall take the densities of radiation on the branes after collision as being given. By imposing Israel matching in both initial and final states, as well as conservation of total energy and momentum, we shall be able to completely fix the state of the outgoing branes and in particular the expansion rate of the extra dimension $H_5(out)$, in terms of $H_5(in)$. The initial state of empty branes with tensions T and $-T$, and with corresponding velocities $v_+ < 0$ and $v_- > 0$ obeys

$$T\sqrt{1-v_+^2} = T\sqrt{1-v_-^2}\,; \quad E_{tot} = \frac{T}{\sqrt{1-v_+^2}} - \frac{T}{\sqrt{1-v_-^2}}\,; \quad P_{tot} = \frac{Tv_+}{\sqrt{1-v_+^2}} - \frac{Tv_-}{\sqrt{1-v_-^2}}\,. \quad (1.29)$$

The first equation follows from Israel matching on the two branes as the approach, and equating the kinks in the brane scale factors. The second and third equations are the definitions of the total energy and momentum. The three equations (1.29) imply that the incoming, empty state has $v_+ = -v_-$, $E_{tot} = 0$ and that the total momentum is

$$P_{tot} = \frac{TLH_5(in)}{\sqrt{1 - \frac{1}{4}(LH_5(in))^2}} < 0, \quad (1.30)$$

where we identify $v_+ - v_-$ with the contraction speed of the fifth dimension, $|LH_5(in)|$. For the eq. (1.15), we can rewrite the eq. (1.30) also as follow:

$$P_{tot} = TL\left(-\frac{\sqrt{8}}{c}\frac{|V_{\min}|^{\frac{1}{2}} e^{\sqrt{\frac{3}{2}}\phi_{\min}}}{\sqrt{1-6c^{-2}}} \right) \frac{1}{\sqrt{1 - \frac{1}{4}\left[L\left(-\frac{\sqrt{8}}{c}\frac{|V_{\min}|^{\frac{1}{2}} e^{\sqrt{\frac{3}{2}}\phi_{\min}}}{\sqrt{1-6c^{-2}}} \right)\right]^2}} < 0. \quad (1.30b)$$

The corresponding equations for the outgoing state are easily obtained, by replacing T with $T + \rho_+ \equiv T_+$ for the positive tension brane, and $-T$ with $-T + \rho_- \equiv -T_-$ for the negative tension brane, assuming the densities of radiation produced at the collision on each brane, ρ_+ and ρ_- respectively, are given from a microphysical calculation, and are both positive.
Writing $v_\pm(out) = \tanh(\theta_\pm)$, where $\theta_\pm$ are the associated rapidities, one obtains two solutions

$$\sinh\theta_+ = -\frac{1}{2T_-}\left(|P_{tot}| + |P_{tot}|^{-1}\left(T_+^2 - T_-^2\right)\right); \quad \sinh\theta_- = \pm\frac{1}{2T_+}\left(|P_{tot}| - |P_{tot}|^{-1}\left(T_+^2 - T_-^2\right)\right), \quad (1.31)$$

where $T_+ \equiv T + \rho_+$, $T_- \equiv T - \rho_-$ with ρ_+ and ρ_- the densities of radiation on the positive and negative tension branes respectively, after collision. Both ρ_+ and ρ_- are assumed to be positive. In the first solution, with signs (– +), the velocities of the positive and negative tension branes are the same after the collision as they were before it. In the second, with signs (– –), the positive tension brane continues in the negative y direction but the negative tension brane is also moving in the negative y direction. The corresponding values for $v_\pm(out)$ and V are

$$v_+(out) = -\frac{|P_{tot}| + |P_{tot}|^{-1}\left(T_+^2 - T_-^2\right)}{\sqrt{P_{tot}^2 + 2\left(T_+^2 + T_-^2\right) + P_{tot}^{-2}\left(T_+^2 - T_-^2\right)^2}}, \quad v_-(out) = \pm\frac{|P_{tot}| - |P_{tot}|^{-1}\left(T_+^2 - T_-^2\right)}{\sqrt{P_{tot}^2 + 2\left(T_+^2 + T_-^2\right) + P_{tot}^{-2}\left(T_+^2 - T_-^2\right)^2}},$$

$$V = -\frac{\sqrt{P_{tot}^2 + 2\left(T_+^2 + T_-^2\right) + P_{tot}^{-2}\left(T_+^2 - T_-^2\right)^2}}{|P_{tot}|\left(T_+^2 + T_-^2\right)/\left(T_+^2 - T_-^2\right) + |P_{tot}|^{-1}\left(T_+^2 - T_-^2\right)}, \text{ or } V = -\frac{\sqrt{P_{tot}^2 + 2\left(T_+^2 + T_-^2\right) + P_{tot}^{-2}\left(T_+^2 - T_-^2\right)^2}}{|P_{tot}| + |P_{tot}|^{-1}\left(T_+^2 + T_-^2\right)}, \quad (1.32)$$

where the first solution for V holds for the (– +) case, and the second for the (– –) case. We are interested in the relative speed of the branes in the outgoing state, since that gives the expansion rate of the extra dimension, $-v_+(out) + v_-(out) = LH_5(out)$, compared to their relative speed $-2v_+ = -LH_5(in)$ in the incoming state. We find in the (– +) solution,

$$\left|\frac{H_5(out)}{H_5(in)}\right| = \frac{v_+(out) - v_-(out)}{2v_+} = \sqrt{\frac{P_{tot}^2 + 4T^2}{P_{tot}^2 + 2\left(T_+^2 + T_-^2\right) + P_{tot}^{-2}\left(T_+^2 - T_-^2\right)^2}}\ , \quad (1.33)$$

and in the (– –) solution

$$\left|\frac{H_5(out)}{H_5(in)}\right| = \frac{\left(T_+^2 - T_-^2\right)}{P_{tot}^2}\sqrt{\frac{P_{tot}^2 + 4T^2}{P_{tot}^2 + 2\left(T_+^2 + T_-^2\right) + P_{tot}^{-2}\left(T_+^2 - T_-^2\right)^2}}\ . \quad (1.34)$$

with P_{tot} given by (1.30) in both cases. We note that we can rewrite the relation above mentioned, i.e. $-v_+(out) + v_-(out) = LH_5(out)$ also as follow:

$$-v_+(out) + v_-(out) = LH_5(out) = L\left[-(1+\chi)\left(-\frac{\sqrt{8}}{c}\frac{\left|V_{\min}\right|^{\frac{1}{2}} e^{\sqrt{\frac{3}{2}}\phi_{\min}}}{\sqrt{1-6c^{-2}}}\right)\right]. \quad (1.34b)$$

At this point we need to consider how the densities of radiation ρ_+ and ρ_- depend on the relative speed of approach of the branes. At very low speeds, $\left|LH_5(in)\right| << 1$, one expects the outer brane collision to be nearly adiabatic and an exponentially small amount of radiation to be produced. The (– +) solution has the speeds of both branes nearly equal before and after collision: we assume that it is this solution, rather than the (– –) solution which is realised in this low velocity limit. As $\left|LH_5(in)\right|$ is increased, we expect ρ_+ and ρ_- to grow. Now, if we consider ρ_+ and ρ_- to be both $<< P_{tot} << T$, then the second term in the denominator dominates. If more radiation is produced on the negative tension brane, $\rho_- > \rho_+$, then

$$\left|\frac{H_5(out)}{H_5(in)}\right| \equiv (1+\chi) \approx \left[1 + \frac{(\rho_- - \rho_+)}{2T}\right] \quad (1.34c)$$

and so χ is small and positive. This is the condition necessary to obtain cyclic behaviour. Conceivably, the brane tension can change from T to $T' = T - t$ at collision. Then, we obtain

$$(1+\chi) \approx \left[1 + \frac{(\rho_- - \rho_+ + 2t)}{2T}\right]. \quad (1.34d)$$

For the (– +) solution, we can straightforwardly determine an upper limit for $\left|H_5(out)/H_5(in)\right| \equiv (1+\chi)$. Consider, for example, the case there the brane tension in unchanged at collision, $t = 0$. The expression in (1.33) gives $\left|H_5(out)/H_5(in)\right|$ as a function of T_+, T_- and P_{tot}. It is greatest, at fixed T_- and P_{tot}, when $T_+ = T$, its smallest value. For $P_{tot}^2 < T^2$, it is maximized for $T_-^2 = T^2 - P_{tot}^2$, and equal to $\sqrt{\frac{1+P_{tot}^2}{\left(4T^2\right)}}$ when equality holds. For $P_{tot} \geq T^2$, it is maximized when $T_- = 0$, its smallest value, and $P_{tot}^2 = 2T^2$, when it is equal to $\sqrt{\frac{4}{3}} = 1{,}154700538$. This is more than enough for us to obtain the small values of χ needed to make the cyclic scenario work. A reduction in brane tension at collisions $t > 0$ further increases the maximal value of the ratio. To obtain cyclic behaviour, we need χ to be constant from bounce to bounce. That is, compared to the tension before collision, the fractional change in tension and the fractional production of radiation must be constant.

We note that for $\sqrt{\frac{4}{3}} = 1{,}154700538$, we have the following mathematical connections with the Aurea ratio:

$$\sqrt{\frac{4}{3}} = 1{,}154700538 = (\Phi)^{-1/7} + (\Phi)^{-22/7} = 0{,}933565132 + 0{,}220384833 = 1{,}1539499\ ;$$

$$\sqrt{\frac{4}{3}} = 1{,}154700538 = (\Phi)^{-7/7} + (\Phi)^{-9/7} = 0{,}6180339887 + 0{,}5386437257 = 1{,}156677713\ .$$

2. **On some equations concerning cosmological perturbations in a Big Crunch/Big Bang space-time and M-Theory model of a Big Crunch/Big Bang transition.** [2] [3] [4] [5] [16]

We consider a positive or negative tension brane with cosmological symmetry but which moves through the five-dimensional bulk. The motion through the warped bulk induces expansion or contraction of the scale factor on the brane. The scale factor on the brane obeys a "modified Friedmann" equation,

$$H_{\pm}^{2} = \pm \frac{1}{3M_{5}^{3}L}\rho_{\pm} + \frac{\rho_{\pm}}{36M_{5}^{6}} - \frac{K}{b_{\pm}^{2}} + \frac{\mathcal{C}}{b_{\pm}^{4}}\ , \quad (2.1)$$

where $\rho_{\pm}$ is the density (not including the tension) of matter or radiation confined to the brane, $b_{\pm}$ is the brane scale factor, and $H_{\pm}$ is the induced Hubble constant on the positive (negative) tension brane. Choosing conformal time on each brane, and neglecting the ρ^{2} terms equations (2.1) become

$$b'^{2}_{+} = +\frac{1}{3M_{5}^{3}L}\rho_{+}b_{+}^{4} - Kb_{+}^{2} + \mathcal{C}\ , \quad b'^{2}_{-} = -\frac{1}{3M_{5}^{3}L}\rho_{-}b_{-}^{4} - Kb_{-}^{2} + \mathcal{C}\ . \quad (2.2)$$

where prime denotes conformal time derivative. The corresponding acceleration equations for b''_{+} and b''_{-}, from which $\mathcal{C}$ disappears, are derived by differentiating equations (2.2) and using $d\left(\rho b^{4}\right) = b^{3}(\rho - 3P)db$ with P being the pressure of matter or radiation on the brines. We now show that these two equations can be derived from a single action provided we equate the conformal times on each brane. Consider the action

$$S = \int dtNd^{3}x\left[-3M_{5}^{3}L\left(N^{-2}b'^{2}_{+} - Kb_{+}^{2}\right) - \rho_{+}b_{+}^{4} + 3L\left(N^{-2}b'^{2}_{-} - Kb_{+}^{2}\right) - \rho_{-}b_{-}^{4}\right], \quad (2.3)$$

where N is a lapse function introduced to make the action time reparameterization invariant. Varying with respect to $b_{\pm}$ and then setting $N = 1$ gives the correct acceleration equations for b''_{+} and b''_{-} following from (2.2). These equations are equivalent to (2.2) up to two integration constants.

We rewrite the action (2.3) in terms of a four-dimensional effective scale factor a and a scalar field ϕ, defined by

$$b_+ = a\cosh\left(\frac{\phi}{\sqrt{6}}\right), \quad b_- = -a\sinh\left(\frac{\phi}{\sqrt{6}}\right).$$

Clearly, a and ϕ transform as a scale factor and as a scalar field under rescalings of the spatial coordinates $\vec{x}$. To interpret ϕ more physically, note that for static branes the bulk space-time is perfect Anti-de Sitter space with line element $dY^2 + e^{2Y/L}\left(-dt^2 + d\vec{x}^2\right)$. The separation between the branes is given by

$$d = L\ln\left(\frac{a_+}{a_-}\right) = L\ln\left[\frac{-\coth}{\left(\frac{\phi}{\sqrt{6}}\right)}\right],$$

so d tends from zero to infinity as ϕ tends from minus infinity to zero. In terms of a and ϕ, the action (2.3) becomes

$$S = \int dt d^3x\left[-3M_5^3L\left(\dot{a}^2 - Ka^2\right) + \frac{1}{2}a^2\dot{\phi}^2\right] + S_m, \quad (2.4)$$

which is recognized as the action for Einstein gravity with line element $a^2(t)\left(-dt^2 + \gamma_{ij}dx^idx^j\right)$, γ_{ij} being the canonical metric on H^3, S^3 or E^3 with curvature K, and a minimally coupled scalar field ϕ. The matter action S_m is conventional, except that the scale factor appearing is not the Einstein-frame scale factor but instead $b_+ = a\cosh\left(\phi/\sqrt{6}\right)$ and $b_- = -a\sinh\left(\phi/\sqrt{6}\right)$ on the positive and negative tension branes respectively.

Now we wish to make use of two very powerful principles. The first is the assertion that even in the absence of symmetry, the low energy modes of the five-dimensional theory should be describable with a four-dimensional effective action. The second is that since the original theory was coordinate invariant, the four dimensional effective action must be coordinate invariant too. Since the five-dimensional theory is local and causal, it is reasonable to expect these properties in the four-dimensional theory. If furthermore the relation between the four-dimensional induced metrics on the branes and the four-dimensional fields is local, then covariance plus agreement with the above results forces the relation to be

$$g^+_{\mu\nu} = \left(\cosh\left(\phi/\sqrt{6}\right)\right)^2 g^{4d}_{\mu\nu} \qquad g^-_{\mu\nu} = \left(-\sinh\left(\phi/\sqrt{6}\right)\right)^2 g^{4d}_{\mu\nu}. \quad (2.5)$$

When we couple matter to the brane metrics, these expressions should enter the action for matter confined to the positive and negative tension branes respectively. Likewise we can from (2.4) and covariance immediately infer the effective action for the four-dimensional theory:

$$S = \int d^4x\sqrt{-g}\left(\frac{M_4^2}{2}R - \frac{1}{2}\left(\partial_\mu\phi\right)^2\right) + S_m^-\left[g^-\right] + S_m^+\left[g^+\right], \quad (2.6)$$

where we have defined the effective four-dimensional Planck mass $M_4^2 = \left(8\pi G_4\right)^{-1} = M_5^3L$.

The two brane geometries are determined according to the formulae (2.5), and the background solution relevant post-collision is assumed to consist of two flat, parallel branes with radiation densities $\rho_\pm$. The corresponding four-dimensional effective theory has radiation density ρ_r, and a massless scalar field with kinetic energy density ρ_ϕ. The four-dimensional Friedmann equation in conformal time then reads

$$a'^2 = \frac{1}{3}\left(\rho_r a^4 + \rho_\phi a^4\right) \equiv 4A_4\left(r_4 + \frac{A_4}{a^2}\right), \quad (2.7)$$

where we have defined the constants A_4 and r_4, and used the fact that the massless scalar kinetic energy $\rho_\phi \propto a^{-6}$. The solution to (2.7) and the massless scalar field equation $\left(a^2\phi'\right)' = 0$ is:

$$a^2 = 4A_4\tau\left(1 + r_4\tau\right), \qquad \phi = \sqrt{\frac{3}{2}}\ln\left(\frac{A_4\tau}{\left(1 + r_4\tau\right)}\right). \quad (2.8)$$

From these solutions, we reconstruct the scale factors on the branes according to (2.5), obtaining:

$$b_\pm = 1 \pm A_4\tau + r_4\tau\ , \quad (2.9)$$

so we see that with the choice of normalization for the scale factor a made in (2.7), the brane scale factors are unity at collision. We may now directly compare the predictions (2.9) with the exact five-dimensional solution, equating the terms linear in τ to obtain

$$A_4 = \left(1/L\right)\left(1 + \frac{L^2\left(r_+ - r_-\right)}{12}\right)\tanh\left(y_0/2\right), \quad r_4 = \frac{L\left(r_+ + r_-\right)}{12\tanh\left(y_0/2\right)}, \quad (2.10)$$

where y_0 is the rapidity associated with the relative velocity of the branes at collision $V = \tanh\left(y_0\right)$ and $r_\pm$ is the value of the radiation density $\rho_\pm$ on each brane at collision. Thence, the eq. (2.9) can be rewritten also:

$$b_\pm = 1 \pm \left(1/L\right)\left(1 + \frac{L^2\left(r_+ - r_-\right)}{12}\right)\tanh\left(y_0/2\right)\tau + \frac{L\left(r_+ + r_-\right)}{12\tanh\left(y_0/2\right)}\tau\ . \quad (2.10b)$$

Furthermore, we define the fractional density mismatch on the two branes as

$$f = \frac{r_+ - r_-}{r_+ + r_-}, \quad (2.11)$$

so that we have

$$r_+ - r_- = \frac{12 f r_4}{L}\tanh\left(y_0/2\right). \quad (2.12)$$

Now, we describe the perturbations of the brane-world system in terms of the four-dimensional effective theory. We shall now describe the scalar perturbations, in longitudinal gauge with a spatially flat background where the scale factor and the scalar field are given by (2.8). The perturbed line element is

$$ds^2 = a^2(\tau)\left[-\left(1 + 2\Phi\right)d\tau^2 + \left(1 - 2\Psi\right)d\vec{x}^2\right]. \quad (2.13)$$

Since there are no anisotropic stresses in the linearized theory, we have $\Phi = \Psi$.
A complete set of perturbation equations consists of the radiation fluid equations, the scalar field equation of motion and the Einstein momentum constraint:

$$\delta'_r = -\frac{4}{3}\left(k^2 v_r - 3\Phi'\right) \quad v'_r = \frac{1}{4}\delta_r + \Phi \quad (\delta\phi)'' + 2\mathcal{H}(\delta\phi)' = -k^2(\delta\phi) + 4\phi'\Phi$$
$$\Phi' + \mathcal{H}\Phi = \frac{2}{3}a^2\rho_r v_r + \frac{1}{2}\phi'(\delta\phi), \quad (2.14)$$

where primes denote τ derivatives, δ_r is the fractional perturbation in the radiation density, v_r is the scalar potential for its velocity i.e. $\vec{v}_r = \vec{\nabla} v_r$, $\delta\phi$ is the perturbation in the scalar field, and from (2.8) we have the background quantities

$$\mathcal{H} \equiv \frac{a'}{a} = \frac{(1+2r_4\tau)}{[2\tau(1+r_4\tau)]}, \text{ and } \sqrt{\frac{2}{3}}\phi' = \frac{1}{[\tau(1+r_4\tau)]}.$$

We are interested in solving these equations in the long wavelength limit, $|k\tau| << 1$. Solving all the above equations for $\delta \ln a$, one finds

$$\frac{\delta_i}{(1+w_i)} \approx \frac{\delta}{(1+w)}, \quad i = 1,...N, \quad (2.15)$$

for adiabatic perturbations. The components of the background energy density in the four-dimensional effective theory are scalar kinetic energy, with $w_\phi = 1$, and radiation, with $w_r = \frac{1}{3}$. It follows that for adiabatic perturbations, at long wavelengths we must have

$$\delta_\phi \approx \frac{3}{2}\delta_r. \quad (2.16)$$

In longitudinal gauge, the fractional energy density perturbation and the velocity potential perturbation in the scalar field (considered as a fluid with $w = 1$) are given by

$$\delta_\phi = 2\left(\frac{(\delta\phi)'}{\phi'} - \Phi\right), \quad v_\phi = \frac{\delta\phi}{\phi'}. \quad (2.17)$$

From the equations (2.14) above (and using $\phi' \propto a^{-2}$) it follows that

$$\left(\delta_\phi - \frac{3}{2}\delta_r\right)' = 2k^2\left(v_r - \frac{\delta\phi}{\phi'}\right). \quad (2.18)$$

Maintaining the adiabaticity condition (2.16) up to order $(k\tau)^2$ then requires that the fractional velocity perturbations for the scalar field and the radiation should be equal: $v_r \approx \delta\phi/\phi'$. Expressing the radiation velocity in terms of $\delta\phi$, the momentum constraint then yields

$$\delta\phi \approx \left(1+\frac{2}{3}\frac{\rho_r}{\rho_\phi}\right)^{-1}\left(\frac{2(\Phi'+\mathcal{H}\Phi)}{\phi'}\right), \quad (2.19)$$

where $\rho_\phi = \frac{1}{2}\phi'^2 a^{-2}$.

The above equations may be used to determine the leading terms in an expansion in $|k\tau|$ of all the quantities of interest about the singularity. We shall choose to parameterize the expressions in terms of the parameters describing the comoving energy density perturbation, $\varepsilon_m = -\frac{2}{3}\mathcal{H}^{-2}k^2\Phi$, which has the following series expansion about $\tau = 0$:

$$\varepsilon_m = \varepsilon_0 D(\tau) + \varepsilon_2 E(\tau), \quad (2.20)$$

where ε_0 and ε_2 are arbitrary constants, and

$$D(\tau) = 1 - 2r_4\tau - \frac{1}{2}k^2\tau^2\ln|k\tau| + \dots, \qquad E(\tau) = \tau^2 + \dots. \quad (2.21)$$

For adiabatic perturbations, we obtain

$$\delta_\phi = \varepsilon_0\left(-\frac{9}{4k^2\tau^2} - \frac{3}{8}\ln|k\tau| + \frac{1}{4} - \frac{3}{4}\frac{r_4^2}{k^2}\right) + \varepsilon_2\frac{3}{4k^2} + O(\tau, \tau\ln|k\tau|), \quad (2.22a)$$

$$v_\phi = \varepsilon_0\left(\frac{3}{4k^2\tau}(1 - r_4\tau)\right) + O(\tau, \tau\ln|k\tau|), \qquad \delta_r = \frac{2}{3}\delta_\phi + O(\tau^2, \tau^2\ln|k\tau|), \qquad v_r = v_\phi + O(\tau, \tau\ln|k\tau|),$$

$$\Phi = \varepsilon_0\left(-\frac{3}{8k^2\tau^2} + \frac{3}{2\cdot 8}\ln|k\tau| + \frac{3\cdot 5}{8}\frac{r_4^2}{k^2}\right) - \varepsilon_2\frac{3}{8k^2} + O(\tau, \tau\ln|k\tau|), \quad (2.22b)$$

$$\frac{(\delta\phi)}{\sqrt{6}} = \varepsilon_0\left(\frac{3}{8k^2\tau^2}(1 - 2r_4\tau) + \frac{1}{2\cdot 8}\ln|k\tau| + \frac{1}{8} + \frac{13}{8}\frac{r_4^2}{k^2}\right) - \varepsilon_2\frac{1}{8k^2} + O(\tau, \tau\ln|k\tau|), \quad (2.22c)$$

$$\zeta_{4,M} = -\frac{1}{2k^2}\varepsilon_2 + \varepsilon_0\left(\frac{1}{8k^2}(k^2 + 16r_4^2) + \frac{1}{4}\ln|k\tau|\right) + O(\tau, \tau\ln|k\tau|), \quad (2.22d)$$

where $\zeta_{4,M}$ is the curvature perturbation on comoving slices introduced by Mukhanov.

With regard the (2.22c) we note that is possible the following mathematical connection with the Aurea ratio:

$$\frac{1}{\sqrt{6}} = 0{,}40824829 = (\Phi)^{-13/7} = \left(\frac{\sqrt{5}+1}{2}\right)^{-13/7} = 0{,}4091477 \cong 0{,}409.$$

Furthermore, in the eqs. (2.22b-2.22c) there is the number 8, that is related to the “modes” that correspond to the physical vibrations of a superstring by the following Ramanujan function

$$8=\frac{1}{3}\frac{4\left[anti\log\frac{\int_0^{\infty}\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]},$$

and that 2, 3, 5, 8 and 13 are Fibonacci's numbers.

Now, we consider the propagation of metric perturbations through a collision of tensionless branes where the background space-time is precisely $\mathcal{M}^C / Z_2 \times R^3$. The form that we take for the five-dimensional cosmological background metric is

$$ds^2 = n^2(t,y)\left(-dt^2 + t^2 dy^2\right) + b^2(t,y)\delta_{ij}dx^i dx^j, \quad (2.23)$$

and we write the most general scalar metric perturbation about this as

$$ds^2 = n^2(t,y)\left(-(1+2\Phi)dt^2 - 2Wdtdy + t^2(1-2\Gamma)dy^2 - 2\nabla_i \alpha dx^i dt + 2t^2\nabla_i \beta dy dx^i\right) + \\ + b^2(t,y)\left((1-2\Psi)\delta_{ij} - 2\nabla_i\nabla_j\chi\right)dx^i dx^j. \quad (2.24)$$

For perturbations on $\mathcal{M}^C \times R^3$ it is straightforward to find a gauge in which the metric takes the form

$$ds^2 = \left(1+\frac{4}{3}k^2\chi\right)\left(-dt^2 + t^2 dy^2\right) + \left(\left(1-\frac{2}{3}k^2\chi\right)\delta_{ij} + 2k_i k_j \chi\right)dx^i dx^j, \quad (2.25)$$

and χ satisfies a massless scalar equation of motion on $\mathcal{M}^C \times R^3$. To be precise, the gauge is

$$\alpha=\beta=0, \quad \Gamma=\Phi-\Psi-k^2\chi, \quad \Phi=\frac{2}{3}k^2\chi, \quad \Psi=\frac{1}{3}k^2\chi, \quad W=0. \quad (2.26)$$

Notice that the non-zero variables can all be related to χ according to

$$(\Gamma,\Phi,\Psi)=\left(-\frac{2}{3},+\frac{2}{3},+\frac{1}{3}\right)k^2\chi. \quad (2.27)$$

We shall, henceforth, refer to these as the "Milne ratio conditions". Furthermore, imposing the Z_2 symmetry, we obtain Neumann boundary conditions on χ,

$$\chi'(y_\pm)=0, \quad (2.28)$$

where $y_\pm = \pm y_0/2$ are the location of the two Z_2 fixed points. In the model space-time, the lowest energy mode for χ is y-independent and has the asymptotic form

$$\chi(t,y) = Q + P\ln|kt|, \quad (2.29)$$

with Q and P being arbitrary constants. We have the following relations:

$$Q_{out} = -Q_{in} + 2(\gamma - \ln 2)P_{in}\,, \qquad P_{out} = P_{in}\,. \quad (2.30)$$

These relations are sufficient to determine the metric fluctuations after the bounce. We are only interested in the long-wavelength part of the spectrum, and, for the cases of interest, P is suppressed by k^2 compared to Q. As a result, we obtain the approximate matching rule

$$Q_{out} = -Q_{in}\,, \qquad P_{out} = P_{in}\,. \quad (2.31)$$

The key conditions (2.26) through (2.28) are satisfied precisely for all time in a compactified Milne $\mathrm{mod}\, Z_2$ background.

Now, we wish to use the four-dimensional effective (moduli) theory to infer the boundary data for the five dimensional bulk perturbations. In any four-dimensional gauge, the four-dimensional metric perturbation $h_{\mu\nu}$ and scalar field perturbation $\delta\phi$ determined the induced metric perturbations on the branes via the formulae (2.5):

$$h_{\mu\nu}^{\pm} = h_{\mu\nu} + 2(\ln\Omega_{\pm})_{,\phi}\,\delta\phi g_{\mu\nu}\,, \quad (2.32)$$

where $\Omega_{+} = \cosh(\phi/\sqrt{6})$ and $\Omega_{-} = -\sinh(\phi/\sqrt{6})$ and the metric perturbations are fractional i.e. $\delta g_{\mu\nu} = a^2 h_{\mu\nu}$, $\delta g_{\mu\nu}^{\pm} = b_{\pm}^2 h_{\mu\nu}^{\pm}$. This formula is particularly easy to use in five-dimensional longitudinal gauge. This gauge may always be chosen, and it is completely gauge fixed. In this gauge the five-dimensional metric takes the form

$$ds^2 = n^2(t,y)\left(-(1+2\Phi_L)dt^2 - 2W_L dtdy + t^2(1-2\Gamma_L)dy^2\right) + b^2(t,y)\left((1-2\Psi_L)\delta_{ij}\right)dx^i dx^j\,. \quad (2.33)$$

In the absence of anisotropic stresses the brane trajectories are unperturbed in this gauge. An immediate consequence is that the four-dimensional longitudinal gauge scalar perturbation variables $\Phi_{\pm}$ and $\Psi_{\pm}$ describing perturbations of the induced geometry on each brane

$$ds_{\pm}^2 = b_{\pm}^2(\tau_{\pm})\left(-(1+2\Phi_{\pm})d\tau_{\pm}^2 + (1-2\Psi_{\pm})d\vec{x}^2\right), \quad (2.34)$$

are precisely the boundary values of the five-dimensional longitudinal gauge perturbations $\Phi_{\pm} \equiv \Phi_L(y_{\pm})$ and $\Psi_{\pm} \equiv \Psi_L(y_{\pm})$. Using (2.32) and (2.34), we find for the induced perturbations

$$\Phi_{+} = \Phi_4 + \frac{1}{\sqrt{6}}\tanh\left(\frac{\phi}{\sqrt{6}}\right)\delta\phi\,, \quad \Psi_{+} = \Phi_4 - \frac{1}{\sqrt{6}}\tanh\left(\frac{\phi}{\sqrt{6}}\right)\delta\phi\,,$$
$$\Phi_{-} = \Phi_4 + \frac{1}{\sqrt{6}}\coth\left(\frac{\phi}{\sqrt{6}}\right)\delta\phi\,, \quad \Psi_{-} = \Phi_4 - \frac{1}{\sqrt{6}}\coth\left(\frac{\phi}{\sqrt{6}}\right)\delta\phi\,. \quad (2.35)$$

The brane conformal times may be expressed in terms of t by integrating,

$$\tau_{\pm} = \int_0^t \frac{dt}{q(t,y_{\pm})}\,, \quad (2.36)$$

where $q \equiv b/n$. So for example the boundary value of the bulk metric perturbation Φ_L on the positive tension brane is given explicitly by

$$\Phi_L(t, y_+) = \Phi_4\left(\int q(t, y_+)^{-1} dt\right) + \frac{1}{\sqrt{6}} \tanh\left(\phi\left(\int q(t, y_+)^{-1} dt\right)/\sqrt{6}\right)\delta\phi\left(\int q(t, y_+)^{-1} dt\right), \quad (2.37)$$

where y_+ is the location of the positive tension brane.
Also in these equations (2.35-2.37), we have the following connection with the Aurea ratio:

$$\frac{1}{\sqrt{6}} = 0{,}40824829 = (\Phi)^{-13/7} = \left(\frac{\sqrt{5}+1}{2}\right)^{-13/7} = 0{,}4091477 \cong 0{,}409 .$$

Now, using (2.22) and the following equation

$$\Psi(t) = \zeta_{4,M}(t) + \frac{\varepsilon_0 \tanh(y_0/2)}{32k^2L^2\cosh^2(y_0/2)}\left[18(y_0 - \sinh(y_0)) - L^2(r_+ - r_-)(-3y_0 + \sinh(y_0))\right] + O\left(\rho_\pm^2 L^2, t, t\ln|kt|\right) \quad (2.38)$$

to find Q and P before and after the bounce for all components of the metric perturbations and matching according to the rule given in equation (2.31) results in $\zeta_{4,M}$ inheriting two separate scale-invariant long wavelength contributions in the post-singularity state. The first occurs as a direct consequence of the sign change in (2.31), and is independent of the amount of radiation generated at the singularity. The second is proportional to the difference in the densities of the radiation on the two branes. At leading order in velocities we have

$$\Delta\zeta_{4,M} = \frac{3}{64}\frac{\varepsilon_0}{k^2L^2}\left(V_{in}^4 + V_{out}^4\right) - \frac{(r_+ - r_-)\varepsilon_0 V_{out}^2}{32k^2} + O\left(r_\pm V^3, V^5 L^{-2}, \rho_\pm^2 L^2\right), \quad (2.39)$$

where V_{in} and V_{out} are the relative velocities of the branes before and after collision. Note that since $P \propto \varepsilon_0$, matching P is in fact equivalent to matching ε_0 across the collision. In terms of four-dimensional parameters including r_4 given in (2.12) defining the abundance of the radiation and the fractional density mismatch f defined in (2.11), we find again at leading order in velocities

$$\Delta\zeta_{4,M} = \frac{3}{64}\frac{\varepsilon_0}{k^2L^2}\left(V_{in}^4 + V_{out}^4\right) - \frac{3\varepsilon_0}{2\cdot 8k^2}\frac{f r_4 V_{out}^3}{L} . \quad (2.40)$$

This is the final result, relevant to tracking perturbations across the singularity in the ekpyrotic and cyclic models.
The result for the long wavelength curvature perturbation amplitude in the four-dimensional effective theory, propagated into the hot Big Bang after the brane collision is:

$$\zeta_M = \frac{9\varepsilon_0}{16k^2L^2}\frac{\tanh(\theta/2)}{\cosh^2(\theta/2)}(\theta - \sinh\theta) \approx \frac{3\varepsilon_0 V_{coll}^4}{64k^2L^2} \quad (2.41)$$

where θ is the rapidity corresponding to the relative speed V_{coll} of the branes at collision, and the second formula assumes V_{coll} is small. L is the bulk curvature scale, and $\frac{\varepsilon_0}{16k^2}$ has a scale invariant power spectrum.

