On the physical interpretation of the Riemann zeta function, the Rigid Surface Operators in Gauge Theory, the adeles and ideles groups applied to various formulae regarding the Riemann zeta function and the Selberg trace formula, p-adic strings, zeta strings and p-adic cosmology and mathematical connections with some sectors of String Theory and Number Theory.

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Abstract

This paper is a review of some interesting results that has been obtained in the study of the physical interpretation of the Riemann zeta function as a FZZT Brane Partition Function associated with a matrix/gravity correspondence and some aspects of the Rigid Surface Operators in Gauge Theory. Furthermore, we describe the mathematical connections with some sectors of String Theory (p-adic and adelic strings, p-adic cosmology) and Number Theory.

In the **Section 1** we have described various mathematical aspects of the Riemann Hypothesis, matrix/gravity correspondence and master matrix for FZZT brane partition functions. In the **Section 2**, we have described some mathematical aspects of the rigid surface operators in gauge theory and some mathematical connections with various sectors of Number Theory, principally with the Ramanujan's modular equations (thence, prime numbers, prime natural numbers, Fibonacci's numbers, partitions of numbers, Euler's functions, etc…) and various numbers and equations related to the Lie Groups. In the **Section 3**, we have described some very recent mathematical results concerning the adeles and ideles groups applied to various formulae regarding the Riemann zeta function and the Selberg trace formula (connected with the Selberg zeta function), hence, we have obtained some new connections applying these results to the adelic strings and zeta strings. In the **Section 4** we have described some equations concerning p-adic strings, p-adic and adelic zeta functions, zeta strings and p-adic cosmology (with regard the p-adic cosmology, some equations concerning a general class of cosmological models driven by a nonlocal scalar field inspired by string field theories). In conclusion, in the **Section 5**, we have showed various and interesting mathematical connections between some equations concerning the **Section 1, 3** and **4**.

1. **On some equations concerning the physical interpretation of the Riemann zeta function as a FZZT Brane Partition Function associated with a matrix/gravity correspondence and the master matrix of the (2,1) minimal and (3,1) minimal matrix model**. [1] [2] [3]

If one can find a special infinite Hermitian matrix M_0 such that:

$$\Xi(z) = \det(M_0 - zI), \quad (1.1)$$

where

$$\Xi(z) = \zeta\left(iz + \frac{1}{2}\right)\Gamma\left(\frac{z}{2} + \frac{1}{4}\right)\pi^{-1/4}\pi^{-iz/2}\left(-\frac{z^2}{2} - \frac{1}{8}\right), \quad (1.2)$$

then the Riemann hypothesis would be true. This is because this function can be written in product form as:

$$\Xi(z) = \frac{1}{2}\prod_n\left(1 - \frac{iz + 1/2}{\rho_n}\right). \quad (1.3)$$

Thence, we can rewritten the eq. (1.2) also in the following form:

$$\frac{1}{2}\prod_n\left(1 - \frac{iz + 1/2}{\rho_n}\right) = \zeta\left(iz + \frac{1}{2}\right)\Gamma\left(\frac{z}{2} + \frac{1}{4}\right)\pi^{-1/4}\pi^{-iz/2}\left(-\frac{z^2}{2} - \frac{1}{8}\right), \quad (1.3b)$$

The eigenvalues of the Hermitian matrix M_0 are denoted by λ_n and are related to the Riemann zeros via $\rho_n = i\lambda_n + 1/2$. Then the product becomes:

$$\Xi(z) = \frac{1}{2}\prod_n\left(1 - \frac{iz + 1/2}{i\lambda_n + 1/2}\right) = \frac{1}{2}\prod_n\frac{\lambda_n - z}{\lambda_n - i/2}. \quad (1.4)$$

This vanishes at the values λ_n just as the formal determinant expression. The λ_n are real if the matrix M_0 is Hermitian and thus the Riemann Hypothesis would be true.
For a general matrix model with potential $V(M)$ the master matrix can be written:

$$M_0 = S^{-1}TS = S^{-1}\left(a + \sum_{n=0}^{\infty} t_n a^{+n}\right)S \quad (1.5)$$

where the similarity transformation S is defined so that M_0 is Hermitian and the operators a, a^+ obey $\left[a, a^+\right] = I$. One can expand the master matrix as a function of the Hermitian operator $\hat{x} = a + a^+$ as:

$$M_0(\hat{x}) = g_1\hat{x} + g_2\hat{x}^2 + \ldots \quad (1.6)$$

One can also define an associated complex function:

$$M_0(y) = \frac{1}{y} + \sum_{n=0}^{\infty} t_n y^n \,, \quad (1.7)$$

as well as a conjugate matrix P_0 that satisfies:

$$[P_0, M_0] = I \,. \quad (1.8)$$

The Master matrix can be determined from the equation:

$$\left(V'(M_0(\hat{x})) + 2P_0\right)|0\rangle = 0 \,. \quad (1.9)$$

Here $|0\rangle$ is the vacuum state annihilated by a. The master matrix is closely connected with the resolvent $R(z)$ and eigenvalue density $\rho(x)$ through:

$$R(z) = Tr\left(\frac{1}{z - M_0}\right) = \int dx \frac{\rho(x)}{z - x} = -\oint_C \frac{dw}{2\pi i} \log(z - M_0(w)) \,. \quad (1.10)$$

The associated function $M_0(y)$ obeys the relation:

$$R(M_0(y)) = M_0(R(y)) = y \,. \quad (1.11)$$

The function $yM_0(y)$ is the generating functional of connected Green functions for the generalized matrix model.

One observable of matrix models is the exponentiated macroscopic loop or FZZT brane partition function. This is given by:

$$B(z) = \det(M - zI) \,. \quad (1.12)$$

This is the characteristic polynomial associated with the matrix M. It's argument z can be complex. In the context of the Riemann zeta function $\zeta(s)$ the variable is related to the usual argument of the zeta function by $s = iz + \frac{1}{2}$. Another observable is the macroscopic loop which is the transform of the Wheeler-DeWitt wave function defined on the gravity side of the correspondence

$$W(z) = -Tr\log(M - zI) = \lim_{\varepsilon \to 0}\left[\int_{\varepsilon}^{\infty} \frac{d\ell}{\ell} Tr\left(e^{\ell(-zI + M)}\right) + \log\varepsilon\right], \quad (1.13)$$

where ε is a UV cutoff. The resolvent observable mentioned above is defined by:

$$R(z) = \frac{\partial W(z)}{\partial z} = Tr\left(\frac{1}{M - zI}\right). \quad (1.14)$$

Thence, from the eqs. (1.10) and (1.13), we obtain:

$$R(z) = \frac{\partial}{\partial z}\left[\lim_{\varepsilon \to 0}\left(\int_{\varepsilon}^{\infty} \frac{d\ell}{\ell} Tr\left(e^{\ell(-zI+M)}\right) + \log \varepsilon\right)\right] = -\oint_C \frac{1}{2\pi i} dw \log(z - M_0(w)) = Tr\left(\frac{1}{M - zI}\right). \quad (1.14b)$$

If a special master matrix M_0 can be found then expectation values such as

$$\langle B(z) \rangle = \langle \det(M - zI) \rangle = \int DM \det(M - zI) e^{-V(M)} = \det(M_0 - zI) \quad (1.15)$$

reduce to evaluating the observable at M_0. In the context of the $\Xi(z)$ function the desired relation is of the form:

$$\Xi(z) = \det(M_0 - zI) = \langle B(z) \rangle = \langle \det(M - zI) \rangle = \int DM \det(M - zI) e^{-V(M)}. \quad (1.16)$$

Some matrix potentials that have been considered are

$$V(M) = Tr(M^2) \quad (1.17)$$

which describes 2d topological gravity or the (2,1) minimal string theory. A quartic potential:

$$V(M) = Tr(-M^2 + gM^4), \quad (1.18)$$

is used to describe minimal superstring theory. A more complicated matrix potentials is

$$V(M) = -Tr(M + \log(I - M)) = \sum_{m=2}^{\infty} \frac{1}{m} Tr(M^m), \quad (1.19)$$

which defines the Penner matrix model and is used to compute the Euler characteristic of the moduli space of Riemann surfaces. Another matrix model that has been introduced is the Liouville matrix model with potential given by:

$$V(M) = Tr(\alpha M + \mu e^M), \quad (1.20)$$

with cosmological constant μ so that:

$$e^{-V(M)} = e^{-\alpha TrM} e^{-\mu Tre^M}. \quad (1.21)$$

In this section we will encounter the matrix potential determined by:

$$e^{-U(M)} = \sum_{q=1}^{\infty}\left(q^4 \pi^2 e^{2TrM} - \frac{3}{2} q^2 \pi e^{TrM}\right) e^{-q^2 \pi Tr\left(e^M\right)}. \quad (1.22)$$

The partition function for this matrix model can be seen as a superposition of partition functions of Liouville matrix models with cosmological constants of the form:

$$\mu = q^2\pi , \quad (1.23)$$

for integer q.

Now we describe the origin of this particular matrix model and it's relation to the zeta function. To see how the matrix potential (1.22) arises it is helpful consider how the coefficients of the characteristic polynomial observable $B(z)$ can be determined by expanding as a series in z. If the function $\Xi(z)$ is interpreted as a characteristic polynomial then one can obtain these coefficients from the expansion:

$$\Xi(z) = \sum_{n=0}^{\infty} a_{2n} \frac{(-1)^n}{(2n)!} z^{2n} , \quad (1.24)$$

where

$$a_{2n} = 4\int_1^{\infty} d\ell \left(\ell^{-1/4} f(\ell) \left(\frac{1}{2}\log \ell \right)^{2n} \right), \quad (1.25)$$

and

$$f(\ell) = \sum_{q=1}^{\infty} \left(q^4\pi^2\ell - \frac{3}{2}q^2\pi \right) \ell^{1/2} e^{-q^2\pi\ell} . \quad (1.26)$$

Inserting the coefficients a_{2n} into $\Xi(z)$ and summing over n we can represent $\Xi(z)$ as an integral transform:

$$\Xi(z) = 4\int_1^{\infty} \frac{d\ell}{\ell} \ell^{(iz+1/2)/2} \sum_{q=1}^{\infty} \left(q^4\pi^2\ell^2 - \frac{3}{2}q^2\pi\ell \right) e^{-q^2\pi\ell} = 4\int_1^{\infty} \frac{d\ell}{\ell} \ell^{(iz+1/2)/2} \ell^{1/2} f(\ell). \quad (1.27)$$

Defining the variable ϕ by $\ell = e^{\phi}$ we have:

$$\Xi[z] = \int d\phi e^{iz\phi} \sum_{k=1}^{\infty} \left(\pi^2 k^4 e^{2\phi} - \frac{3}{2}\pi k^2 e^{\phi} \right) e^{-\pi k^2 e^{\phi}} , \quad (1.28)$$

which is a well known integral expression for the function $\Xi(z)$. For the simple potential $V(M) = Tr(M^2)$ the exponentiated macroscopic loop observable (FZZT brane) can be computed. It is given by the Airy function:

$$Ai(z) = \int DM \det(M - zI) e^{-Tr(M^2)} = \int d\phi e^{iz\phi + i\phi^3 \frac{1}{3}} . \quad (1.29)$$

Because this function is associated with an Hermitian matrix model it's zeros are real. This is the analog of the Riemann hypothesis for $V(M) = Tr(M^2)$. The similarity between the integral representations of (1.28) and (1.29) suggest an analogy between the Airy and zeta functions.