With regard the eqs. (2.40) and (2.41), we have the following mathematical connections with the Aurea ratio and the Fibonacci's numbers:

64 = 48 + 16 = 24 + 24 + 8 + 8; $64 = 8^2 = 34 + 21 + 8 + 1$ that are Fibonacci's numbers;
$24 = 3 \times 8$; $64 = (3 \times 8) + (3 \times 8) + 8 + 8$, with 3 and 8 that are Fibonacci's numbers;
$8 \cong (\Phi)^{-53/7} \cdot 306{,}342224$.

Furthermore, in the eq. (2.40) 2, 3 and 8 are Fibonacci's numbers and 8 is connected with the "modes" that correspond to the physical vibrations of a superstring by the following Ramanujan function:

$$8 = \frac{1}{3} \frac{4\left[anti\log \frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4} w'} \phi_{w'}(itw')}\right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}.$$

The d + 1-dimensional space-time we consider is a direct product of d – 1-dimensional Euclidean space, R^{d-1}, and a two-dimensional time-dependent space-time known as compactified Milne space-time, or $\mathcal{M}_C$. The line element for $\mathcal{M}_C \times R^{d-1}$ is thus

$$ds^2 = -dt^2 + t^2 d\theta^2 + d\vec{x}^2, \qquad 0 \le \theta \le \theta_0, \qquad -\infty < t < \infty, \quad (2.42)$$

where $\vec{x}$ are Euclidean coordinates on R^{d-1}, θ parameterizes the compact dimension and t is the time. The compact dimension may either be a circle, in which case we identify θ with $\theta + \theta_0$, or a Z_2 orbifold in which case we identify θ with $\theta + 2\theta_0$ and further identify θ with $2\theta_0 - \theta$. The fixed points $\theta = 0$ and $\theta = \theta_0$ are then interpreted as tensionless Z_2-branes approaching at rapidity θ_0, colliding at $t = 0$ to re-emerge with the same relative rapidity. The orbifold reduction is the case of prime interest in the ekpyrotic/cyclic models, originally motivated by the construction of heterotic M theory from eleven dimensional supergravity. In these models, the boundary branes possess nonzero tension. However, the tension is a subdominant effect near $t = 0$ and the brane collision is locally well-modelled by $\mathcal{M}_C \times R^{d-1}$.

Now consider a string loop of radius R in M theory frame. Its mass M is $2\pi R$ times the effective string tension $\mu_2 L$, where L is the size of the extra dimension. The effective Einstein-frame gravitational coupling is given by $\kappa_d^2 = \kappa_{d+1}^2 / L$. The gravitational potential produced by such a loop in d spacetime dimensions is:

$$\Phi = -\kappa_d^2 \frac{M}{(d-2)A_{d-2}R^{d-3}} \quad (2.43)$$

where A_D is the area of the unit D-sphere, $A_D = 2\pi \frac{D+1}{2} / \Gamma((D+1)/2)$. Specializing to the case of interest, namely 2-branes in eleven-dimensional M theory, the tension μ_2 is related to the eleven dimensional gravitational coupling by a quantization condition relating to the four-form flux, reading

$$\mu_2^3 = 2\pi^2 / \left(n\kappa_{11}^2\right) \quad (2.44)$$

with n an integer. Equations (2.43) and (2.44) then imply that the typical gravitational potential around a string loop is

$$\Phi = -\frac{105}{64\pi\mu_2^2 R^6 n} \approx -\left(\mu_2^2 R^6 n\right)^{-1} \approx -\theta_0^2 / n \quad (2.45)$$

up to numerical factors.
With regard the (2.45) we note that are possible the following mathematical connections with the Aurea ratio and the Fibonacci's numbers:

$$105 = 89 + 16 = 55 + 34 + 8 + 8; \quad 105 = 21 \times 5;$$
$$64 = 8^2 = 34 + 21 + 8 + 1 = 48 + 16 = 24 + 24 + 8 + 8 = (3 \times 8) + (3 \times 8) + 8 + 8;$$

and 3, 5, 8, 21, 34, 55 and 89 are Fibonacci's numbers. Furthermore, we have that

$$\frac{105}{64} = 1{,}640625 \cong 1{,}640; \quad (\Phi)^{5/7} + (\Phi)^{-21/7} = \left(\frac{\sqrt{5}+1}{2}\right)^{5/7} + \left(\frac{\sqrt{5}+1}{2}\right)^{-21/7} = 1{,}64625 \cong 1{,}646.$$

Thence, the gravitational potential on the scale of the loops is of order θ_0^2 and therefore is consistently small for small collision rapidity. Since the mean separation of the loops when they are produced is of order their size R, this potential Φ is the typical gravitational potential throughout space. Multiplying the tt component of the background metric (2.42) by $1 + 2\Phi$ and redefining t, we conclude that the outgoing metric has an expansion rapidity of order $\approx \theta_0\left(1 + C\theta_0^2\right)$ with C a constant of order unity. We conclude that for small θ_0 the gravitational back-reaction due to string loop productions is small.

2.1 On some equations concerning the solution of a braneworld Big Crunch/Big Bang Cosmology.

We shall employ a coordinate system in which the five-dimensional line element for the background takes the form

$$ds^2 = n^2(t,y)\left(-dt^2 + t^2 dy^2\right) + b^2(t,y) d\vec{x}^2, \quad (2.46)$$

where y parameterizes the fifth dimension and x^i, $i = 1,2,3,$ the three non-compact dimensions. Cosmological isotropy excludes $dtdx^i$ or $dydx^i$ terms, and homogeneity ensures n and b are independent of $\vec{x}$. The t, y part of the background metric may then be taken to be conformally flat.

We find it simplest to work in coordinates in which the brane locations are fixed but the bulk evolves. The bulk metric is therefore given by (2.46), with the brane locations fixed at $y = \pm y_0$ for all time t. The five-dimensional solution then has to satisfy both the Einstein equations and the Israel matching conditions on the branes. The bulk Einstein equations read $G_\mu^\nu = -\Lambda \delta_\mu^\nu$, where the bulk cosmological constant is $\Lambda = -6/L^2$. Evaluating the linear combinations $G_0^0 + G_5^5$ and $G_0^0 + G_5^5 - (3/2)G_i^i$, we find

$$\beta_{,\tau\tau} - \beta_{,yy} + \beta_{,\tau}^2 - \beta_{,y}^2 + 12e^{2\nu} = 0, \quad (2.47) \qquad \nu_{,\tau\tau} - \nu_{,yy} + \frac{1}{3}\left(\beta_{,y}^2 - \beta_{,\tau}^2\right) - 2e^{2\nu} = 0, \quad (2.48)$$

where $(t/L) = e^\tau$, $\beta \equiv 3\ln b$ and $\nu \equiv \ln(nt/L)$. The Israel matching conditions on the branes read

$$\frac{b_{,y}}{b} = \frac{n_{,y}}{n} = \frac{nt}{L}, \quad (2.49)$$

where all quantities are to be evaluated at the brane locations $y = \pm y_0$.

Now we express the metric as a series of Dirichlet or Neumann polynomials in y_0 and y, bounded at order n by a constant times y_0^n, such that the series satisfies the Israel matching conditions exactly at every order in y_0. To implement this, we first change variables from b and n to those obeying Neumann boundary conditions. From (2.49), b/n is Neumann. Likewise, if we define $N(t,y)$ by

$$nt = \frac{1}{N(t,y) - y}, \quad (2.50)$$

then one can easily check that $N(t,y)$ is also Neumann on the branes. Since N and b/n obey Neumann boundary conditions on the branes, we can expand both in a power series

$$N = N_0(t) + \sum_{n=3}^{\infty} N_n(t) P_n(y), \qquad b/n = q_0(t) + \sum_{n=3}^{\infty} q_n(t) P_n(y), \quad (2.51)$$

where $P_n(y)$ are polynomials

$$P_n(y) = y^n - \frac{n}{n-2} y^{n-2} y_0^2, \qquad n = 3,4,\ldots \quad (2.52)$$

satisfying Neumann boundary conditions and each bounded by $|P_n(y)| < 2y_0^n/(n-2)$, for the relevant range of y. Note that the time-dependent coefficients in this ansatz may also be expanded as a power series in y_0. By construction, our ansatz satisfies the Israel matching conditions exactly at each order in the expansion. Substituting the series ansatze (2.51) into the background Einstein equations (2.47) and (2.48), we may determine the solution order by order in the rapidity y_0. At each order in y_0, one generically obtains a number of linearly independent algebraic equations, and at most one ordinary differential equation in t.

The first few terms of the solution are

$$N_0 = \frac{1}{t} - \frac{1}{2}ty_0^2 + \frac{1}{24}t\left(8 - 9t^2\right)y_0^4 + \dots \quad (2.53) \qquad N_3 = -\frac{1}{6} + \left(\frac{5}{72} - 2t^2\right)y_0^2 + \dots \quad (2.54)$$

and

$$q_0 = 1 - \frac{3}{2}t^2y_0^2 + \left(t^2 - \frac{7}{8}t^4\right)y_0^4 + \dots \quad (2.55) \qquad q_3 = -2t^3y_0^2 + \dots \quad (2.56).$$

With regard the eqs. (2.54-2.55) we note that are possible the following mathematical connections with the Aurea ratio and the Fibonacci's numbers:

$$\frac{5}{72} = 0{,}06944 \cong (\Phi)^{-39/7} = \left(\frac{\sqrt{5}+1}{2}\right)^{-39/7} = 0{,}06849207\,;$$

$$\frac{7}{8} = 0{,}875 \cong (\Phi)^{-2/7} = \left(\frac{\sqrt{5}+1}{2}\right)^{-2/7} = 0{,}87154\,.$$

Furthermore, $72(= 24 \times 3)$ and 8 are connected with the "modes" that correspond to the physical vibrations of the bosonic strings and to the physical vibrations of a superstring by the following Ramanujan functions:

$$8 = \frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]},$$

$$24 = \frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}.$$

To calculate the affine distance between the branes along a spacelike geodesic we must solve the geodesic equations in the bulk. Let us first consider the situation in Birkhoff-frame coordinates for which the bulk metric is static and the branes are moving. The Birkhoff-frame metric takes the form

$$ds^2 = dY^2 - N^2(Y)dT^2 + A^2(Y)d\vec{x}^2, \quad (2.57)$$

where for Schwarzschild-AdS with a horizon at $Y = 0$,

$$A^2(Y)=\frac{\cosh(2Y/L)}{\cosh(2Y_0/L)}, \qquad N^2(Y)=\frac{\cosh(2Y_0/L)}{\cosh(2Y/L)}\left[\frac{\sinh(2Y/L)}{\sinh(2Y_0/L)}\right]^2. \quad (2.58)$$

At $T=0$, the Y-coordinate of the branes is represented by the parameter Y_0; their subsequent trajectories $Y_\pm(T)$ can then be determined by integrating the Israel matching conditions, which read $\tanh(2Y_\pm/L)=\pm\sqrt{1-V_\pm^2}$, where $V_\pm=(dY_\pm/dT)/N(Y_\pm)$ are the proper speeds of the positive- and negative-tension branes respectively. From this, it further follows that Y_0 is related to the rapidity y_0 of the collision by $\tanh y_0=\operatorname{sech}(2Y_0/L)$.

For the purpose of measuring the distance between the branes, a natural choice is to use spacelike geodesics that are orthogonal to the four translational Killing vectors of the static bulk, corresponding to shifts in $\vec{x}$ and T. Taking the $\vec{x}$ and T coordinates to be fixed along the geodesic then, we find that $Y_{,\lambda}$ is constant for an affine parameter λ along the geodesic. To make the connection to our original brane-static coordinate system, recall that the metric function $b^2(t,y)=A^2(Y)$, and thus

$$Y_{,\lambda}^2=\frac{\left(bb_{,t}t_{,\lambda}+bb_{,y}y_{,\lambda}\right)^2}{b^4-\theta^2}=n^2\left(-t_{,\lambda}^2+t^2y_{,\lambda}^2\right), \quad (2.59)$$

where we have introduced the constant $\theta=\tanh y_0=V/c$. Adopting y now as the affine parameter, we have

$$0=\left(b_{,t}^2b^2+n^2\left(b^4-\theta^2\right)\right)t_{,y}^2+2b_{,t}b_{,y}b^2t_{,y}+\left(b_{,y}^2b^2-n^2t^2\left(b^4-\theta^2\right)\right), \quad (2.60)$$

where t is to be regarded now as a function of y. We can solve this equation order by order in y_0 using the series ansatz

$$t(y)=\sum_{n=0}^{\infty}c_ny^n, \quad (2.61)$$

where the constants c_n are themselves series in y_0. Using the series solution for the background geometry given in the eqs. (2.53)-(2.56), and imposing the boundary condition that $t(y_0)=t_0$, we obtain

$$c_0=t_0+\frac{t_0y_0^2}{2}-2t_0^2y_0^3+\frac{\left(t_0+36t_0^3\right)y_0^4}{24}-t_0^2\left(1+5t_0^2\right)y_0^5+\left(\frac{t_0}{720}+\frac{17t_0^3}{4}+4t_0^5\right)y_0^6-\frac{t_0^2\left(13+250t_0^2+795t_0^4\right)y_0^7}{60}+O\left(y_0^8\right) \quad (2.62)$$

$$c_1=2t_0^2y_0^2+\left(\frac{5t_0^2}{3}+5t_0^4\right)y_0^4-8t_0^3y_0^5+\left(\frac{91t_0^2}{180}+\frac{23t_0^4}{6}+\frac{53t_0^6}{4}\right)y_0^6+O\left(y_0^7\right) \quad (2.63)$$

$$c_2=-\frac{t_0}{2}-\frac{t_0\left(1+6t_0^2\right)y_0^2}{4}+t_0^2y_0^3-\left(\frac{t_0}{48}-2t_0^3+4t_0^5\right)y_0^4+\frac{\left(t_0^2+23t_0^4\right)y_0^5}{2}+O\left(y_0^6\right) \quad (2.64)$$

$$c_3=-\frac{5t_0^2y_0^2}{3}-\frac{t_0^2\left(25+201t_0^2\right)y_0^4}{18}+O\left(y_0^5\right) \quad (2.65)$$

$$c_4=\frac{5t_0}{24}+\left(\frac{5t_0}{48}+\frac{7t_0^3}{4}\right)y_0^2-\frac{5t_0^2y_0^3}{12}+O\left(y_0^4\right) \quad (2.66)$$

$c_5 = \frac{61 t_0^2 y_0^2}{60} + O(y_0^3) \quad (2.67)$

$c_6 = -\frac{61 t_0}{720} + O(y_0^2) \quad (2.68)$

$c_7 = 0 + O(y_0). \quad (2.69)$

Substituting $t_0 = x_0 / y_0$ and $y = \omega y_0$ we find $x(\omega) = x_0 / y_0 + O(y_0)$, i.e. to lowest order in y_0, the geodesics are trajectories of constant time lying solely along the ω direction. Hence in this limit, the affine and metric separation of the branes, defined with the following equation

$$d_m = L \int_{-1}^{1} \frac{x_4}{1 - \omega x_4} d\omega + O(y_0^2) = L \ln\left(\frac{1 + x_4}{1 - x_4}\right) + O(y_0^2), \quad (2.70)$$

must necessarily agree. To check this, the affine distance between the branes is given by

$$\frac{d_a}{L} = \int_{-y_0}^{y_0} n\sqrt{t^2 - t'^2}\, dy = 2 t_0 y_0 + \frac{(t_0 + 5 t_0^3) y_0^3}{3} - 4 t_0^2 y_0^4 + \frac{(t_0 - 10 t_0^3 + 159 t_0^5) y_0^5}{60} - \frac{2(t_0^2 + 30 t_0^4) y_0^6}{3} + \\ + \frac{(t_0 + 31115 t_0^3 - 5523 t_0^5 + 12795 t_0^7) y_0^7}{2520} + O(y_0^8), \quad (2.71)$$

which to lowest order in y_0 reduces to

$$\frac{d_a}{L} = 2 x_0 + \frac{5 x_0^3}{3} + \frac{53 x_0^5}{20} + \frac{853 x_0^7}{168} + O(x_0^8) + O(y_0^2), \quad (2.72)$$

in agreement with the series expansion of (2.70).
We obtain also the following equation:

$$\frac{\delta d_m}{L} = \int_{-1}^{1} \frac{n x_4 \xi_{40}(x_4)}{b^2} e^{-\frac{1}{2} x_4^2} d\omega = \frac{2 x_4 \xi_{40}(x_4)}{(1 - x_4^2)^2} = \\ = \frac{1}{(x_4^2 - 1)} \left[\frac{8}{3} \tilde{k} x_4 \left(A_0 J_0(\tilde{k} x_4) + B_0 Y_0(\tilde{k} x_4) \right) - 4 \left(A_0 J_1(\tilde{k} x_4) + B_0 Y_1(\tilde{k} x_4) \right) \right]. \quad (2.72b)$$

To evaluate the perturbation δd_a in the affine distance between the branes, consider

$$\delta \int \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}\, d\lambda = \frac{1}{2} \int \frac{d\lambda}{\sqrt{g_{\rho\sigma} \dot{x}^\rho \dot{x}^\sigma}} \left(\delta g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + g_{\mu\nu,\kappa} \delta x^\kappa \dot{x}^\mu \dot{x}^\nu + 2 g_{\mu\nu} \dot{x}^\mu \delta \dot{x}^\nu \right) = \left[\frac{\dot{x}_\nu \delta x^\nu}{\sqrt{g_{\rho\sigma} \dot{x}^\rho \dot{x}^\sigma}} \right] + \frac{1}{2} \int \frac{\delta g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}{\sqrt{g_{\rho\sigma} \dot{x}^\rho \dot{x}^\sigma}} d\lambda$$

(2.73)

where dots indicate differentiation with respect to the affine parameter λ, and in going to the second line we have integrated by parts and made use of the background geodesic equation $\ddot{x}_\sigma = \frac{1}{2} g_{\mu\nu,\sigma} \dot{x}^\mu \dot{x}^\nu$ and the constraint $g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 1$. If the endpoints of the geodesics on the branes are unperturbed, this expression is further simplified by the vanishing of the surface term. Converting to coordinates where $t_0 = x_0 / y_0$ and $y = \omega y_0$, to lowest order in y_0 the unperturbed geodesics lie purely in the ω direction, and so the perturbed affine distance is identical to the following perturbed metric distance

$$\frac{\delta d_m}{L} = \int_{-y_0}^{y_0} nt\Gamma_L dy = \int_{-1}^{1} \frac{nx_4\xi_4}{b^2} e^{-\frac{1}{2}x_4^2} d\omega . \quad (2.74)$$

Explicitly, we find

$$\frac{\delta d_a}{L} = -\frac{2\left(B + At_0^2\right)y_0}{t_0} - \left[\frac{B\left(4+3t_0^2\right)}{12t_0} + \frac{A\left(t_0 + 9t_0^3\right)}{3}\right]y_0^3 + \left(-4B + 4At_0^2\right)y_0^4 +$$
$$-\left[\frac{B\left(2+2169t_0^2+135t_0^4\right) + 2At_0^2\left(1+1110t_0^2+375t_0^4\right)}{120t_0}\right]y_0^5 + \left[\frac{4At_0^2\left(1+42t_0^2\right) - B\left(4+57t_0^2\right)}{6}\right]y_0^6 +$$
$$-\left[\frac{B\left(4+88885t_0^2+952866t_0^4+28875t_0^6\right) + 4At_0^2\left(1-152481t_0^2+293517t_0^4+36015t_0^6\right)}{10080t_0}\right]y_0^7 + O\left(y_0^8\right), \quad (2.75)$$

which, substituting $t_0 = x_0 / y_0$ and dropping terms of $O\left(y_0^2\right)$, reduces to

$$\frac{\delta d_a}{L} = -\frac{2\tilde{B}}{x_0} - 2Ax_0 - \frac{\tilde{B}}{4}x_0 - 3Ax_0^3 - \frac{9}{8}\tilde{B}x_0^3 - \frac{25}{4}Ax_0^5 - \frac{275}{96}\tilde{B}x_0^5 - \frac{343}{24}Ax_0^7 + O\left(x_0^8\right), \quad (2.76)$$

where $\tilde{B} = By_0^2$. Also this expression is in accordance with the series expansion of (2.74). However, the perturbed affine and metric distance do not agree at $O\left(y_0^2\right)$.

With regard the eqs. (2.72) and (2.76) we have the following mathematical connections with Aurea ratio and with Fibonacci's numbers:

$$\frac{5}{3} = 1{,}666 \cong (\Phi)^{5/7} + (\Phi)^{-20/7} = \left(\frac{\sqrt{5}+1}{2}\right)^{5/7} + \left(\frac{\sqrt{5}+1}{2}\right)^{-20/7} = 1{,}663054757;$$

$$\frac{53}{20} = 2{,}65 \cong (\Phi)^{12/7} + (\Phi)^{-14/7} = \left(\frac{\sqrt{5}+1}{2}\right)^{12/7} + \left(\frac{\sqrt{5}+1}{2}\right)^{-14/7} = 2{,}663697$$

$$\frac{853}{168} = 5{,}077381 \cong (\Phi)^{21/7} + (\Phi)^{-16/7} + (\Phi)^{-10/7} = \left(\frac{\sqrt{5}+1}{2}\right)^{21/7} + \left(\frac{\sqrt{5}+1}{2}\right)^{-16/7} + \left(\frac{\sqrt{5}+1}{2}\right)^{-10/7} = 5{,}0718$$

$$\frac{9}{8} = 1{,}125 \cong (\Phi)^{-7/7} + (\Phi)^{-10/7} = \left(\frac{\sqrt{5}+1}{2}\right)^{-7/7} + \left(\frac{\sqrt{5}+1}{2}\right)^{-10/7} = 1{,}12089;$$

$$\frac{25}{4} = 6{,}25 \cong (\Phi)^{28/7} - (\Phi)^{-7/7} = \left(\frac{\sqrt{5}+1}{2}\right)^{28/7} - \left(\frac{\sqrt{5}+1}{2}\right)^{-7/7} = 6{,}236067977;$$

$$\frac{275}{96} = 2{,}864583 \cong (\Phi)^{14/7} + (\Phi)^{-20/7} = \left(\frac{\sqrt{5}+1}{2}\right)^{14/7} + \left(\frac{\sqrt{5}+1}{2}\right)^{-20/7} = 2{,}870901;$$

$$\frac{343}{24} = 14{,}2916 \cong (\Phi)^{35/7} + (\Phi)^{17/7} = \left(\frac{\sqrt{5}+1}{2}\right)^{35/7} + \left(\frac{\sqrt{5}+1}{2}\right)^{17/7} = 14{,}3078 .$$

Furthermore, we have that:

53 = 21 + 34 – 2; 853 = 2 + 8 + 233 + 610; where 2, 8, 21, 34, 233 and 610 are Fibonacci's numbers;

$9 = 3^2$; $25 = 34 - 3^2$; 275 = 233 + 34 + 8; 343 = 233 + 34 + 8 + 13 + 55; where 3, 8, 13, 34, 55 and 233 are Fibonacci's numbers.

3. On some equations concerning the generating ekpyrotic curvature perturbations before the Big Bang. [6] [7] [8] [16]

With regard the ekpyrotic perturbations including gravity, we consider the action for N decoupled fields interacting only through gravity:

$$\int d^4x\sqrt{-g}\left[\frac{1}{2}R - \frac{1}{2}\sum_{i=1}^{N}(\partial\phi_i)^2 - \sum_{i=1}^{N}V_i(\phi_i)\right], \quad (3.1)$$

where we have chosen units in which $8\pi G \equiv M_{Pl}^{-2} = 1$. In a flat Friedmann-Robertson-Walker background with line element $ds^2 = -dt^2 + a^2(t)d\,\mathbf{x}^2$, the scalar field and Friedmann equations are given by

$$\ddot{\phi}_i + 3H\dot{\phi}_i + V_{i,\phi_i} = 0 \quad (3.2)$$

and

$$H^2 = \frac{1}{3}\left[\frac{1}{2}\sum_i \dot{\phi}_i^2 + \sum_i V_i(\phi_i)\right], \quad (3.3)$$

where $H = \dot{a}/a$ and $V_{i,\phi_i} = (\partial V_i/\partial\phi_i)$ with no summation implied. Another useful relation is

$$\dot{H} = -\frac{1}{2}\sum_i \dot{\phi}_i^2. \quad (3.4)$$

If all the fields have negative exponential potentials $V_i(\phi_i) = -V_i e^{-c_i\phi_i}$ then as is well-known, the Einstein-scalar equations admit the scaling solution

$$a = (-t)^p, \qquad \phi_i = \frac{2}{c_i}\ln(-A_i t), \qquad V_i = \frac{2A_i^2}{c_i^2}, \qquad p = \sum_i \frac{2}{c_i^2}. \quad (3.5)$$

Thus, if $c_i >> 1$ for all i, we have a very slowly contracting universe with $p << 1$.
We focus on the entropy perturbation since this is a local, gauge-invariant quantity, and on the case of only two scalar fields. The entropy perturbation equation

$$\ddot{\delta s}+\left[k^2+\left(\frac{\dot{\phi}_2^2 V_{,\phi_1\phi_1}-2\dot{\phi}_1\dot{\phi}_2 V_{,\phi_1\phi_2}+\dot{\phi}_1^2 V_{,\phi_2\phi_2}}{\dot{\phi}_1^2+\dot{\phi}_2^2}\right)+3\left(\frac{\dot{\phi}_2 V_{,\phi_1}-\dot{\phi}_1 V_{,\phi_2}}{\dot{\phi}_1^2+\dot{\phi}_2^2}\right)^2\right]\delta s=0 \quad (3.6)$$

in flat spacetime is replaced by

$$\ddot{\delta s}+3H\dot{\delta s}+\left[\frac{k^2}{a^2}+\left(\frac{\dot{\phi}_2^2 V_{,\phi_1\phi_1}-2\dot{\phi}_1\dot{\phi}_2 V_{,\phi_1\phi_2}+\dot{\phi}_1^2 V_{,\phi_2\phi_2}}{\dot{\phi}_1^2+\dot{\phi}_2^2}\right)+3\left(\frac{\dot{\phi}_2 V_{,\phi_1}-\dot{\phi}_1 V_{,\phi_2}}{\dot{\phi}_1^2+\dot{\phi}_2^2}\right)^2\right]\delta s=\frac{4k^2\dot{\theta}}{a^2\sqrt{\dot{\phi}_1^2+\dot{\phi}_2^2}}\Phi . \quad (3.7)$$

For simplicity we will focus attention on straight line trajectories in scalar field space. Since $\dot{\theta}=0$, the entropy perturbation is not sourced by the Newtonian potential Φ and we can solve the equations rather simply. We shall assume that the background solution obeys scaling symmetry so that $\dot{\phi}_2=\gamma\dot{\phi}_1$. Denoting τ derivatives with primes, and introducing the re-scaled entropy field

$$\delta S=a(\tau)\delta s , \quad (3.8)$$

eq. (3.7) becomes

$$\delta S''+\left(k^2-\frac{a''}{a}+a^2V_{,\phi\phi}\right)\delta S=0 . \quad (3.9)$$

The crucial term governing the spectrum of the perturbations is then

$$\tau^2\left(\frac{a''}{a}-V_{,\phi\phi}a^2\right). \quad (3.10)$$

When this quantity is approximately 2, we will again get nearly scale-invariant perturbations. It is customary to define the quantity

$$\varepsilon\equiv\frac{3}{2}(1+w)\equiv\frac{\dot{\phi}_1^2+\dot{\phi}_2^2}{2H^2}=\frac{(1+\gamma^2)\dot{\phi}^2}{2H^2} . \quad (3.11)$$

In the background scaling solution,

$$\varepsilon=\frac{c^2}{2(1+\gamma^2)} . \quad (3.12)$$

We proceed by evaluating the quantity in (3.10) in an expansion in inverse powers of ε and its derivatives with respect to N, where $N=\ln(a/a_{end})$, where a_{end} is the value of a at the end of the ekpyrotic phase. Note that N decreases as the fields roll downhill and the contracting ekpyrotic phase proceeds. We obtain the first term in (3.10) by differentiating (3.4), obtaining

$$\frac{a''}{a}=2H^2a^2\left(1-\frac{1}{2}\varepsilon\right). \quad (3.13)$$

The second term in (3.10) is found by differentiating (3.11) twice with respect to time and using the background equations and the definition of N. We obtain

$$a^2V_{,\phi\phi} = -a^2H^2\left(2\varepsilon^2 - 6\varepsilon - \frac{5}{2}\varepsilon_{,N}\right) + O\left(\varepsilon^0\right). \quad (3.14)$$

Finally, need to express $\mathcal{H} \equiv (a'/a) = aH$ in terms of the conformal time τ. From (3.13) we obtain

$$\mathcal{H}' = \mathcal{H}^2(1-\varepsilon), \quad (3.15)$$

which integrates to

$$\mathcal{H}^{-1} = \int_0^\tau d\tau(\varepsilon - 1). \quad (3.16)$$

Now, inserting $1 = d(\tau)/d\tau$ under the integral and using integration by parts we can re-write this as

$$\mathcal{H}^{-1} = \varepsilon\tau\left(1 - \frac{1}{\varepsilon} - (\varepsilon\tau)^{-1}\int_0^\tau \varepsilon'\tau d\tau\right). \quad (3.17)$$

Using the same procedure once more, the integral in this expression can be written as

$$(\varepsilon\tau)^{-1}\int_0^\tau \varepsilon'\tau d\tau = \frac{\varepsilon'\tau}{\varepsilon} - (\varepsilon\tau)^{-1}\int_0^\tau \frac{d}{d\tau}(\varepsilon'\tau)\tau d\tau. \quad (3.18)$$

Now using the fact that $\varepsilon' = \mathcal{H}\varepsilon_{,N}$, and that to leading order in $1/\varepsilon$, $\mathcal{H}$ can be replaced by its value in the scaling solution (with constant ε), $\mathcal{H}\tau = \varepsilon^{-1}$, we can re-write the second term on the right-hand side as

$$-(\varepsilon\tau)^{-1}\int_0^\tau \frac{d}{d\tau}(\varepsilon'\tau)\tau d\tau = -(\varepsilon\tau)^{-1}\int_0^\tau d\tau \frac{1}{\varepsilon}\left(\frac{\varepsilon_{,N}}{\varepsilon}\right)_{,N}, \quad (3.19)$$

which shows that this term is of order $1/\varepsilon^2$ and can thus be neglected. Altogether we obtain

$$\mathcal{H}^{-1} = \int_0^\tau d\tau(\varepsilon - 1) \approx \varepsilon\tau\left(1 - \frac{1}{\varepsilon} - \frac{\varepsilon_{,N}}{\varepsilon^2}\right). \quad (3.20)$$

Using (3.13) and (3.14) with (3.20) we can calculate the crucial term entering the entropy perturbation equation,

$$\tau^2\left(\frac{a''}{a} - V_{,\phi\phi}a^2\right) = 2\left(1 - \frac{3}{2\varepsilon} + \frac{3}{4}\frac{\varepsilon_{,N}}{\varepsilon^2}\right). \quad (3.21)$$

The deviation from scale-invariance in the spectral index of the entropy perturbation is then given by

$$n_s - 1 = \frac{2}{\varepsilon} - \frac{\varepsilon_{,N}}{\varepsilon^2}. \quad (3.22)$$

The first term on the right-hand side is the gravitational contribution, which, being positive, tends to make the spectrum blue. The second term is the non-gravitational contribution, which tends to make the spectrum red.

Now defining $\mathcal{R}$ to be the curvature perturbation on comoving spatial slices, for N scalar fields with general Kahler metric $g_{ij}(\phi)$ on scalar field space, the linearized Einstein-scalar field equations lead to

$$\dot{\mathcal{R}} = -\frac{H}{\dot{H}}\left(g_{ij}\frac{D^2\phi^i}{Dt^2}s^j - \frac{k^2}{a^2}\Psi \right), \quad (3.23)$$

where the $N-1$ entropy perturbations

$$s^i = \delta\phi^i - \dot{\phi}^i \frac{g_{jk}(\phi)\dot{\phi}^j\delta\phi^k}{g_{lm}(\phi)\dot{\phi}^l\dot{\phi}^m} \quad (3.24)$$

are just the components of $\delta\phi^i$ orthogonal to the background trajectory, and the operator D^2/Dt^2 is just the geodesic operator on scalar field space. Things simplify because the scalar field space is flat, so the metric is $g_{ij} = \delta_{ij}$, and D/Dt reduces to an ordinary time derivative. Considering only two scalar fields, we have

$$s^1 = -\dot{\phi}_2\delta s/\sqrt{\dot{\phi}_1^2+\dot{\phi}_2^2}\,, \qquad s^2 = +\dot{\phi}_1\delta s/\sqrt{\dot{\phi}_1^2+\dot{\phi}_2^2}\,. \quad (3.25)$$

For a straight line trajectory in field space, the right-hand side of (3.24) vanishes even if the entropy perturbation is nonzero.

We assume that the scalar field bounce occurs after the ekpyrotic potentials are turned off, so that the universe is kinetic-dominated from the 4d point-of-view. The scalar field trajectory is $\dot{\phi}_2 = -\tilde{\gamma}\dot{\phi}_1$, for $t < t_b$, and $\dot{\phi}_2 = \tilde{\gamma}\dot{\phi}_1$, for $t > t_b$, with $\dot{\phi}_1$ constant and negative in the vicinity of the bounce. The bounce leads to a delta function on the right-hand side of (3.23),

$$\frac{D^2\phi_2}{Dt^2} = \delta(t-t_b)2\dot{\phi}_2(t_b^+), \quad (3.26)$$

where t_b is the time of the bounce of the negative-tension brane. As can be readily seen from (3.23), if the entropy perturbations already have acquired a scale-invariant spectrum by the time t_b, then the bounce leads to their instantaneous conversion into curvature perturbations with precisely the same long wavelength spectrum. We can estimate the amplitude of the resulting curvature perturbation by integrating equation (3.24) using (3.26). Since we have assumed the universe is kinetic-dominated at this time, $H = 1/(3t)$. Since the entropy perturbation

$$\delta s \equiv (\dot{\phi}_1\delta\phi_2 - \dot{\phi}_2\delta\phi_1)/\sqrt{\dot{\phi}_1^2+\dot{\phi}_2^2} \quad (3.27)$$

is canonically normalized, its spectrum is given by

$$\langle \delta\phi^2 \rangle = \hbar\int \frac{k^2dk}{4\pi^2}\frac{1}{k^3t^2} \quad (3.28)$$

up to non-scale invariant corrections. This expression only holds as long as the ekpyrotic behaviour is still underway: the ekpyrotic phase ends at a time t_{end} approximately given by $|V_{\min}| = 2/(c^2t_{end}^2)$. After t_{end}, the entropy perturbation obeys $\ddot{\delta s} + t^{-1}\dot{\delta s} = 0$, which has the solution $\delta s = A + B\ln(-t)$.

Matching this solution to the growing mode solution t^{-1} in the ekpyrotic phase, one finds that by t_b the entropy grows by an additional factor of $1+\ln(t_{end}/t_b)$. Employing the Friedmann equation to relate $\dot{\phi}_2 = \tilde{\gamma}\dot{\phi}_1$ to H, putting everything together and restoring the Planck mass, we find for the variance of the spatial curvature perturbation in the scale-invariant case,

$$\left\langle \mathcal{R}^2 \right\rangle = \hbar \frac{c^2 |V_{\min}|}{3\pi^2 M_{Pl}^2} \frac{\tilde{\gamma}^2}{\left(1+\tilde{\gamma}^2\right)^2} \left(1+\ln(t_{end}/t_b)\right)^2 \int \frac{dk}{k} \equiv \int \frac{dk}{k} \Delta_{\mathcal{R}}^2(k) \quad (3.29)$$

for the perfectly scale-invariant case. Notice that the results depends only logarithmically on t_b: the main dependence is on the minimum value of the effective potential and the parameter c. Observations on the current Hubble horizon indicate $\Delta_{\mathcal{R}}^2(k) \approx 2.2 \times 10^{-9}$. Ignoring the logarithm in (3.29), this requires $c|V_{\min}|^{\frac{1}{2}} \approx 10^{-3} M_{Pl}$, or approximately the GUT scale. This is of course entirely consistent with the heterotic M-theory.

With regard the eqs. (3.29) we have the following mathematical connections with the Aurea ratio:

$\frac{1}{3\pi^2} = 0{,}03377 \cong \frac{1}{2\cdot 5}(\Phi)^{-16/7} = \frac{1}{2\cdot 5}\left(\frac{\sqrt{5}+1}{2}\right)^{-16/7} = 0{,}03329$. If we take $M_{Pl} = 0{,}4340$, then we have that

$\frac{1}{3\pi^2 M_{Pl}^2} = 0{,}179307948 \cong (\Phi)^{-25/7} = \left(\frac{\sqrt{5}+1}{2}\right)^{-25/7} = 0{,}1793145665$. Furthermore, we obtain

$$\arcsin(0{,}1793145665)\cdot\frac{180}{\pi} + \arccos(0{,}1793145665)\cdot\frac{180}{\pi} = 10{,}3289 + 79{,}6702 \cong 90 = 55 + 34 + 1,$$

with 34 and 55 Fibonacci's numbers.

If the entropic perturbations are suddenly converted to curvature perturbations, the curvature perturbations inherit the spectral tilt given in (3.22). We now begin by re-expressing eq. (3.22) in terms of $\mathcal{N}$, the number of e-folds before the end of the ekpyrotic phase (where $d\mathcal{N} = (\varepsilon - 1)N$ and $\varepsilon >> 1$):

$$n_s - 1 = \frac{2}{\varepsilon} - \frac{d\ln\varepsilon}{d\mathcal{N}}. \quad (3.30)$$

This expression is identical to the case of the Newtonian potential perturbations, except that the first term has the opposite sign. In this expression, $\varepsilon(\mathcal{N})$ measures the equation of state during the ekpyrotic phase, which must decrease from a value much greater than unity to a value of order unity in the last $\mathcal{N}$ e-folds. If we estimate $\varepsilon \approx \mathcal{N}^\alpha$, then the spectral tilt is

$$n_s - 1 \approx \frac{2}{\mathcal{N}^\alpha} - \frac{\alpha}{\mathcal{N}}. \quad (3.31)$$

Here we see that the sign of the tilt is sensitive to α. For nearly exponential potentials $(\alpha \approx 1)$, the spectral tilt is $n_s \approx 1 + 1/N \approx 1.02$, slightly blue, because the first term dominates. However, there are well-motivated examples in which the equation of state does not decrease linearly with $\mathcal{N}$. We have introduced α to parameterize these cases. If $\alpha > 0.14$, the spectral tilt is red. For example,

$n_s = 0.97$ for $\alpha \approx 2$. These examples represent the range that can be achieved for the entropically-induced curvature perturbations in the simplest models, roughly $0.97 < n_s < 1.02$.

For comparison, if we use the same estimating procedure for the Newtonian potential fluctuations in the cyclic model (assuming they converted to curvature fluctuations before the bounce through 5d effects), we obtain $0.95 < n_s < 0.97$. This range agrees with the estimate obtained by an independent analysis based on studying inflaton potentials directly.
With regard the values of n_s, i.e. 0.95, 0.97 and 1.02, we have the following mathematical connections with the Aurea ratio:

$$0{,}95 \cong 0{,}9552046220 = (\Phi)^{-0{,}67/7} = \left(\frac{\sqrt{5}+1}{2}\right)^{-0{,}67/7};$$

$$0{,}97 \cong 0{,}9773457024 = (\Phi)^{-0{,}33/7} = \left(\frac{\sqrt{5}+1}{2}\right)^{-0{,}33/7};$$

$$1{,}02 \cong 1{,}0231794109 = (\Phi)^{0{,}33/7} = \left(\frac{\sqrt{5}+1}{2}\right)^{0{,}33/7};$$

A second way of analyzing the spectral tilt is to assume a form for the scalar field potential. Consider the case where the two fields have steep potentials that can be modelled as $V(\phi_1) = -V_0 e^{-\int c d\phi}$ and $\dot{\phi}_2 = \gamma\dot{\phi}_1$. Then eq. (3.22) becomes

$$n_s - 1 = \frac{4(1+\gamma^2)}{c^2 M_{Pl}^2} - \frac{4c_{,\phi}}{c^2}, \quad (3.32)$$

where we have used the fact that $c(\phi)$ has the dimensions of inverse mass and restored the factors of Planck mass. The presence of M_{Pl} clearly indicates that the first term on the right is a gravitational term. It is also the piece that makes a blue contribution to the spectral tilt. The second term is the non-gravitational term and agrees precisely with the following flat space-time result

$$n_s - 1 = -4\frac{c_{,\phi}}{c^2}, \quad (3.33)$$

although the agreement is not at all obvious at intermediate steps of the calculation. For a pure exponential potential, which has $c_{,\phi} = 0$, the non-gravitational contribution is zero, and the spectrum is slightly blue. For plausible values of $c = 20$ and $\gamma = 1/2$, say, the gravitational piece is about one percent and the spectral tilt is $n_s \approx 1.01$, also consistent with our earlier estimate.
We note that $c = 20$, is related with the Aurea ratio by the following mathematical formula:

$$\left[(\Phi)^{35/7} + (\Phi)^{14/7}\right]\cdot\frac{3}{2} = \left[\left(\frac{\sqrt{5}+1}{2}\right)^{35/7} + \left(\frac{\sqrt{5}+1}{2}\right)^{14/7}\right]\cdot\frac{3}{2} \cong 11{,}090 + 2{,}618 = 13{,}708\cdot\frac{3}{2} = 20{,}562.$$

In the cyclic model, the steepness of the potential must decrease as the field rolls downhill in order that the ekpyrotic phase comes to an end, which corresponds to $c_{,\phi} > 0$. If $c(\phi)$ changes from some initial value $\bar{c} >> 1$ to some value of order unity at the end of the ekpyrotic phase after ϕ changes by an amount $\Delta\phi$, then $c_{,\phi} \approx \bar{c}/\Delta\phi$. When c is large, the non-gravitational term in eq. (3.32) typically dominates and the spectral tilt is a few per cent towards the red.
For example, suppose $c \propto \phi^{\beta}$ and $\int c(\phi)d\phi \approx 125$; then, the spectral tilt is

$$n_s - 1 = -0.03\frac{\beta}{1+\beta}, \quad (3.34)$$

which corresponds to $0.97 < n_s < 1$ for positive $0 < \beta < \infty$, in agreement with our earlier estimate.
With regard the value 125, we have the following mathematical connection with the Fibonacci's numbers:

125 = 2 + 5 + 8 + 21 + 34 +55.

We note that negative potentials of this type with very large values of c have been argued to arise naturally in string theory. Our expression for the spectral tilt of the entropically induced curvature spectrum can also be expressed in terms of the customary "fast-roll" parameters

$$\bar{\varepsilon} \equiv \left(\frac{V}{V_{,\phi}}\right)^2 = \frac{1}{c^2} \qquad \bar{\eta} \equiv \left(\frac{V}{V_{,\phi}}\right)_{,\phi}. \quad (3.35)$$

Note that $\bar{\varepsilon} = 1/\left(2\left(1+\gamma^2\right)\varepsilon\right)$. Then, the spectral tilt is

$$n_s - 1 = \frac{4\left(1+\gamma^2\right)}{M_{Pl}^2}\bar{\varepsilon} - 4\bar{\eta}. \quad (3.36)$$

This result can be compared with the spectral index of the time-delay (Newtonian potential) perturbation, where the corresponding formula is

$$n_s - 1 = -\frac{4}{M_{Pl}^2}\bar{\varepsilon} - 4\bar{\eta}. \quad (3.37)$$

Here, the first term is again gravitational, but it has the opposite sign of the gravitational contribution to the entropically induced fluctuation spectrum. So, the tilt is typically a few per cent redder. Finally, for inflation, the spectral tilt is

$$n_s - 1 = -6\varepsilon + 2\eta \quad (3.38)$$

where the result is expressed in terms of the slow-roll parameters $\varepsilon \equiv (1/2)\left(M_{Pl}V_{,\phi}/V\right)^2$ and $\eta \equiv M_{Pl}^2 V_{,\phi\phi}/V$. Here we have revealed the factors of M_{Pl} to illustrate that both inflationary contributions are gravitational in origin. This gives the same range for n_s as the Newtonian potential perturbations in the cyclic model.

3.1 On some equations concerning the effective five-dimensional theory of the strongly coupled heterotic string as a gauged version of N = 1 five-dimensional supergravity with four-dimensional boundaries.

We will now briefly review the effective description of strongly coupled heterotic string theory as 11-dimensional supergravity with boundaries given by Horava and Witten. The bosonic part of the action is of the form

$$S = S_{SG} + S_{YM} \quad (3.39)$$

where S_{SG} is the familiar 11-dimensional supergravity

$$S_{SG} = -\frac{1}{2\kappa^2}\int_{M^{11}}\sqrt{-g}\left[R + \frac{1}{24}G_{IJKL}G^{IJKL} + \frac{\sqrt{2}}{1728}\varepsilon^{I_1 \ldots I_{11}}C_{I_1 I_2 I_3}G_{I_4 \ldots I_7}G_{I_8 \ldots I_{11}}\right] \quad (3.40)$$

and S_{YM} are the two E_8 Yang-Mills theories on the orbifold planes explicitly given by

$$S_{YM} = -\frac{1}{8\pi\kappa^2}\left(\frac{\kappa}{4\pi}\right)^{2/3}\int_{M_{10}^{(1)}}\sqrt{-g}\left\{tr\left(F^{(1)}\right)^2 - \frac{1}{2}trR^2\right\} - \frac{1}{8\pi\kappa^2}\left(\frac{\kappa}{4\pi}\right)^{2/3}\int_{M_{10}^{(2)}}\sqrt{-g}\left\{tr\left(F^{(2)}\right)^2 - \frac{1}{2}trR^2\right\}.$$

(3.41)

Here $F_{IJ}^{(i)}$ are the two E_8 gauge field strengths and C_{IJK} is the 3-form with field strength $G_{IJKL} = 24\partial_{[I}C_{JKL]}$. In order for the above theory to be supersymmetric as well as anomaly free, the Bianchi identity for G should receive a correction such that

$$(dG)_{11\overline{IJKL}} = -\frac{1}{2\sqrt{2}\pi}\left(\frac{\kappa}{4\pi}\right)^{2/3}\left\{J^{(1)}\delta\left(x^{11}\right) + J^{(2)}\delta\left(x^{11} - \pi\rho\right)\right\}_{\overline{IJKL}} \quad (3.42)$$

where the sources are given by

$$J^{(i)} = \left(trF^{(i)} \wedge F^{(i)} - \frac{1}{2}trR \wedge R\right). \quad (3.43)$$

With regard the eqs. (3.40) and (3.42), we have the following mathematical connections with the aurea ratio and the Ramanujan modular equations:

$$\sqrt{2} = 1{,}414213562 \cong (\Phi)^{5/7} = \left(\frac{\sqrt{5}+1}{2}\right)^{5/7} = 1{,}4101875; \quad 1728 = 432 \cdot 4 = 24 \cdot 24 \cdot 3 = 24^2 \cdot 3;$$

$$432 = 306{,}342Hz \cdot 1{,}4101875817; \quad 432 = 2 \cdot 3^3 \cdot 8;$$

$$\frac{1}{2\sqrt{2}\pi} = 0{,}112539539 \cong (\Phi)^{-31,67/7} = \left(\frac{\sqrt{5}+1}{2}\right)^{-31,67/7} = 0{,}1133912969$$ We note that 2, 3 and 8 are Fibonacci's numbers, while 24 is related to the physical vibrations of the bosonic strings by the following Ramanujan function:

$$24 = \frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}.$$

While the standard embedding of the spin connection into the gauge connection

$$trF^{(1)} \wedge F^{(1)} = trR \wedge R \quad (3.43b)$$

leads to vanishing source terms in the weakly coupled heterotic string Bianchi identity, in the present case, one is left with non-zero sources $\pm trR \wedge R$ on the two hyperplanes. As a result, the antisymmetric tensor field G and, hence, the second term in the gravitino supersymmetry variation

$$\delta\Psi_I = D_I\eta + \frac{\sqrt{2}}{288}\left(\Gamma_{IJKLM} - 8g_{IJ}\Gamma_{KLM}\right)G^{JKLM}\eta + ... \quad (3.44)$$

do not vanish.
With regard the mathematical connections with the Aurea ratio and Fibonacci's numbers, we note that:

$432 = 306,342\cdot(\Phi)^{5/7} = 306,342\cdot\left(\frac{\sqrt{5}+1}{2}\right)^{5/7} = 306,342\cdot 1,4101875817$; 432 – 288 = 144 and 144 is a Fibonacci's number. Furthermore, $288 = 24\cdot 12$ and 24 is related to the physical vibrations of the bosonic strings by the following Ramanujan function:

$$24 = \frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}.$$

Now, let us start with the zeroth order metric

$$ds_{11}^2 = \eta_{\mu\nu}dx^\mu dx^\nu + R_0^2\left(dx^{11}\right)^2 + V_0^{1/3}\Omega_{AB}dx^A dx^B, \quad (3.45)$$

where Ω_{AB} is a Calabi-Yau metric with Kahler form $\omega_{a\bar{b}} = i\Omega_{a\bar{b}}$. (Here a and $\bar{b}$ are holomorphic and anti-holomorphic indices). To keep track of the scaling properties of the solution, we have introduced moduli V_0 and R_0 for the Calabi-Yau volume and the orbifold radius, respectively. To order $\kappa^{2/3}$, the metric can be written in the form

$$ds_{11}^2 = \left(1+\hat{b}\right)\eta_{\mu\nu}dx^\mu dx^\nu + R_0^2(1+\hat{\gamma})\left(dx^{11}\right)^2 + V_0^{1/3}\left(\Omega_{AB} + h_{AB}\right)dx^A dx^B \quad (3.46)$$

where the functions $\hat{b}$, $\hat{\gamma}$ and h_{AB} depend on x^{11} and the Calabi-Yau coordinates. Furthermore, G_{ABCD} and G_{ABC11} receive a contribution of order $\kappa^{2/3}$ from the Bianchi identity source terms. The general explicit form of the corrections are

$$\hat{b} = -\frac{\sqrt{2}}{3} R_0 V_0^{-2/3} \alpha \left(\left| x^{11} \right| - \pi\rho/2 \right) \quad (3.47a) \qquad \hat{\gamma} = \frac{2\sqrt{2}}{3} R_0 V_0^{-2/3} \alpha \left(\left| x^{11} \right| - \pi\rho/2 \right) \quad (3.47b)$$

$$h_{AB} = \frac{\sqrt{2}}{3} R_0 V_0^{-2/3} \alpha \left(\left| x^{11} \right| - \pi\rho/2 \right) \Omega_{AB} \quad (3.47c) \qquad G_{ABCD} = \frac{1}{6} \alpha \varepsilon_{ABCD}{}^{EF} \omega_{EF} \varepsilon\left(x^{11}\right) \quad (3.47d)$$

$$G_{ABC11} = 0 \quad (3.47e)$$

with

$$\alpha = -\frac{1}{8\sqrt{2}\pi v} \left(\frac{\kappa}{4\pi} \right)^{2/3} \int_X \omega \wedge trR^{(\Omega)} \wedge R^{(\Omega)}, \qquad v = \int_X \sqrt{\Omega}\,. \quad (3.48)$$

Here $\varepsilon\left(x^{11}\right)$ is the step function which is $+1(-1)$ for x^{11} positive (negative).
With regard the eqs. (3.47) and (3.48), we have the following mathematical connections with the Aurea ratio:

$$\frac{\sqrt{2}}{3} = 0,47140452 \cong (\Phi)^{-11/7} = \left(\frac{\sqrt{5}+1}{2} \right)^{-11/7} = 0,469451;$$

$$\frac{2\sqrt{2}}{3} = 0,942809041 \cong (\Phi)^{-7/7} + (\Phi)^{-35/7} = \left(\frac{\sqrt{5}+1}{2} \right)^{-7/7} + \left(\frac{\sqrt{5}+1}{2} \right)^{-35/7} = 0,618034 + 0,090170 =$$

$$= 0,708204; \quad 0,708204 \cdot \frac{4}{3} = 0,944272;$$

$$\frac{1}{6} = 0,1666 \cong (\Phi)^{-26/7} = \left(\frac{\sqrt{5}+1}{2} \right)^{-26/7} = 0,1674018269;$$

$$\frac{1}{8\sqrt{2}\pi} = 0,0281348 \cong \frac{1}{2 \cdot 5} (\Phi)^{-18,33/7} = \frac{1}{2 \cdot 5} \left(\frac{\sqrt{5}+1}{2} \right)^{-18,33/7} = 0,02835642.$$

In the five-dimensional space M_5 of the reduced theory, the orbifold fixed planes constitute four-dimensional hypersurfaces which we denote by $M_4^{(i)}$, $i = 1,2$. There will be an E_6 gauge field $A_\mu^{(1)}$ accompanied by gauginos and gauge matter fields on the orbifold plane $M_4^{(1)}$. We will set these gauge matter fields to zero in the following. The field content of the orbifold plane $M_4^{(2)}$ consists of an E_8 gauge field $A_\mu^{(2)}$ and the corresponding gauginos. In addition, there is another important boundary effect which results from the non-zero internal gauge field and gravity curvatures. More precisely, note that

$$\int_X \sqrt{\Omega} trF_{AB}^{(1)} F^{(1)AB} = \int_X \sqrt{\Omega} trR_{AB} R^{AB} = -16\sqrt{2}\pi v \left(\frac{4\pi}{\kappa} \right)^{2/3} \alpha\,, \qquad F_{AB}^{(2)} = 0\,. \quad (3.49)$$

In view of the boundary action (3.41), it follows that we will retain cosmological type terms with opposite signs on the two boundaries. Note that the size of those terms is set by the same constant α, given by eq. (3.48), which determines the magnitude of the non-zero mode. The boundary cosmological terms are another important ingredient in reproducing the 11-dimensional background as a solution of the five-dimensional theory.
We can perform the Kaluza-Klein reduction on the metric

$$ds_{11}^2 = V^{-2/3} g_{\alpha\beta} dx^\alpha dx^\beta + V^{1/3} \Omega_{AB} dx^A dx^B . \quad (3.50)$$

The complete configuration for the antisymmetric tensor field that we use in the reduction is given by

$$C_{\alpha\beta\gamma}\,, \qquad G_{\alpha\beta\gamma\delta} = 24\partial_{[\alpha} C_{\beta\gamma\delta]}\,, \qquad C_{\alpha AB} = \frac{1}{6} \mathrm{A}_\alpha \omega_{AB}\,, \qquad G_{\alpha\beta AB} = \mathcal{F}_{\alpha\beta}\omega_{AB}\,, \qquad \mathcal{F}_{\alpha\beta} = \partial_\alpha \mathrm{A}_\beta - \partial_\beta \mathrm{A}_\alpha\,,$$
$$C_{ABC} = \frac{1}{6}\xi\omega_{ABC}\,, \qquad G_{\alpha ABC} = \partial_\alpha \xi \omega_{ABC} \quad (3.51)$$

and the non-zero mode is

$$G_{ABCD} = \frac{\alpha}{6} \varepsilon_{ABCD}{}^{EF} \omega_{EF} \varepsilon\left(x^{11}\right), \quad (3.52)$$

where α was defined in eq. (3.48).
We can now compute the five-dimensional effective action of Horava-Witten theory. Using the field configuration (3.49) – (3.52) we find from the action (3.39) – (3.41) that

$$S_5 = S_{grav} + S_{hyper} + S_{bound} \quad (3.53)$$

where

$$S_{grav} = -\frac{1}{2\kappa_5^2}\int_{M_5}\sqrt{-g}\left[R + \frac{3}{2}\mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta} + \frac{1}{\sqrt{2}}\varepsilon^{\alpha\beta\gamma\delta\varepsilon}\mathrm{A}_\alpha \mathcal{F}_{\beta\gamma}\mathcal{F}_{\delta\varepsilon}\right] \quad (3.54a)$$

$$S_{hyper} = -\frac{1}{2\kappa_5^2}\int_{M_5}\sqrt{-g}\left[\frac{1}{2}V^{-2}\partial_\alpha V \partial^\alpha V + 2V^{-1}\partial_\alpha \xi \partial^\alpha \bar{\xi} + \frac{1}{24}V^2 G_{\alpha\beta\gamma\delta}G^{\alpha\beta\gamma\delta} + \right.$$
$$\left. + \frac{\sqrt{2}}{24}\varepsilon^{\alpha\beta\gamma\delta\varepsilon} G_{\alpha\beta\gamma\delta}\left(i\left(\xi\partial_\varepsilon\bar{\xi} - \bar{\xi}\partial_\varepsilon \xi\right) + 2\alpha \mathrm{A}_\varepsilon\right) + \frac{1}{3}V^{-2}\alpha^2\right] \quad (3.54b)$$

$$S_{bound} = -\frac{1}{2\kappa_5^2}\left\{-2\sqrt{2}\int_{M_4^{(1)}}\sqrt{-g}V^{-1}\alpha + 2\sqrt{2}\int_{M_4^{(2)}}\sqrt{-g}V^{-1}\alpha\right\} - \frac{1}{16\pi\alpha_{GUT}}\sum_{i=1}^{2}\int_{M_4^{(i)}}\sqrt{-g}V trF_{\mu\nu}^{(i)^2} . \quad (3.54c)$$

In this expression, we have now dropped higher-derivative terms. The 4-form field strength $G_{\alpha\beta\gamma\delta}$ is subject to the Bianchi identity

$$\left(dG\right)_{11\mu\nu\rho\sigma} = -\frac{\kappa_5^2}{4\sqrt{2}\pi\alpha_{GUT}}\left\{J^{(1)}\delta\left(x^{11}\right) + J^{(2)}\delta\left(x^{11} - \pi\rho\right)\right\}_{\mu\nu\rho\sigma} \quad (3.55)$$

which follows directly from the 11-dimensional Bianchi identity (3.42). The currents $J^{(i)}$ have been defined in eq. (3.43). The five-dimensional Newton constant κ_5 and the Yang-Mills coupling α_{GUT} are expressed in terms of 11-dimensional quantities as

$$\kappa_5^2 = \frac{\kappa^2}{v}, \qquad \alpha_{GUT} = \frac{\kappa^2}{2v}\left(\frac{4\pi}{\kappa}\right)^{2/3}. \quad (3.56)$$

Since we have compactified on a Calabi-Yau space, we expect the bulk part of the above action to have eight preserved supercharges and, therefore, to correspond to minimal $N = 1$ supergravity in five dimensions. Accordingly, let us compare the result (3.54) to the known $N = 1$ supergravity-matter theories in five dimensions. In these theories, the scalar fields in the universal hypermultiplet parameterize a quaternionic manifold with coset structure $\mathcal{M}_Q = SU(2,1)/SU(2)\times U(1)$. Hence, to compare our action to these we should dualize the three-form $C_{\alpha\beta\gamma}$ to a scalar field σ by setting (in the bulk)

$$G_{\alpha\beta\gamma\delta} = \frac{1}{\sqrt{2}} V^{-2} \varepsilon_{\alpha\beta\gamma\delta\varepsilon}\left(\partial^\varepsilon \sigma - i\left(\xi\partial^\varepsilon\bar{\xi} - \bar{\xi}\partial^\varepsilon \xi\right) - 2\alpha \mathrm{A}^\varepsilon\right). \quad (3.57)$$

Then the hypermultiplet part of the action (3.54b) can be written as

$$S_{hyper} = -\frac{v}{2\kappa^2}\int_{M_5}\sqrt{-g}\left[h_{uv}\nabla_\alpha q^u \nabla^\alpha q^v + \frac{1}{3}V^{-2}\alpha^2\right] \quad (3.58)$$

where $q^u = \left(V, \sigma, \xi, \bar{\xi}\right)$. The covariant derivative ∇_α is defined as $\nabla_\alpha q^u = \partial_\alpha q^u + \alpha \mathrm{A}_\alpha k^u$ with $k^u = (0,-2,0,0)$. The sigma model metric $h_{uv} = \partial_u \partial_v K_Q$ can be computed from the Kahler potential

$$K_Q = -\ln\left(S + \bar{S} - 2C\bar{C}\right), \qquad S = V + \xi\bar{\xi} + i\sigma, \qquad C = \xi. \quad (3.59)$$

Consequently, the hypermultiplet scalars q^u parameterize a Kahler manifold with metric h_{uv}. It can be demonstrated that k^u is a Killing vector on this manifold.

To analyze the supersymmetry properties of the solutions shortly to be discussed, we need the supersymmetry variations of the fermions associated with the theory (3.53). They can be obtained either by a reduction of the 11-dimensional gravitino variation (3.44) or by generalizing the known five-dimensional transformations by matching onto gauged four-dimensional $N = 2$ theories. It is sufficient to keep the bosonic terms only. Both approaches lead to

$$\begin{aligned}
\delta\psi_\alpha^i &= D_\alpha \varepsilon^i + \frac{\sqrt{2}i}{8}\left(\gamma_\alpha^{\beta\gamma} - 4\delta_\alpha^\beta \gamma^\gamma\right)\mathcal{F}_{\beta\gamma}\varepsilon^i - \frac{1}{2}V^{-1/2}\left(\partial_\alpha \xi\left(\tau_1 - i\tau_2\right)^i_j - \partial_\alpha\bar{\xi}\left(\tau_1 + i\tau_2\right)^i_j\right)\varepsilon^j + \\
&- \frac{\sqrt{2}i}{96}V\varepsilon_\alpha^{\beta\gamma\delta\varepsilon}G_{\beta\gamma\delta\varepsilon}\left(\tau_3\right)^i{}_j\varepsilon^j - \frac{\sqrt{2}}{12}\alpha V^{-1}\varepsilon\left(x^{11}\right)\gamma_\alpha\left(\tau_3\right)^i{}_j\varepsilon^j \\
\delta\zeta^i &= \frac{\sqrt{2}}{48}V\varepsilon^{\alpha\beta\gamma\delta\varepsilon}G_{\alpha\beta\gamma\delta}\gamma_\varepsilon\varepsilon^i - \frac{i}{2}V^{-1/2}\gamma^\alpha\left(\partial_\alpha\xi\left(\tau_1 - i\tau_2\right)^i_j + \partial_\alpha\bar{\xi}\left(\tau_1 + i\tau_2\right)^i_j\right)\varepsilon^j + \\
&+ \frac{i}{2}V^{-1}\gamma_\beta\partial^\beta V\varepsilon^i - \frac{i}{\sqrt{2}}\alpha V^{-1}\varepsilon\left(x^{11}\right)\left(\tau_3\right)^i{}_j\varepsilon^j \quad (3.60)
\end{aligned}$$

where τ_i are the Pauli spin matrices. Thence, we see that the relevant five-dimensional effective theory for the reduction of Horava-Witten theory is a gauged $N = 1$ supergravity theory with bulk and boundary potentials.