The integral representation of the Airy function has a matrix integral generalization. The matrix potential is defined from:

$$e^{-U(\Phi)} = e^{i\frac{1}{3}Tr\left(\Phi^3\right)}. \quad (1.30)$$

The matrix generalized Airy function is given by:

$$Ai(Z) = \int d\Phi e^{iTr(Z\Phi)} e^{-U(\Phi)}. \quad (1.31)$$

Thence, from the eqs. (1.30) and (1.31), we can write also:

$$Ai(Z) = \int d\Phi e^{iTr(Z\Phi)}\ e^{i\frac{1}{3}Tr\left(\Phi^3\right)}. \quad (1.31b)$$

In the above Φ and Z are $n \times n$ matrices. The interpretation of this matrix integral is that it describes n FZZT brines. The matrix Φ in the Kontsevich integrand is an effective degree of freedom describing open strings stretched between n FZZT branes. One can try to interpret the integrand of the $\Xi(z)$ function in a similar manner. In that case the analog of the potential defined by:

$$e^{-U(\Phi)} = \sum_{k=1}^{\infty}\left(\pi^2 k^4 e^{2Tr\Phi} - \frac{3}{2}\pi k^2 e^{Tr\Phi}\right)e^{-\pi k^2 Tr e^{\Phi}}\ , \quad (1.32)$$

and the analog of the matrix integral describing n FZZT branes is:

$$\Xi[Z] = \int D\Phi e^{iTr(Z\Phi)} \sum_{k=1}^{\infty}\left(\pi^2 k^4 e^{2Tr\Phi} - \frac{3}{2}\pi k^2 e^{Tr\Phi}\right)e^{-\pi k^2 Tr e^{\Phi}}. \quad (1.33)$$

The Airy function is the FZZT partition function for the (2,1) minimal matrix model. The FZZT partition function for the generalized $(p,1)$ minimal matrix model with parameters s_k is given by:

$$B(z) = \frac{1}{2\pi}\int d\phi e^{iz\phi - \frac{1}{p+1}(i\phi)^{p+1} + \sum_{k=1}^{p-2} s_k \frac{1}{k+1}(i\phi)^{k+1}}. \quad (1.34)$$

Unlike the (2,1) matrix model the definition of the generalized $(p,1)$ matrix model requires a two matrix integral of the form:

$$Z_{(p,1)}(g) = \int DMDAe^{-\frac{1}{g}(V(M+I) - AM)}. \quad (1.35)$$

Comparison with the integral representation of the $\Xi(z)$ function shows that a generalized matrix model for large p can be constructed as an approximation. One writes:

$$\log\left(\sum_{k=1}^{\infty}\left(\pi^2 k^4 e^{2\phi} - \frac{3}{2}\pi k^2 e^{\phi}\right)e^{-\pi k^2 e^{\phi}}\right) = -\frac{1}{p+1}(i\phi)^{p+1} + \sum_{k=1}^{p-2} s_k \frac{1}{k+1}(i\phi)^{k+1}. \quad (1.36)$$

In the above formula the function on the left is expanded to order $p+1$ in the variable ϕ. We denote this terminated expansion by $\Xi_p(z)$. Another way to compute the coefficients s_k is to differentiate the left hand side and set:

$$s_k = \frac{i^{-(k+1)}}{k!}\partial_\phi^k \log\left(\sum_{k=1}^{\infty}\left(\pi^2 k^4 e^{2\phi} - \frac{3}{2}\pi k^2 e^{\phi}\right)e^{-\pi k^2 e^{\phi}}\right)\Big|_{\phi=0} . \quad (1.37)$$

From the integral representation one has:

$$Q\Xi_p(z) = z\Xi_p(z), \quad P\Xi_p(z) = -\partial_z \Xi_p(z), \quad (1.38)$$

where:

$$Q = \left(P^p + \sum_{k=0}^{p-1} s_k P^k\right). \quad (1.39)$$

Inserting this operator into the above equation one has the generalization of the Airy equation given by:

$$\left(P^p + \sum_{k=0}^{p-1} s_k P^k\right)\Xi_p(z) = z\Xi_p(z). \quad (1.40)$$

To recover the equation for the full $\Xi(z)$ function one has to take p to infinity which agrees with the fact that the zeta function does not obey a finite order differential equation. Note that z and ϕ are in some sense canonically conjugate. Denote the Fourier transform of the $\Xi(z)$ function as $\tilde{\Xi}(p)$ then:

$$\Xi(z) = \int d\phi e^{i\phi z}\tilde{\Xi}(\phi). \quad (1.41)$$

The generalized Airy equation then becomes in Fourier space:

$$\left(\phi^p + \sum_{k=0}^{p-1} s_k \phi^k\right)\tilde{\Xi}_p(\phi) = Q\tilde{\Xi}_p(\phi). \quad (1.42)$$

This can be written:

$$(U'(\phi) - Q)\tilde{\Xi}(\phi) = 0, \quad (1.43)$$

where:

$$e^{-U(\phi)} = \sum_{k=1}^{\infty}\left(\pi^2 k^4 e^{2\phi} - \frac{3}{2}\pi k^2 e^{\phi}\right)e^{-\pi k^2 e^{\phi}} . \quad (1.44)$$

Equation (1.43) is very similar to the equation for the master matrix. Indeed if we set:

$$\phi = M_0(y), \quad z = P_0(y), \quad (1.45)$$

we see that y can be thought of as coordinates of a parametrization of the Riemann surface $M_{p,1}$ which is determined from the ϕ and z constraint $U'(\phi) - z = 0$. If we make these variables into operators through:

$$\hat{\phi} = \hat{M}_0(a, a^+), \quad \hat{z} = \hat{P}_0(a, a^+), \quad (1.46)$$

this classical surface is turned into a quantum Riemann surface similar to those studied using noncommutative geometry. Once one has obtained the coefficients s_k one can define matrix potential associated with a finite N theory as:

$$V(M)=\lim_{p\to\infty} Tr\left(V_p(M)+\sum_{k=1}^{p-2} s_k V_k(M)\right), \quad (1.47)$$

where:

$$V_k(M)=\sum_{j=1}^{p}\frac{1}{j}\left(M^j-I\right). \quad (1.48)$$

Thence, we can write also:

$$V(M)=\lim_{p\to\infty} Tr\left(V_p(M)+\sum_{k=1}^{p-2} s_k \sum_{j=1}^{p}\frac{1}{j}\left(M^j-I\right)\right). \quad (1.48b)$$

A set of orthogonal polynomials with this matrix potential through the integral equation:

$$B_n(z)=\frac{n!}{2\pi i}\oint e^{-\lim_{p\to\infty}\left(V_p(y+1)+\sum_{k=1}^{p-2} s_k V_k(y+1)\right)+2zy}\frac{1}{y^{n+1}}dy\,. \quad (1.49)$$

Or equivalently though the generating function definition:

$$e^{-\lim_{p\to\infty}\left(V_p(y+1)+\sum_{k=1}^{p-2} s_k V_k(y+1)\right)+2zy}=\sum_{n=0}^{\infty} B_n(z)\frac{y^n}{n!}\,. \quad (1.50)$$

These are the generalizations of the integral and generating function definitions of the Hermite polynomials associated with the (2,1) minimal model.
Most of this analysis has centred on the matrix side of the matrix/gravity correspondence. The gravity side is related through an integral transform. For example the macroscopic loop observable associated with the Riemann zeta function is given by:

$$\log\zeta(iz+1/2)=\int_0^{\infty}\ell^{-iz-1/2}W(\ell)d\ell\,. \quad (1.51)$$

In terms of the λ_n this observable takes the form:

$$W(\ell)=\frac{1}{\log\ell}-\sum_n\frac{2\cos(\lambda_n\log\ell)}{\ell^{1/2}\log\ell}-\frac{1}{\ell(\ell^2-1)\log\ell}\,. \quad (1.52)$$

Thence, the eq. (1.51) can be written also:

$$\log\zeta(iz+1/2)=\int_0^{\infty}\ell^{-iz-1/2}\ \frac{1}{\log\ell}-\sum_n\frac{2\cos(\lambda_n\log\ell)}{\ell^{1/2}\log\ell}-\frac{1}{\ell(\ell^2-1)\log\ell}d\ell\,. \quad (1.52b)$$

The indefinite integral of this Wheeler-DeWitt wave function is connected to the prime numbers p through:

$$\int_2^{\ell} W(\ell')d\ell' = \frac{1}{2}\left(\sum_{p^n < x} \frac{1}{n} + \sum_{p^n \le \ell} \frac{1}{n} \right). \quad (1.53)$$

The FZZT brane partition function can also be represented by prime numbers as:

$$\log \zeta(iz + 1/2) = \sum_p \sum_n \frac{1}{n} p^{-n(iz+1/2)}. \quad (1.54)$$

Thence, the eq. (1.51) can be written also:

$$\sum_p \sum_n \frac{1}{n} p^{-n(iz+1/2)} = \int_0^{\infty} \ell^{-iz-1/2} \frac{1}{\log \ell} - \sum_n \frac{2\cos(\lambda_n \log \ell)}{\ell^{1/2} \log \ell} - \frac{1}{\ell(\ell^2 - 1)\log \ell} d\ell. \quad (1.54b)$$

Both of the above formulas follow from the Euler product formula of the zeta function. Much of the physical intuition about the meaning of the FZZT brane and the Wheeler-DeWitt wave function occurs on the gravity side of the correspondence. Thus the connection of Number Theory and Gravity in this context is quite intriguing.
The (2,1) minimal model is defined by the partition function:

$$\int dMdP e^{-V(M)+Tr(PM)}, \quad (1.55)$$

with:

$$V(M) = \frac{1}{g} Tr(M^2), \quad (1.56)$$

and g is the coupling constant. We define a master matrix associated with the model as a matrix whose characteristic polynomial is equal to the matrix integral:

$$\int dMdP \det(M - zI) e^{-V(M)+Tr(PM)}, \quad (1.57)$$

which is the FZZT partition function.
The master matrix for the (2,1) minimal model is given by:

$$M = \sqrt{\frac{g}{2}} \begin{pmatrix} 0 & \sqrt{1} & 0 & \cdots & 0 \\ \sqrt{1} & 0 & \sqrt{2} & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & \sqrt{N-2} & 0 & \sqrt{N-1} \\ 0 & \cdots & 0 & \sqrt{N-1} & 0 \end{pmatrix}. \quad (1.58)$$