The theory (3.53) has all of the prerequisites necessary for such a three-brane solution to exist. Generally, in order to have a $(D-2)$-brane in a D-dimensional theory, one needs to have a $(D-1)$-form field, or, equivalently, a cosmological constant. This cosmological term is provided by the bulk potential term in the action (3.53). From the viewpoint of the bulk theory, we could have multi three-brane solutions with an arbitrary number of parallel branes located at various places in the x^{11} direction. As is well known, however, elementary brane solutions have singularities at the location of the branes, needing to be supported by source terms. The natural candidates for those source terms, are the boundary actions. Given the anomaly-cancellation requirements, this restricts the possible solutions to those representing a pair of parallel three-branes corresponding to the orbifold planes. It is clear that in order to find a three-brane solution, we should start with the Ansatz

$$ds_5^2 = a(y)^2 dx^\mu dx^\nu \eta_{\mu\nu} + b(y)^2 dy^2 \quad (3.61)$$
$$V = V(y)$$

where a and b are functions of $y = x^{11}$ and all other field vanish. The general solution for this Ansatz, satisfying the equations of motion derived from the action (3.53), is given by

$$a = a_0 H^{1/2}, \qquad b = b_0 H^2, \qquad V = b_0 H^3 \qquad \text{and} \qquad H = \frac{\sqrt{2}}{3}\alpha|y| + c_0 \quad (3.62)$$

where a_0, b_0 and c_0 are constants. We note that the boundary source terms have fixed the form of the harmonic function H in the above solution. Without specific information about the sources, the function H would generically be glued together from an arbitrary number of linear pieces with slopes $\pm\frac{\sqrt{2}}{3}\alpha$. The edges of each piece would then indicate the location of the source terms. The necessity of matching the boundary sources at $y = 0$ and $\pi\rho$, however, has forced us to consider only two such linear pieces, namely $y \in [0, \pi\rho]$ and $y \in [-\pi\rho, 0]$. These pieces are glued together at $y = 0$ and $\pi\rho$. Therefore, we have

$$\partial_y^2 H = \frac{2\sqrt{2}}{3}\alpha(\delta(y) - \delta(y - \pi\rho)) \quad (3.63)$$

which shows that the solution represents two parallel three-branes located at the orbifold planes. We stress that this solution solves the five-dimensional theory (3.53) exactly, whereas the original deformed Calabi-Yau solution was only an approximation to order $\kappa^{2/3}$. It is straightforward to show that the linearized version of (3.62), that is, the expansion to first order in $\alpha = O(\kappa^{2/3})$, coincides with Witten's solution (3.46) – (3.47) upon appropriate matching of the integration constants. Hence, we have found an exact generalization of the linearized 11-dimensional solution. We still have to check that our solution preserves half of the supersymmetries. When $g_{\alpha\beta}$ and V are the only non-zero fields, the supersymmetry transformations (3.60) simplify to

$$\delta\psi_\alpha^i = D_\alpha\varepsilon^i - \frac{\sqrt{2}}{12}\alpha\varepsilon(y)V^{-1}\gamma_\alpha(\tau_3)^i{}_j\varepsilon^j \qquad \delta\zeta^i = \frac{i}{2}V^{-1}\gamma_\beta\partial^\beta V\varepsilon^i - \frac{i}{\sqrt{2}}\alpha\varepsilon(y)V^{-1}(\tau_3)^i{}_j\varepsilon^j. \quad (3.64)$$

With regard the eqs. (3.54), (3.55), (3.57), (3.60) and (3.62-3.64), we have the following mathematical connections with Aurea ratio and with the Ramanujan's modular equations:

$$\frac{1}{\sqrt{2}} = 0{,}707106781 \cong (\Phi)^{-5/7} = \left(\frac{\sqrt{5}+1}{2}\right)^{-5/7} = 0{,}709125;$$

$$\frac{\sqrt{2}}{24} = 0{,}058925565 \cong \frac{1}{2\cdot 5}(\Phi)^{-7{,}67/7} = \frac{1}{2\cdot 5}\left(\frac{\sqrt{5}+1}{2}\right)^{-7{,}67/7} = 0{,}0590213661;$$

$$2\sqrt{2} = 2{,}828427125 \cong (\Phi)^{15/7} = \left(\frac{\sqrt{5}+1}{2}\right)^{15/7} = 2{,}8043399;$$

$$\frac{1}{4\sqrt{2}} = 0{,}17677 \cong (\Phi)^{-25{,}33/7} = \left(\frac{\sqrt{5}+1}{2}\right)^{-25{,}33/7} = 0{,}17525;$$

$$\frac{1}{4\sqrt{2}\pi} = 0{,}056269769 \cong \frac{1}{2\cdot 5}(\Phi)^{-8{,}33/7} = \frac{1}{2\cdot 5}\left(\frac{\sqrt{5}+1}{2}\right)^{-8{,}33/7} = 0{,}05640;$$

$$\frac{\sqrt{2}}{8} = 0{,}176776695 \cong (\Phi)^{-25/7} = \left(\frac{\sqrt{5}+1}{2}\right)^{-25/7} = 0{,}179314;$$

$$\frac{\sqrt{2}}{12} = 0{,}11785113 \cong (\Phi)^{-31/7} = \left(\frac{\sqrt{5}+1}{2}\right)^{-31/7} = 0{,}118708;$$

$$\frac{\sqrt{2}}{48} = 0{,}029462782 \cong \frac{1}{2\cdot 5}(\Phi)^{-18/7} = \frac{1}{2\cdot 5}\left(\frac{\sqrt{5}+1}{2}\right)^{-18/7} = 0{,}0290137;$$

$$\frac{\sqrt{2}}{96} = 0{,}014731391 \cong \frac{1}{2\cdot 5}(\Phi)^{-28/7} = \frac{1}{2\cdot 5}\left(\frac{\sqrt{5}+1}{2}\right)^{-28/7} = 0{,}0145898;$$

$$\frac{\sqrt{2}}{3} = 0{,}47140452 \cong (\Phi)^{-11/5} = \left(\frac{\sqrt{5}+1}{2}\right)^{-11/5} = 0{,}469451.$$

Furthermore, the number 8, 12, 24, 48 and 96 are connected with the "modes" that correspond to the physical vibrations of a superstring and to physical vibrations of the bosonic strings by the following Ramanujan functions:

$$8 = \frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]},$$

$$24 = \frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}.$$

We note also that 8 is a Fibonacci's number.

The Killing spinor equations $\delta\psi_\alpha^i = 0$, $\delta\zeta^i = 0$ are satisfied for the solution (3.62) if we require that the spinor ε^i is given by

$$\varepsilon^i = H^{1/4}\varepsilon_0^i, \quad \gamma_{11}\varepsilon_0^i = (\tau_3)^i_j\,\varepsilon_0^j \quad (3.65)$$

where ε_0^i is a constant symplectic Majorana spinor. This shows that we have indeed found a BPS solution preserving four of the eight bulk supercharges.

3.2 On some equations concerning the colliding Branes and the Origin of the Hot Big Bang

We have derived the five-dimensional effective action of heterotic M-theory in the precedent subsection (**3.1**). Now, we shall use a simplified action describing gravity $g_{\gamma\delta}$, the universal "breathing" modulus of the Calabi-Yau three-fold ϕ, a four-form gauge field $\mathrm{A}_{\gamma\delta\varepsilon\zeta}$ with field strength $\mathcal{F} = d\mathrm{A}$ and a single bulk M5-brane. It is given by

$$S = \frac{M_5^3}{2}\int_{\mathcal{M}_5} d^5x\sqrt{-g}\left(\mathcal{R} - \frac{1}{2}(\partial\phi)^2 - \frac{3}{2}\frac{e^{2\phi}\mathcal{F}^2}{5!}\right) +$$
$$-3\sum_{i=1}^{3}\alpha_i M_5^3\int_{\mathcal{M}_4^{(i)}} d^4\xi_{(i)}\left(\sqrt{-h_{(i)}}e^{-\phi} - \frac{\varepsilon^{\mu\nu\kappa\lambda}}{4!}\mathrm{A}_{\gamma\delta\varepsilon\zeta}\partial_\mu X_{(i)}^\gamma\partial_\nu X_{(i)}^\delta\partial_\kappa X_{(i)}^\varepsilon\partial_\lambda X_{(i)}^\zeta\right), \quad (3.66)$$

where $\gamma,\delta,\varepsilon,\zeta = 0,...,4,\ \mu,\nu.... = 0,...,3.$ The space-time is a five-dimensional manifold $\mathcal{M}_5$ with coordinates x^γ. The four-dimensional manifolds $\mathcal{M}_4^{(i)}$, $i = 1,2,3$ are the visible, hidden, and bulk branes respectively, and have internal coordinates $\xi_{(i)}^\mu$ and tension $\alpha_i M_5^3$. Note that α_i has dimension of mass. If we denote $\alpha_1 \equiv -\alpha$, $\alpha_2 \equiv \alpha - \beta$, and $\alpha_3 \equiv \beta$, then the visible brane has tension $-\alpha M_5^3$, the hidden brane $(\alpha-\beta)M_5^3$, and the bulk brane βM_5^3. It is straightforward to show that the tension of the bulk brane, βM_5^3, must always be positive. Furthermore, one can easily deduce that the tension on the visible brane, $-\alpha M_5^3$, can be either positive or negative. We will take $\alpha > 0$, so that the tension on the visible brane is negative. Furthermore, we will choose β such that $\alpha - \beta > 0$, that is, the tension of the hidden brane is positive. The tensor $h_{\mu\nu}^{(i)}$ is the induced metric on $\mathcal{M}_4^{(i)}$. The functions $X_{(i)}^\gamma\left(\xi_{(i)}^\mu\right)$ are the coordinates in $\mathcal{M}_5$ of a point on $\mathcal{M}_4^{(i)}$ with coordinates $\xi_{(i)}^\mu$. In other words, $X_{(i)}^\gamma\left(\xi_{(i)}^\mu\right)$ describe the embedding of the branes into $\mathcal{M}_5$. The BPS solution of Lukas, Ovrut and Waldram is then given by

$$ds^2 = D(y)\left(-N^2 d\tau^2 + A^2 d\vec{x}^2\right) + B^2 D^4(y) dy^2; \qquad e^{\phi} = BD^3(y);$$
$$\mathcal{F}_{0123Y} = -\alpha A^3 N B^{-1} D^{-2}(y) \qquad \text{for} \quad y < Y$$
$$\mathcal{F}_{0123Y} = -(\alpha - \beta) A^3 N B^{-1} D^{-2}(y) \qquad \text{for} \quad y > Y, \qquad (3.67)$$

where

$$D(y) = \alpha y + C \qquad \text{for} \quad y < Y$$
$$D(y) = (\alpha - \beta) y + C + \beta Y \qquad \text{for} \quad y > Y, \qquad (3.68)$$

and A, B, C, N and Y are constants. Note that A, B, C, N are dimensionless and Y has the dimension of length. The visible and hidden boundary branes are located at $y = 0$ and $y = R$, respectively, and the bulk brane is located at $y = Y$, $0 \le Y \le R$. We assume that $C > 0$ so that the curvature singularity at $D = 0$ does not fall between the boundary branes. Note that $y = 0$ lies in the region of smaller volume while $y = R$ lies in the region of larger volume. Note that inserting the solution of the four-form equation of motion into eq. (3.66) yields precisely the bulk action with charge $-\alpha$ in the interval $0 \le y \le Y$ and charge $-\alpha + \beta$ in the interval $Y \le y \le R$. The formulation of the action eq. (3.66) using the four-form A is particularly useful when the theory contains bulk branes, as is the case in ekpyrotic theory.
The following equation

$$|\delta_k| = 4 a_c H_c \left(1 - \frac{2\dot{H}_c}{a_c H_c^2}\right) |\Delta\tau(k)|, \qquad (3.69)$$

expresses the density perturbation in terms of the time delay at the time of collision, $\Delta\tau(k)$. If we consider the exponential potential $V = -v e^{-m\alpha Y}$, then the eq. (3.69) yields

$$|\delta_k| \approx \frac{4m^2 \alpha \sqrt{2v}}{mC + 2} |\Delta\tau(k)|. \qquad (3.70)$$

Now we compute the spectrum of quantum fluctuations of the brane δY_k and use the result to compute the time delay, $\Delta\tau(k)$.
For the calculation of quantum fluctuations, it is sufficient to work at the lowest order in β/α. Without loss of generality, we can therefore set $A = N = 1$. In that case, the bulk brane Lagrangian is given by

$$\mathcal{L}_\beta = 3\beta M_5^3 B \left[\frac{1}{2} D(Y)^2 \eta^{\mu\nu} \partial_\mu Y \partial_\nu Y - V(Y)\right]. \qquad (3.71)$$

Note that this agrees with $\mathcal{L}_\beta$ given in the following equation

$$\mathcal{L}_\beta = \frac{3\beta M_5^3 A^3 B}{N} \left[\frac{1}{2} D^2(Y) \dot{Y}^2 - N^2 V(Y)\right], \qquad (3.72)$$

when we set $A = N = 1$ and spatial gradients of Y to zero. Let us first consider the spatially homogeneous motion of the brane which will be described by $Y_0(\tau)$. It is governed by the following equation of motion

$$\frac{1}{2}D(Y_0)^2\dot{Y}_0^2+V(Y_0)=E\,, \quad (3.73)$$

where E is a constant. Eq. (3.73) is simply the statement that the energy E of the bulk brane is conserved to this order in β/α. Since we have chosen the visible brane to lie at $y=0$ and the hidden universe to lie at $y=R$, we focus on the branch $\dot{Y}<0$ in which case the bulk brane moves towards the visible brane. The solution to eq. (3.73) is then given by

$$(-\tau)=\int_0^{Y_0}\frac{D(Y')dY'}{\sqrt{2(E-V(Y'))}} \quad (3.74)$$

with $\tau\le 0$, and with the collision occurring at $\tau=0$. Let us now consider fluctuations around the background solution $Y_0(\tau)$. Namely, if $Y=Y_0(\tau)+\delta Y(\tau,\vec{x})$, with $\delta Y(\tau,\vec{x})<<Y_0(\tau)$, we can expand the action to quadratic order in δY

$$\mathcal{L}_{fluc}\approx\frac{1}{2}D_0^2\left[-\delta\dot{Y}^2+\left(\vec{\partial}(\delta Y)\right)^2\right]+\left[\alpha^2D_0^{-2}(V_0-E)-\alpha D_0^{-1}\frac{dV_0}{dY_0}+\frac{1}{2}\frac{d^2V_0}{dY_0^2}\right](\delta Y)^2, \quad (3.75)$$

where we have used eq. (3.73), and where we have introduced $D_0\equiv D(Y_0)$ and $V_0\equiv V(Y_0)$ for simplicity. The key relation is the fluctuation equation as derived from the action (3.75)

$$x^2\frac{d^2f_{\vec{k}}}{dx^2}-\left[\frac{\ddot{a}_{pert}}{a_{pert}}\tau^2-x^2\right]f_{\vec{k}}=0\,; \qquad x\equiv\left|\vec{k}\right|(-\tau), \quad (3.76)$$

where $f_{\vec{k}}\equiv D_0\cdot\delta Y_{\vec{k}}$ and where a_{pert} is defined by

$$\frac{\ddot{a}_{pert}}{a_{pert}}\equiv D_0^{-3}\left(\alpha\frac{dV_0}{dY_0}-D_0\frac{d^2V_0}{dY_0^2}\right). \quad (3.77)$$

The fluctuation eq. (3.76), can be compared with the corresponding equation for the perturbations of a scalar field with no potential and minimally coupled to an FRW background with scale factor $a(\tau)$

$$\delta\ddot{\phi}_{\vec{k}}+2\frac{\dot{a}}{a}\delta\dot{\phi}_{\vec{k}}+k^2\delta\phi_{\vec{k}}=0\,. \quad (3.78)$$

Defining $f_{\vec{k}}=a\cdot\delta\phi_{\vec{k}}$, eq. (3.78) becomes

$$x^2\frac{d^2f_{\vec{k}}}{dx^2}-\left[\frac{\ddot{a}}{a}\tau^2-x^2\right]f_{\vec{k}}=0\,. \quad (3.79)$$

Let us now discuss the Hubble horizon for the perturbations. Recall that in usual 4d cosmology (see eq. (3.79)), we have

$$x=k(-\tau)=\left(\frac{k}{a}\right)\cdot a\cdot(-\tau)=k_{phys}a\cdot(-\tau)\approx k_{phys}H^{-1}, \quad (3.80)$$

where $H^{-1} \equiv a^2 / \dot{a}$ is the Hubble radius as derived from the scale-factor a. By definition, a mode is said to be outside the Hubble horizon when its wavelength is larger than the Hubble radius. From eq. (3.80), we see that this occurs when $x < 1$. Therefore, a mode with amplitude $f_{\vec{k}}$ crosses outside the horizon when $x \approx \mathcal{O}(1)$. Similarly, in our scenario we can write

$$x = k(-\tau) = k_{phys} D_0^{1/2}(-\tau) \equiv k_{phys} H_{pert}^{-1}, \quad (3.81)$$

where $k_{phys} = k / D_0^{1/2}$. The role of the Hubble radius is replaced by

$$H_{pert}^{-1} \equiv D_0^{1/2}(-\tau) = D_0^{1/2} \int_0^{Y_0} \frac{D(Y)dY}{\sqrt{2(E - V(Y))}}, \quad (3.82)$$

which is to be thought of as an effective Hubble radius for the perturbations. So, the length scale at which amplitudes freeze depends on a_B (rather than a_{pert}), but the amplitude itself, as derived from eq. (3.76), depends on a_{pert}. The feature of two different scale factors is novel aspect of ekpyrotic scenario. With regard the comparison to inflationary cosmology, we have that in inflation, the wavelengths are stretched superluminally while the horizon is nearly constant. In the ekpyrotic scenario, the wavelengths are nearly constant while the horizon shrinks. We can obtain a spectrum which is scale-invariant. Writing the equation for the perturbations in the form of eq. (3.76) is useful since one can read off from it the spectral slope of the power spectrum. It is determined by the value of $(\ddot{a}_{pert} / a_{pert})\tau^2$. In particular, one obtains a scale-invariant spectrum if $(\ddot{a}_{pert} / a_{pert})\tau^2 = 2$ when the modes observed on the CMB cross outside the horizon.
Combining eqs. (3.74) and (3.77), we find

$$\frac{\ddot{a}_{pert}}{a_{pert}} \tau^2 = D_0^{-3} \left(\alpha \frac{dV_0}{dY_0} - D_0 \frac{d^2 V_0}{dY_0^2} \right) \left[\int_0^Y \frac{D(Y')dY'}{\sqrt{2(E - V(Y'))}} \right]^2. \quad (3.83)$$

The spectrum will be scale-invariant if the right hand side of eq. (3.83) equals 2 when the modes of interest cross outside the horizon. Thence, we have:

$$\frac{\ddot{a}_{pert}}{a_{pert}} \tau^2 = D_0^{-3} \left(\alpha \frac{dV_0}{dY_0} - D_0 \frac{d^2 V_0}{dY_0^2} \right) \left[\int_0^Y \frac{D(Y')dY'}{\sqrt{2(E - V(Y'))}} \right]^2 = 2. \quad (3.83b)$$

With regard the eq. (3.83b), we have the following mathematical connections with the Aurea ratio:

$$(\Phi)^{14/7} + (\Phi)^{-14/7} = \left(\frac{\sqrt{5}+1}{2} \right)^{14/7} + \left(\frac{\sqrt{5}+1}{2} \right)^{-14/7} = 2{,}618034 + 0{,}381966 = 3; \quad \frac{2}{3} \cdot 3 = 2;$$

$$(\Phi)^{21/7} + (\Phi)^{7/7} + (\Phi)^{-28/7} = \left(\frac{\sqrt{5}+1}{2} \right)^{21/7} + \left(\frac{\sqrt{5}+1}{2} \right)^{7/7} + \left(\frac{\sqrt{5}+1}{2} \right)^{-28/7} =$$

$= 4{,}236068 + 1{,}618034 + 0{,}145898 = 6; \quad \frac{1}{3} \cdot 6 = 2$. We note also that 2 is prime number and Fibonacci's number.

Thence we can rewrite the eq. (3.83b) also as follow:

$$\frac{\ddot{a}_{pert}}{a_{pert}}\tau^2 = D_0^{-3}\left(\alpha\frac{dV_0}{dY_0} - D_0\frac{d^2V_0}{dY_0^2}\right)\left[\int_0^Y \frac{D(Y')dY'}{\sqrt{2(E - V(Y'))}}\right]^2 =$$

$$= \frac{2}{3}[(\Phi)^{14/7} + (\Phi)^{-14/7}] = \frac{2}{3}\left[\left(\frac{\sqrt{5}+1}{2}\right)^{14/7} + \left(\frac{\sqrt{5}+1}{2}\right)^{-14/7}\right] = \frac{2}{3}(2{,}618034 + 0{,}381966) = 2\,;$$

We can add a potential $V(Y)$ of the form that might result from the exchange of wrapped M2-branes. We would like to think of V as the potential derived from the superpotential W for the modulus Y in the 4d low energy theory. Typically, superpotentials for such moduli are of exponential form, for example,

$$W \approx e^{-cY}\,, \quad (3.84)$$

where c is a positive parameter with dimension of mass. The corresponding potential is constructed from W and the Kahler potential K according to the usual prescription

$$V = e^{K/M_{pl}^2}\left[K^{ij}D_iWD_j\overline{W} - \frac{3}{M_{pl}^2}W\overline{W}\right]. \quad (3.85)$$

where $D_i = \partial/\partial\phi^i + K_i/M_{pl}^2$ is the Kahler covariant derivative, $K_i = \partial K/\partial\phi^i$, $K_{ij} = \partial^2 K/\partial\phi^i\partial\phi^j$ and a sum over each superfield ϕ_i is implicit. Eqs. (3.84) and (3.85) imply that V decays exponentially with Y. Here it will suffice to perform the calculation using a simple exponential potential, namely

$$V(Y) = -ve^{-m\alpha Y}\,, \quad (3.86)$$

where v and m are positive, dimensionless constants. Note that, in the case where the potential is generated by the exchange of wrapped M2-branes, the parameter m is of the form $m = cT_3\nu/\alpha$, where c is a constant, T_3 is the tension of the M2-brane, and ν is the volume of the curve on which it is wrapped. The perturbation modes of interest are those which are within the current Hubble horizon. As the wavelengths corresponding to those modes passed outside the effective Hubble horizon on the moving bulk-brane, the amplitudes became fixed. Scale invariance will require $mD >> 1$ during this period. We know that, if the potential V is negligible compared to E, the spectrum of fluctuations is not scale-invariant. Hence, we consider the limit where $|E| << |V_0|$. This condition, as seen from the equation of motion for Y_0, eq. (3.73), is satisfied if $\dot{Y}_0 = 0$ initially, or, equivalently, if the bulk brane begins nearly at rest. For the brane to be nearly at rest, one must have $|E| \approx |V_0|$ initially. As the brane traverses the fifth dimension, $|V|$ increases exponentially, whereas E is constant. Hence, the condition $|E| << |V_0|$ is automatically satisfied. The bulk brane beginning nearly at rest is precisely what we expect for a nearly BPS initial state. Applying the condition $|E| << |V_0|$, eq. (3.74) reduces to

$$\tau^2 \approx \frac{1}{2v}\left[\int_0^{Y_0} D(Y')e^{m\alpha Y'/2}dY'\right]^2 \approx \frac{2D_0^2}{m^2\alpha^2 ve^{-m\alpha Y_0}}\left(1-\frac{2}{mD_0}\right), \quad (3.87)$$

where we have neglected the endpoint contribution at $Y = 0$. On the other hand, eq. (3.77) gives

$$\frac{\ddot{a}_{pert}}{a_{pert}} = \frac{m^2\alpha^2 ve^{-m\alpha Y_0}}{D_0^2}\left(1+\frac{1}{mD_0}\right). \quad (3.88)$$

Combining the above two expressions, we obtain:

$$\frac{D_0^2}{m^2\alpha^2 ve^{-m\alpha Y_0}} = \left(1+\frac{1}{mD_0}\right)/\frac{\ddot{a}_{pert}}{a_{pert}}; \qquad \tau^2 = 2\left(1+\frac{1}{mD_0}\right)/\frac{\ddot{a}_{pert}}{a_{pert}}\left(1-\frac{2}{mD_0}\right);$$

$$\frac{\ddot{a}_{pert}}{a_{pert}}\tau^2 = 2\left(1+\frac{1}{mD_0}\right)\left(1-\frac{2}{mD_0}\right). \quad (3.89)$$

The right hand side of eq. (3.89) is approximately equal to 2 in the limit of large mD_0. Thence, we have

$$\frac{\ddot{a}_{pert}}{a_{pert}}\tau^2 = 2\left(1+\frac{1}{mD_0}\right)\left(1-\frac{2}{mD_0}\right) = 2. \quad (3.89b)$$

Also here, with regard the numerical result of eq. (3.89b), we have the following mathematical connections with Aurea ratio:

$$(\Phi)^{14/7} + (\Phi)^{-14/7} = \left(\frac{\sqrt{5}+1}{2}\right)^{14/7} + \left(\frac{\sqrt{5}+1}{2}\right)^{-14/7} = 2{,}618034 + 0{,}381966 = 3; \quad \frac{2}{3}\cdot 3 = 2;$$

$$(\Phi)^{21/7} + (\Phi)^{7/7} + (\Phi)^{-28/7} = \left(\frac{\sqrt{5}+1}{2}\right)^{21/7} + \left(\frac{\sqrt{5}+1}{2}\right)^{7/7} + \left(\frac{\sqrt{5}+1}{2}\right)^{-28/7} =$$

$$= 4{,}236068 + 1{,}618034 + 0{,}145898 = 6; \quad \frac{1}{3}\cdot 6 = 2.$$

Hence, the exponential potential of eq. (3.86) results in a nearly scale-invariant spectrum of perturbations provided that $|E| << |V_0|$ and $mD_0 >> 1$ are satisfied when modes pass outside the effective Hubble horizon.

We next compute the perturbation amplitude, by using eq. (3.76) to calculate $|\Delta Y_k|$. The conditions $|E| << |V_0|$ and $mD_0 >> 1$ must be satisfied when wavelengths pass outside the horizon. These conditions can be relaxed once the mode is well outside the horizon. In the limit that $mD_0 >> 1$ when the relevant modes cross outside the horizon, eq. (3.76) reduces to

$$x^2\frac{d^2 f_{\vec{k}}}{dx^2} - \left[2 - x^2\right]f_{\vec{k}} = 0, \quad (3.90)$$

with solution

$$f_k = x^{1/2}\left(C_1(k)J_{3/2}(x) + C_2(k)J_{-3/2}(x)\right), \quad (3.91)$$

where $J_{\pm 3/2}$ are Bessel functions. The coefficients $C_1(k)$ and $C_2(k)$ are fixed by requiring that modes well-within the horizon (i.e., $x >> 1$) be Minkowskian vacuum fluctuations, that is

$$f_{\vec{k}} = \frac{1}{\sqrt{6k\beta M_5^3 B}} e^{-ik\tau} \quad \text{for} \quad x >> 1. \quad (3.92)$$

Using this initial condition, we find the following amplitude for modes outside the horizon (with $x << 1$)

$$\Delta f_k \equiv \frac{k^{3/2} f_k}{(2\pi)^{3/2}} = \frac{-i}{(-\tau)(2\pi)^{3/2}\sqrt{6\beta M_5^3 B}}. \quad (3.93)$$

Substituting eq. (3.87) and using $f_k = D_0 \delta Y_k$, we find

$$\Delta Y_k = \frac{m\alpha}{2(2\pi)^{3/2}\sqrt{3\beta M_5^3 B}} \frac{\sqrt{ve^{-m\alpha Y_0}}}{D_0^2}. \quad (3.94)$$

Finally, we define the time-delay $\Delta\tau(k)$ by

$$|\Delta\tau(k)| = \left|\frac{\Delta Y_k}{\dot{Y}_0}\right| = \frac{m^2\alpha}{16\pi^{3/2}\sqrt{3\beta M_5^3 B}}\left(\frac{2}{mD_0}\right), \quad (3.95)$$

where we have used the equation of motion for Y_0, eq. (3.73). Note that the time-dependence of $\Delta\tau(k)$ is mild, a necessary condition for the validity of the time-delay formalism. The factor of $mD_0 \equiv mD(Y_0(\tau))$ is to be evaluated at time τ when a given mode crosses outside the horizon during the motion of the bulk brane. Let D_k denote the value of D_0 at horizon crossing for mode k. Since horizon crossing occurs when $x = 1$, or, equivalently, when $(-\tau) = k^{-1}$, eq. (3.87) gives

$$D_k \approx \frac{2}{m}\log\left(\frac{m^2\alpha}{2k}\sqrt{\frac{ve^{mC}}{2}}\right). \quad (3.96)$$

Substituting eqs. (3.95) and (3.96) into eq. (3.70), we find

$$|\delta_k| = \frac{\alpha^2 m^4\sqrt{2v}}{4\pi^{3/2}\sqrt{3\beta M_5^3 B}(mC+2)}\left(\frac{2}{mD_k}\right). \quad (3.97)$$

This expression for $|\delta_k|$ increases gradually with increasing k, corresponding to a spectrum tilted slightly towards the blue. The blue tilt is due to the fact that, in this example, D is decreasing as the brane moves. That is, the spectral index,

$$n_s \equiv 1 + \frac{d \log |\delta_k|^2}{d \log k} \approx 1 + \frac{4}{mD_k}, \quad (3.98)$$

exceeds unity. The current CMB data constrains the spectral index to lie in the range about $0.8 < n_s < 1.2$. Therefore, for our results to be consistent with experiments, we must have

$$mD_k > 20, \quad (3.99)$$

a constraint that is easily satisfied.
With regard the value s of n_s, i.e. 0.8 and 1.2, we have the following mathematical connections with Aurea ratio:

$$(\Phi)^{7/7} = \left(\frac{\sqrt{5}+1}{2}\right)^{7/7} = 1{,}618033987 \cong 1{,}618034; \quad 1{,}618034 \cdot \frac{3}{4} = 1{,}2135255;$$

$$(\Phi)^0 + (\Phi)^{-42/7} + (\Phi)^{-56/7} = \left(\frac{\sqrt{5}+1}{2}\right)^0 + \left(\frac{\sqrt{5}+1}{2}\right)^{-42/7} + \left(\frac{\sqrt{5}+1}{2}\right)^{-56/7} = 1 + 0{,}055728 + 0{,}021286 =$$

$$= 1{,}077014; \quad 1{,}077014 \cdot \frac{3}{4} = 0{,}8077605.$$

For the value of the eq. (3.99), we have the following mathematical connections:

$$(\Phi)^{35/7} + (\Phi)^{21/7} + (\Phi)^{-14/7} = \left(\frac{\sqrt{5}+1}{2}\right)^{35/7} + \left(\frac{\sqrt{5}+1}{2}\right)^{21/7} + \left(\frac{\sqrt{5}+1}{2}\right)^{-14/7} = 11{,}090170 + 4{,}236068 + 0{,}381966 =$$

$$= 15{,}708204 \cdot \frac{4}{3} = 20{,}944272.$$

Thence, we obtain:

$$mD_k > 20 =$$

$$= (\Phi)^{35/7} + (\Phi)^{21/7} + (\Phi)^{-14/7} = \left(\frac{\sqrt{5}+1}{2}\right)^{35/7} + \left(\frac{\sqrt{5}+1}{2}\right)^{21/7} + \left(\frac{\sqrt{5}+1}{2}\right)^{-14/7} = 11{,}090170 + 4{,}236068 + 0{,}381966 =$$

$= 15{,}708204 \cdot \frac{4}{3} = 20{,}944272$. We note that $20{,}944272 \cong 21$; 21 = 13 + 8, where 8 and 13 are Fibonacci's numbers. Furthermore, the number 8 is connected with the "modes" that correspond to the physical vibrations of a superstring by the following Ramanujan function:

$$8 = \frac{1}{3} \frac{4\left[anti\log \frac{\int_0^\infty \frac{\cos \pi t x w'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4} w'} \phi_{w'}(itw')}\right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}.$$

Furthermore, we consider the power-law potential

$$V(Y) = -vD(Y)^q = -v(\alpha Y + C)^q, \quad (3.100)$$

where $v > 0$ and $q < 0$ are constants. In this case, eq. (3.83) gives

$$\frac{\ddot{a}_{pert}}{a_{pert}}\tau^2 \approx 2\frac{\left(1-\frac{2}{q}\right)}{\left(1-\frac{4}{q}\right)^2} \approx 2 \quad (3.101)$$

for $|q| >> 1$. Hence, a power-law potential can also lead to a nearly scale-invariant spectrum provided that its exponent is sufficiently large. We can straightforwardly extend our analysis to an arbitrary potential $V(Y)$. Let us suppose that $V(Y)$ satisfies

$$\left|D(Y)\frac{dV}{dY}\right| >> \alpha|V(Y)|, \qquad \left|D(Y)\frac{d^2V}{dY^2}\right| >> \alpha\left|\frac{dV}{dY}\right|. \quad (3.102)$$