Which for $N = 8$ is given by:

$$\sqrt{g/2}\begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{1} & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{4} & 0 & \sqrt{5} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{5} & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{6} & 0 & \sqrt{7} \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{7} & 0 \end{pmatrix}. \quad (1.59)$$

The FZZT partition function for the (2,1) minimal model is:

$$\left(\frac{g}{4}\right)^{N/2} H_N\left(z/\sqrt{g}\right). \quad (1.60)$$

This coincides with the characteristic polynomial of the master matrix. For the case $N=8$ this is:

$$\frac{105}{19}g^4 - \frac{105}{2}g^3z^2 + \frac{105}{2}g^2z^4 - 14gz^6 + z^8. \quad (1.61)$$

The master matrix (1.58) agrees with the master matrix of the Gaussian matrix model which is has the same partition function as the (2,1) minimal model after integration over P.
Because the master matrix is manifestly Hermitian it's eigenvalues are real. The large N limit of FZZT partition function corresponds to:

$$g \to \frac{1}{N}, \quad z \to -1 + \frac{1}{N^{1/3}}z, \quad (1.62)$$

and leads to the Airy function $Ai(z)$. This function is given by the contour integral:

$$\Phi(z) = \int_{C_0} \frac{d\varphi}{2\pi i} e^{\varphi^3/3 - z\varphi}, \quad (1.63)$$

with contour C_0 starting at infinity with argument $-\pi/3$ and ending at infinity with argument $\pi/3$. It has the series expansion:

$$Ai(z) = \sum_{n=0}^{\infty} \frac{1}{3^{2/3}\pi} \frac{\Gamma((n+1)/3)}{n!} \sin(2(n+1)\pi/3)\left(3^{1/3}z\right)^n. \quad (1.64)$$

The Airy function obeys the differential equation:

$$Ai''(z) - zAi(z) = 0. \quad (1.65)$$

The Airy function has all it's zeros on the real axis and this is a manifestation of the Hermitian nature of the master matrix in (1.58).
The (3,1) minimal model is defined by the partition function with matrix potential:

$$V(M)=\frac{1}{g}\left(\frac{3}{2}Tr\left(M^2+\frac{1}{3}Tr\left(M^3\right)\right)\right). \quad (1.66)$$

The master matrix of the (3,1) minimal model is the matrix M with nonzero components:

$$M_{i,j}=(i-1)(i-2)\delta_{i,j+2}+3(i-1)\delta_{i,j+1}+g\delta_{i+1,j}, \quad (1.67)$$

which is of the form:

$$M=\begin{pmatrix} 0 & g & 0 & \cdots & 0 \\ 3 & 0 & g & \ddots & \vdots \\ 2 & \ddots & \ddots & \ddots & 0 \\ \vdots & (N-2)(N-3) & 3(N-2) & 0 & g \\ 0 & \cdots & (N-1)(N-2) & 3(N-1) & 0 \end{pmatrix}. \quad (1.68)$$

For $N=8$ this is given by:

$$\begin{pmatrix} 0 & g & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & g & 0 & 0 & 0 & 0 & 0 \\ 2 & 6 & 0 & g & 0 & 0 & 0 & 0 \\ 0 & 6 & 9 & 0 & g & 0 & 0 & 0 \\ 0 & 0 & 12 & 12 & 0 & g & 0 & 0 \\ 0 & 0 & 0 & 20 & 15 & 0 & g & 0 \\ 0 & 0 & 0 & 0 & 30 & 18 & 0 & g \\ 0 & 0 & 0 & 0 & 0 & 42 & 21 & 0 \end{pmatrix}. \quad (1.69)$$

The characteristic polynomial of this master matrix for $g=1/N$ is given by:

$$\frac{8085}{4096}-\frac{945z}{256}-\frac{175z^2}{8}+\frac{105z^3}{16}+\frac{945z^4}{32}-\frac{7z^5}{4}-\frac{21z^6}{2}+z^8, \quad (1.70)$$

and this correspond to the FZZT partition function of the (3,1) minimal model.

$$Q_N(z)=g^N(\partial_x)^N e^{-\frac{1}{g}\left(\frac{1}{3}x^3+\frac{3}{2}x^2-xz\right)}\Big|_{x=0}, \quad (1.71)$$

for $N=8$. The expression for $Q_N(z)$ can be written using the residue theorem as:

$$Q_N(z)=(-g)^N N!\frac{1}{2\pi i}\oint\frac{d\varphi}{\varphi^{N+1}}e^{-V(\varphi)-\varphi z}. \quad (1.72)$$

After taking the large N limit:

$$g \to \frac{1}{N}, \quad z \to -1 + \frac{1}{N^{1/4}} z, \quad (1.73)$$

one obtains a generalized Airy function $\Phi(z)$ defined by the integral:

$$\Phi(z) = \int_{C_0} \frac{d\varphi}{2\pi i} e^{\varphi^4/4 - z\varphi} . \quad (1.74)$$

The generalized Airy function obeys the differential equation:

$$\Phi'''(z) + z\Phi(x) = 0, \quad (1.75)$$

with solutions:

$$\Phi(z) = A\left({}_0F_2\left(\{\ \},\left\{\frac{1}{2},\frac{3}{4}\right\},-\frac{z^4}{64}\right)\right) + B\left(z^2\left({}_0F_2\left(\{\ \},\left\{\frac{5}{4},\frac{3}{2}\right\},-\frac{z^4}{64}\right)\right)\right) + C\left(z\left({}_0F_2\left(\{\ \},\left\{\frac{3}{4},\frac{5}{4}\right\},-\frac{z^4}{64}\right)\right)\right),$$

(1.76)

for constants A, B, and C where pFq is a generalized hypergeometric function. We note, in this expression, that $64 = 8^2$ and that 8 is a Fibonacci's number.

Modifying the contour to be along the imaginary axis we can define a modified generalized Airy function $\Psi(z)$ by:

$$\Psi(z) = \int_{-\infty}^{\infty} e^{-\frac{1}{4}\phi^4 + i\phi z} d\phi, \quad (1.77)$$

with a series expansion given by:

$$\Psi(z) = \frac{1}{\sqrt{2}} \sum_{k=0}^{\infty} \frac{(-2)^k}{(2k)!} \Gamma\left(\frac{1}{4} + \frac{k}{2}\right) z^{2k} . \quad (1.78)$$

This modified generalized Airy function obeys the differential equation:

$$\Psi'''(z) - z\Psi(x) = 0, \quad (1.79)$$

with solution:

$$\Psi(z) = \frac{1}{\sqrt{2}}\left(\Gamma\left(\frac{1}{4}\right){}_0F_2\left(\{\ \},\left\{\frac{1}{2},\frac{3}{4}\right\},\frac{z^4}{64}\right) - z^2\Gamma\left(\frac{3}{4}\right){}_0F_2\left(\{\ \},\left\{\frac{5}{4},\frac{3}{2}\right\},\frac{z^4}{64}\right)\right). \quad (1.80)$$

We note that also in this expression $64 = 8^2$ and that 8 is a Fibonacci's number.

The Riemann Ξ function is defined by:

$$\Xi(z) = \zeta\left(iz + \frac{1}{2}\right)\Gamma\left(i\frac{z}{2} + \frac{1}{4}\right)\pi^{-1/4}\pi^{-iz/2}\left(-\frac{z^2}{2} - \frac{1}{8}\right). \quad (1.81)$$

Also in this expression, we note easily that we have the Fibonacci's number 8.

It is even and can be expressed as an integral along the imaginary axis as:

$$\Xi(z)=\int_{-\infty}^{\infty}e^{-U(\phi)+i\phi z}d\phi\,,\quad (1.82)$$

where:

$$U(\phi)=-\log\left(\sum_{k=1}^{\infty}\left(\pi^2k^4e^{2\phi}-\frac{3}{2}\pi k^2e^{\phi}\right)e^{-\pi k^2e^{\phi}}\right).\quad (1.83)$$

This function plays the same role for the Ξ function as the Konsevich potential $\phi^3/3$ plays for the Airy function and $\phi^4/4$ for the Φ function. For small ϕ one can develop an expansion:

$$U(\phi)=9.36345\phi^2+5.95896\phi^4-2.15104\phi^6+O(\phi^8),\quad (1.84)$$

which is probably why the (3,1) minimal model modified FZZT partition function shares some of the characteristics of the Ξ function. The Ξ function itself can be expanded as:

$$\Xi(z)=\sum_{n=0}^{\infty}a_{2n}\frac{(-1)^n}{(2n)!}z^{2n}\,,\quad (1.85)$$

where

$$a_{2n}=4\int_1^{\infty}d\ell\left(\ell^{-1/4}f(\ell)\left(\frac{1}{2}\log\ell\right)^{2n}\right)\quad (1.86)$$

and

$$f(\ell)=\sum_{q=1}^{\infty}\left(q^4\pi^2\ell-\frac{3}{2}q^2\pi\right)\ell^{1/2}e^{-q^2\pi\ell}\,.\quad (1.87)$$

Thence, we can write the eq. (1.85) also:

$$\Xi(z)=\sum_{n=0}^{\infty}4\int_1^{\infty}d\ell\left(\ell^{-1/4}\sum_{q=1}^{\infty}\left(q^4\pi^2\ell-\frac{3}{2}q^2\pi\right)\ell^{1/2}e^{-q^2\pi\ell}\left(\frac{1}{2}\log\ell\right)^{2n}\right)\frac{(-1)^n}{(2n)!}z^{2n}\,.\quad (1.87b)$$

Thus like the Ψ function one can think of the $\Xi(z)$ function as an infinite order polynomial expanded in even powers of z.
Now we take the pure numbers of the expression (1.84). We obtain an interesting mathematical connection with the aurea section and the aurea ratio. Indeed, we have that:

9.36345+5.95896 – 2.15104 = 13.17137 $\cong$ **13.17**;

$$\left(\frac{\sqrt{5}+1}{2}\right)^5+\left(\frac{\sqrt{5}+1}{2}\right)^2-\left(\frac{\sqrt{5}-1}{2}\right)=13.0901667\cong\mathbf{13.09}$$

Nevertheless keeping the first two terms in the expansion for $U(\phi)$ one can derive the following approximate equation for small z:

$$4(5.95896)\Xi(z)'''-2(9.36345)\Xi(z)'-z\Xi(z)\approx 0. \quad (1.88)$$

Rescaling the argument of $\Xi(z)$ we define:

$$\Xi_*(z)=\Xi\left(\sqrt{2}(5.95896)^{1/4}\right)z. \quad (1.89)$$

So that one has the following approximate equation for small z:

$$\Xi_*(z)'''-s_1\Xi_*(z)'-z\Xi_*(z)\approx 0, \quad (1.90)$$

where

$$s_1=\frac{9.36345}{\sqrt{5.95896}}=3.835753241. \quad (1.91)$$

This appear related to the deformed (3,1) minimal model with deformation parameter s_1. The solution to the equation for $\Xi*$ is denoted by $\Psi(z,s_1)$ and is:

$$\Psi(z,s_1)=\int_{-\infty}^{\infty}e^{-\frac{1}{4}\phi^4-\frac{1}{2}s_1\phi^2+i\phi z}d\phi. \quad (1.92)$$