Then, eq. (3.83) reduces to

$$\frac{\ddot{a}_{pert}}{a_{pert}}\tau^2 \approx 2\left(\frac{VV''}{V'^2}\right). \quad (3.103)$$

Hence, the conditions for scale invariance are eqs. (3.102) as well as

$$\frac{VV''}{V'^2} \approx 1. \quad (3.104)$$

Also for the numerical value of eqs. (3.101) and (3.104), we have the following mathematical connections with the Aurea ratio:

$$(\Phi)^{14/7} + (\Phi)^{-14/7} = \left(\frac{\sqrt{5}+1}{2}\right)^{14/7} + \left(\frac{\sqrt{5}+1}{2}\right)^{-14/7} = 2{,}618034 + 0{,}381966 = 3; \quad \frac{2}{3}\cdot 3 = 2;$$

$$(\Phi)^{21/7} + (\Phi)^{7/7} + (\Phi)^{-28/7} = \left(\frac{\sqrt{5}+1}{2}\right)^{21/7} + \left(\frac{\sqrt{5}+1}{2}\right)^{7/7} + \left(\frac{\sqrt{5}+1}{2}\right)^{-28/7} =$$

$$= 4{,}236068 + 1{,}618034 + 0{,}145898 = 6; \quad \frac{1}{3}\cdot 6 = 2;$$

$$(\Phi)^{14/7} + (\Phi)^{-14/7} = \left(\frac{\sqrt{5}+1}{2}\right)^{14/7} + \left(\frac{\sqrt{5}+1}{2}\right)^{-14/7} = 2{,}618034 + 0{,}381966 = 3; \quad \frac{1}{3}\cdot 3 = 1.$$

If $\bar{g}_{\mu\nu}$ is the unperturbed, homogeneous metric (see eq. (3.67) with A and N functions of time), the perturbed 5d metric can be written as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + A^2(t)D(y,t)h_{\mu\nu}(\vec{x},t), \quad (3.105)$$

where $\mu,\nu = 0,...,3$. We can treat the tensor perturbations $h_{\mu\nu}$ as functions of $\vec{x}$ and t only. We are interested in the tensor perturbations which satisfy the conditions: $h_{0\mu} = 0$, $h^i_j = 0$, and $\partial^i h_{ij} = 0$. The perturbed 5d Einstein action to quadratic order is

$$S^T_{fluct} \equiv \frac{M_5^3}{2}\int d^5x\sqrt{-g}R = \frac{M_5^2}{8}\int d^4xa^2\left(\dot{h}^\mu_\nu \dot{h}^\nu_\mu - \partial_i h^\mu_\nu \partial^i h^\nu_\mu\right) \quad (3.106)$$

where the second expression is obtained by integrating over y. The tensor action is analogous to the scalar action given in eq. (3.75). From the action, we can derive the tensor analogue of the scalar fluctuation equation of motion, eq. (3.76)

$$x^2\frac{d^2 f^T_{\vec{k}}}{dx^2} - \left[\frac{\ddot{a}}{a}\tau^2 - x^2\right]f^T_{\vec{k}} = 0, \quad (3.107)$$

where

$$h^\nu_\mu \equiv \int \frac{d^3k}{(2\pi)^3}\varepsilon^\mu_\nu h_k(\tau) \quad (3.108)$$

and

$$f^T_{\vec{k}} \equiv ah_{\vec{k}}. \quad (3.109)$$

The critical difference between this tensor equation and the scalar fluctuation equation, eq. (3.76), is that the effective scale factor a_{pert} in eq. (3.76) has been replaced by a.
We introduced a potential to insure that a_{pert} led to a nearly scale-invariant spectrum, $(\ddot{a}_{pert}/a_{pert})\tau^2 \approx 2$. However, $a(\tau)$ in the tensor equation is approximately constant (recall that $a = \left(BI_3^{(0)}M_5\right)^{1/2} + O(\beta/\alpha)$). Consequently, the root mean square tensor fluctuation amplitude

$$\left|\Delta h_{\vec{k}}\right| \equiv \frac{k^{3/2}h_{\vec{k}}}{(2\pi)^{3/2}} \approx \frac{k}{(2\pi)^{3/2}}. \quad (3.110)$$

is not scale-invariant.
With regard the eq. (3.110), we have obtained the following mathematical connections with the Aurea ratio:

$$(2\pi)^{3/2} = 15,74960995;$$

$$(\Phi)^{-32/7} + (\Phi)^{40/7} = \left(\frac{\sqrt{5}+1}{2}\right)^{-32/7} + \left(\frac{\sqrt{5}+1}{2}\right)^{40/7} = 15,75004243 \cong 15,75;$$

$$(\Phi)^{-19/7} = \left(\frac{\sqrt{5}+1}{2}\right)^{-19/7} = 0{,}2708618458; \quad \arcsin(0{,}2708618458)\cdot\frac{180}{\pi} = 15{,}715558;$$

$$(\Phi)^{28/7} + (\Phi)^{14/7} + (\Phi)^{0} = \left(\frac{\sqrt{5}+1}{2}\right)^{28/7} + \left(\frac{\sqrt{5}+1}{2}\right)^{14/7} + \left(\frac{\sqrt{5}+1}{2}\right)^{0} = 10{,}472136;$$

$$10{,}472136\cdot\frac{3}{2} = 15{,}708204.$$

The cyclic story can be described in terms of an ordinary four-dimensional field theory, which can be obtained by taking the long wavelength limit of the brane picture. The distance between branes becomes a moduli (scalar) field ϕ. The interbrane interaction is replaced by a scalar field potential, $V(\phi)$. The different stages in the cyclic model in the brane picture are in one-to-one correspondence to the motion of the scalar field along the potential. Then, the action S describing gravity, the scalar field ϕ, and the matter-radiation fluid is:

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{16\pi G}\mathcal{R} - \frac{1}{2}(\partial\phi)^2 - V(\phi) + \beta^4(\phi)\rho_R\right], \quad (3.111)$$

where g is the determinant of the Friedmann-Robertson-Walker metric $g_{\mu\nu}$, G is Newton's constant and $\mathcal{R}$ is the Ricci scalar.

The β factor has the property that $\beta \to \infty$ as $a \to 0$ such that $a\beta \to$ constant. The revised solution to the equation of motion is $\rho_R \propto 1/(a\beta)^4$ which approaches a constant as $a \to 0$. The energy, once thinned out during the dark energy dominated phase, remains thinned out at the bounce. The β-factor simply reflects the fact that the extra-dimension collapses but our three-dimensions do not. As a result, entropy produced during one cycle is not concentrated at the crunch and does not contribute significantly to the entropy density at the beginning of the next cycle. Hence, cycles can continue for an arbitrarily long time and there is no practical way of distinguishing one cycle from the next.

If the cyclic model can be described in terms of ordinary field theory, then it may seem surprising that it is possible to generate a nearly scale invariant spectrum density perturbations. There are actually three distinct ways of producing a nearly scale-invariant spectrum, and that inflation represents only one of them. The three ways can be characterized by

$$w \equiv \frac{\left(\frac{1}{2}\dot{\phi}^2 - V\right)}{\left(\frac{1}{2}\dot{\phi}^2 + V\right)},$$

the effective equation of state of the scalar field. Case I is where $w \approx 1$ and the universe is expanding, the example of inflation. Case II is a contracting universe with $w \approx 0$. Case III is a contracting universe with $w >> 1$, that is the situation that applies in the cyclic model.

What is required to obtain $w >> 1$? From the expression for w, it is apparent that this is only possible if the potential is negative. In particular, for a negative exponentially steep potential $V \approx -\exp(c\phi)$, the solutions to the equation of motion have a scaling solution in which $\dot{\phi}^2/2V$ is constant and approximately -1. Consequently, w is much greater than unity and nearly constant. The generation of fluctuations for $w >> 1$ can be understood heuristically by examining the perturbed Klein-Gordon equation:

$$\delta\phi''_k = -\left(k^2 + \frac{a''}{a} + V_{,\phi\phi}\right)\delta\phi_k \qquad (3.112)$$

where $\phi(\mathbf{x},t)$ has been expanded in Fourier components $\delta\phi_k(t)$ with wavenumber k and prime is derivative with respect to conformal time η. The a''/a term is due to gravitational expansion, and the last term is due to the self-interaction of the scalar field. **This equation applies equally to inflation and to cyclic models**. The cyclic model corresponds to the limit where the gravity term is negligible and, instead, the perturbation equation is driven by the potential term. For the negative exponential potential, for example, the scaling solution corresponds to $V_{\phi,\phi} \approx 2/\eta^2$.

We have defined that $w >> 1$. We take the following values: $w = 4.97$ and $w = 4.23$. We obtain the following mathematical connections with the Aurea ratio:

$$(\Phi)^{7/7} + (\Phi)^{-7/7} + (\Phi)^{-21/7} + (\Phi)^{-63/7} = 1{,}618034 + 0{,}618034 + 0{,}236068 + 0{,}013156 = 2{,}485292;$$

$$2{,}485292 \cdot 2 = 4{,}970584 \quad (\text{for } \Phi = \left(\frac{\sqrt{5}+1}{2}\right));$$

$$(\Phi)^{21/7} = (\Phi)^3 = \left(\frac{\sqrt{5}+1}{2}\right)^{21/7} = 4{,}236067977; \quad (\Phi)^{-21/7} = (\Phi)^{-3} = \left(\frac{\sqrt{5}+1}{2}\right)^{-3} = 0{,}236067977;$$

$$\arcsin(0{,}236067977) \cdot \frac{180}{\pi} = 13{,}654585; \quad \arcsin(0{,}23) \cdot \frac{180}{\pi} = 13{,}29 \cong 13.$$

We note that 3, 21 and 13 are Fibonacci's numbers.

For **inflation**, the most stringent constraints are on the flat part of the potential, the range of the inflaton field where the density perturbations are generated. The constraints are commonly expressed as bounds on two "slow-roll" parameters:

$$\varepsilon = \left(\frac{V'}{V}\right)^2 << 1 \qquad (3.113)$$

and

$$\eta = \frac{V''}{V} << 1. \qquad (3.114)$$

For the **cyclic model**, the analogous constraints are on the steep portion of the potential where perturbations are generated. The constraints can be expressed in terms of two "fast-roll" parameters:

$$\bar{\varepsilon} = \left(\frac{V}{V'}\right)^2 << 1 \qquad (3.115)$$

and

$$\bar{\eta} = 1 - \frac{V''V}{(V')^2} << 1. \qquad (3.116)$$

The first constraint forces the slope to be steep and the second fixes the curvature, where each applies to the range of ϕ where the fluctuations are generated that are within the horizon today. The result is that **the constraints in the two models are remarkably similar**.

We note that if we take for $\overline{\eta} = 0{,}090170$, we have the following mathematical connections with the Aurea ratio:

$$(\Phi)^{-21/7} + (\Phi)^{-49/7} = \left(\frac{\sqrt{5}+1}{2}\right)^{-21/7} + \left(\frac{\sqrt{5}+1}{2}\right)^{-49/7} = 0{,}236068 + 0{,}034442 = 0{,}270510\,;$$

$$0{,}270510 \cdot \frac{1}{3} = 0{,}090170\,;$$

$(\Phi)^{-35/7} = \left(\frac{\sqrt{5}+1}{2}\right)^{-35/7} = 0{,}090169944$; $\quad \arcsin(0{,}090169944) \cdot \frac{180}{\pi} = 5{,}173384 \cong 5$ that is a Fibonacci's number.

4. On some equations concerning the "null energy condition" (NEC) violation regarding the inflationary models. [9] [10] [16]

The metric of the higher dimensional theory is $\mathcal{R}$-flat (RF) or $\mathcal{R}$-flat up to a conformal factor (CRF):

$$ds^2 = e^{2\Omega}(-dt^2 + \overline{a}^2(t) d\,\mathbf{x}^2) + g_{mn} dy^m dy^n\,, \quad (4.1)$$

where the **x** are the non-compact spatial dimensions; $y \equiv \{y^m\}$ are the extra dimensions; $\overline{a}(t)$ is the usual FRW scale factor; and

$$g_{mn}(t, y) = e^{-2\overline{\Omega}} \overline{g}_{mn} \quad (4.2)$$

where $\overline{g}_{mn}$ has Ricci (scalar) curvature $\mathcal{R} = 0$, as evaluated in the compact dimensions. We call the metric $\mathcal{R}$-flat (RF) if $\overline{\Omega} = \text{const.}$ and conformally $\mathcal{R}$-flat (CRF) if $\Omega(t, y) = \overline{\Omega}(t, y)$.
Now we develop some basic relations that make it possible to detect easily when a higher dimensional theory is forced to violate the NEC.
To describe a spatially-flat FRW spacetime after dimensional reduction, the metric $g_{mn}(t, y)$ and warp function $\Omega(t, y)$ must be functions of time t and extra-dimensional coordinates y^m only. We parameterize the rate of change of g_{mn} using quantities ξ and σ_{mn} defined by

$$\frac{1}{2}\frac{dg_{mn}}{dt} = \frac{1}{k}\xi g_{mn} + \sigma_{mn} \quad (4.3)$$

where $g^{mn}\sigma_{mn} = 0$ and where ξ and σ are functions of time and the extra dimensions.
The space-space components of the energy-momentum tensor are block diagonal with a 3×3 block describing the energy-momentum in the three non-compact dimensions and $k \times k$ block for the k compact directions. The 0-0 component is the higher dimensional energy density ρ.
Associated with the two blocks of space-space components of T_{IJ} are two trace averages:

$$p_3 \equiv \frac{1}{3}\gamma_3^{\mu\nu} T_{\mu\nu} \quad \text{and} \quad p_k \equiv \frac{1}{k}\gamma_k^{mn} T_{mn}\,, \quad (4.4)$$

where $\gamma_{3,k}$ are respectively the 3×3 and $k\times k$ blocks of the higher dimensional space-time metric. Violating the NEC means that $T_{MN}n^M n^N < 0$ for at least one null vector n^M and at least one space-time point. We find simple methods for identifying a subset of cases where the NEC must be violated. For this purpose, the following two lemmas are very useful:

Lemma 1: If $\rho + p_3$ or $\rho + p_k$ is less than zero for any space-time point, then the NEC is violated.

The second lemma utilizes the concept of A-averaged quantities:

$$\langle Q\rangle_A = \left(\int Q e^{A\Omega}\sqrt{g}\,d^k y\right)/\left(\int e^{A\Omega}\sqrt{g}\,d^k y\right); \quad (4.5)$$

that is, quantities averaged over the extra dimensions with weight factor $e^{A\Omega}$ where, for simplicity, we restrict ourselves to constant A. Using the fact that the weight function in the A-average is positive definite, a straightforward consequence is:

Lemma 2: If $\langle \rho + p_3\rangle_A < 0$ or $\langle \rho + p_k\rangle_A < 0$ for any A and any $\{t, \mathbf{x}\}$, then the NEC must be violated.

To illustrate the utility of A-averaging, we introduce the CRF metric into the higher-dimensional Einstein equations, and then try to express terms dependent on $\bar{a}$ in terms of the 4d effective scale factor using the relation $a(t) \equiv e^{\phi/2}\bar{a}(t)$, where:

$$e^{\phi} \equiv \ell^{-k}\int e^{2\Omega}\sqrt{g}\,d^k y \quad (4.6)$$

and ℓ is the 4+k-dimensional Planck length. The 4d effective scale factor, $a(t)$, obeys the usual 4d Friedmann equations:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}\rho_{4d} \quad (4.7)$$

$$\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{\ddot{a}}{a} = -p_{4d}. \quad (4.8)$$

Note that the 4d effective energy density ρ_{4d} and pressure p_{4d} are generally different from ρ and p_3 in the higher dimensional theory if the warp factor is non-trivial. Then, using the Einstein equations, we obtain

$$e^{-\phi}\left\langle e^{2\Omega}(\rho + p_3)\right\rangle_A = (\rho_{4d} + p_{4d}) - \frac{k+2}{2k}\langle\xi\rangle_A^2 - \frac{k+2}{2k}\left\langle\left(\xi - \langle\xi\rangle_A\right)^2\right\rangle_A - \left\langle\sigma^2\right\rangle_A \quad (4.9)$$

$$e^{-\phi}\left\langle e^{2\Omega}(\rho + p_k)\right\rangle_A = \frac{1}{2}(\rho_{4d} + 3p_{4d}) + 2\left(\frac{A}{4} - 1\right)\frac{k+2}{2k}\left\langle\left(\xi - \langle\xi\rangle_A\right)^2\right\rangle_A - \frac{k+2}{2k}\langle\xi\rangle_A^2 - \left\langle\sigma^2\right\rangle_A +$$
$$+\left[-5 + \frac{10}{k} + k + A\left(-3 + \frac{6}{k}\right)\right]\left\langle e^{2\Omega}(\partial\Omega)^2\right\rangle_A + \frac{k+2}{2k}\frac{1}{a^3}\frac{d}{dt}\left(a^3\langle\xi\rangle_A\right). \quad (4.10)$$

There is a range where

$$4 \geq A \geq \frac{10-5k+k^2}{3k-6} \equiv A_*, \quad (4.11)$$

which is the case for $13 \geq k \geq 3$ (for CRF). Some theorems below rely on choosing $A=2$; for this value to be within the range given in eq. (4.11), it is necessary that $8 \geq k \geq 3$. Since this includes the relevant string and M-theory models, we will implicitly assume this range of k for CRF models.
We note that 3, 8 and 13 are Fibonacci's numbers.
The two relations in eq. (4.9) can be rewritten

$$e^{-\phi}\left\langle e^{2\Omega}(\rho+p_3)\right\rangle_A = \rho_{4d}(1+w) - \frac{k+2}{2k}\langle\xi\rangle_A^2 + \text{non-positive terms for } \textit{all } A \quad (4.12)$$

$$e^{-\phi}\left\langle e^{2\Omega}(\rho+p_k)\right\rangle_A = \frac{1}{2}\rho_{4d}(1+3w) + \frac{k+2}{2k}\frac{1}{a^3}\frac{d}{dt}\left(a^3\langle\xi\rangle_A\right) + \text{ non-positive terms for } \textit{some } A, \quad (4.13)$$

where the values of A that make the last term non-positive are those that are in the range in eq. (4.11). Recall that w represents the ratio of the total 4d effective pressure p_{4d} to the total 4d effective energy density ρ_{4d}.
On the left hand side of eqs. (4.12) and (4.13), both ϕ and $\langle ... \rangle_A$ depend on the warp factor, Ω, but the combination is invariant under shifts $\Omega \to \Omega + C$, where C is a constant. Furthermore, the combination tends to have a weak dependence on Ω. For example, if $\rho + p_k$ is homogeneous in $\{y^m\}$, the left hand side reduces to $K(\rho + p_k)$, where the dimensionless coefficient K is not very sensitive to Ω or A; in particular,

$$K = \ell^k I(A+2)/I(A)I(2),$$

where

$$I(\bar{A}) \equiv \int e^{\bar{A}\Omega}\sqrt{g}\,d^k y. \quad (4.14)$$

In this notation, the k-dimensional volume of the compact space is $V_k = I(0)$; then, K is equal to ℓ^k / V_k, a coefficient which is strictly less than unity. Similarly, if $\rho + p_k$ is smooth and Ω has a sharp maximum on some subspace of dimension m and volume v_m, then the left hand side of eq. (4.13) is $\mathrm{O}(1)(\ell^m / v_m)(\rho + p_k)_{\max}$, where $(\rho + p_k)_{\max}$ is the value of $\rho + p_k$ evaluated on the subspace where Ω is maximal.
If the NEC is violated, it must be violated in the compact dimensions; it must be violated strongly (w_k significantly below the minimally requisite value for NEC violation); and the violation in the compact dimensions must vary with time in a manner that precisely tracks the equation-of-state in the 4d effective theory. The magnitude of the NEC violation is proportional to ρ_{4d} according to eq. (4.13), which is roughly 10^{100} times greater during the inflationary epoch than during the present dark energy dominated epoch. Hence, the source of NEC violation for inflation must be different and 10^{100} stronger.
The fact that NEC violation is required to have inflation in theories with extra dimensions is unexpected since this was not a requirement in the original inflationary models based on four dimensions only. Curiously, a criticism raised at times **about models with bounces from a contracting phase to an expanding phase, such as the ekpyrotic and cyclic alternatives to inflationary cosmology, is that the bounce requires a violation of the NEC** (or quantum gravity corrections to GR as the FRW scale factor $a(t) \to 0$ that serve the same function).

If is true that the violation of the condition NEC (condition of null energy) is required for the inflationary universe model and for the cyclic universe model, then it is possible that for the cyclic model the acceleration and initial exponential expansion of the inflationary phase, is equivalent to the collision between the two Brane-worlds and to the consequent acceleration of the expansion of space immediately after the Big Bang. ***This could be the explanation of the various cosmological and mathematical connections between the two models. Then, the inflation and the Big Bang would be only phases of the cyclic universe. Every cycle has its phase of Big Bang and its phase of inflation***.

4.1 On some equations concerning the evolution to a smooth universe in an ekpyrotic contracting phase with $w > 1$.

With regard the evolution to a smooth universe in an ekpyrotic contracting phase with $w > 1$, we find that the ekpyrotic smoothing mechanism is robust in the sense that the ratio ot the proper volume of the smooth region to the mixmaster-like region grows exponentially fast along time slices of constant mean curvature.

In this system the spacetime is described in terms of a coordinate system (t, x^i) and a tetrad ($\mathbf{e}_0$, $\mathbf{e}_\alpha$) where both the spatial coordinate index i and the spatial tetrad index α go from 1 to 3. Choose $\mathbf{e}_0$ to be hypersurface orthogonal with the relation between tetrad and coordinates of the form $\mathbf{e}_0 = N^{-1}\partial_t$ and $\mathbf{e}_\alpha = e_\alpha{}^i \partial_i$ where N is the lapse and the shift is chosen to be zero. Choose the spatial frame $\{e_\alpha\}$ to be Fermi propagated along the integral curves of $\mathbf{e}_0$. The commutators of the tetrad components are decomposed as follows:

$$[e_0, e_\alpha] = \dot{u}_\alpha e_0 - \left(H\delta_\alpha^\beta + \sigma_\alpha^\beta\right) e_\beta \quad (4.15) \qquad [e_\alpha, e_\beta] = \left(2a_{[\alpha}\delta_{\beta]}{}^\gamma + \varepsilon_{\alpha\beta\delta} n^{\delta\gamma}\right) e_\gamma \quad (4.16)$$

where $n^{\alpha\beta}$ is symmetric, and $\sigma^{\alpha\beta}$ is symmetric and trace free. The scale invariant tetrad variables are defined by $\partial_0 \equiv e_0 / H$ and $\partial_\alpha \equiv e_\alpha / H$ while scale invariant versions of the other gravitational variables are given by

$$\{E_\alpha^i, \Sigma_{\alpha\beta}, A^\alpha, N_{\alpha\beta}\} \equiv \{e_\alpha^i, \sigma_{\alpha\beta}, a^\alpha, n_{\alpha\beta}\} / H \ . \quad (4.17)$$

Note that the relation between the scale invariant tetrad variables and the coordinate derivatives is

$$\partial_0 = \mathcal{N}^{-1}\partial_t \quad (4.18) \qquad \partial_\alpha = E_\alpha^i \partial_i \, , \quad (4.19)$$

where $\mathcal{N} = NH$ is the scale invariant lapse. The matter model is a scalar field ϕ with potential V of the form

$$V(\phi) = -V_0 e^{-c\phi} \, , \quad (4.20)$$

where V_0 and c are positive constants. The scale invariant matter variables are given by

$$W = \partial_0 \phi \quad (4.21) \qquad S_\alpha = \partial_\alpha \phi \quad (4.22) \qquad \overline{V} = V / H^2 \, . \quad (4.23)$$

The time coordinate t is chosen so that

$$e^{-t} = 3H \ . \quad (4.24)$$

Note that this means that the surfaces of constant time are constant mean curvature surfaces. Note also that the singularity is approached as $t \to -\infty$. Due to equation (4.24) the scale invariant lapse satisfies an elliptic equation

$$-\partial^{\alpha}\partial_{\alpha}\mathcal{N} + 2A^{\alpha}\partial_{\alpha}\mathcal{N} + \mathcal{N}\left(3 + \Sigma_{\alpha\beta}\Sigma^{\alpha\beta} + W^{2} - \overline{V}\right) = 3 . \quad (4.25)$$

We note that 3 is a Fibonacci's number. Furthermore, we have the following mathematical connection with the Aurea ratio:

$$(\Phi)^{14/7} + (\Phi)^{0} + (\Phi)^{-14/7} = \left(\frac{\sqrt{5}+1}{2}\right)^{14/7} + \left(\frac{\sqrt{5}+1}{2}\right)^{0} + \left(\frac{\sqrt{5}+1}{2}\right)^{-14/7} = 2{,}618034 + 1 + 0{,}381966 = 4 ;$$

$$4 \cdot \frac{3}{4} = 3 .$$

The gravitational quantities E_{α}^{i}, A_{α}, $N^{\alpha\beta}$ and $\Sigma_{\alpha\beta}$ satisfy the following hyperbolic evolution equations

$$\partial_{t}E_{\alpha}^{i} = E_{\alpha}^{i} - \mathcal{N}\left(E_{\alpha}^{i} + \Sigma_{\alpha}^{\beta}E_{\beta}^{i}\right) \quad (4.26)$$

$$\partial_{t}A_{\alpha} = A_{\alpha} + \frac{1}{2}\Sigma_{\alpha}^{\beta}\partial_{\beta}\mathcal{N} - \partial_{\alpha}\mathcal{N} + \mathcal{N}\left(\frac{1}{2}\partial_{\beta}\Sigma_{\alpha}^{\beta} - A_{\alpha} - \Sigma_{\alpha}^{\beta}A_{\beta}\right) \quad (4.27)$$

$$\partial_{t}N^{\alpha\beta} = N^{\alpha\beta} - \varepsilon^{\gamma\delta(\alpha}\Sigma_{\delta}^{\beta)}\partial_{\gamma}\mathcal{N} + \mathcal{N}\left(-N^{\alpha\beta} + 2N^{(\alpha}{}_{\gamma}\Sigma^{\beta)\gamma} - \varepsilon^{\gamma\delta(\alpha}\partial_{\gamma}\Sigma_{\delta}^{\beta)}\right) \quad (4.28)$$

$$\partial_{t}\Sigma_{\alpha\beta} = \Sigma_{\alpha\beta} + \partial_{<\alpha}\partial_{\beta>}\mathcal{N} + A_{<\alpha}\partial_{\beta>}\mathcal{N} + \varepsilon_{\gamma\delta(\alpha}N_{\beta)}{}^{\delta}\partial^{\gamma}\mathcal{N} + \mathcal{N}\Big[-3\Sigma_{\alpha\beta} - \partial_{<\alpha}A_{\beta>} - 2N_{<\alpha}{}^{\gamma}N_{\beta>\gamma} + N^{\gamma}{}_{\gamma}N_{<\alpha\beta>} + \varepsilon_{\gamma\delta(\alpha}\left(\partial^{\gamma}N_{\beta)}{}^{\delta} - 2A^{\gamma}N_{\beta)}{}^{\delta}\right) + S_{<\alpha}S_{\beta>}\Big]. \quad (4.29)$$

Here parentheses around a pair of indices denote the symmetric part, while angle brackets denote the symmetric trace-free part. The equations of motion for the matter variables are as follows:

$$\partial_{t}\phi = \mathcal{N}W \quad (4.30)$$

$$\partial_{t}S_{\alpha} = S_{\alpha} + W\partial_{\alpha}\mathcal{N} + \mathcal{N}\left[\partial_{\alpha}W - \left(S_{\alpha} + \Sigma_{\alpha}^{\beta}S_{\beta}\right)\right] \quad (4.31)$$

$$\partial_{t}W = W + S^{\alpha}\partial_{\alpha}\mathcal{N} + \mathcal{N}\left(\partial^{\alpha}S_{\alpha} - 3W - 2A^{\alpha}S_{\alpha} - \frac{\partial\overline{V}}{\partial\phi}\right). \quad (4.32)$$

In addition, the variables are subject to the vanishing of the following constraint quantities

$$(\mathcal{C}_{com})^{\lambda i} = \varepsilon^{\alpha\beta\lambda}\left[\partial_{\alpha}E_{\beta}^{i} - A_{\alpha}E_{\beta}^{i}\right] - N^{\lambda\gamma}E_{\gamma}^{i} \quad (4.33)$$

$$(\mathcal{C}_{J})^{\gamma} = \partial_{\alpha}N^{\alpha\gamma} + \varepsilon^{\alpha\beta\gamma}\partial_{\alpha}A_{\beta} - 2A_{\alpha}N^{\alpha\gamma} \quad (4.34)$$

$$(\mathcal{C}_{C})_{\alpha} = \partial_{\beta}\Sigma_{\alpha}^{\beta} - 3\Sigma_{\alpha}^{\beta}A_{\beta} - \varepsilon_{\alpha\beta\gamma}N^{\beta\delta}\Sigma_{\delta}^{\gamma} - WS_{\alpha} \quad (4.35)$$

$$\mathcal{C}_{G} = 1 + \frac{2}{3}\partial_{\alpha}A^{\alpha} - A^{\alpha}A_{\alpha} - \frac{1}{6}N^{\alpha\beta}N_{\alpha\beta} + \frac{1}{12}\left(N^{\gamma}{}_{\gamma}\right)^{2} - \frac{1}{6}\Sigma^{\alpha\beta}\Sigma_{\alpha\beta} - \frac{1}{6}W^{2} - \frac{1}{6}S^{\alpha}S_{\alpha} - \frac{1}{3}\overline{V} \quad (4.36)$$

$$(\mathcal{C}_{S})_{\alpha} = S_{\alpha} - \partial_{\alpha}\phi . \quad (4.37)$$

With regard the value 12 of eq. (4.36) we have the mathematical connection with the following Ramanujan's modular equation:
The number 12 = 24 / 2, is related to the physical vibrations of the bosonic strings by the following Ramanujan function:

$$24 = \frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}.$$

For simplicity, we choose the initial conformal metric to be flat and (x, y, z) to be the usual cartesian coordinates for that metric, and we choose the spatial triad to lie along those spatial directions. Thus, the scale free spatial triad becomes

$$E_\alpha^i = H^{-1}\psi^{-2}\delta_\alpha^i. \quad (4.38)$$

It then follows from equation (4.16) that

$$A_\alpha = -2\psi^{-1}E_\alpha^i\partial_i\psi \quad (4.39) \qquad N_{\alpha\beta} = 0. \quad (4.40)$$

The shear is essentially the trace-free part of the extrinsic curvature, and the constraint equations simplify for a particular rescaling of the trace-free part of the extrinsic curvature with the conformal factor. We therefore introduce the quantity $Z_{\alpha\beta}$ defined by

$$\Sigma_{\alpha\beta} = \psi^{-6}Z_{\alpha\beta}. \quad (4.41)$$

Similar considerations apply to the matter variables, leading us to define the quantity Q given by

$$W = \psi^{-6}Q. \quad (4.42)$$

Here we will specify Q, ϕ and a part of Z_{ik} and solve the constraint equations for the conformal factor ψ and the rest of Z_{ik}. From equation (4.35) and our ansatz for the scale invariant variables we obtain

$$\partial^i Z_{ik} = Q\partial_k\phi. \quad (4.43)$$

In the vacuum case this equation simply becomes the conditions that Z_{ik} is divergence-free, which is in turn simply an algebraic condition on the Fourier coefficients of Z_{ik}. Note that since $\Sigma_{\alpha\beta}$ must be trace-free, so must Z_{ik}. A simple, but still fairly general divergence-free and trace-free Z_{ik} is the following:

$$Z_{ik} = \begin{pmatrix} b_2 & \kappa & 0 \\ \kappa & a_1 \cos x + b_1 & a_2 \cos x \\ 0 & a_2 \cos x & -b_1 - b_2 - a_1 \cos x \end{pmatrix}, \quad (4.44)$$

where κ, a_1, a_2, b_1 and b_2 are constants. We still keep this divergence-free part of Z_{ik} but now add to it a piece that has a non-zero divergence. We simply specify the Fourier coefficients of ϕ and Q via

$$Q(x, t=0) = \frac{f_1}{H} \cos(m_1 x + d_1) \quad (4.45) \qquad \phi(x, t=0) = f_2 \cos(m_2 x + d_2), \quad (4.46)$$

where f_1, m_1, d_1, f_2, m_2 and d_2 are constants. This turns equation (4.43) into an algebraic equation for the Fourier coefficients of this non-zero divergence piece of Z_{ik} which we then solve. Now imposing equation (4.36) our ansatz yields

$$\partial^i \partial_i \psi = \left(\frac{3}{4} H^2 - \frac{1}{4} V \right) \psi^5 - \frac{1}{8} \left(\partial^i \phi \partial_i \phi \right) \psi - \frac{1}{8} \left(Q^2 + Z^{ik} Z_{ik} \right) H^2 \psi^{-7}, \quad (4.47)$$

which is solved for the conformal factor ψ using the numerical methods.
With regard the eq. (4.47), the number 8 is connected with the "modes" that correspond to the physical vibrations of a superstring by the following Ramanujan function:

$$8 = \frac{1}{3} \frac{4 \left[anti \log \frac{\int_0^\infty \frac{\cos \pi t x w'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4} w'} \phi_{w'}(itw')} \right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log \left[\sqrt{\left(\frac{10 + 11\sqrt{2}}{4} \right)} + \sqrt{\left(\frac{10 + 7\sqrt{2}}{4} \right)} \right]}.$$