We note that the value of s_1 is related with the following expressions:

$$\left(\frac{\sqrt{5}+1}{2}\right)^3-\left(\frac{\sqrt{5}-1}{2}\right)+\frac{1}{3}\left(\frac{\sqrt{5}-1}{2}\right)=3.82404468\cong 3.82$$
$$\left(\frac{\sqrt{5}+1}{2}\right)^{19/7}+\left(\frac{\sqrt{5}-1}{2}\right)^{29/7}=3.8281\cong 3.83$$

One can improve the approximate equation (1.88) by including higher order terms in the ϕ expansion of $U(\phi)$. Keeping terms up to ϕ^6 in (1.84) one obtains the approximate differential equation:

$$6(2.15104)\Xi'''''(z)+4(5.95896)\Xi(z)'''-2(9.36345)\Xi(z)'-z\Xi(z)\approx 0. \quad (1.93)$$

Now rescaling can put the equation in the form:

$$\Xi_{**}(z)'''''+s_3\Xi_{**}(z)'''-s_1\Xi_{**}(z)'-z\Xi_{**}(z)\approx 0, \quad (1.94)$$

with deformation parameters s_1 and s_3. Finally we can define a function $\Phi(z,s_1)$ as the solution to:

$$\Phi(z,s_1)'''-s_1\Phi(z,s_1)'+z\Phi(z,s_1)=0, \quad (1.95)$$

which is real on the real axis and decays non-oscillatory for large positive y. We have that $\Phi(z,s_1)$ is the FZZT partition function associated with the matrix potential:

$$V(x)=\frac{N}{1+\frac{s_1}{\sqrt{N}}}\left[3\left(-1+\frac{x}{N^{1/4}}\right)+\frac{3}{2}\left(-1+\frac{x}{N^{1/4}}\right)^2+\frac{1}{3}\left(-1+\frac{x}{N^{1/4}}\right)^3+\frac{s_1}{\sqrt{N}}\left(-1+\frac{x}{N^{1/4}}\right)\right]. \quad (1.96)$$

After rescaling and shifting the point of origin of the potential one can define polynomials for the matrix model deformed by the parameter s_1 through:

$$Q_N(z,s_1)=\left(\frac{1+\frac{s_1}{\sqrt{N}}}{N}\right)^N \partial_x^N\left(\exp\left(-\frac{N}{1+\frac{s_1}{\sqrt{N}}}\left(3x+\frac{3}{2}x^2+\frac{1}{3}x^3+\frac{s_1}{\sqrt{N}}x-xz\right)\right)\right)\Bigg|_{x=0}. \quad (1.97)$$

The master matrix which has this as characteristic polynomial is a simple rescaling of the coupling constant of the master matrix of the (3,1) minimal model and is given by:

$$M_{i,j}=(i-1)(i-2)\delta_{i,j+2}+3(i-1)\delta_{i,j+1}+\frac{1}{N}\left(1+\frac{s_1}{\sqrt{N}}\right)\delta_{i+1,j}. \quad (1.98)$$

This master matrix can develop complex eigenvalues for large enough N and s_1. In particular for $N\geq 34$ (note that 34 is a Fibonacci's number) and s_1 given by (1.91) the eigenvalues are complex. However the function $\Psi(z,s_1)$ obtained from changing the sign of z in the third term in (1.95) is very different from $\Phi(z,s_1)$ in this respect. It would be of interest to determine the master matrix associated with $\Psi(z,s_1)$ and it's corrections for terms involving s_3 and higher, which should in principle converge to the Riemann Ξ function.
With regard the eq. (1.93), we note that the pure number 2.15104 is related to the following expressions:

$$\left(\frac{\sqrt{5}+1}{2}\right)^{11/7}=2.13014\cong 2.13\cong 2.15104;$$

$$\left(\frac{\sqrt{5}-1}{2}\right)^4+\left(\frac{\sqrt{5}-1}{2}\right)^2+\left(\frac{\sqrt{5}+1}{2}\right)=2.1458\cong 2.15\cong 2.15104$$

Furthermore, from the eq. (1.93) we have also that:

$$\left(\frac{\sqrt{5}+1}{2}\right)^{26/7}\cong 5.97\cong 5.95896;$$

$$6(2.15104)+4(5.95896)-2(9.36345)=12.90624+23.83584-18.7269=18.01518\cong\left(\frac{\sqrt{5}+1}{2}\right)^{42/7}\cong 17.94$$

$$\left(\frac{\sqrt{5}+1}{2}\right)^{37/7}=12.72\cong 12.90; \quad \left(\frac{\sqrt{5}+1}{2}\right)^{46/7}=23.62\cong 23.83; \quad \left(\frac{\sqrt{5}+1}{2}\right)^{42/7}=17.94\cong 18.72$$

The Ξ function can be expressed also as Meixner-Pollaczek polynomials. Thence, we have that:

$$p_n(z) = n!\frac{(2n+1)!!}{(2n)!!} i^n {}_2F_1\left(-n, \frac{3}{4} + i\frac{1}{2}z; \frac{3}{2}; 2\right). \quad (1.99)$$

These polynomials are the characteristic polynomial of a matrix with nonzero components:

$$M_{i,j} = i\left(i + \frac{1}{2}\right)\delta_{i,j+1} + \delta_{i+1,j}. \quad (1.100)$$

For $N = 8$ this matrix is given by:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3/2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 21/2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 18 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 55/2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 36 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 105/2 & 0 \end{pmatrix}. \quad (1.100b)$$

The characteristic polynomial of this matrix is:

$$\frac{363825}{16} - \frac{74247}{2}z^2 + \frac{10493}{2}z^4 - 154z^6 + z^8, \quad (1.101)$$

which agrees with (1.99) for $N = 8$. The expansion of the Ξ function with an exponential factor can be expanded in terms of the Meixner-Pollaczek polynomials as:

$$\Xi(z)e^{-\pi z/4} = \sum_{n=0}^{\infty} b_n p_n(z). \quad (1.102)$$

Terminating this series at N one can write this expansion as the characteristic polynomial of a $N \times N$ matrix. For $N = 8$ this is given by:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3/2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 21/2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 18 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 55/2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 36 & 0 & 1 \\ b_0/b_8 & b_1/b_8 & b_2/b_8 & b_3/b_8 & b_4/b_8 & b_5/b_8 & 105/2 + b_6/b_8 & b_7/b_8 \end{pmatrix}. \quad (1.103)$$

When b_n are taken to zero this reproduces the matrix (1.100). The coefficients b_n are linearly related to the integrals:

$$I_n = \int_0^\infty \frac{\sin\left(\frac{y^2}{2}+\frac{\pi}{8}\right)}{e^{2\sqrt{\pi y}}+1} y^{2n+1} dy . \quad (1.104)$$

With regard the pure numbers of the matrix (1.100) and (1.103) we note that:

3, 5, 21, 55 are Fibonacci's numbers, while 18 = 13 + 5; 36 = 2 + 34 and 105 = 3 + 13 + 89, thence are sum of Fibonacci's numbers.

1.1 On some equations concerning the partition functions of the rigid string and membrane at any temperature.

The first two terms in the loop expansion

$$S_{eff} = S_0 + S_1 + \ldots \quad (1.105)$$

of the effective action corresponding to the rigid string

$$S = \frac{1}{2\alpha_0}\int d^2\sigma\left[\rho^{-1}\partial^2 X^\mu \partial^2 X_\mu + \lambda^{ab}\left(\partial_a X^\mu \partial_b X_\mu - \rho\delta_{ab}\right)\right] + \mu_0 \int d^2\sigma\rho , \quad (1.106)$$

where α_0 is the dimensionless, asymptotically free coupling constant, ρ the intrinsic metric, μ_0 the explicit string tension (important at low energy) and λ^{ab}, $a,b = 1,2$, the usual Lagrange multipliers, are given – in the world sheet $0 \le \sigma^1 \le L$ and $0 \le \sigma^2 \le \beta t$ – by

$$S_0 = \frac{L\beta t}{2\alpha_0}\left[\lambda^{11} + \lambda^{22}t^{-2} + \rho\left(2\alpha_0\mu_0 - \lambda^{aa}\right)\right] \quad (1.107)$$

at tree level, and by

$$S_1 = \frac{d-2}{2} L \sum_{n=-\infty}^{\infty} \int_{-\infty}^{+\infty} \frac{1}{2\pi} dk \ln\left[\left(k^2 + \frac{4\pi^2 n^2}{\beta^2 t^2}\right)^2 + \rho\left(\lambda^{11}k^2 + \frac{4\pi^2\lambda^{22}n^2}{\beta^2 t^2}\right)\right] \quad (1.108)$$

at one-loop order, respectively. Of course, to make sense, this last expression needs to be regularized and its calculation is highly non-trivial. We shall make use of the zeta function procedure and thence, one can write the expression for S_1 also:

$$S_1 = -(d-2)L\frac{d}{ds}\zeta_A(s/2)\Big|_{s=0} , \quad \zeta_A(s/2) = \sum_{n=-\infty}^{\infty}\int_{-\infty}^{+\infty}\frac{1}{2\pi}dk\left(k^2+y_+^2\right)^{-s/2}\left(k^2+y_-^2\right)^{-s/2} , \quad (1.109)$$

where

$$y_{\pm} = \frac{a}{t}\left[n^2 + \frac{\rho t^2 \lambda^{11}}{2a^2} \pm \sqrt{\rho}\,\frac{t}{a}\left(\left(\lambda^{11} - \lambda^{22}\right) n^2 + \frac{\rho t^2 \lambda^{11^2}}{4a^2} \right)^{1/2} \right]^{1/2} , \quad a \equiv \frac{2\pi}{\beta}. \quad (1.110)$$

We may consider two basic approximations of overlapping validity: one for low temperature, $\beta^{-2} << \mu_0$, and the other for high temperature, $\beta^{-2} >> \alpha_0 \mu_0$. Both these approximations can be obtained from the expression above, which on its turn can be written in the form

$$\zeta_A(s/2) = \frac{1}{2\sqrt{\pi}} \frac{\Gamma(s-1/2)}{\Gamma(s)} \sum_{n=-\infty}^{\infty} \frac{y_-}{(y_+ y_-)^s} F(s/2, 1/2; s; 1-\eta), \quad \eta \equiv \frac{y_-^2}{y_+^2}. \quad (1.111)$$

This is an exact formula.
With regard the low temperature case, the term $n = 0$ in (1.111) is non-vanishing and must be treated separately from the rest. It gives

$$\zeta_A^{n=0}(s/2) = \frac{1}{2\pi} \frac{\Gamma[(1-s)/2]\Gamma(s-1/2)}{\Gamma(s/2)} \left(\lambda^{11}\rho\right)^{1/2-s}, \quad \frac{1}{2} < R(s) < 1. \quad (1.112)$$