The constraint equations (4.33) and (4.34) are automatically satisfied by this ansatz. We then satisfy equation (4.37) by using the given value of ϕ to compute the initial value of S_α.
Now we show results from a single example that demonstrates the generic behaviour: evolution from a highly inhomogeneous, anisotropic universe with significant curvature at the initial time to a universe containing distinct volumes of either smooth, homogeneous $w >> 1$ matter dominated regions, or $w = 1$ mixmaster-like regions. Whenever a $w >> 1$ region forms it grows exponentially fast in proper volume relative to $w = 1$ regions. The particular initial conditions for this example are (4.44 – 4.46)
$a_1 = 0.70$, $a_2 = 0.10$, $\kappa = 0.01$, $b_1 = 1.80$, $b_2 = -0.15$, $f_1 = 2.00$,
$m_1 = 1$, $d_1 = -1.7$, $f_2 = 0.15$, $m_2 = 2$, $d_2 = -1.0$, and $V_0 = 0.1$, $c = 10$ (4.48)

for the scalar field potential parameters (4.20).
With regard the values of (4.48), we take the following: 0,10 0,15 0,70 1,70 1,80 and 10. We have the following mathematical connections with the Aurea ratio:

$$(\Phi)^{-35/7}+(\Phi)^{-49/7}+(\Phi)^{-63/7}=\left(\frac{\sqrt{5}+1}{2}\right)^{-35/7}+\left(\frac{\sqrt{5}+1}{2}\right)^{-49/7}+\left(\frac{\sqrt{5}+1}{2}\right)^{-63/7}=$$

$$=0{,}090170+0{,}034442+0{,}013156=0{,}137767;\quad 0{,}137767\cdot\frac{3}{4}=0{,}103326;$$

$$(\Phi)^{-28/7}+(\Phi)^{-42/7}+(\Phi)^{-56/7}+(\Phi)^{-84/7}=\left(\frac{\sqrt{5}+1}{2}\right)^{-28/7}+\left(\frac{\sqrt{5}+1}{2}\right)^{-42/7}+\left(\frac{\sqrt{5}+1}{2}\right)^{-56/7}+\left(\frac{\sqrt{5}+1}{2}\right)^{-84/7}=$$

$$=0{,}145898+0{,}055728+0{,}021286+0{,}003106=0{,}226018\cdot\frac{2}{3}=0{,}150679;$$

$$(\Phi)^{-7/7}+(\Phi)^{-21/7}+(\Phi)^{-35/7}=\left(\frac{\sqrt{5}+1}{2}\right)^{-7/7}+\left(\frac{\sqrt{5}+1}{2}\right)^{-21/7}+\left(\frac{\sqrt{5}+1}{2}\right)^{-35/7}=$$

$$=0{,}618034+0{,}236068+0{,}090170=0{,}944272\cdot\frac{3}{4}=0{,}708204;$$

$$(\Phi)^{7/7}+(\Phi)^{-7/7}+(\Phi)^{-42/7}=\left(\frac{\sqrt{5}+1}{2}\right)^{7/7}+\left(\frac{\sqrt{5}+1}{2}\right)^{-7/7}+\left(\frac{\sqrt{5}+1}{2}\right)^{-42/7}=$$

$$=1{,}618034+0{,}618034+0{,}055728=2{,}291796\cdot\frac{3}{4}=1{,}718847;$$

$$(\Phi)^{-7/7}+(\Phi)^{-21/7}+(\Phi)^{-49/7}+(\Phi)^{-84/7}=\left(\frac{\sqrt{5}+1}{2}\right)^{-7/7}+\left(\frac{\sqrt{5}+1}{2}\right)^{-21/7}+\left(\frac{\sqrt{5}+1}{2}\right)^{-49/7}+\left(\frac{\sqrt{5}+1}{2}\right)^{-84/7}=$$

$$=0{,}618034+0{,}236068+0{,}034442+0{,}003106=0{,}891649\cdot 2=1{,}783298\cong 0{,}90\cdot 2=1{,}80;$$

$$(\Phi)^{35/7}+(\Phi)^{14/7}=\left(\frac{\sqrt{5}+1}{2}\right)^{35/7}+\left(\frac{\sqrt{5}+1}{2}\right)^{14/7}=11{,}090170+2{,}618034=13{,}708204\cdot\frac{3}{4}=10{,}281153.$$

Now, let $\chi_1(n)$ be a complex character to the modulus 5 such that $\chi_1(2)=i$, and let

$$\kappa=\frac{\sqrt{10-2\sqrt{5}}-2}{\sqrt{5}-1}=0{,}284078227. \quad (4.48b)$$

The function

$$f(s)=\frac{1-i\kappa}{2}L(s,\chi_1)+\frac{1+i\kappa}{2}L(s,\overline{\chi}_1), \quad (4.48c) \quad \text{where} \quad L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s},$$

is called the Davenport-Heilbronn function and satisfies the Riemann-type equation

$$\left(\frac{\pi}{5}\right)^{-s/2}\Gamma\left(\frac{(s+1)}{2}\right)f(s)=g(1-s). \quad (4.48d)$$

We note that $10{,}281153-\kappa\cong 10$. Furthermore:

$(\Phi)^{-14/7} = \left(\frac{\sqrt{5}+1}{2}\right)^{-14/7} = 0{,}381966$; $\quad 0{,}381966 \cdot \frac{3}{4} = 0{,}286475$. Thence, we can write also:

$$\kappa = \frac{\sqrt{10-2\sqrt{5}}-2}{\sqrt{5}-1} = 0{,}284078227 \cong 0{,}2841 \cong \left(\frac{\sqrt{5}+1}{2}\right)^{-14/7} \cdot \frac{3}{4} \cong 0{,}2865 .$$

It is enlightening to visualize the evolution via the behaviour of the matter (Ω_m), shear (Ω_s) and curvature (Ω_k) contributions to the normalized energy density, defined as

$$\Omega_m \equiv \frac{1}{6}W^2 + \frac{1}{6}S^\alpha S_\alpha + \frac{1}{3}\overline{V} \quad (4.49) \qquad \Omega_s \equiv \frac{1}{6}\Sigma^{\alpha\beta}\Sigma_{\alpha\beta} \quad (4.50)$$

$$\Omega_k \equiv -\frac{2}{3}\partial_\alpha A^\alpha + A^\alpha A_\alpha + \frac{1}{6}N^{\alpha\beta}N_{\alpha\beta} - \frac{1}{12}\left(N^\gamma{}_\gamma\right)^2 , \quad (4.51)$$

where $\Omega_m + \Omega_s + \Omega_k = 1$ by (4.36).

We note that the eq. (4.51), i.e. 12 = 24/2, is related to the physical vibrations of the bosonic strings by the following Ramanujan function:

$$24 = \frac{4\left[anti\log \frac{\int_0^\infty \frac{\cos \pi t x w'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4} w'} \phi_{w'}(itw')}\right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} .$$

The effective equation of state parameter w takes the following form in Hubble normalized variables:

$$w = \frac{\frac{1}{2}W^2 + \frac{1}{2}S^\alpha S_\alpha - \overline{V}}{\frac{1}{2}W^2 + \frac{1}{2}S^\alpha S_\alpha + \overline{V}} . \quad (4.52)$$

It is evident that at late times the region that has smoothed out and become matter dominated coincides with $w >> 1$, whereas the mixmaster-like regime evolves to $w = 1$. We can calculate the behaviour of the solution in the asymptotic matter dominated region as follows. At late times, all spatial derivatives have become negligible. The constraint (4.36) then reduces to

$$\frac{W^2 + 2\overline{V}}{6} - 1 \approx 0 , \quad (4.53)$$

and slicing condition for $\mathcal{N}$ (4.25) becomes

$$3\mathcal{N} \approx \frac{3}{3-\overline{V}}, \quad (4.54)$$

Furthermore, $\overline{V}$ is finite and non-zero. This implies from (4.20, 4.23, 4.24) that ϕ takes the asymptotic form

$$\phi(x,t) \approx \phi_0(x) + \frac{2t}{c} \quad (4.55)$$

and thus W (4.21) tends to

$$W \approx \frac{2}{c\mathcal{N}}. \quad (4.56)$$

Combining these relations gives

$$W \approx c, \quad (4.57) \qquad \overline{V} \approx 3 - \frac{c^2}{2}, \quad (4.58) \qquad \mathcal{N} \approx \frac{2}{c^2}, \quad (4.59)$$

and from (4.52)

$$w \approx c^2/3 - 1. \quad (4.60)$$

We have that $c = \sqrt{2} = 1{,}414213562$; $W \cong c$ and $\overline{V} = 2$. Thence, we have the following mathematical connections with the Aurea ratio:

$$(\Phi)^{-7/7} + (\Phi)^{-21/7} + (\Phi)^{-35/7} = \left(\frac{\sqrt{5}+1}{2}\right)^{-7/7} + \left(\frac{\sqrt{5}+1}{2}\right)^{-21/7} + \left(\frac{\sqrt{5}+1}{2}\right)^{-35/7} =$$

$$= 0{,}618034 + 0{,}236068 + 0{,}090170 = 0{,}944272 \cdot \frac{3}{2} = 1{,}416408;$$

$$(\Phi)^{14/7} + (\Phi)^{-14/7} = \left(\frac{\sqrt{5}+1}{2}\right)^{14/7} + \left(\frac{\sqrt{5}+1}{2}\right)^{-14/7} = 2{,}618034 + 0{,}381966 = 3; \quad \frac{2}{3} \cdot 3 = 2;$$

$$(\Phi)^{21/7} + (\Phi)^{7/7} + (\Phi)^{-28/7} = \left(\frac{\sqrt{5}+1}{2}\right)^{21/7} + \left(\frac{\sqrt{5}+1}{2}\right)^{7/7} + \left(\frac{\sqrt{5}+1}{2}\right)^{-28/7} =$$

$$= 4{,}236068 + 1{,}618034 + 0{,}145898 = 6; \quad \frac{1}{3} \cdot 6 = 2;$$

Let S denote the proper spatial volume element associated with the spatial metric h_{ij} of $t = \text{const.}$ slices i.e., $S = \sqrt{\det h}$. The fractional change of S with respect to time is

$$\partial_t \ln S = -\frac{1}{2} h_{ij} \partial_t h^{ij}, \quad (4.61)$$

which can be written as

$$\partial_t \ln S = 3\mathcal{N}. \quad (4.62)$$

In the asymptotic regime where spatial gradients are negligible, $\mathcal{N}$ approaches a constant (4.54), and thus (4.62) can be integrated to give

$$S_m \propto e^{6t/c^2}, \quad w >> 1 \quad (4.63) \qquad S_v \propto e^t, \quad w = 1 \quad (4.64)$$

where we have used (4.54) where $w >> 1$, and note that $\bar{V} \approx 0$ when $w = 1$. Thus, at late times the ratio $\mathcal{R}$ of the proper volume of matter to mixmaster-like regions of the universe grows as

$$\mathcal{R} = \frac{\int S_m dx}{\int S_v dx} \propto e^{-t\left(1-6/c^2\right)}. \quad (4.65)$$

Thus, as long as $c > \sqrt{6}$ (which is equivalent to $w > 1$), $\mathcal{R} \to \infty$ as $t \to -\infty$.
We note that $\sqrt{6} = 2,449489743$, is related to the following mathematical connection with the Aurea ratio:

$$(\Phi)^{-7/7} + (\Phi)^{-28/7} + (\Phi)^{-42/7} = \left(\frac{\sqrt{5}+1}{2}\right)^{-7/7} + \left(\frac{\sqrt{5}+1}{2}\right)^{-28/7} + \left(\frac{\sqrt{5}+1}{2}\right)^{-42/7} =$$
$$= 0,618034 + 0,145898 + 0,055728 = 0,819660 \cdot 3 = 2,458980.$$

5. On some equations concerning the approximate inflationary solutions rolling away from the unstable maximum of p-adic string theory. [11] [16]

The action of p-adic string theory is given by

$$S = \frac{m_s^4}{g_p^2}\int d^4x\left(-\frac{1}{2}\phi p^{-\frac{\Box}{2m_s^2}}\phi + \frac{1}{p+1}\phi^{p+1}\right) \equiv \frac{m_s^4}{g_p^2}\int d^4x\left(-\frac{1}{2}\phi e^{-\frac{\Box}{m_p^2}}\phi + \frac{1}{p+1}\phi^{p+1}\right) \quad (5.1)$$

where $\Box = -\partial_t^2 + \nabla^2$ in the flat space and we have defined

$$\frac{1}{g_p^2} \equiv \frac{1}{g_s^2}\frac{p^2}{p-1} \quad \text{and} \quad m_p^2 \equiv \frac{2m_s^2}{\ln p}. \quad (5.2)$$

The dimensionless scalar field $\phi(x)$ describes the open string tachyon, m_s is the string mass scale and g_s is the open string coupling constant. Though the action (5.1) was originally derived for p a prime number, it appears that it can be continued to any positive integer and even makes sense in the limit $p \to 1$. Setting $\Box = 0$ in the action, the resulting potential takes the form

$$V=\left(\frac{m_s^4}{g_p^2}\right)\left(\frac{1}{2}\phi^2-\frac{1}{p+1}\phi^{p+1}\right). \quad (5.2b)$$

The action (5.1) is a simplified model of the bosonic string which only qualitatively reproduces some aspects of a more realistic theory. That being said, there are several nontrivial similarities between p-adic string theory and the full string theory.

The field equation that results from (5.1) is

$$e^{-\Box/m_p^2}\phi=\phi^p \quad (5.3)$$

We are interested in perturbing around the solution $\phi=1$, which is a critical point of the potential, representing the unstable tachyonic maximum.

One may wonder whether the field theory (5.1) naively allows for slow roll inflation in the conventional sense. Naively one might expect that for a slowly rolling field the higher powers of $\Box$ in the kinetic term are irrelevant and one may approximate (5.1) by a local field theory. The action (5.1) can be rewritten as

$$S=\int d^4x\left[\frac{1}{2}\chi\Box\chi-V(\chi)+...\right] \quad (5.4)$$

where we have defined the field χ as

$$\chi=\chi_0\phi \quad (5.5) \qquad \chi_0=\frac{p}{g_s}\sqrt{\frac{\ln p}{2(p-1)}}m_s \quad (5.6)$$

and the potential is

$$V(\chi)=\frac{m_s^2}{\ln p}\chi^2-\frac{m_s^4}{g_s^2}\frac{p^2}{p^2-1}\left(\frac{\chi}{\chi_0}\right)^{p+1}. \quad (5.7)$$

In (5.4) the . . . denotes terms with higher powers of $\Box$. Thence, the eq. (5.4) can be rewritten also

$$S=\int d^4x\left[\frac{1}{2}\frac{p}{g_s}\sqrt{\frac{\ln p}{2(p-1)}}m_s\phi\Box\frac{p}{g_s}\sqrt{\frac{\ln p}{2(p-1)}}m_s\phi-\frac{m_s^2}{\ln p}\chi^2-\frac{m_s^4}{g_s^2}\frac{p^2}{p^2-1}\left(\frac{\chi}{\chi_0}\right)^{p+1}+...\right]. \quad (5.7b)$$

Working in the context of the action (5.4) let us consider the slow roll parameters describing the flatness of the potential (5.7) about the unstable maximum $\chi=\chi_0$. It is straightforward to show that

$$\frac{M_p^2}{2}\frac{1}{V(\chi_0)^2}\left(\frac{\partial V(\chi)}{\partial\chi}\right)^2\Big|_{\chi=\chi_0}=0 \quad (5.8)$$

$$M_p^2\frac{1}{V(\chi_0)}\frac{\partial^2V(\chi)}{\partial\chi^2}\bigg|_{\chi=\chi_0}=-\frac{4g_s^2}{\ln p}\frac{p^2-1}{p^2}\left(\frac{M_p}{m_s}\right)^2. \quad (5.9)$$

With regard the approximate solution for the classical background, we must solve the Friedmann equation

$$H^2 = \frac{1}{3M_p^2}\rho_\phi \quad (5.10)$$

to second order in u. To find the energy density ρ_ϕ, we turn to the stress energy tensor for the p-adic scalar field. A convenient expression for $T_{\mu\nu}$ is:

$$T_{\mu\nu} = \frac{m_s^4}{2g_p^2} g_{\mu\nu}\left[\phi e^{-\frac{\Box}{m_p^2}}\phi - \frac{2}{p+1}\phi^{p+1} + \frac{1}{m_p^2}\int_0^1 d\tau \left(\Box e^{-\frac{\tau\Box}{m_p^2}}\phi\right)\left(e^{-\frac{(1-\tau)\Box}{m_p^2}}\phi\right) + \frac{1}{m_p^2}\int_0^1 d\tau \left(\partial_\alpha e^{-\frac{\tau\Box}{m_p^2}}\phi\right)\left(\partial^\alpha e^{-\frac{(1-\tau)\Box}{m_p^2}}\phi\right)\right] +$$
$$- \frac{m_s^4}{m_p^2 g_p^2}\int_0^1 d\tau \left(\partial_\mu e^{-\frac{\tau\Box}{m_p^2}}\phi\right)\left(\partial_\nu e^{-\frac{(1-\tau)\Box}{m_p^2}}\phi\right). \quad (5.11)$$

One may verify that the $T_{\mu\nu}$ is symmetric by changing the dummy integration variable $\tau \to 1-\tau$ in the last term. For homogeneous $\phi(t)$ the above expression simplifies, and for T_{00} we find

$$\rho_\phi = -T_{00} = \frac{m_s^4}{2g_p^2}\left[\phi e^{-\frac{\Box}{m_p^2}}\phi - \frac{2}{p+1}\phi^{p+1} + \frac{1}{m_p^2}\int_0^1 d\tau \left(\Box e^{-\frac{\tau\Box}{m_p^2}}\phi\right)\left(e^{-\frac{(1-\tau)\Box}{m_p^2}}\phi\right) +\right.$$
$$\left. + \frac{1}{m_p^2}\int_0^1 d\tau \partial_t\left(e^{-\frac{\tau\Box}{m_p^2}}\phi\right)\partial_t\left(e^{-\frac{(1-\tau)\Box}{m_p^2}}\phi\right)\right]. \quad (5.12)$$

One can evaluate the above expression term by term, keeping up to $\mathrm{O}\left(e^{2\lambda t}\right) \approx u^2$. The final result reads

$$T_{00} = \frac{m_s^4}{2g_p^2}\left[1 - u\left(1 + e^{\mu_1}\right) - \frac{2\left[1-(p+1)u\right]}{p+1} + u\left(e^{\mu_1} - 1\right)\right] + \mathrm{O}\left(u^2\right) = \frac{m_s^4(p-1)}{2g_p^2(p+1)} + \mathrm{O}\left(u^2\right). \quad (5.13)$$

The $\mathrm{O}(u)$ terms cancel out and matching the coefficients in the Friedmann equation gives us the simple results

$$H_0^2 = \frac{m_s^4}{6M_p^2}\frac{p-1}{g_p^2(p+1)} \quad (5.14)$$

and

$$H_1 = 0 \quad (5.15)$$

for zeroth and first order respectively. The $\mathrm{O}\left(u^2\right)$ contribution to T_{00} is quite complicated but once we use (5.15) it simplifies greatly. Matching coefficient at order $\mathrm{O}\left(u^2\right)$ in the Friedmann equation gives

$$H_2 = \frac{\lambda m_s^4}{4g_p^2 m_p^2 M_p^2} e^{\mu_1} = \frac{1}{8g_s^2}\frac{p^3 \ln p}{p-1}\left(\frac{m_s}{M_p}\right)^2 \lambda . \quad (5.16)$$

We note that the number 8 in the eq (5.16) is a Fibonacci's number and can be connected with the "modes" that correspond to the physical vibrations of a superstring by the following Ramanujan function:

$$8 = \frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}.$$

Because of our sign convention for H_r, the fact that $H_2 > 0$ means that the expansion is slowing as ϕ rolls from the unstable maximum, as one would expect in a conventional inflationary model.
We are approximating the background dynamics as de Sitter which amounts to working in the limit $u \to 0$ so that

$$H^2 \equiv H_0^2 = \frac{m_s^4}{6M_p^2}\frac{p-1}{g_p^2(p+1)} \quad (5.17) \qquad \phi \equiv \phi_0 \equiv 1. \quad (5.18)$$

We expand the p-adic tachyon field in perturbation theory as

$$\phi(t,\vec{x}) = \phi^{(0)}(t) + \delta\phi(t,\vec{x}) = 1 + \delta\phi(t,\vec{x}). \quad (5.19)$$

The perturbed Klein-Gordon equation (5.3) takes the form

$$e^{-\Box/m_p^2}\delta\phi = p\delta\phi. \quad (5.20)$$

One can construct solutions by taking $\delta\phi$ to be an eigenfunction of the $\Box$ operator. If we choose $\delta\phi$ to satisfy

$$-\Box\delta\phi = +B\delta\phi \quad (5.21)$$

then this is also a solution to (5.20) if

$$B = m_p^2 \ln p = 2m_s^2 \quad (5.22)$$

where in the second equality we have used (5.2).
For fields which are on-shell (that is, when (5.21) is solved) the field obeys

$$\left(1-e^{-\Box/m_p^2}\right)\delta\phi = \left(1-e^{B/m_p^2}\right)\delta\phi = \left(1-e^{B/m_p^2}\right)\frac{1}{(-B)}(-B)\delta\phi = \left(1-e^{B/m_p^2}\right)\frac{1}{(-B)}\Box\delta\phi = \frac{p-1}{2m_s^2}\Box\delta\phi. \quad (5.23)$$

Thus, for on-shell fields the kinetic term in the Lagrangian can be written as

$$\mathcal{L}_{on-shell} = \frac{m_s^4}{g_p^2}\frac{1}{2}\phi\left(1-e^{-\Box/m_p^2}\right)\phi + ... = \frac{m_s^4}{g_p^2}\frac{p-1}{2m_s^2}\frac{1}{2}\phi\Box\phi + ... = \frac{1}{2}\varphi\Box\varphi + ... \quad (5.24)$$

In (5.24) we have defined the “canonical” field

$$\varphi \equiv A\phi \quad (5.25)$$

where

$$A \equiv \frac{m_s p}{\sqrt{2} g_s}. \quad (5.26)$$

The field φ has a canonical kinetic term in the action, at least while (5.21) is satisfied. Now, let us return to the task of solving (5.21), bearing in mind that $\delta\varphi = A\delta\phi$ is the appropriate canonically normalized field. We write the quantum mechanical solution in term of annihilation/creation operators as

$$\delta\varphi(t,\vec{x}) = \int \frac{1}{(2\pi)^{3/2}} d^3k \left[a_k \varphi_k(t) e^{ikx} + h.c. \right] \quad (5.27)$$

and the mode functions $\varphi_k(t)$ are given by

$$\varphi_k(t) = \frac{1}{2}\sqrt{\frac{\pi}{a^3 H_0}} e^{\frac{i\pi}{2}(\nu+1/2)} H_\nu^{(1)}\left(\frac{k}{aH_0}\right) \quad (5.27b)$$

where the order of the Hankel functions is

$$\nu = \sqrt{\frac{9}{4} + \frac{B}{H_0^2}} = \sqrt{\frac{9}{4} + \frac{2m_s^2}{H_0^2}} \quad (5.28)$$

and of course $a = e^{H_0 t}$. In the second equality in (5.28) we have used (5.22) and (5.2). In writing (5.27) we have used the usual Bunch-Davies vacuum normalization so that on small scales, $k >> aH_0$, one has

$$|\varphi_k| \cong \frac{a^{-1}}{\sqrt{2k}}$$

which reproduces the standard Minkowski space fluctuations. This is the usual procedure in cosmological perturbation theory. On large scales, $k << aH_0$, the solutions (5.27) behave as

$$|\varphi_k| \cong \frac{H}{\sqrt{2k^3}}\left(\frac{k}{aH_0}\right)^{3/2-\nu}$$

which gives a large-scale power spectrum for the fluctuations

$$P_{\delta\varphi} = \left(\frac{H_0}{2\pi}\right)^2 \left(\frac{k}{aH_0}\right)^{n_s - 1}$$

with spectral index

$$n_s - 1 = 3 - 2\nu .$$

From (5.28) it is clear that to get an almost scale-invariant spectrum we require $m_s << H_0$. In this limit we have

$$n_s - 1 \cong -\frac{4}{3}\left(\frac{m_s}{H_0}\right)^2 \qquad (5.29)$$

which gives a red tilt to the spectrum, in agreement with the latest WMAP data. For $n_s \cong 0.95$ one has $m_s \cong 0.2H_0$. Comparing (5.27) to the corresponding solution in a local field theory we see that the p-adic tachyon field fluctuations evolve as though the mass-squared of the field was $-2m_s^2$ which may be quite different from the mass scale which one would infer by truncating the infinite series of derivatives: $\partial^2 V/\partial\chi^2(\chi = \chi_0)$.

We note that for $n_s = 0.95$, from the eq. (5.29), we obtain that $2\frac{m_s^2}{H_0^2} = 0{,}075$. Thence, we have the following mathematical connections with the Aurea ratio:

$$0{,}95 \cong (\Phi)^{-7/7} + (\Phi)^{-35/7} = \left(\frac{\sqrt{5}+1}{2}\right)^{-7/7} + \left(\frac{\sqrt{5}+1}{2}\right)^{-35/7} = 0{,}618034 + 0{,}90170 = 0{,}708204 \cdot \frac{4}{3} = 0{,}944272$$

From the eq. (5.28), we have that:

$$\nu = \sqrt{\frac{9}{4} + \frac{2m_s^2}{H_0^2}} = \sqrt{\frac{9}{4} + 0{,}075} = \sqrt{2{,}25 + 0{,}075} = \sqrt{2{,}325} = 1{,}524795068\,;$$

$$1{,}52480 \cong (\Phi)^0 + (\Phi)^{-28/7} = \left(\frac{\sqrt{5}+1}{2}\right)^0 + \left(\frac{\sqrt{5}+1}{2}\right)^{-28/7} = 1 + 0{,}145898 = 1{,}145898 \cdot \frac{4}{3} = 1{,}527864\,.$$

We note that for the eqs. (4.48b-4.48c-4.48d), we have that

$$\frac{\kappa}{100} = \frac{\sqrt{10 - 2\sqrt{5}} - 2}{\sqrt{5} - 1} = 0{,}00284078227\,; \quad 1{,}527864 - 0{,}002840 = 1{,}525024.$$

We now want to fix the parameters of the model by comparing to the observed features of the CMB perturbation spectrum. There are three dimensionless parameters, g_s, p and the ratio m_s/M_p. The important question is whether there is a sensible parameter range which can account for CMB observations, i.e., the spectral tilt and the amplitude of fluctuations. Using (5.14) in (5.29), we can relate the tilt to the model parameters via

$$|n_s - 1| = \frac{8(p+1)}{p^2}\left(\frac{M_p}{m_s}\right)^2 g_s^2 \Leftrightarrow \left(\frac{m_s}{M_p}\right)^2 = \frac{8(p+1)}{p^2}\frac{g_s^2}{|n_s - 1|}. \qquad (5.30)$$

Also for this equation, we note that the number 8 is a Fibonacci's number and is connected with the "modes" that correspond to the physical vibrations of a superstring by the following Ramanujan function:

$$8=\frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}.$$

Thus one can have a small tilt while ensuring that the string scale is smaller than the Planck scale, provided that $g_s^2/p<<1$. Henceforth we will use (5.30) to determine m_s/M_p in terms of p, g_s, and $|n_s-1|\cong 0.05$. All the dimensionless parameters in our solution, m_s/H_0, λ/m_s, ϕ_2 and H_2/m_s, are likewise functions of n_s-1, p and g_s. From (5.14) and (5.30) we see that for $p>>1$,

$$\frac{m_s}{H_0}\cong\sqrt{6}g_p\frac{M_p}{m_s}\cong g_s\sqrt{\frac{6}{p}}\frac{M_p}{m_s}\cong\frac{1}{2}\sqrt{3|n_s-1|}. \quad (5.31)$$

It may seem strange to have H exceeding m_s since that means the energy density exceeds the fundamental scale, but this is an inevitable property of the p-adic tachyon at its maximum, as shown in eq. (5.13). This is similar to other attempts to get tachyonic or brane-antibrane inflation from string theory, since the false vacuum energy is just the brane tension which goes like m_s^4/g_s.

Next we determine λ/m_s, where λ is the mass scale appearing in the power series in $e^{\lambda t}$ which provides the ansatz for the background solutions. We consider the following equation for λ in the $H_0>>m_s$ limit

$$e^{\mu_1}=e^{(\lambda^2+3H_0)/m_p^2}=p.$$

The positive root for λ gives

$$\frac{\lambda}{m_s}\cong\sqrt{\frac{|n_s-1|}{3}}. \quad (5.31b)$$

In order to fix the amplitude of the density perturbations we consider the curvature perturbation ζ. We assume that

$$\zeta\approx-\frac{H}{\dot\varphi}\delta\varphi$$

as in conventional inflation models. To evaluate the prefactor $H/\dot\varphi$ we must work beyond zeroth order in the small u expansion. We take $\phi=1-u$ to evaluate the prefactor, even though the perturbation $\delta\chi$ is computed in the limit that $\phi=1$. This should reproduce the full answer up to $O(u)$ corrections. The prefactor is

$$-\frac{H}{\dot\varphi}\cong\frac{H_0}{A\lambda u}\cong\frac{2^{3/2}g_s}{p}\frac{1}{|n_s-1|}\frac{1}{u}m_s^{-1}.$$

With regard the eqs. (5.31) and (5.31b), we have the following mathematical connections with the Aurea ratio:

$$\frac{m_s}{H_0} \cong \sqrt{6} g_p \frac{M_p}{m_s} \cong g_s \sqrt{\frac{6}{p}} \frac{M_p}{m_s} \cong \frac{1}{2}\sqrt{3|n_s - 1|} = 0{,}193649167;$$

$$0{,}19365 \cong (\Phi)^{-28/7} = \left(\frac{\sqrt{5}+1}{2}\right)^{-28/7} = 0{,}145898 \cdot \frac{4}{3} = 0{,}194531.$$

$$\frac{\lambda}{m_s} \cong \sqrt{\frac{|n_s - 1|}{3}} = 0{,}129099444;$$

$$0{,}12901 \cong (\Phi)^{-28/7} + (\Phi)^{-49/7} + (\Phi)^{-63/7} = \left(\frac{\sqrt{5}+1}{2}\right)^{-28/7} + \left(\frac{\sqrt{5}+1}{2}\right)^{-49/7} + \left(\frac{\sqrt{5}+1}{2}\right)^{-63/7} =$$

$$= 0{,}145898 + 0{,}034442 + 0{,}013156 = 0{,}193496 \cdot \frac{2}{3} = 0{,}128997 \cong 0{,}1290.$$

We should evaluate u at the time of horizon crossing, t_*, defined to be approximately 60 e-foldings before the end of inflation t_{end}, assuming that the energy scale of inflation is high. The inflation ends when $u \approx 1/p^{1/2}$. From eqs. (5.31-5.31b) we see that $H_0/\lambda = 2/|n_s - 1|$; therefore we can write the scale factor $a(t) \cong e^{H_0 t}$ in the form

$$a(t) \cong u(t)^{2/|n_s - 1|} \quad (5.32)$$

so that $a_* = e^{-60} a_{end}$ corresponds to

$$u_* = e^{-30|n_s - 1|} u_{end} \equiv e^{-30|n_s - 1|} \frac{1}{p^{1/2}}. \quad (5.33)$$

We note that $H_0/\lambda = 2/|n_s - 1|$ for $n_s = 0.95$ is equal to 40. This value can be related with the following mathematical connections with the Aura ratio:

$$(\Phi)^{14/7} + (\Phi)^{0} + (\Phi)^{-35/7} = \left(\frac{\sqrt{5}+1}{2}\right)^{14/7} + \left(\frac{\sqrt{5}+1}{2}\right)^{0} + \left(\frac{\sqrt{5}+1}{2}\right)^{-35/7} = 2{,}618034 + 1 + 0{,}090170 =$$

$$= 3{,}708204 \cdot \frac{4}{3} = 4{,}944272;$$

$$(\Phi)^{35/7} + (\Phi)^{7/7} = \left(\frac{\sqrt{5}+1}{2}\right)^{35/7} + \left(\frac{\sqrt{5}+1}{2}\right)^{7/7} = 11{,}090170 + 1{,}618034 = 12{,}708204 \cdot \frac{4}{3} = 16{,}944272;$$

$$(\Phi)^{35/7} + (\Phi)^{14/7} = \left(\frac{\sqrt{5}+1}{2}\right)^{35/7} + \left(\frac{\sqrt{5}+1}{2}\right)^{14/7} = 11{,}090170 + 2{,}618034 = 13{,}708204 \cdot \frac{4}{3} = 18{,}277605;$$

$H_0/\lambda = 2/|n_s - 1|$ = 40; 4,9 + 16,9 + 18,2 = 40.

The power spectrum of the curvature perturbation is given by

$$P_\zeta = \left|\frac{H}{\dot{\varphi}}\right|^2 P_{\delta\varphi} \equiv A_\zeta^2 \left(\frac{k}{aH_0}\right)^{n_s - 1} \quad (5.34)$$

where the amplitude of fluctuations A_ζ can now be read off as

$$A_\zeta^2 = \frac{8}{3\pi^2}\frac{g_s^2}{p}\frac{e^{60|n_s-1|}}{|n_s-1|^3}. \quad (5.35)$$

Thence, we can rewrite the eq. (5.34) as follows

$$P_\zeta = \left|\frac{H}{\dot{\varphi}}\right|^2 P_{\delta\varphi} \equiv \frac{8}{3\pi^2}\frac{g_s^2}{p}\frac{e^{60|n_s-1|}}{|n_s-1|^3}. \quad (5.35b)$$

As an example, taking $n_s \cong 0.95$ one can fix the amplitude of the density perturbations $A_\zeta^2 \cong 10^{-10}$ by choosing

$$\frac{g_s}{\sqrt{p}} \cong 0.48 \times 10^{-7}. \quad (5.36)$$

Setting $A_\zeta^2 = 10^{-10}$ and using (5.35) we obtain an expression for g_s in terms of p and $|n_s - 1|$

$$g_s = \sqrt{\frac{3\pi^2}{8}}\sqrt{p}e^{-30|n_s-1|}|n_s-1|^{3/2} \times 10^{-5}. \quad (5.37)$$

Combining (5.37) with (5.30), we also obtain

$$\frac{m_s}{M_p} = \sqrt{3\pi^2}\sqrt{\frac{p+1}{p}}e^{-30|n_s-1|}|n_s-1| \times 10^{-5}. \quad (5.38)$$

The string scale is bounded from above as $m_s / M_p \leq 0.94 \times 10^{-6}$ and that for typical values of p, n_s it is close to $m_s / M_p \cong 0.61 \times 10^{-6}$. Furthermore, from (5.37) that g_s is unconstrained and that g_s, p are not independent parameters.

Now we define the Hubble slow roll parameters ε_H, η_H by

$$\varepsilon_H \equiv \frac{1}{2M_p^2}\frac{\dot{\varphi}^2}{H^2}, \quad (5.39) \qquad \varepsilon_H - \eta_H \equiv \frac{\ddot{\varphi}}{H\dot{\varphi}}. \quad (5.40)$$

These are the appropriate parameters to describe the rate of time variation of the inflaton as compared to the Hubble scale. Using the solution $\phi \cong 1-u$ (recall that $\varphi = A\phi$, $A = m_s p/\left(\sqrt{2}g_s\right)$) we find that

$$\varepsilon_H \cong \frac{1}{2}\frac{p+1}{p}e^{-60|n-1|}|n_s-1|, \quad (5.41) \qquad \eta_H \cong -\frac{|n_s-1|}{2}. \quad (5.42)$$

We see that the Hubble slow-roll parameters are small. This means that p-adic tachyon field rolls slowly in the conventional sense. One reaches the same conclusion if one defines the potential slow roll parameters using the correct canonical field, which is φ (5.25):

$$\frac{M_p^2}{2}\left(\frac{1}{V}\frac{\partial V}{\partial \varphi}\right)^2\Big|_{\varphi=A} = 0, \quad (5.43) \qquad M_p^2 \frac{1}{V}\frac{\partial^2 V}{\partial \varphi^2}\Big|_{\varphi=A} = -\frac{1}{2}|n_s-1|. \quad (5.44)$$

With regard the eqs. (5.42) and (5.44), we have that:

$$-\frac{1}{2}|n_s-1| = 0{,}025;$$

$$(\Phi)^{-21/7} + (\Phi)^{-35/7} + (\Phi)^{-70/7} = \left(\frac{\sqrt{5}+1}{2}\right)^{-21/7} + \left(\frac{\sqrt{5}+1}{2}\right)^{-35/7} + \left(\frac{\sqrt{5}+1}{2}\right)^{-70/7} =$$

$$= 0{,}236068 + 0{,}090170 + 0{,}008131 = 0{,}334369 \cdot \frac{3}{4} = 0{,}250776; \quad \frac{1}{2\cdot 5} \times 0{,}250776 = 0{,}0250776.$$

On the other hand, consider the potential slow roll parameter which one would naively define using the derivative truncated action (5.4):

$$\frac{M_p^2}{2}\left(\frac{1}{V}\frac{\partial V}{\partial \chi}\right)^2\Big|_{\chi=\chi_0} = 0, \quad (5.45) \qquad M_p^2 \frac{1}{V}\frac{\partial^2 V}{\partial \chi^2}\Big|_{\chi=\chi_0} = -\left(\frac{p-1}{\ln p}\right)\frac{1}{2}|n_s-1| \quad (5.46)$$

where in (5.46) we have used equations (5.9) and (5.30). We see that (5.46) can be enormous, though the tachyon field rolls slowly. Taking the largest allowed value of p, $p \approx 10^{14}$, and $n_s \cong 0.95$ we have $M_p^2 V^{-1}\left|\partial^2 V/\partial \chi^2\right| \approx 10^{11}$. Since large values of p are required if one wants to obtain $g_s \approx 1$, it follows that it is somewhat natural for p-adic inflation to operate in the regime where the higher derivative corrections play an important role in the dynamics.

6. On some equations concerning p-adic minisuperspace model, zeta strings, zeta nonlocal scalar fields and p-adic and adelic quantum cosmology. [12] [13] [14] [15] [16]

Consider the standard Minkowski signature minisuperspace model of a homogeneous isotropic universe with a cosmological constant λ. The usual parametrization of the metric

$$ds^2 = -N^2 dt^2 + a^2 d\Omega_3^2 \quad (6.1)$$

leads to classical solutions which are trigonometrical functions of time. In the p-adic case we prefer to work with rational functions. We shall use the following ansatz

$$ds^2 = -\frac{N(t)^2}{q(t)}dt^2 + q(t)d\Omega_3^2 . \quad (6.2)$$

Here N and a are functions of time and $d\Omega_3^2$ is the metric on the unit 3-sphere. The action for this metric is the same as the corresponding usual case

$$S[q(t)]) = \frac{1}{2}\int_{t_1}^{t_2} dtN\left(\frac{\dot{q}^2}{4N^2} - \lambda^2 q - 1\right). \quad (6.3)$$

We assume that the cosmological constant λ is a rational number. The classical equations of motions have the form

$$\ddot{q} = 2\lambda . \quad (6.4)$$

The solution of this equation for the boundary conditions

$$q(0) = q_1 , \quad q(T) = q_2 , \quad (6.4b)$$

is the following

$$q(t) = \lambda t^2 + \left[\frac{q_2 - q_1}{T} - \lambda T\right]t + q_1 . \quad (6.5)$$

Here $q(t),\, p(t) \in Q_p$. The Green function corresponding to the transition from the point q_1 to the point q_2 has the form

$$\mathcal{G}_p(q_1, q_2) = \int_{Q_p} dTK_T(q_1,0|q_2,T) \quad (6.6)$$

where $K_T(q_1,0|q_2,T)$ is the propagator

$$K_T(q_1,0|q_2,T) = \int \chi_p(S)\prod_t dq(t). \quad (6.7)$$

In the path integral one integrates over trajectories with the boundary conditions (6.4b). One can perform the Gaussian path integral (6.7) in the usual way using shifting to the classical solution. One gets

$$K_T(q_1,0|q_2,T) = c(T)\chi_p(S_{cl}) \quad (6.8)$$

where S_{cl} is the action calculated on the trajectories (6.5).

$$S_{cl} = S_{cl}(q_2, q_1, T) = -\frac{\lambda^2 T^3}{24} + [\lambda(q_1 + q_2) - 2]\frac{T}{4} + \frac{(q_1 - q_2)^2}{8T} . \quad (6.9)$$

The factor $c(T)$ is the same as for a free particle

$$c(t)=\frac{\lambda_p(T)}{|T|_p^{1/2}}. \quad (6.9b)$$

Therefore one has the Green function

$$\mathcal{G}_p(q_1,q_2)=\int_{Q_p} dT\,\frac{\lambda_p(T)}{|T|_p^{1/2}}\chi_p\left(-\frac{\lambda^2T^3}{24}+[\lambda(q_1+q_2)-2]\frac{T}{4}+\frac{(q_1-q_2)^2}{8T}\right). \quad (6.10)$$

The corresponding wave function has the form

$$\Psi(q)=\int_{Q_p} dT\,\frac{\lambda_p(T)}{|T|_p^{1/2}}\chi_p\left(\frac{1}{l_{Pl}^2}\left[\frac{q^2}{8T}+(\lambda q-2)\frac{T}{4}-\frac{\lambda^2T^3}{24}\right]\right) \quad (6.11)$$

where we restore the explicit dependence on the Planck length.
We note that the number 24 in the eq. (6.11) can be related to the physical vibrations of the bosonic strings by the following Ramanujan function:

$$24=\frac{4\left[anti\log\frac{\int_0^\infty\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}.$$

Now let us estimate the integral on T applying the stationary phase approximation. The saddle-point equation has the form

$$S'=-\frac{\lambda^2T^2}{8}-\frac{q^2}{8T^2}+\frac{\lambda q}{4}-\frac{1}{2}=0 \quad (6.12)$$

which yields

$$T_{1,2}^2=-\frac{\left(1\pm\sqrt{1-\lambda q}\right)^2}{\lambda^2}. \quad (6.12b)$$

As is known, for $p\equiv 1(\mathrm{mod}\,4)$ there is the square root of -1 in Q_p, so we get nontrivial saddle points. For $p\equiv 3(\mathrm{mod}\,4)$ we have no saddle point at all. To make sense of the saddle points in the case $p\equiv 1(\mathrm{mod}\,4)$ we should be sure that the square root $\sqrt{1-\lambda q}$ also has a sense. For this purpose we have to assume that $|\lambda q|_p<1$. The corresponding actions have the form

$$S=-\frac{1}{3\lambda}\left[1\pm(1-\lambda q)^{3/2}\right], \quad S=\frac{1}{3\lambda}\left[1\pm(1-\lambda q)^{3/2}\right], \quad (6.13)$$

In order that these expressions be rational we have to assume that λ is rational as well as that $\lambda q = \xi^2$ is such a rational that the solution of the equation

$$\xi^2 + \eta^2 = 1 \quad (6.14)$$

in respect to η is also rational. Let us consider the Euclidean metric

$$ds^2 = \frac{N(t)^2}{q(t)} dt^2 + q(t) d\Omega_3^2 . \quad (6.15)$$

The Euclidean action for this metric is the same as the corresponding action in the usual case

$$S[q(t)] = \frac{1}{2}\int_{t_1}^{t_2} dtN\left(-\frac{\dot{q}^2}{4N^2} + \lambda^2 q - 1\right). \quad (6.16)$$

We shall prove that it is possible to restore

$$\Psi_E(q) = \int dT \int \exp\left(\frac{2\pi i}{l_{Pl}^2} S_E\right) \prod dq \quad (6.17)$$

when $l_{Pl} \to 0$ in the corresponding p-adic wave function. Indeed, in the p-adic case for Euclidean metric, we get a basic Green function

$$\mathcal{G}_p(q_1, q_2) = \int dT \frac{\lambda_p(T)}{|T|_p^{1/2}} \chi_p\left(\frac{\lambda^2 T^3}{24} + [\lambda(q_1 - q_2) - 2]\frac{T}{4} - \frac{(q_1 - q_2)^2}{8T}\right). \quad (6.18)$$

Now let us estimate the integral on T in

$$\int_{Q_p} dT \frac{\lambda_p(T)}{|T|^{1/2}} \chi_p\left(\frac{\lambda^2 T^3}{24} + (\lambda q - 2)\frac{T}{4} - \frac{q^2}{8T}\right) \quad (6.19)$$

applying the stationary phase approximation. Formally there are the following saddle points

$$T = \frac{1}{\lambda}\left[1 \pm (1 - \lambda q)^{1/2}\right], \quad T = -\frac{1}{\lambda}\left[1 \pm (1 - \lambda q)^{1/2}\right], \quad (6.19b)$$

for

$$|\lambda q|_p < 1$$

with corresponding actions

$$S = -\frac{1}{3\lambda}\left[1 \pm (1 - \lambda q)^{3/2}\right], \quad S = \frac{1}{3\lambda}\left[1 \pm (1 - \lambda q)^{3/2}\right]. \quad (6.20)$$

Note that now these stationary points have sense for all p and for q satisfying (6.15) according our general formula for the wave function of the universe

$$\Psi\left(\mathcal{G}^3\right)=\sum_{am}\int\prod_p \chi_p\left(S_p\right)\mathcal{D}\left(g_{\mu\nu}\right)_p , \quad (6.21)$$

we write

$$\Psi_E(q)=\prod_{p=2,3,5,\ldots}\frac{\lambda_p|\lambda|_p}{\left|\left(1-\lambda q_p\left(\pm\frac{1}{3\lambda}\right)\right)\right|}\left[\left(1-\lambda q_1\right)^{3/2}\pm 1\right]\Big) = \frac{\left|\left(1-\lambda q\right)^{1/2}\pm 1\right|}{|\lambda|}\exp\left\{\pm\frac{2\pi i}{3\lambda}\left[\left(1-\lambda q_1\right)^{3/2}\pm 1\right]\right\}.$$

(6.22)

6.1 Zeta strings and zeta nonlocal scalar fields.

The exact tree-level Lagrangian for effective scalar field φ which describes open p-adic string tachyon is

$$\mathcal{L}_p=\frac{1}{g^2}\frac{p^2}{p-1}\left[-\frac{1}{2}\varphi p^{-\frac{\Box}{2}}\varphi+\frac{1}{p+1}\varphi^{p+1}\right], \quad (6.23)$$

where p is any prime number, $\Box=-\partial_t^2+\nabla^2$ is the D-dimensional d'Alambertian and we adopt metric with signature $(-+\ldots+)$. Now, we want to show a model which incorporates the p-adic string Lagrangians in a restricted adelic way. Let us take the following Lagrangian

$$L=\sum_{n\geq 1}C_n\mathcal{L}_n=\sum_{n\geq 1}\frac{n-1}{n^2}\mathcal{L}_n=\frac{1}{g^2}\left[-\frac{1}{2}\phi\sum_{n\geq 1}n^{-\frac{\Box}{2}}\phi+\sum_{n\geq 1}\frac{1}{n+1}\phi^{n+1}\right]. \quad (6.24)$$

Recall that the Riemann zeta function is defined as

$$\zeta(s)=\sum_{n\geq 1}\frac{1}{n^s}=\prod_p\frac{1}{1-p^{-s}}, \quad s=\sigma+i\tau , \quad \sigma>1. \quad (6.25)$$

Employing usual expansion for the logarithmic function and definition (6.25) we can rewrite (6.24) in the form

$$L=-\frac{1}{g^2}\left[\frac{1}{2}\phi\zeta\left(\frac{\Box}{2}\right)\phi+\phi+\ln(1-\phi)\right], \quad (6.26)$$

where $|\phi|<1$. $\zeta\left(\frac{\Box}{2}\right)$ acts as pseudodifferential operator in the following way:

$$\zeta\left(\frac{\Box}{2}\right)\phi(x)=\frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk , \quad -k^2=k_0^2-\vec{k}^2>2+\varepsilon , \quad (6.27)$$

where $\tilde{\phi}(k) = \int e^{(-ikx)}\phi(x)dx$ is the Fourier transform of $\phi(x)$.
Dynamics of this field ϕ is encoded in the (pseudo)differential form of the Riemann zeta function. **When the d'Alambertian is an argument of the Riemann zeta function we shall call such string a "zeta string".** Consequently, the above ϕ is an open scalar zeta string. The equation of motion for the zeta string ϕ is

$$\zeta\left(\frac{\Box}{2}\right)\phi = \frac{1}{(2\pi)^D}\int_{k_0^2 - \vec{k}^2 > 2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \frac{\phi}{1-\phi} \quad (6.28)$$

which has an evident solution $\phi = 0$.
For the case of time dependent spatially homogeneous solutions, we have the following equation of motion

$$\zeta\left(\frac{-\partial_t^2}{2}\right)\phi(t) = \frac{1}{(2\pi)}\int_{|k_0| > \sqrt{2}+\varepsilon} e^{-ik_0 t}\zeta\left(\frac{k_0^2}{2}\right)\tilde{\phi}(k_0)dk_0 = \frac{\phi(t)}{1-\phi(t)}. \quad (6.29)$$

With regard the open and closed scalar zeta strings, the equations of motion are

$$\zeta\left(\frac{\Box}{2}\right)\phi = \frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \sum_{n\geq 1}\theta^{\frac{n(n-1)}{2}}\phi^n, \quad (6.30)$$

$$\zeta\left(\frac{\Box}{4}\right)\theta = \frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{4}\right)\tilde{\theta}(k)dk = \sum_{n\geq 1}\left[\theta^{n^2} + \frac{n(n-1)}{2(n+1)}\theta^{\frac{n(n-1)}{2}-1}\left(\phi^{n+1}-1\right)\right], \quad (6.31)$$

and one can easily see trivial solution $\phi = \theta = 0$.

The exact tree-level Lagrangian of effective scalar field φ, which describes open p-adic string tachyon, is:

$$\mathcal{L}_p = \frac{m_p^D}{g_p^2}\frac{p^2}{p-1}\left[-\frac{1}{2}\varphi p^{-\frac{\Box}{2m_p^2}}\varphi + \frac{1}{p+1}\varphi^{p+1}\right], \quad (6.32)$$

where p is any prime number, $\Box = -\partial_t^2 + \nabla^2$ is the D-dimensional d'Alambertian and we adopt metric with signature $(-+\ldots+)$, as above. Now, we want to introduce a model which incorporates all the above string Lagrangians (6.32) with p replaced by $n \in N$. Thence, we take the sum of all Lagrangians $\mathcal{L}_n$ in the form

$$L = \sum_{n=1}^{+\infty} C_n \mathcal{L}_n = \sum_{n=1}^{+\infty} C_n \frac{m_n^D}{g_n^2}\frac{n^2}{n-1}\left[-\frac{1}{2}\phi n^{-\frac{\Box}{2m_n^2}}\phi + \frac{1}{n+1}\phi^{n+1}\right], \quad (6.33)$$

whose explicit realization depends on particular choice of coefficients C_n, masses m_n and coupling constants g_n.
Now, we consider the following case

$$C_n = \frac{n-1}{n^{2+h}}, \quad (6.34)$$

where h is a real number. The corresponding Lagrangian reads

$$L_h = \frac{m^D}{g^2}\left[-\frac{1}{2}\phi\sum_{n=1}^{+\infty} n^{-\frac{\Box}{2m^2}-h}\phi + \sum_{n=1}^{+\infty}\frac{n^{-h}}{n+1}\phi^{n+1}\right] \quad (6.35)$$

and it depends on parameter h. According to the Euler product formula one can write

$$\sum_{n=1}^{+\infty} n^{-\frac{\Box}{2m^2}-h} = \prod_p \frac{1}{1-p^{-\frac{\Box}{2m^2}-h}}. \quad (6.36)$$

Recall that standard definition of the Riemann zeta function is

$$\zeta(s) = \sum_{n=1}^{+\infty}\frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}, \quad s = \sigma + i\tau, \quad \sigma > 1, \quad (6.37)$$

which has analytic continuation to the entire complex s plane, excluding the point $s=1$, where it has a simple pole with residue 1. Employing definition (6.37) we can rewrite (6.35) in the form

$$L_h = \frac{m^D}{g^2}\left[-\frac{1}{2}\phi\zeta\left(\frac{\Box}{2m^2}+h\right)\phi + \sum_{n=1}^{+\infty}\frac{n^{-h}}{n+1}\phi^{n+1}\right]. \quad (6.38)$$

Here $\zeta\left(\frac{\Box}{2m^2}+h\right)$ acts as a pseudodifferential operator

$$\zeta\left(\frac{\Box}{2m^2}+h\right)\phi(x) = \frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2m^2}+h\right)\tilde{\phi}(k)dk, \quad (6.39)$$

where $\tilde{\phi}(k) = \int e^{(-ikx)}\phi(x)dx$ is the Fourier transform of $\phi(x)$.

We consider Lagrangian (6.38) with analytic continuations of the zeta function and the power series $\sum\frac{n^{-h}}{n+1}\phi^{n+1}$, i.e.

$$L_h = \frac{m^D}{g^2}\left[-\frac{1}{2}\phi\zeta\left(\frac{\Box}{2m^2}+h\right)\phi + AC\sum_{n=1}^{+\infty}\frac{n^{-h}}{n+1}\phi^{n+1}\right], \quad (6.40)$$

where AC denotes analytic continuation.

Potential of the above zeta scalar field (6.40) is equal to $-L_h$ at $\Box = 0$, i.e.

$$V_h(\phi) = \frac{m^D}{g^2}\left(\frac{\phi^2}{2}\zeta(h) - AC\sum_{n=1}^{+\infty}\frac{n^{-h}}{n+1}\phi^{n+1}\right), \quad (6.41)$$

where $h \neq 1$ since $\zeta(1) = \infty$. The term with ζ-function vanishes at $h = -2, -4, -6, \ldots$. The equation of motion in differential and integral form is

$$\zeta\left(\frac{\Box}{2m^2} + h\right)\phi = AC\sum_{n=1}^{+\infty} n^{-h}\varphi^n , \quad (6.42)$$

$$\frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2} + h\right)\tilde{\phi}(k)dk = AC\sum_{n=1}^{+\infty} n^{-h}\phi^n , \quad (6.43)$$

respectively.

Now, we consider five values of h, which seem to be the most interesting, regarding the Lagrangian (6.40): $h = 0$, $h = \pm 1$, and $h = \pm 2$. For $h = -2$, the corresponding equation of motion now read:

$$\zeta\left(\frac{\Box}{2m^2} - 2\right)\phi = \frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2} - 2\right)\tilde{\phi}(k)dk = \frac{\phi(\phi+1)}{(1-\phi)^3}. \quad (6.44)$$

This equation has two trivial solutions: $\phi(x) = 0$ and $\phi(x) = -1$. Solution $\phi(x) = -1$ can be also shown taking $\tilde{\phi}(k) = -\delta(k)(2\pi)^D$ and $\zeta(-2) = 0$ in (6.44).

For $h = -1$, the corresponding equation of motion is:

$$\zeta\left(\frac{\Box}{2m^2} - 1\right)\phi = \frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2} - 1\right)\tilde{\phi}(k)dk = \frac{\phi}{(1-\phi)^2}. \quad (6.45)$$

where $\zeta(-1) = -\frac{1}{12}$.

The equation of motion (6.45) has a constant trivial solution only for $\phi(x) = 0$.

For $h = 0$, the equation of motion is

$$\zeta\left(\frac{\Box}{2m^2}\right)\phi = \frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2}\right)\tilde{\phi}(k)dk = \frac{\phi}{1-\phi}. \quad (6.46)$$

It has two solutions: $\phi = 0$ and $\phi = 3$. The solution $\phi = 3$ follows from the Taylor expansion of the Riemann zeta function operator

$$\zeta\left(\frac{\Box}{2m^2}\right) = \zeta(0) + \sum_{n\geq 1}\frac{\zeta^{(n)}(0)}{n!}\left(\frac{\Box}{2m^2}\right)^n , \quad (6.47)$$

as well as from $\tilde{\phi}(k) = (2\pi)^D 3\delta(k)$.

For $h = 1$, the equation of motion is:

$$\frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2} + 1\right)\tilde{\phi}(k)dk = -\frac{1}{2}\ln(1-\phi)^2 , \quad (6.48)$$

where $\zeta(1) = \infty$ gives $V_1(\phi) = \infty$.

In conclusion, for $h = 2$, we have the following equation of motion:

$$\frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2}+2\right)\tilde{\phi}(k)dk = -\int_0^{\phi}\frac{\ln(1-w)^2}{2w}dw . \quad (6.49)$$

Since holds equality

$$-\int_0^1 \frac{\ln(1-w)}{w}dw = \sum_{n=1}^{\infty}\frac{1}{n^2} = \zeta(2)$$

one has trivial solution $\phi = 1$ in (6.49).

Now, we want to analyze the following case: $C_n = \frac{n^2-1}{n^2}$. In this case, from the Lagrangian (6.33), we obtain:

$$L = \frac{m^D}{g^2}\left[-\frac{1}{2}\phi\left\{\zeta\left(\frac{\Box}{2m^2}-1\right)+\zeta\left(\frac{\Box}{2m^2}\right)\right\}\phi + \frac{\phi^2}{1-\phi}\right]. \quad (6.50)$$

The corresponding potential is:

$$V(\phi) = -\frac{m^D}{g}\frac{31-7\phi}{24(1-\phi)}\phi^2 . \quad (6.51)$$

The equation of motion is:

$$\left[\zeta\left(\frac{\Box}{2m^2}-1\right)+\zeta\left(\frac{\Box}{2m^2}\right)\right]\phi = \frac{\phi\left[(\phi-1)^2+1\right]}{(\phi-1)^2}. \quad (6.52)$$

Its weak field approximation is:

$$\left[\zeta\left(\frac{\Box}{2m^2}-1\right)+\zeta\left(\frac{\Box}{2m^2}\right)-2\right]\phi = 0 , \quad (6.53)$$

which implies condition on the mass spectrum

$$\zeta\left(\frac{M^2}{2m^2}-1\right)+\zeta\left(\frac{M^2}{2m^2}\right) = 2 . \quad (6.54)$$

From (6.54) it follows one solution for $M^2 > 0$ at $M^2 \approx 2.79m^2$ and many tachyon solutions when $M^2 < -38m^2$.

With regard the extension by ordinary Lagrangian, we have the Lagrangian, potential, equation of motion and mass spectrum condition that, when $C_n = \frac{n^2-1}{n^2}$, are:

$$L = \frac{m^D}{g^2}\left[\frac{\phi}{2}\left\{\frac{\Box}{m^2}-\zeta\left(\frac{\Box}{2m^2}-1\right)-\zeta\left(\frac{\Box}{2m^2}\right)-1\right\}\phi + \frac{\phi^2}{2}\ln\phi^2 + \frac{\phi^2}{1-\phi}\right], \quad (6.55)$$

$$V(\phi) = \frac{m^D}{g^2}\frac{\phi^2}{2}\left[\zeta(-1)+\zeta(0)+1-\ln\phi^2 - \frac{1}{1-\phi}\right], \quad (6.56)$$

$$\left[\zeta\left(\frac{\Box}{2m^2}-1\right)+\zeta\left(\frac{\Box}{2m^2}\right)-\frac{\Box}{m^2}+1\right]\phi=\phi\ln\phi^2+\phi+\frac{2\phi-\phi^2}{(1-\phi)^2}, \quad (6.57)$$

$$\zeta\left(\frac{M^2}{2m^2}-1\right)+\zeta\left(\frac{M^2}{2m^2}\right)=\frac{M^2}{m^2}. \quad (6.58)$$

In addition to many tachyon solutions, equation (6.58) has two solutions with positive mass: $M^2 \approx 2.67m^2$ and $M^2 \approx 4.66m^2$.

Now, we describe the case of $C_n = \mu(n)\frac{n-1}{n^2}$. Here $\mu(n)$ is the Mobius function, which is defined for all positive integers and has values 1, 0, – 1 depending on factorization of n into prime numbers p. It is defined as follows:

$$\mu(n)=\begin{cases}0, & n=p^2m\\ (-1)^k, & n=p_1p_2...p_k, p_i\neq p_j\\ 1, & n=1,(k=0).\end{cases} \quad (6.59)$$

The corresponding Lagrangian is

$$L_\mu = C_0\mathcal{L}_0+\frac{m^D}{g^2}\left[-\frac{1}{2}\phi\sum_{n=1}^{+\infty}\frac{\mu(n)}{n^{\frac{\Box}{2m^2}}}\phi+\sum_{n=1}^{+\infty}\frac{\mu(n)}{n+1}\phi^{n+1}\right] \quad (6.60)$$

Recall that the inverse Riemann zeta function can be defined by

$$\frac{1}{\zeta(s)}=\sum_{n=1}^{+\infty}\frac{\mu(n)}{n^s}, \quad s=\sigma+it, \quad \sigma>1. \quad (6.61)$$

Now (6.60) can be rewritten as

$$L_\mu = C_0\mathcal{L}_0+\frac{m^D}{g^2}\left[-\frac{1}{2}\phi\frac{1}{\zeta\left(\frac{\Box}{2m^2}\right)}\phi+\int_0^\infty \mathcal{M}(\phi)d\phi\right], \quad (6.62)$$

where $\mathcal{M}(\phi)=\sum_{n=1}^{+\infty}\mu(n)\phi^n=\phi-\phi^2-\phi^3-\phi^5+\phi^6-\phi^7+\phi^{10}-\phi^{11}-...$ The corresponding potential, equation of motion and mass spectrum formula, respectively, are:

$$V_\mu(\phi)=-L_\mu(\Box=0)=\frac{m^D}{g^2}\left[\frac{C_0}{2}\phi^2(1-\ln\phi^2)-\phi^2-\int_0^\phi \mathcal{M}(\phi)d\phi\right], \quad (6.63)$$

$$\frac{1}{\zeta\left(\frac{\Box}{2m^2}\right)}\phi-\mathcal{M}(\phi)-C_0\frac{\Box}{m^2}\phi-2C_0\phi\ln\phi=0, \quad (6.64)$$

$$\frac{1}{\zeta\left(\frac{M^2}{2m^2}\right)} - C_0\frac{M^2}{m^2} + 2C_0 - 1 = 0, \quad |\phi| << 1, \quad (6.65)$$

where usual relativistic kinematic relation $k^2 = -k_0^2 + \vec{k}^2 = -M^2$ is used.