This is again an exact expression, that yields

$$\frac{d}{ds}\zeta_A^{n=0}(s/2)\Big|_{s=0} = -\frac{1}{2}\sqrt{\rho\lambda^{11}} \quad (1.113)$$

and

$$S_1^{(n=0)} = \frac{d-2}{2}\sqrt{\rho\lambda^{11}}\,. \quad (1.114)$$

For high temperature, the ordinary expansion of the confluent hypergeometric function F of eq. (1.111) is in order

$$F(s/2, 1/2; s; 1-\eta) = \sum_{k=0}^{\infty} \frac{(s/2)_k (1/2)_k}{k!(s)_k} (1-\eta)^k, \quad (1.115)$$

$(s)_k = s(s+1)...(s+k-1)$ being Pochhamer’s symbol (the rising factorial).
Now we shall consider the case of the pure bosonic membrane and corresponding p-brane. The tree level action similar to (1.107) is

$$S_0^{(m)} = \kappa L^2 \beta t \left[(1+\sigma_0)^{1/2}(1+\sigma_1) - \left(\frac{1}{2}\lambda_0\sigma_0 + \lambda_1\sigma_1 \right) \right] \quad (1.116)$$

where λ_0 and λ_1 are Lagrange multipliers and σ_0 and σ_1 are composite fields. The one-loop contribution to the action can be written formally as follows

$$S_1^{(bm)} = \frac{(d-3)L^2}{2} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{+\infty} \frac{dk_1 dk_2}{(2\pi)^2} \left[\lambda_1\left(k_1^2 + k_2^2\right) + \frac{4\pi^2 \lambda_0 n^2}{\beta^2 t^2} \right]. \quad (1.117)$$

As in the string case (eq. 1.109), we choose the zeta function method. Calling ζ_2 the corresponding zeta function, we have

$$S_1^{(bm)} = -\frac{(d-3)L^2}{2}\frac{d}{ds}\zeta_2(s)\Big|_{s=0}, \quad \zeta_2(s) = \sum_{n=-\infty}^{\infty}\int_{-\infty}^{+\infty}\frac{dk_1 dk_2}{(2\pi)^2}\ln\left[\lambda_1\left(k_1^2+k_2^2\right)+\frac{4\pi^2\lambda_0 n^2}{\beta^2 t^2}\right]^{-s}. \quad (1.118)$$

After some calculations, we get

$$\zeta_2(s) = \frac{1}{4\pi(s-1)\lambda_1}\left(\frac{4\pi^2\lambda_0}{\beta^2 t^2}\right)^{1-s}\zeta_R(2s-2), \quad (1.119)$$

where ζ_R is Riemann's zeta function. We thus obtain

$$S_1^{(bm)} = -\frac{2(d-3)\pi\lambda_0 L^2}{\lambda_1\beta^2 t^2}\zeta_R'(-2). \quad (1.120)$$

In the case of the bosonic p-brane, the corresponding expressions are

$$S_0^{(p)} = \kappa L^p \beta t\left[(1+\sigma_0)^{1/2}(1+\sigma_1)^{p/2} - \frac{1}{2}(\lambda_0\sigma_0 + p\lambda_1\sigma_1)\right] \quad (1.121)$$

and

$$S_1^{(bp)} = -\frac{(d-p-1)L^p}{2}\frac{d}{ds}\zeta_p(s)\Big|_{s=0},$$

$$\zeta_p(s) = \sum_{n=-\infty}^{\infty}\int_{-\infty}^{+\infty}\frac{dk_1 \dots dk_p}{(2\pi)^p}\ln\left[\lambda_1\left(k_1^2+\dots+k_p^2\right)+\frac{4\pi^2\lambda_0 n^2}{\beta^2 t^2}\right]^{-s} = \frac{V_p\Gamma(p/2)\Gamma(s-p/2)}{(2\pi)^p\lambda_1^{p/2}\Gamma(s)}\left(\frac{4\pi^2\lambda_0}{\beta^2 t^2}\right)^{p/2-s}\zeta_R(2s-p)$$
$$(1.122)$$

where V_p is the "volume" of the $p-1$-dimensional unit sphere.

Now we shall consider the case of the bosonic membrane with rigid term and the corresponding p-brane. The tree level action is the same as before, eq. (1.116). The one-loop order contribution for the bosonic membrane is

$$S_1^{(rm)} = -\frac{(d-3)L^2}{2}\frac{d}{ds}\zeta_2^r(s)\Big|_{s=0},$$

$$\zeta_2^r(s) = \sum_{n=-\infty}^{\infty}\int_{-\infty}^{+\infty}\frac{1}{(2\pi)^2}dk_1 dk_2\left[\frac{1}{\rho^2}\left(k_1^2+k_2^2+\frac{4\pi^2 n^2}{\beta^2 t^2}\right)^2 + \kappa\left(\lambda_1\left(k_1^2+k_2^2\right)+\frac{4\pi^2\lambda_0 n^2}{\beta^2 t^2}\right)\right]^{-s}, \quad (1.123)$$

where the label r means *rigid*. We can write

$$\zeta_2^r(s) = \frac{1}{4\pi}\rho^{2s}\sum_{n=-\infty}^{\infty}\int_0^{\infty}dk(k+c_+)^{-s}(k+c_-)^{-s}, \quad (1.124)$$

with

$$c_{\pm} = \frac{4\pi^2 n^2}{\beta^2 t^2} + \frac{\kappa\rho^2\lambda_1}{2} \pm \sqrt{\kappa}\rho\left[(\lambda_1-\lambda_0)\frac{4\pi^2 n^2}{\beta^2 t^2} + \frac{\kappa\rho^2\lambda_1^2}{4}\right]^{1/2}. \quad (1.125)$$

Here, in analogy with the rigid string case, the term corresponding to $n=0$ must be treated separately. It yields a beta function. Also as in the rigid string case, the remaining series can be written in terms of a confluent hypergeometric function. The complete result is:

$$\zeta_2^r(s) = \frac{\rho^{2-2s}(\kappa\lambda_1)^{1-2s}\Gamma(1-s)\Gamma(2s-1)}{4\pi\Gamma(s)} + \frac{\rho^{2s}\Gamma(2s-1)}{2\pi\Gamma(2s)}\sum_{n=1}^{\infty} c_-^{1-2s} F(s,2s-1;2s;1-c_+/c_-). \quad (1.126)$$

For the general case of the p-brane with rigid term, the one-loop contribution to the action is

$$S_1^{(rp)} = -\frac{(d-p-1)L^p}{2}\frac{d}{ds}\zeta_p^r(s)\Big|_{s=0},$$

$$\zeta_p^r(s) = \sum_{n=-\infty}^{\infty}\int_{-\infty}^{+\infty}\frac{1}{(2\pi)^p}dk_1...dk_p\left[\frac{1}{\rho^2}\left(k_1^2+...+k_p^2+\frac{4\pi^2 n^2}{\beta^2 t^2}\right)^2 + \kappa\left(\lambda_1\left(k_1^2+...+k_p^2\right)+\frac{4\pi^2\lambda_0 n^2}{\beta^2 t^2}\right)\right]^{-s} =$$

$$= \frac{V_p\rho^{2s}}{2(2\pi)^p}\sum_{n=-\infty}^{\infty}\int_0^{\infty} dk\, k^{p/2-1}(k+c_+)^{-s}(k+c_-)^{-s}, \quad (1.127)$$

where V_p is again the "volume" of the p – 1-dimensional unit sphere and the $c_{\pm}$ are again given by (1.125). Considering the $n=0$ term separately, we obtain the following generalization of the formula corresponding to the rigid membrane:

$$\zeta_p^r(s) = \frac{V_p\rho^{p-2s}(\kappa\lambda_1)^{p/2-2s}}{2(2\pi)^p}\frac{\Gamma(p/2-s)\Gamma(2s-p/2)}{\Gamma(s)} + \frac{V_p\rho^{2s}}{(2\pi)^p}\times$$
$$\times\frac{\Gamma(2s-p/2)\Gamma(p/2)}{\Gamma(2s)}\sum_{n=1}^{\infty} c_-^{p/2-2s}F(s,2s-p/2;2s;1-c_+/c_-). \quad (1.128)$$

In the case of the rigid membrane we get the rather simpler result

$$\frac{d}{ds}\zeta_2^r(s)\Big|_{s=0} = \frac{1}{4\pi}\kappa\rho^2\lambda_1\left[\ln\left(\kappa\rho^2\lambda_1(\beta t)^2\right)-1\right] - \frac{8\pi\zeta'(-2)}{(\beta t)^2}\frac{1}{4\pi}\kappa\rho^2(\lambda_1-\lambda_0). \quad (1.129)$$

The one loop action for the rigid membrane is readily obtained from (1.129)

$$S_1^{(rm)} = \frac{(d-3)L^2}{8\pi}\kappa\rho^2\lambda_1\left[1-\ln\left(\kappa\rho^2\lambda_1(\beta t)^2\right)\right] + \frac{4\pi(d-3)L^2}{(\beta t)^2}\zeta'(-2) + \frac{(d-3)L^2}{8\pi}\kappa\rho^2(\lambda_1-\lambda_0). \quad (1.130)$$

Here, terms up to $k=2$ in the expansion (1.115) of the hypergeometric function of (1.128) have been taken into account. We note that all higher-order terms would be easy to obtain from (1.128) and that a consistent loop expansion to any desired order can in fact performed. The conditions for extremum of $S^{(rm)} = S_0^{(m)} + S_1^{(rm)}$, eqs. (1.116) and (1.130), are obtained by taking the derivatives with respect to the parameters $\lambda_0, \lambda_1, \sigma_0$ and σ_1. The result is

$$\sigma_0 = -\frac{d-3}{4\pi}\frac{\rho^2}{\beta t}, \quad \sigma_1 = \frac{(d-3)\rho^2}{8\pi\beta t}\left[1-\ln\left(\kappa\rho^2\lambda_1(\beta t)^2\right)\right],$$
$$\lambda_0 = (1+\sigma_1)(1+\sigma_0)^{-1/2}, \quad \lambda_1 = (1+\sigma_0)^{1/2}. \quad (1.131)$$

We can easily identify here the transition that also takes place for the rigid string: for values of the temperature higher than the one coming from the expression

$$\beta_c^{-1} = \frac{4\pi t}{(d-3)\rho^2}, \quad (1.132)$$

the values of the parameters, and hence of the action and of the winding soliton mass squared, acquire an imaginary part. Guided by the fact that in the rigid string case this temperature lies above the Hagedorn temperature, we conclude that in order that the whole scheme of the string case can be translated to the membrane situation we must demand that $\rho^2\mu$ be small.

For the Hagedorn temperature, defined as the value for which the winding soliton mass

$$M_1^{(rm)} \equiv \frac{S_{eff}^{(rm)}}{L^2} \quad (1.133)$$

vanishes, we find

$$\beta_H^{-1} \cong \frac{4\pi\sqrt{-\zeta'(-2)}\,t}{\rho\sqrt{-\kappa\ln(\kappa\rho^6)}}. \quad (1.134)$$

The values of the constants which determine the leading behaviour of the effective action at high temperature, namely the derivative of the zeta function at the point -2 (in general, $-p$, respectively), have been calculated. In particular, we have that

$$\zeta'(-1) = -0{,}16542115, \qquad \zeta'(-2) = -0{,}03049103. \quad (1.135)$$

Also here, we note the mathematical connection with the aurea section, i.e. $\phi = \frac{\sqrt{5}-1}{2}$. Indeed, we have that:

$$-(\phi)^{26/7} = -\left(\frac{\sqrt{5}-1}{2}\right)^{26/7} = -0{,}16740; \qquad -(\phi)^{51/7} = -\left(\frac{\sqrt{5}-1}{2}\right)^{51/7} = -0{,}030017. \quad (1.136)$$

Now, we take the pure numbers of the eqs. (1.61), (1.70) and (1.101). We have the following sequence:

2, 4, 7, 8, 14, 16, 21, 32, 105, 154, 175, 256, 945, 4096, 8085, 10493, 74247, 363825.

We note that:

$2=2$, $4=2^2$, $7=7\times1=7$, $8=2^3$, $14=2\times7$, $16=2^4=2\times8$, $21=3\times7=21$,
$32=2^5=4\times8$, $105=3\times5\times7=21\times5$, $154=2\times7\times11$, $175=5^2\times7$, $256=2^8=4\times8\times8$,
$945=3^3\times5\times7=3^2\times5\times21$, $4096=2^{12}=8^2\times8^2$, $8085=3\times5\times7^2\times11=5\times7\times11\times21$,
$10493=7\times1499$, $74247=3\times24749$, $363825=3^3\times5^2\times7^2\times11=3\times5^2\times11\times21^2$.

Here, 5, 7 and 11 are prime natural numbers and 2, 3, 5, 8 and 21 are Fibonacci's numbers. The number 8 is also connected with the "modes" that correspond to the physical vibrations of a superstring by the following Ramanujan function:

$$8=\frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (1.137)$$

With regard the numbers 2, 3, 5, 7 and 11, these are also factors of the numbers of dimension of the Lie's Groups, connected to the string theory. Indeed, we observe that:

L1 = 2×7; L2 = $2^2\times3$; L3 = $2\times3\times13$; L4 = 7×19; L5 = $2^3\times31$; L6 = $2^4\times3^2\times5\times11$;
L7 = $2^3\times3\times5\times7\times11\times19$; L8 = $2^{10}\times3^3\times5\times7\times11\times23$; L9 = $2^4\times3^6\times5^2\times839^2$, with
$839=140\times6-1$ and $140=144-3-1$ (that are Fibonacci's numbers);
L10 = $2^{46}\times3^{20}\times5^9\times7^6\times11^2\times13^3\times17\times19\times23\times29\times31\times41\times47\times59\times71$.

We have also that:

$46\cong(34+55)/2$; $20=21-1$; $9=8+1$; $6=5+1$; 2 = 2; 3 = 3; with 2, 3, 5, 8, 21, 34 and 55 that are Fibonacci's numbers.
With regard the prime natural numbers, we have that: 5, 7, 11, 13, 17, 19, 29, 31 and 47 are of the form $6f\pm1$ with $f=1,2,3,5$ and 8 that are Fibonacci's numbers. With regard the numbers 59 and 71, we have that: $59=6\times10-1$, with 10 = 8 + 2 (8 and 2 are Fibonacci's numbers), while $71=6\times12-1$, with 12 = 13 – 1 (1 and 13 are Fibonacci's numbers).
We note also that for the Lie's Groups G_2, F_4, E_6, E_7 and E_8, that have dimensions 14, 52, 78, 133 and 248, we have that:

$14=2\times7$, with $7=6\times1+1$; $52=4\times13$, with $13=6\times2+1$; $78=6\times13$, with $13=6\times2+1$;
$133=7\times19$, with $19=6\times3+1$; $248=8\times31$, with $31=6\times5+1$. (1, 2, 3 and 5 are Fibonacci's numbers, while 7, 13, 19 and 31 are prime natural numbers. Furthermore, also here there is the numbers 8 that is related to the physical vibrations of a superstring by eq. (1.137)).

Furthermore, we have that:

$10493 = 7 \times 1499$, with 1499 = (1499 + 1)/6 = 250 = 233 + 17 = 233 + [(21 + 13)/2]. Note that 13, 21 and 233 are Fibonacci's numbers and 17 = 34/2 with 34 = Fibonacci's number.

$74247 = 3 \times 24749$, with 24749 = $6 \times 4125 - 1$, with $4125 = 3 \times 5^3 \times 11 = 4181 - 56$, with 56 = 55 + 1. Note that 1, 55 and 4181 are Fibonacci's numbers. We have also that 25087 – 24749 = 338; $338 = 6 \times 55 + 8$, with 8 and 55 that are Fibonacci's numbers. Furthermore, 25087 is a prime natural number. Indeed, $25087 = 6 \times 4181 + 1$.

2. On some mathematical aspects concerning the rigid surface operators in gauge theory [4]

Now, in this chapter, we describe some interesting aspects of the rigid surface operators in gauge theory for the connections with the Geometric Langlands Program.

The familiar examples of non-local operators in four-dimensional gauge theory include line operators, such as Wilson and 't Hooft operators, supported on a one-dimensional curve L in the space-time manifold M. While a Wilson operator labelled by a representation R of the gauge group G can be defined by modifying the measure in the path integral, namely by inserting a factor

$$W_R(L) = Tr_R \operatorname{Hol}_L(A) = Tr_R\left(P \exp \oint_L A\right), \quad (2.1)$$

an 't Hooft operator is defined by modifying the space of fields over which one performs the path integral. Similarly, a surface operator in four-dimensional gauge theory is an operator supported on a two-dimensional submanifold $D \subset M$ in the space-time manifold M.

Four-dimensional gauge theories admit surface operators, and in the supersymmetric case, they often admit supersymmetric surface operators, that is, surface operators that preserve some of the supersymmetry. Now, we consider some mathematical aspects of $N = 4$ super Yang-Mills theory in four dimensions, the maximally supersymmetric case. This theory has many remarkable properties, including electric-magnetic duality, and has been extensively studied in the context of string dualities, in particular in the AdS/CFT correspondence.

Hitchin's equations are equations in the $x^2 - x^3$ plane that can be written as follows:

$$F_A - \phi \wedge \phi = 0, \quad d_A \phi = 0, \quad d_A * \phi = 0. \quad (2.2)$$

To define a supersymmetric surface operator, one picks a solution of Hitchin's equations with a singularity along D, and one requires that quantization of $N = 4$ super Yang-Mills theory should be carried out for fields with precisely this kind of singularity. It is natural to look for surface operators that are invariant under rotations of the $x^2 - x^3$ plane. If we set $x^2 + ix^3 = re^{i\theta}$, then the most general possible rotation-invariant ansatz is

$$A = a(r)d\theta + f(r)\frac{dr}{r}, \quad \phi = b(r)\frac{dr}{r} - c(r)d\theta. \quad (2.3)$$

Setting $f(r) = 0$ by a gauge transformation and introducing a new variable $s = -\ln r$, we can write the supersymmetry equations (2.2) in the form of Nahm's equations:

$$\frac{da}{ds} = [b, c], \quad \frac{db}{ds} = [c, a], \quad \frac{dc}{ds} = [a, b]. \quad (2.4)$$

The most general conformally invariant solution is obtained by setting a,b,c to constant elements α,β,γ of the Lie algebra g of G. The equations imply that $\alpha,\beta,$ and γ must commute, so we can conjugate them to the Lie algebra t of a maximal torus T of G. The resulting singular solution of Hitchin's equations then takes the simple form

$$A = \alpha d\theta\,, \quad \phi = \beta\frac{dr}{r} - \gamma d\theta\,. \quad (2.5)$$

The definition of the surface operator is that A and ϕ have singularities proportional to α,β,γ modulo terms that are less singular than $1/r$. Generically, for $\alpha,\beta,\gamma \to 0$, we conclude that A and ϕ are less singular than $1/r$. In fact, Hitchin's equations do have a rotationally symmetric solution that is singular at $r=0$ but less singular than $1/r$. The Nahm equations (2.4) are solved with

$$a = -\frac{t_1}{s+1/f}\,, \quad b = -\frac{t_2}{s+1/f}\,, \quad c = -\frac{t_3}{s+1/f} \quad (2.6)$$

where $t_1,t_2,$ and t_3 are elements of the Lie algebra g, which satisfy the usual $su(2)$ commutation relations, $[t_1,t_2]=t_3$, etc. Moreover, f is an arbitrary non-negative constant. Since we are taking $G = SU(2)$, the matrices t_i, if nonzero, correspond to the two-dimensional representation of $SU(2)$. So the surface operator that we get from the ansatz (2.6), with f allowed to fluctuate, is actually conformally invariant.
A convenient way to describe this surface operator is to say that the fields behave near $r=0$ as

$$A = \frac{t_1 d\theta}{\ln r} + \ldots\,, \quad \phi = \frac{t_2 dr}{r\ln r} - \frac{t_3 d\theta}{\ln r} + \ldots\,, \quad (2.7)$$

where the ellipses refer to terms that are less singular (at most of order $1/r\ln^2 r$) at $r=0$. The complex-valued flat connection $\mathrm{A} = A + i\phi$ is invariant under part of the supersymmetry preserved by the surface operator. Hence the conjugacy class of the monodromy

$$U = P\exp\left(-\int_\ell \mathrm{A}\right) \quad (2.8)$$

is a supersymmetric observable. Here ℓ is a contour surrounding the singularity. Hitchin's equations imply that the curvature of A, namely $F = d\mathrm{A} + \mathrm{A}\wedge\mathrm{A}$, is equal to zero. So if Hitchin's equations are obeyed, then the conjugacy class of U is invariant under deformations of ℓ. Of course, U is an element of G_C, the complexification of G. For a generic surface operator with parameters $\alpha,\beta,\gamma,$ we set $\xi = \alpha - i\gamma$. Then $\mathrm{A} = \xi d\theta$, and the monodromy is hence

$$U = \exp(-2\pi\xi)\,. \quad (2.9)$$

Thence, from the eq. (2.8), we can write also:

$$U = P\exp\left(-\int_\ell \mathrm{A}\right) = \exp(-2\pi\xi)\,. \quad (2.9b)$$

This is independent of the choice of ℓ. On the other hand, for the solution (2.6), we find $A = -d\theta(t_1 - it_3)/(s + 1/f)$. If we take ℓ to be the circle $s = s_1$, the monodromy comes out to be

$$U' = \exp[-2\pi(t_1 - it_3)/(s_1 + 1/f)]. \quad (2.10)$$

We note that U can also be diagonalized, with eigenvalues $\exp(\pm 2\pi\xi_0)$:

$$U = \begin{pmatrix} \exp(-2\pi\xi_0) & 0 \\ 0 & \exp(2\pi\xi_0) \end{pmatrix}. \quad (2.11)$$

As long as $\xi_0 \neq 0$, this matrix is conjugate to

$$U_w = \begin{pmatrix} \exp(-2\pi\xi_0) & 0 \\ w & \exp(2\pi\xi_0) \end{pmatrix}, \quad (2.12)$$

so it does not matter if w is zero or not.