Now, we take the pure numbers concerning the eqs. (6.54) and (6.58). They are: 2.79, 2.67 and 4.66. We note that all the numbers are related with $\Phi = \frac{\sqrt{5}+1}{2}$, thence with the aurea ratio, by the following expressions:

$$2{,}79 \cong (\Phi)^{15/7}; \qquad 2{,}67 \cong (\Phi)^{13/7} + (\Phi)^{-21/7}; \qquad 4{,}66 \cong (\Phi)^{22/7} + (\Phi)^{-30/7}. \quad (6.66)$$

6.2 p-Adic and adelic quantum cosmology

Adelic quantum cosmology is an application of adelic quantum theory to the universe as a whole. In the path integral approach to standard quantum cosmology starting point is Feynman's idea that the amplitude to go from one state with intrinsic metric h_{ij}, and matter configuration ϕ on an initial hypersurface Σ, to another state with metric h'_{ij}, and matter configuration ϕ' on a final hypersurface Σ', is given by a functional integral of $\chi_\infty(-S_\infty[g_{\mu\nu},\Phi])$ over all four-geometries $g_{\mu\nu}$, and matter configurations Φ, which interpolate between the initial and final configurations, i.e.

$$\langle h'_{ij},\phi',\Sigma' | h_{ij},\phi,\Sigma\rangle_\infty = \int \mathcal{D}(g_{\mu\nu})_\infty \mathcal{D}(\Phi)_\infty \chi_\infty(-S_\infty[g_{\mu\nu},\Phi]). \quad (6.67)$$

The $S_\infty[g_{\mu\nu},\Phi]$ is the usual Einstein-Hilbert action

$$S[g_{\mu\nu},\Phi] = \frac{1}{16\pi G}\left(\int_M d^4x\sqrt{-g}(R-2\Lambda) + 2\int_{\partial M} d^3x\sqrt{h}K\right) - \frac{1}{2}\int_M d^4x\sqrt{-g}\left[g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + V(\Phi)\right] \quad (6.68)$$

for the gravitational field and matter fields Φ. In (6.68), R is scalar curvature of four-manifold M, Λ is cosmological constant, K is trace of the extrinsic curvature K_{ij} at the boundary ∂M of the manifold M. To perform p-adic and adelic generalization we first make p-adic counterpart of the action (6.68) using form-invariance under change of real to the p-adic number fields. Then we generalize (6.67) and introduce p-adic complex-valued cosmological amplitude

$$\langle h'_{ij},\phi',\Sigma' | h_{ij},\phi,\Sigma\rangle_p = \int \mathcal{D}(g_{\mu\nu})_p \mathcal{D}(\Phi)_p \chi_p(-S_p[g_{\mu\nu},\Phi]). \quad (6.69)$$

The space of all 3-metrics and matter field configurations $(h_{ij}(\vec{x}),\phi(\vec{x}))$ on a 3-surface is called superspace (this is the configuration space in quantum cosmology). Superspace is the infinite dimensional one with a finite number of coordinates $(h_{ij}(\vec{x}),\phi(\vec{x}))$ at each point $\vec{x}$ of the 3-surface. One useful approximation is to truncate the infinite degrees of freedom to a finite number, thereby obtaining some particular minisuperspace model. Usually, one restricts the four-metric to be of the form $ds^2 = -N^2(t)dt^2 + h_{ij}dx^i dx^j$, where $N(t)$ is the laps function. For such minisuperspaces, functional integrals (6.67) and (6.69) are reduced to functional integration over three-metrics, matter configurations and to one usual integral over the laps function. If one takes boundary condition $q^\alpha(t_2) = q_2^\alpha, q^\alpha(t_1) = q_1^\alpha$ then integral in (6.67) and (6.69), in the gauge $N = 0$, is a minisuperspace propagator. In this case it holds

$$\left\langle q_2^\alpha \middle| q_1^\alpha \right\rangle_v = \int dN \mathcal{K}_v\left(q_2^\alpha, N \middle| q_1^\alpha, 0\right), \quad (6.70)$$

where

$$\mathcal{K}_v\left(q_2^\alpha, N \middle| q_1^\alpha, 0\right) = \int \mathcal{D}q^\alpha \chi_v\left(-S_v\left[q^\alpha\right]\right) \quad (6.71)$$

is an ordinary quantum-mechanical propagator between fixed q^α in fixed time N. For quadratic classical action $S_p^{cl}(q_2, N|q_1, 0)$, (6.71) becomes

$$\mathcal{K}_p\left(q_2, N|q_1, 0\right) = \lambda_p\left(-\frac{\partial^2 S_p^{cl}}{2\partial q_2 \partial q_1}\right)\left|\frac{\partial^2 S_p^{cl}}{\partial q_2 \partial q_1}\right|_p^{1/2} \chi_p\left(-S_p^{cl}\left(q_2, N|q_1, 0\right)\right). \quad (6.72)$$

If system has n decoupled degrees of freedom, its p-adic kernel is a product

$$\mathcal{K}_p\left(q_2, N|q_1, 0\right) = \prod_{\alpha=1}^{n} \lambda_p\left(-\frac{\partial^2 S_p^{cl}}{2\partial q_2^\alpha \partial q_1^\alpha}\right)\left|\frac{\partial^2 S_p^{cl}}{\partial q_2^\alpha \partial q_1^\alpha}\right|_p^{1/2} \chi_p\left(-S_p^{cl}\left(q_2^\alpha, N\middle|q_1^\alpha, 0\right)\right). \quad (6.73)$$

p-Adic and adelic wave functions of the universe may be found by means of the following equation

$$U(t)\psi_{\alpha\beta}(x) = \chi(E_\alpha t)\psi_{\alpha\beta}(x), \quad (6.73b)$$

where $\psi_{\alpha\beta}(x)$ are adelic wave eigenfunctions, $E = (E_\infty, E_2, ..., E_p, ...)$ is the corresponding adelic energy, $\alpha = (\alpha_\infty, \alpha_2, ..., \alpha_p, ...)$ and $\beta = (\beta_\infty, \beta_2, ..., \beta_p, ...)$ are indicies for energy levels and their degeneration, respectively.
The corresponding adelic eigenstates have the form

$$\Psi(q^\alpha) = \psi_\infty(q_\infty^\alpha)\prod_{p\in S}\psi_p(q_p^\alpha)\prod_{p\notin S}\Omega\left(\left|q_p^\alpha\right|_p\right). \quad (6.74)$$

A necessary condition to construct an adelic model is existence of the p-adic (vacuum) state $\Omega\left(\left|q^\alpha\right|_p\right)$, which satisfies

$$\int_{\left|q_1^\alpha\right|_p \le 1} \mathcal{K}_p\left(q_2^\alpha, N\middle|q_1^\alpha, 0\right) dq_1^\alpha = \Omega\left(\left|q_2^\alpha\right|_p\right) \quad (6.75)$$

for all but a finite number of p.
Now we describe the p-adic and adelic model with cosmological constant in D = 3 dimensions. This model have the metric

$$ds^2 = \sigma^2\left(-N^2(t)dt^2 + a^2(t)\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right), \quad (6.76)$$

where $\sigma = G$. The corresponding v-adic action is

$$S_v[a] = \frac{1}{2}\int_0^1 dtNa^2(t)\left(-\frac{\dot{a}^2}{N^2a^2} + \frac{1}{a^2} - \lambda\right), \quad (6.77)$$

where $\lambda = \Lambda\sigma^2$. The Euler-Lagrange equation of motion

$$\ddot{a} - N^2 a\lambda = 0$$

has the solution

$$a(t) = \frac{1}{2\sinh\left(N\sqrt{\lambda}\right)}\left(\left(a_2 - a_1e^{-N\sqrt{\lambda}}\right)e^{N\sqrt{\lambda}t} + \left(a_1e^{N\sqrt{\lambda}} - a_2\right)e^{-N\sqrt{\lambda}t}\right), \quad (6.78)$$

where the boundary conditions are $a(0) = a_1$, $a(1) = a_2$. For the classical action it gives

$$S_v^{cl}(a_2, N|a_1, 0) = \frac{1}{2\sqrt{\lambda}}\left[N\sqrt{\lambda} + \lambda\left(\frac{2a_1a_2}{\sinh\left(N\sqrt{\lambda}\right)} - \frac{a_1^2 + a_2^2}{\tanh\left(N\sqrt{\lambda}\right)}\right)\right]. \quad (6.79)$$

Quantum-mechanical propagator has the form

$$\mathcal{K}_v(a_2, N|a_1, 0) = \lambda_v\left(-\frac{\sqrt{\lambda}}{2\sinh\left(N\sqrt{\lambda}\right)}\right)\left|\frac{\sqrt{\lambda}}{\sinh\left(N\sqrt{\lambda}\right)}\right|_v^{1/2}\chi_v\left(-S_v^{cl}(a_2, N|a_1, 0)\right). \quad (6.80)$$

The equation (6.75), in a more explicit form, reads

$$\Omega\left(|a_2|_p\right) = \lambda_p\left(-\frac{\sqrt{\lambda}}{2\sinh\left(N\sqrt{\lambda}\right)}\right)\left|\frac{\sqrt{\lambda}}{\sinh\left(N\sqrt{\lambda}\right)}\right|_p^{1/2}\chi_p\left(-\frac{N}{2} + \frac{\sqrt{\lambda}}{2\tanh\left(N\sqrt{\lambda}\right)}a_2^2\right)\times$$
$$\times\int_{|a_1|_p \le 1}\chi_p\left(\frac{\sqrt{\lambda}}{2\tanh\left(N\sqrt{\lambda}\right)}a_1^2 - \frac{\sqrt{\lambda}}{\sinh\left(N\sqrt{\lambda}\right)}a_2a_1\right)da_1. \quad (6.81)$$

We note that the p-adic Gauss integral over the region of integration $|x|_p \le p^{-v}$ is

$$\int_{|x|_p \le p^{-v}}\chi_p\left(\alpha x^2 + \beta x\right)dx = p^{-v}\Omega\left(p^{-v}|\beta|_p\right), |\alpha|_p p^{-2v} \le 1; \quad (6.81a)$$

$$\int_{|x|_p \le p^{-v}}\chi_p\left(\alpha x^2 + \beta x\right)dx = \lambda_p(\alpha)|2\alpha|_p^{-1/2}\chi_p\left(-\frac{\beta^2}{4\alpha}\right)\Omega\left(p^v\left|\frac{\beta}{2\alpha}\right|_p\right), |\alpha|_p p^{-2v} > 1; \quad (6.81b)$$

where $\Omega(u)$ is defined as follows:

$$\Omega(u)=1,\ u\leq 1;\quad \Omega(u)=0,\ u>1.$$

Using (6.81b), for $v=0$, we obtain

$$\Omega(|a_2|_p)=\chi_p\left(-\frac{N}{2}+\frac{\sqrt{\lambda}}{2}\tanh\left(N\sqrt{\lambda}\right)a_2^2\right)\Omega(|a_2|_p)\quad (6.82)$$

with condition $\left|\frac{\sqrt{\lambda}}{2\tanh\left(N\sqrt{\lambda}\right)}\right|_p>1$, i.e. $|N|_p<1$. For $p\neq 2$, left hand side is equal to $\Omega(|a_2|_p)$ if $|\lambda|_p\leq 1$ holds. Applying also the (6.81a) to (6.81), we have

$$\Omega(|a_2|_p)=\lambda_p\left(-\frac{\sqrt{\lambda}}{2\sinh\left(N\sqrt{\lambda}\right)}\right)\left|\frac{\sqrt{\lambda}}{\sinh\left(N\sqrt{\lambda}\right)}\right|_p^{1/2}\times\chi_p\left(-\frac{N}{2}+\frac{\sqrt{\lambda}a_2^2}{2\coth\left(N\sqrt{\lambda}\right)}\right)\Omega\left(\left|\frac{\sqrt{\lambda}a_2}{\sinh\left(N\sqrt{\lambda}\right)}\right|_p\right).\quad (6.83)$$

It becomes an equality if condition $|N|_p\leq 1$ take place.
Thence, we can rewrite the eq. (6.81) as follow:

$$\Omega(|a_2|_p)=\lambda_p\left(-\frac{\sqrt{\lambda}}{2\sinh\left(N\sqrt{\lambda}\right)}\right)\left|\frac{\sqrt{\lambda}}{\sinh\left(N\sqrt{\lambda}\right)}\right|_p^{1/2}\chi_p\left(-\frac{N}{2}+\frac{\sqrt{\lambda}}{2\tanh\left(N\sqrt{\lambda}\right)}a_2^2\right)\times$$
$$\times\int_{|a_1|_p\leq 1}\chi_p\left(\frac{\sqrt{\lambda}}{2\tanh\left(N\sqrt{\lambda}\right)}a_1^2-\frac{\sqrt{\lambda}}{\sinh\left(N\sqrt{\lambda}\right)}a_2a_1\right)da_1=$$
$$=\lambda_p\left(-\frac{\sqrt{\lambda}}{2\sinh\left(N\sqrt{\lambda}\right)}\right)\left|\frac{\sqrt{\lambda}}{\sinh\left(N\sqrt{\lambda}\right)}\right|_p^{1/2}\times\chi_p\left(-\frac{N}{2}+\frac{\sqrt{\lambda}a_2^2}{2\coth\left(N\sqrt{\lambda}\right)}\right)\Omega\left(\left|\frac{\sqrt{\lambda}a_2}{\sinh\left(N\sqrt{\lambda}\right)}\right|_p\right).\quad (6.84)$$

The de Sitter minisuperspace model in quantum cosmology is the simplest, nontrivial and exactly soluble model. This model is given by the Einstein-Hilbert action with cosmological term (6.68) without matter fields, and by Robertson-Walker metric

$$ds^2=\sigma^2\left(-N^2(t)dt^2+a^2(t)d\Omega_3^2\right),\quad (6.85)$$

where $\sigma^2=\frac{2G}{3\pi}$ and $a(t)$ is the scale factor. Instead of (6.85) we shall use

$$ds^2=\sigma^2\left(-\frac{N^2(t)}{q(t)}dt^2+q(t)d\Omega_3^2\right).\quad (6.86)$$

The corresponding v-adic action for this one-dimensional minisuperspace model is

$$S_v[q]=\frac{1}{2}\int_{t_1}^{t_2}dtN\left(-\frac{\dot{q}^2}{4N^2}-\lambda q+1\right),\quad (6.87)$$

where $\lambda = \frac{\Lambda\sigma^2}{3}$. The classical equation of motion ($N = 1$) $\ddot{q} = 2\lambda$, with the boundary conditions $q(0) = q_1$ and $q(T) = q_2$, $(T = t_2 - t_1)$ gives

$$q(t) = \lambda t^2 + \left(\frac{q_2 - q_1}{T} - \lambda T \right) t + q_1 . \quad (6.88)$$

After substitution (6.88) into (6.87) and integration, one obtains that the classical action is

$$S_v^{cl}(q_2, T|q_1, 0) = \frac{\lambda^2 T^3}{24} - [\lambda(q_1 + q_2) - 2]\frac{T}{4} - \frac{(q_2 - q_1)^2}{8T} . \quad (6.89)$$

Since (6.89) is quadratic in q_2 and q_1, quantum-mechanical propagator has the form

$$\mathcal{K}_v(q_2, T|q_1, 0) = \frac{\lambda_v(-8T)}{|4T|_v^{1/2}} \chi_v\left(-S_v^{cl}(q_2, T|q_1, 0)\right). \quad (6.90)$$

The equation (6.75) reads

$$\Omega\left(|q_2|_p\right) = \frac{\lambda_p(-8T)}{|4T|_p^{1/2}} \chi_p\left(-\frac{\lambda^2 T^3}{24} - \frac{T}{2} + \frac{\lambda q_2 T}{4} + \frac{q_2^2}{8T} \right) \times \int_{|q_1|_p \le 1} \chi_p\left[\frac{q_1^2}{8T} + \left(\frac{\lambda T}{4} - \frac{q_2}{4T} \right) q_1 \right] dq_1 . \quad (6.91)$$

7. Mathematical connections.

Now, we describe some possible mathematical connections. We take the eq. (1.26) of **Section 1**. We note that can be related with the eqs. (5.11), (5.12) of **Section 5**, hence we have the following connections:

$$N_e = \int d\phi \frac{V}{V_{,\phi}} \approx \frac{e^{c\phi_C}}{c^2} \Rightarrow$$

$$\Rightarrow \frac{m_s^4}{2g_p^2} g_{\mu\nu} \left[\phi e^{-\frac{\Box}{m_p^2}} \phi - \frac{2}{p+1}\phi^{p+1} + \frac{1}{m_p^2}\int_0^1 d\tau \left(\Box e^{-\frac{\tau\Box}{m_p^2}} \phi \right)\left(e^{-\frac{(1-\tau)\Box}{m_p^2}} \phi \right) + \frac{1}{m_p^2}\int_0^1 d\tau \left(\partial_\alpha e^{-\frac{\tau\Box}{m_p^2}} \phi \right)\left(\partial^\alpha e^{-\frac{(1-\tau)\Box}{m_p^2}} \phi \right) \right] +$$

$$- \frac{m_s^4}{m_p^2 g_p^2} \int_0^1 d\tau \left(\partial_\mu e^{-\frac{\tau\Box}{m_p^2}} \phi \right)\left(\partial_\nu e^{-\frac{(1-\tau)\Box}{m_p^2}} \phi \right);$$

$$N_e = \int d\phi \frac{V}{V_{,\phi}} \approx \frac{e^{c\phi_C}}{c^2} \Rightarrow \frac{m_s^4}{2g_p^2} \left[\phi e^{-\frac{\Box}{m_p^2}} \phi - \frac{2}{p+1}\phi^{p+1} + \frac{1}{m_p^2}\int_0^1 d\tau \left(\Box e^{-\frac{\tau\Box}{m_p^2}} \phi \right)\left(e^{-\frac{(1-\tau)\Box}{m_p^2}} \phi \right) \right.$$

$$\left. + \frac{1}{m_p^2}\int_0^1 d\tau \partial_t \left(e^{-\frac{\tau\Box}{m_p^2}} \phi \right) \partial_t \left(e^{-\frac{(1-\tau)\Box}{m_p^2}} \phi \right) \right]. \quad (7.1)$$

Thence, mathematical connections between the slow-roll formula regarding the number of e-foldings N_e of inflation and the equations of the stress energy tensor for the p-adic scalar field in p-adic inflation.
Now, we take the eqs. (2.4), (2.6), (2.72b) and (2.74) of **Section 2**. We note that can be related with the eq. (6.91) of **Section 6**, hence we obtain the following connections:

$$S = \int dt d^3 x \left[-3M_5^3 L\left(\dot{a}^2 - Ka^2\right) + \frac{1}{2}a^2\dot{\phi}^2 \right] + S_m \Rightarrow$$

$$\Rightarrow \frac{\lambda_p(-8T)}{|4T|_p^{1/2}} \chi_p\left(-\frac{\lambda^2 T^3}{24} - \frac{T}{2} + \frac{\lambda q_2 T}{4} + \frac{q_2^2}{8T} \right) \times \int_{|q_1|_p \le 1} \chi_p\left[\frac{q_1^2}{8T} + \left(\frac{\lambda T}{4} - \frac{q_2}{4T} \right) q_1 \right] dq_1 ; \quad (7.2)$$

$$S = \int d^4 x \sqrt{-g}\left(\frac{M_4^2}{2} R - \frac{1}{2}\left(\partial_\mu \phi\right)^2 \right) + S_m^-\left[g^-\right] + S_m^+\left[g^+\right] \Rightarrow$$

$$\Rightarrow \frac{\lambda_p(-8T)}{|4T|_p^{1/2}} \chi_p\left(-\frac{\lambda^2 T^3}{24} - \frac{T}{2} + \frac{\lambda q_2 T}{4} + \frac{q_2^2}{8T} \right) \times \int_{|q_1|_p \le 1} \chi_p\left[\frac{q_1^2}{8T} + \left(\frac{\lambda T}{4} - \frac{q_2}{4T} \right) q_1 \right] dq_1 ; \quad (7.3)$$

$$\frac{\delta d_m}{L} = \int_{-1}^{1} \frac{n x_4 \xi_{40}(x_4)}{b^2} e^{-\frac{1}{2}x_4^2} d\omega = \frac{2 x_4 \xi_{40}(x_4)}{\left(1 - x_4^2\right)^2} =$$

$$= \frac{1}{\left(x_4^2 - 1\right)} \left[\frac{8}{3}\tilde{k} x_4 \left(A_0 J_0\left(\tilde{k} x_4\right) + B_0 Y_0\left(\tilde{k} x_4\right)\right) - 4\left(A_0 J_1\left(\tilde{k} x_4\right) + B_0 Y_1\left(\tilde{k} x_4\right)\right)\right] \Rightarrow$$

$$\Rightarrow \frac{\lambda_p(-8T)}{|4T|_p^{1/2}} \chi_p\left(-\frac{\lambda^2 T^3}{24} - \frac{T}{2} + \frac{\lambda q_2 T}{4} + \frac{q_2^2}{8T} \right) \times \int_{|q_1|_p \le 1} \chi_p\left[\frac{q_1^2}{8T} + \left(\frac{\lambda T}{4} - \frac{q_2}{4T} \right) q_1 \right] dq_1 ; \quad (7.4)$$

$$\frac{\delta d_m}{L} = \int_{-y_0}^{y_0} n t \Gamma_L dy = \int_{-1}^{1} \frac{n x_4 \xi_4}{b^2} e^{-\frac{1}{2}x_4^2} d\omega \Rightarrow$$

$$\Rightarrow \frac{\lambda_p(-8T)}{|4T|_p^{1/2}} \chi_p\left(-\frac{\lambda^2 T^3}{24} - \frac{T}{2} + \frac{\lambda q_2 T}{4} + \frac{q_2^2}{8T} \right) \times \int_{|q_1|_p \le 1} \chi_p\left[\frac{q_1^2}{8T} + \left(\frac{\lambda T}{4} - \frac{q_2}{4T} \right) q_1 \right] dq_1 ; \quad (7.5)$$

Thence, mathematical connections between some equations concerning cosmological perturbations in a Big Crunch/Big Bang space-time and M-theory model of a Big Crunch/Big Bang transition (2.4-2.6), some equations concerning the solution of a braneworld Big Crunch/Big Bang cosmology (2.72b-2.74) and the equation concerning the de Sitter minisuperspace model in p-adic quantum cosmology (6.91).

Now, we take the eqs. (5.11), (5.12), (5.29), (5.35) and (5.46) of **Section 5**. We note that can be related with the eqs. (3.30), (3.32), (3.34), (3.87), (3.96) and (3.98) of **Section 3**, hence we obtain the following mathematical connections:

$$\tau^2 \approx \frac{1}{2v}\left[\int_0^{Y_0} D(Y')e^{m\alpha Y'/2}dY'\right]^2 \approx \frac{2D_0^2}{m^2\alpha^2 v e^{-m\alpha Y_0}}\left(1-\frac{2}{mD_0}\right) \Rightarrow$$

$$\Rightarrow \frac{m_s^4}{2g_p^2} g_{\mu\nu}\left[\phi e^{-\frac{\Box}{m_p^2}}\phi - \frac{2}{p+1}\phi^{p+1} + \frac{1}{m_p^2}\int_0^1 d\tau\left(\Box e^{-\frac{\tau\Box}{m_p^2}}\phi\right)\left(e^{-\frac{(1-\tau)\Box}{m_p^2}}\phi\right) + \frac{1}{m_p^2}\int_0^1 d\tau\left(\partial_\alpha e^{-\frac{\tau\Box}{m_p^2}}\phi\right)\left(\partial^\alpha e^{-\frac{(1-\tau)\Box}{m_p^2}}\phi\right)\right] +$$

$$-\frac{m_s^4}{m_p^2 g_p^2}\int_0^1 d\tau\left(\partial_\mu e^{-\frac{\tau\Box}{m_p^2}}\phi\right)\left(\partial_\nu e^{-\frac{(1-\tau)\Box}{m_p^2}}\phi\right) \Rightarrow \frac{m_s^4}{2g_p^2}\left[\phi e^{-\frac{\Box}{m_p^2}}\phi - \frac{2}{p+1}\phi^{p+1} + \frac{1}{m_p^2}\int_0^1 d\tau\left(\Box e^{-\frac{\tau\Box}{m_p^2}}\phi\right)\left(e^{-\frac{(1-\tau)\Box}{m_p^2}}\phi\right) +\right.$$

$$\left.+\frac{1}{m_p^2}\int_0^1 d\tau\,\partial_t\left(e^{-\frac{\tau\Box}{m_p^2}}\phi\right)\partial_t\left(e^{-\frac{(1-\tau)\Box}{m_p^2}}\phi\right)\right]\Bigg]; \quad (7.6)$$

$$n_s - 1 \cong -\frac{4}{3}\left(\frac{m_s}{H_0}\right)^2 \Rightarrow \frac{2}{\varepsilon} - \frac{d\ln\varepsilon}{d\mathcal{N}} \Rightarrow -0.03\frac{\beta}{1+\beta} \Rightarrow 1 + \frac{d\log|\delta_k|^2}{d\log k} \approx 1 + \frac{4}{m}\times$$

$$\times 1/[\frac{2}{m}\log\left(\frac{m^2\alpha}{2k}\sqrt{\frac{ve^{mC}}{2}}\right)] \Rightarrow \frac{1}{2v}\left[\int_0^{Y_0} D(Y')e^{m\alpha Y'/2}dY'\right]^2 \approx \frac{2D_0^2}{m^2\alpha^2 v e^{-m\alpha Y_0}}\left(1-\frac{2}{mD_0}\right); \quad (7.7)$$

$$A_\zeta^2 = \frac{8}{3\pi^2}\frac{g_s^2}{p}\frac{e^{60|n_s-1|}}{|n_s-1|^3} \Rightarrow \frac{2}{\varepsilon} - \frac{d\ln\varepsilon}{d\mathcal{N}} \Rightarrow \frac{4(1+\gamma^2)}{c^2 M_{Pl}^2} - \frac{4c_{,\phi}}{c^2} \Rightarrow -0.03\frac{\beta}{1+\beta} \Rightarrow$$

$$\Rightarrow 1 + \frac{d\log|\delta_k|^2}{d\log k} \approx 1 + \frac{4}{m}\times 1/[\frac{2}{m}\log\left(\frac{m^2\alpha}{2k}\sqrt{\frac{ve^{mC}}{2}}\right)] \Rightarrow$$

$$\Rightarrow \frac{1}{2v}\left[\int_0^{Y_0} D(Y')e^{m\alpha Y'/2}dY'\right]^2 \approx \frac{2D_0^2}{m^2\alpha^2 v e^{-m\alpha Y_0}}\left(1-\frac{2}{mD_0}\right); \quad (7.8)$$

$$M_p^2 \frac{1}{V}\frac{\partial^2 V}{\partial\chi^2}\Big|_{\chi=\chi_0} = -\left(\frac{p-1}{\ln p}\right)\frac{1}{2}|n_s - 1| \Rightarrow \frac{2}{\varepsilon} - \frac{d\ln\varepsilon}{d\mathcal{N}} \Rightarrow \frac{4(1+\gamma^2)}{c^2 M_{Pl}^2} - \frac{4c_{,\phi}}{c^2} \Rightarrow -0.03\frac{\beta}{1+\beta} \Rightarrow$$

$$\Rightarrow 1 + \frac{d\log|\delta_k|^2}{d\log k} \approx 1 + \frac{4}{m}\times 1/[\frac{2}{m}\log\left(\frac{m^2\alpha}{2k}\sqrt{\frac{ve^{mC}}{2}}\right)] \Rightarrow$$

$$\Rightarrow \frac{1}{2v}\left[\int_0^{Y_0} D(Y')e^{m\alpha Y'/2}dY'\right]^2 \approx \frac{2D_0^2}{m^2\alpha^2 v e^{-m\alpha Y_0}}\left(1-\frac{2}{mD_0}\right). \quad (7.9)$$

Thence, mathematical connections between some equations concerning the generating ekpyrotic curvature perturbations before the Big Bang, some equations concerning the colliding branes and the origin of the hot Big Bang and some equations concerning the approximate inflationary solutions rolling away from the unstable maximum of p-adic string theory.

Now, we take the eqs. (5.2), (5.11) and (5.12) of **Section 5** and the eqs. (6.28), (6.43) and (6.62), of **Section 6**. We obtain the following mathematical connections:

$$S = \frac{m_s^4}{g_p^2}\int d^4x\left(-\frac{1}{2}\phi p^{-\frac{\Box}{2m_s^2}}\phi + \frac{1}{p+1}\phi^{p+1}\right) \equiv \frac{m_s^4}{g_p^2}\int d^4x\left(-\frac{1}{2}\phi e^{-\frac{\Box}{m_p^2}}\phi + \frac{1}{p+1}\phi^{p+1}\right) \Rightarrow$$

$$\Rightarrow \frac{m_s^4}{2g_p^2}g_{\mu\nu}\left[\phi e^{-\frac{\Box}{m_p^2}}\phi - \frac{2}{p+1}\phi^{p+1} + \frac{1}{m_p^2}\int_0^1 d\tau\left(\Box e^{-\frac{\tau\Box}{m_p^2}}\phi\right)\left(e^{-\frac{(1-\tau)\Box}{m_p^2}}\phi\right) + \frac{1}{m_p^2}\int_0^1 d\tau\left(\partial_\alpha e^{-\frac{\tau\Box}{m_p^2}}\phi\right)\left(\partial^\alpha e^{-\frac{(1-\tau)\Box}{m_p^2}}\phi\right)\right] +$$

$$-\frac{m_s^4}{m_p^2 g_p^2}\int_0^1 d\tau\left(\partial_\mu e^{-\frac{\tau\Box}{m_p^2}}\phi\right)\left(\partial_\nu e^{-\frac{(1-\tau)\Box}{m_p^2}}\phi\right) \Rightarrow \frac{m_s^4}{2g_p^2}\left[\phi e^{-\frac{\Box}{m_p^2}}\phi - \frac{2}{p+1}\phi^{p+1} + \frac{1}{m_p^2}\int_0^1 d\tau\left(\Box e^{-\frac{\tau\Box}{m_p^2}}\phi\right)\left(e^{-\frac{(1-\tau)\Box}{m_p^2}}\phi\right) +\right.$$

$$\left. + \frac{1}{m_p^2}\int_0^1 d\tau\partial_t\left(e^{-\frac{\tau\Box}{m_p^2}}\phi\right)\partial_t\left(e^{-\frac{(1-\tau)\Box}{m_p^2}}\phi\right)\right] \Rightarrow \zeta\left(\frac{\Box}{2}\right)\phi = \frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \frac{\phi}{1-\phi} \Rightarrow$$

$$\Rightarrow \frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2}+h\right)\tilde{\phi}(k)dk = AC\sum_{n=1}^{+\infty} n^{-h}\phi^n \Rightarrow$$

$$\Rightarrow C_0\mathcal{L}_0 + \frac{m^D}{g^2}\left[-\frac{1}{2}\phi\frac{1}{\zeta\left(\frac{\Box}{2m^2}\right)}\phi + \int_0^\infty \mathcal{M}(\phi)d\phi\right]. \quad (7.10)$$

Thence, mathematical connections between some equations concerning the approximate inflationary solutions rolling away from the unstable maximum of p-adic string theory and some equations concerning the zeta strings and the zeta nonlocal scalar fields.

In conclusion, with regard the **Section 6** we have the following mathematical connections between the eqs. (6.84) and (6.91) and the eq. (3.87) of the **Section 3**:

$$\Omega\left(|a_2|_p\right) = \lambda_p\left(-\frac{\sqrt{\lambda}}{2\sinh\left(N\sqrt{\lambda}\right)}\right)\left|\frac{\sqrt{\lambda}}{\sinh\left(N\sqrt{\lambda}\right)}\right|_p^{1/2}\chi_p\left(-\frac{N}{2}+\frac{\sqrt{\lambda}}{2\tanh\left(N\sqrt{\lambda}\right)}a_2^2\right)\times$$

$$\times\int_{|a_1|_p\le 1}\chi_p\left(\frac{\sqrt{\lambda}}{2\tanh\left(N\sqrt{\lambda}\right)}a_1^2 - \frac{\sqrt{\lambda}}{\sinh\left(N\sqrt{\lambda}\right)}a_2 a_1\right)da_1 =$$

$$= \lambda_p\left(-\frac{\sqrt{\lambda}}{2\sinh\left(N\sqrt{\lambda}\right)}\right)\left|\frac{\sqrt{\lambda}}{\sinh\left(N\sqrt{\lambda}\right)}\right|_p^{1/2}\times\chi_p\left(-\frac{N}{2}+\frac{\sqrt{\lambda}a_2^2}{2\coth\left(N\sqrt{\lambda}\right)}\right)\Omega\left(\left|\frac{\sqrt{\lambda}a_2}{\sinh\left(N\sqrt{\lambda}\right)}\right|_p\right) \Rightarrow$$

$$\Rightarrow \frac{1}{2v}\left[\int_0^{Y_0} D(Y')e^{m\alpha Y'/2}dY'\right]^2 \approx \frac{2D_0^2}{m^2\alpha^2 v e^{-m\alpha Y_0}}\left(1-\frac{2}{mD_0}\right); \quad (7.11)$$

$$\Omega\left(|q_2|_p\right) = \frac{\lambda_p(-8T)}{|4T|_p^{1/2}}\chi_p\left(-\frac{\lambda^2 T^3}{24}-\frac{T}{2}+\frac{\lambda q_2 T}{4}+\frac{q_2^2}{8T}\right)\times\int_{|q_1|_p\le 1}\chi_p\left[\frac{q_1^2}{8T}+\left(\frac{\lambda T}{4}-\frac{q_2}{4T}\right)q_1\right]dq_1 \Rightarrow$$

$$\Rightarrow \frac{1}{2v}\left[\int_0^{Y_0} D(Y')e^{m\alpha Y'/2}dY'\right]^2 \approx \frac{2D_0^2}{m^2\alpha^2 v e^{-m\alpha Y_0}}\left(1-\frac{2}{mD_0}\right). \quad (7.12)$$

Thence, mathematical connections between the some equations concerning the p-adic quantum cosmology and the fundamental equation concerning the colliding branes and the origin of the hot Big Bang.

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