Let $\mathcal{C}_\xi$ be the conjugacy class in SL(2,C) that contains the element $U = \exp(-2\pi\xi)$, with generic ξ. Then $\mathcal{C}_\xi$ is of complex dimension two. Indeed, U commutes only with a one-parameter subgroup of diagonal matrices, so its orbit in the three-dimensional group SL(2,C) is two-dimensional.

Unipotent elements U of G_C correspond naturally to nilpotent elements n of the Lie algebra g_C of G_C, via $U = \exp(n)$. It is convenient to think in terms of the Lie algebra. A natural source of nilpotent elements of G_C comes by picking an embedding of Lie algebras $\rho : sl(2,C) \to g_C$.

Then the raising (or lowering) operator for this embedding gives us a nilpotent element $n \in g_C$.

Conversely, the Jacobson-Morozov theorem states that every nilpotent element $n \in g_C$ is the raising operator for some $sl(2,C)$ embedding. In fact, up to conjugacy, every nilpotent element is the raising operator of some unitary embedding

$$\rho : su(2) \to g \quad (2.13)$$

of the real Lie algebra of SU(2) to that of the compact form of G.

Now it is clear how to make a surface operator associated with any unipotent conjugacy class $\mathcal{C} \subset G_C$. We pick an SU(2) embedding $\rho : su(2) \to g$, and define the surface operator using eqn. (2.6), where $t_1, t_2,$ and t_3 are now the images of the standard SU(2) generators under the chosen embedding.

Rigid unipotent conjugacy classes or rigid nilpotent orbits also exist in exceptional groups. A (noncentral) unipotent conjugacy class of minimal dimension in a complex semisimple Lie group is always rigid, except for A_N. It is convenient to be able to compute the dimension of a unipotent conjugacy class in G_C, or equivalently of a nilpotent orbit in g_C. So we pause to explain how to do this. Let d be the complex dimension of G_C, and let s be the complex dimension of the subgroup $G_C^n \subset G_C$ of elements that commute with a given $n \in g_C$. The dimension of the orbit of n (or of $\exp(n)$) is $d - s$. So it suffices to compute s.

The element n is the raising operator for some embedding $\rho : su(2) \to g$. We decompose g in irreducible representations $\mathcal{R}_i$ of $su(2)$:

$$g = \oplus_{i=1}^{s} \mathcal{R}_i . \quad (2.14)$$

The subspace of g that commutes with the raising operator n is precisely the space of highest weight vectors for the action of $su(2)$. Each irreducible summand $\mathcal{R}_i$ has a one-dimensional space of highest weight vectors. So the subspace of g that commutes with n is of dimension equal to s, the number of summands in (2.14).

For example, one can use this method to compute the dimensions of the minimal unipotent conjugacy classes in SO(N,C) or Sp(2N,C). For another important example, we re-examine the regular unipotent orbit of SL(N,C). This corresponds to an irreducible N-dimensional representation of $su(2)$, and the summands in (2.14) are of dimension 3, 5, 7,…, 2N – 1. There are N – 1 summands. This shows that the subgroup of SL(N,C) that commutes with a principal unipotent element has dimension N – 1. The number N – 1 equals the dimension of the maximal torus, showing that a principal unipotent orbit has the same dimension as a generic semisimple orbit.

Now we will describe a gauge theory singularity in real codimension 2 associated with a rigid semisimple element of G. We take the singularity to be at $x^2 = x^3 = 0$, and we use polar coordinates $x^2 + ix^3 = re^{i\theta}$. In the absence of any singularity, an adjoint-valued field on the $x^2 - x^3$ plane can be represented by an adjoint-valued function $\Phi(r,\theta)$ that obeys $\Phi(r,\theta + 2\pi) = \Phi(r,\theta)$. If S is any element of the gauge group G, we can modify this condition to

$$\Phi(r,\theta + 2\pi) = S\Phi(r,\theta)S^{-1}. \quad (2.15)$$

Since G is a symmetry group of $N = 4$ super Yang-Mills theory, it makes sense to formulate $N = 4$ super Yang-Mills theory for fields that have this sort of behaviour, near a codimension two surface D in spacetime. If we impose this condition, then along D, we should divide only by gauge transformations that commute with S. This recipe gives a surface operator that makes sense for any $S \in G$. It varies smoothly as long as the centralizer G^S of S in G does not change. To get a rigid surface operator, we must pick S to be rigid, meaning that G^S jumps if S is changed at all.

As in eqn. (2.5), we considered a gauge singularity of the form $A = \alpha d\theta$. One quantizes $N = 4$ super Yang-Mills theory for fields with this type of singularity, dividing by gauge transformations that at $z = 0$ are valued in G^α, the centralizer of α in G. Let us call this type of surface operator a generic one. A generic surface operator behaves well as α is varied as long as the centralizer of α is the same as the centralizer of the monodromy $S = \exp(-2\pi\alpha)$. We are precisely in the situation in which this is not the case, for if S is strongly rigid (and noncentral) then the centralizer of S is strictly larger than the centralizer of any $\alpha \in g$ such that $S = \exp(-2\pi\alpha)$.

To being with, any element $V \in G_C$ can be written as $V = SU$, where S is semisimple, U is unipotent, and S commutes with U. Moreover, let G_C^S be the centralizer of S in G_C, so $U \in G_C^S$. Then the condition for $V = SU$ to be rigid (or strongly rigid) in G_C is that S must be rigid (or strongly rigid) in G_C and U must be rigid in G_C^S. To construct a surface operator with monodromy $V = SU$, we combine the two constructions as follows. First we require that near $r = 0$, all fields of $N = 4$ super Yang-Mills theory obey $\Phi(r,\theta + 2\pi) = S\Phi(r,\theta)S^{-1}$, as in (2.15). Second, we also pick a homomorphism $\rho : su(2) \to g^S$ (here g^S is the Lie algebra of G^S) and we require that the fields have a singularity near $r = 0$ that is given by the solution (2.7) of Nahm's equations:

$$A = \frac{t_1 d\theta}{\ln r} + \ldots, \quad \phi = \frac{t_2 dr}{r \ln r} - \frac{t_3 d\theta}{\ln r} + \ldots, \quad (2.16)$$

where the ellipses denote terms that are less singular at $r = 0$. Because ρ commutes with S, this condition on the fields is compatible with (2.15). The combined condition defines a surface operator with the monodromy

$$V = SU\ . \quad (2.17)$$

There is no need here for V to be rigid. For every conjugacy class G_C, a construction along these lines gives a surface operator of monodromy V.

Now we describe rigid nilpotent orbits in exceptional cases. In such cases, the appropriate language to classify nilpotent orbits is based on Bala-Carter theory. According to Bala and Carter, nilpotent orbits in g_C are in one-to-one correspondence with pairs (l, p_l), where $l \subset g$ is a Levi subalgebra, and p_l is a distinguished parabolic subalgebra of the semisimple algebra $[l, l]$. Such pairs can be conveniently labelled as $X_N(a_i)$ where X_N is the Cartan type of the semisimple part of l and i is the number of simple roots in any Levi subalgebra of p_l. If $i = 0$ one simply writes X_N, and if a simple component of a Levi subalgebra l involves short roots (when g has two root lengths) then one labels its Cartan type with a tilde. Using this notation, below we list rigid nilpotent orbits in G_2:

orbit c	$\dim(c)$	$\pi_1(c)$
A_1	6	1
$\tilde{A}_1$	8	1

We note that 6 = 5 + 1 and that 8 = 3 + 5, with 1, 3, 5 and 8 that are Fibonacci's numbers. Furthermore, 8 is the number related to the physical vibrations of the superstrings by the following Ramanujan function:

$$8 = \frac{1}{3} \frac{4\left[anti\log \frac{\int_0^\infty \frac{\cos \pi t x w'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4} w'} \phi_{w'}(itw')} \right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10 + 11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10 + 7\sqrt{2}}{4}\right)} \right]} . \quad (2.18)$$

These are the only nilpotent orbits in G_2 which are not special.

We omit the trivial orbit, and in the last column we also list the G_{sc}-equivariant fundamental group of c (defined as $\pi_1(c) = G_{sc}(c)/G_{sc}(c)^o$, where $G_{sc}(c)$ is the centralizer of c in the simply-connected form of G). The G_{ad}-equivariant fundamental group, usually denoted $A(c)$, is the same as $\pi_1(c)$ in types G_2, F_4 and E_8. In the following table we list rigid nilpotent orbits in F_4:

orbit c	$\dim(c)$	$\pi_1(c)$
A_1	16	1
$\tilde{A}_1$	22	S_2

$A_1 + \tilde{A}_1$	28	1
$A_2 + \tilde{A}_1$	34	1
$\tilde{A}_2 + A_1$	36	1

We have that: 16 = 24 – 8 = 3 + 5 + 8; 22 = 21 + 1 = 13 + 8 + 1; 28 = 13 + 8 + 5 + 2;
34 = 13 + 21 = 5 + 8 + 21 = 24 + 8 + 2; 36 = 34 + 2 = 24 + 12.
We note that 1, 2, 3, 5, 8, 13, 21 and 34 are Fibonacci's numbers. With regard the numbers 8 and 24 (and 12 = 24 / 2) they are related to the physical vibrations of the superstrings and of the bosonic strings by the eq. (2.18) and by the following Ramanujan function:

$$24 = \frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (2.19)$$

All of these orbits, except for $\tilde{A}_1$ and $A_1 + \tilde{A}_1$, are not special. In type E_6, rigid nilpotent orbits are the following:

orbit c	$\dim(c)$	$\pi_1(c)$
A_1	22	1
$3A_1$	40	1
$2A_2 + A_1$	54	Z_3

Here, we have that: 40 = 16 + 24 = 34 + 5 + 1 = 21 + 8 + 5 + 6; 54 = 34 + 13 + 5 + 2 = 48 + 6.
Also in this case we have that 1, 2, 5, 8, 13, 21 and 34 are Fibonacci's numbers and that the numbers 8, 16, 24 and 48 are related to the physical vibrations of the superstrings (8) and of the bosonic strings (24) and hence to the eqs. (2.18) and (2.19).
In type E_7, rigid nilpotent orbits are the following:

orbit c	$\dim(c)$	$\pi_1(c)$
A_1	34	1
$2A_1$	52	1
$(3A_1)'$	64	1
$4A_1$	70	1
$A_2 + 2A_1$	82	1
$2A_2 + A_1$	90	1
$(A_3 + A_1)'$	92	1

Here, we have that: 52 = 34 + 13 + 5 = 12 + 16 + 24; 64 = 48 + 16 = 55 + 8 + 1;
70 = 55 + 13 + 2 = 48 + 13 + 8 + 1; 82 = 55 + 21 + 5 + 1 = 48 + 21 + 8 + 5;
90 = 55 + 34 + 1 = 48 + 24 + 13 + 5; 92 = 55 + 34 + 3 = 48 + 24 + 13 + 5 + 2.

Also in this case we have that 1, 2, 5, 8, 13, 21, 34 and 55 are Fibonacci's numbers and that the numbers 8, 16, 24 and 48 are related to the physical vibrations of the superstrings (8) and of the bosonic strings (24 and 12 = 24 / 2) and hence to the eqs. (2.18) and (2.19).
All of these orbits have $A(c)=1$. Among these, the orbits $A_1, 2A_1,$ and A_2+2A_1 are special. Finally, in the following table we list rigid nilpotent orbits in E_8:

orbit c	$\dim(c)$	$\pi_1(c)$
A_1	58	1
$2A_1$	92	1
$3A_1$	112	1
$4A_1$	128	1
A_2+A_1	136	S_2
A_2+2A_1	146	1
A_2+3A_1	154	1
$2A_2+A_1$	162	1
A_3+A_1	164	1
$2A_2+2A_1$	168	1
A_3+2A_1	172	1
$D_4(a_1)+A_1$	176	S_3
$A_3+A_2+A_1$	182	1
$2A_3$	188	1
A_4+A_3	200	1
A_5+A_1	202	1
$D_5(a_1)+A_2$	202	1

The only special orbits in this list are $A_1, 2A_1, A_2+A_1, A_2+2A_1, D_4(a_1)+A_1$.
Here, we have that: 58 = 21+ 34 + 3 = 48 + 8 + 2 = 48 + 5 + 3 + 2;
112 = 144(12×12) – 21 – 8 – 3; 128 = 144 – 16 = 144 – 8 – 5 – 3; 136 = 144 – 8;
146 = 144 + 2; 154 = 144 + 5 + 3 + 2; 162 = 144 + 13 + 5; 164 = 144 + 13 + 5 + 2;
168 = 144 + 21 + 3 = 144 + 24; 172 = 144 + 21 + 5 + 2 = 144 + 24 + 3 + 1;
176 = 144 + 21 + 8 + 3 = 144 + 24 + 8; 182 = 144 + 34 + 3 + 1 = 144 + 24 + 8 + 5 + 1;
188 = 144 + 34 + 8 + 2 = 144 + 24 + 16 + 3 + 1; 200 = 144 + 48 + 8 = 144 + 34 + 21 + 1;
202 = 144 + 48 + 8 + 2 = 144 + 34 + 21 + 3.
Also in this case we have that 1, 2, 5, 8, 13, 21, 34 and 144 (89 + 55) are Fibonacci's numbers and that the numbers 8, 16, 24, 48 and 144 (144 = 12×12) are related to the physical vibrations of the superstrings (8) and of the bosonic strings (24) (12 = 24 / 2) and hence to the eqs. (2.18) and (2.19).
Furthermore, related to the numbers of this list, we have the following forms:

6n – 2 for n = 3, 4, 5, 6, 7, 9, 10, 11, 12, 14, 19 and 23, we obtain the numbers: 16, 22, 28, 34, 40, 52, 58, 64, 70, 82, 112, 136.

6n for n = 1, 6, 9 and 15, we obtain the numbers: 6, 36, 54, 90.

6n + 2 for n = 1, 15, 21 and 24, we obtain the numbers: 8, 92, 128, 146.

8n – 2 for n = 1, 3, 7, 9 and 23, we obtain the numbers: 6, 22, 54, 70, 182.

8n for n = 1, 2, 5, 8, 14, 16, 17, 21, 22 and 25, we obtain the numbers: 8, 16, 40, 64, 112, 128, 136, 168, 176, 200.

8n + 2 for n = 4, 7, 10, 11, 18, 19, 20 and 25, we obtain the numbers: 34, 58, 82, 90, 146, 154, 162, 202.

We note that for 6n – 2 we have 12 numbers and for 8n we have 10 numbers. The form 6n – 2, is related to the form $6n \pm 1$ that is connected to the generation of the prime numbers. Furthermore, 6 is the number of dimensions compactified concerning the superstring theory. The form 8n, is related to the Fibonacci's number 8 and to the physical vibrations of a superstring, hence to the Ramanujan modular equation (2.18).
With regard the Lucas's series, we have obtained the following interesting connections, with the forms 6n, 8n, 6n – 2 and $8n \pm 2$. Indeed, we have:

6n for n = 1:

$$6 \cdot 1 = 6;$$

8n for n = 1, 2:

$$8 \cdot 1 = 8, \quad 8 \cdot 2 = 16;$$

6n – 2 for n = 3, 4, 7 and 11:

$$6 \cdot 3 - 2 = 16, \quad 6 \cdot 4 - 2 = 22, \quad 6 \cdot 7 - 2 = 40, \quad 6 \cdot 11 - 2 = 64;$$

$8n \pm 2$ for n = 3, 4, 7, 11 and 18:

$$8 \cdot 3 - 2 = 22, \quad 8 \cdot 4 + 2 = 34, \quad 8 \cdot 7 - 2 = 54, \quad 8 \cdot 11 + 2 = 90, \quad 8 \cdot 18 + 2 = 146.$$

We note that 2, 1, 3, 4, 7, 11 and 18 are Lucas's numbers.
In conclusion of this chapter, we observe also that the numbers 6, 8, 16, 22, 28, 36, 40, 52, 54, 58, 64, 70, 82 and 90 are values of the Eulero's phi function $\varphi(n)$. We note that a Dirichlet series that gives the $\varphi(n)$ is:

$$\sum_{n=0}^{\infty} \frac{\varphi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)}, \quad (2.20)$$

where ζ is the Riemann zeta function.

3. On some equations concerning the study of the Riemann zeta function and the Selberg trace formula. [5]

Let k denote the field of rational numbers. For every place v, we denote by k_v, O_v, and P_v the completion of k at v, the maximal compact subring of k_v, and the unique maximal ideal of O_v, respectively. The **adele group** A of k is the restricted direct product of the additive groups k_v relative to subgroups O_v, and is denoted by A. For every place v of k we denote by $| \, |_v$ the valuation of k normalized so that $| \, |_v$ is the ordinary absolute value if v is real, and $|\pi_v|_v = 1/p$ if O_v / P_v contains p elements where $P_v = \pi_v O_v$. In this chapter, v and p always correspond to each

other this way. The **idele group** J of k is the restricted direct product of the multiplicative groups k_v^* relative to subgroups O_v^* of units of k_v.
Let J^1 be the set of **ideles** $\alpha = (\alpha_v)$ such that $\prod |\alpha_v|_v = 1$. We denote by C for the **idele class group** J/k^*. We define a map $x \to \lambda_v(x)$ of k_v into the set of reals modulo 1. Then

$$\psi_v : x \to e^{2\pi i \lambda_v(x)} \quad (3.1)$$

is a character on the additive group k_v. It is trivial on O_v, and is nontrivial on $\pi_v^{-1} O_v$ for $v \neq \infty$.
Let G be a locally compact abelian group with a nontrivial proper continuous homomorphism

$$G \to R_+^*, \quad g \to |g| \quad (3.2)$$

whose range is cocompact in R_+^*. There exists a unique Haar measure $d^* g$ on G such that

$$\int_{|g| \in [1,\Lambda]} d^* g \approx \log \Lambda \quad (3.3)$$

when $\Lambda \to \infty$. Let $G_0 = \{ g \in G : |g| = 1 \}$. We identify G/G_0 with the range N of the module. Choose a measure $d^* n$ on N such that (3.3) holds for the measure $d^* g$ given by

$$\int_G f(g) d^* g = \int_N \left(\int_{G_0} f(n g_0) d g_0 \right) d^* n , \quad (3.4)$$

where the Haar measure dg_0 is normalized so that

$$\int_{G_0} d g_0 = 1 \quad (3.5)$$

In particular, for $N = R_+^*$ the unique Haar measure on G satisfying (3.3) is

$$d^* g = d^* n d g_0 \quad (3.6) \qquad \text{with } d^* n = \frac{dn}{n} .$$

If $N = q^Z$, the unique Haar measure on G satisfying (3.3) is given by

$$\int_G f(g) d^* g = \log q \sum_{n \in Z} \int_{G_0} f(q^n g_0) d g_0 . \quad (3.7)$$

Let $h \in C_0^\infty(0,\infty)$ be a smooth complex-valued function with compact support in $(0,\infty)$. Then

$$\sum_\rho \tilde{h}(\rho) = \int_0^\infty h(x) dx + \int_0^\infty h^*(x) dx - \sum_{n=1}^\infty \Lambda(n) \{ h(n) + h^*(n) \} - (\log \pi + \gamma) h(1) -$$
$$- \int_1^\infty \left\{ h(x) + h^*(x) - \frac{2}{x^2} h(1) \right\} \frac{x dx}{x^2 - 1} , \quad (3.8)$$

where the sum on ρ ranges over all complex zeros of $\zeta(s)$ and where γ is Euler's constant.
Let g_0 be a real-valued function in $C_0^\infty(0,\infty)$. We define

$$h_0(x)=\int_0^\infty g_0(xy)g_0(y)dy\,. \quad (3.9)$$

Then

$$\tilde{h}_0(s)=\tilde{g}_0(s)\tilde{g}_0(1-s)\,. \quad (3.10)$$

Since g_0 has a compact support in $(0,\infty)$, there is a number μ satisfying $0<\mu<1$ such that the support of g_0 is contained in $\left[\sqrt{\mu},\mu^{-1/2}\right]$. It follows that

$$h_0(x)=0 \quad (3.11)$$

for all $x\notin\left[\mu,\mu^{-1}\right]$.

Theorem 1.

Let h_0 be given as in (3.9). Then

$$\sum_\rho \tilde{h}_0(\rho)=\tilde{h}_0(0)+\tilde{h}_0(1)-\sum_v \int_{k_v^*}' \frac{h_0\left(|u|_v^{-1}\right)}{|1-u|_v}d^*u\,, \quad (3.12)$$

where the sum on ρ is over all nontrivial zeros of $\zeta(s)$, the sum on v is over all places of k, and the principal value $\int'$ is uniquely determined by the unique distribution on k_v^* which agrees with $\frac{d^*u_v}{|1-u|_v}$ for $u\neq 1$ and whose Fourier transform vanishes at 1.

If v is a finite place of k, then

$$\int_{k_v^*}' \frac{h_0\left(|u|^{-1}\right)}{|1-u|_v}d^*u=-\int_{k_v^*}\hat{g}(u)\log|u|_v\,du \quad (3.13)$$

where

$$g(u)=\frac{h_0\left(|u+1|^{-1}\right)}{|u+1|_v}\,.$$

If v is the infinite place of k, then

$$\int_{R^*}' \frac{h_0\left(|u|^{-1}\right)}{|1-u|}d^*u=(\gamma+\log(2\pi))h_0(1)+\lim_{\varepsilon\to 0}\left(\int_{|1-u|\geq\varepsilon}\frac{h_0\left(|u|^{-1}\right)}{|1-u|}d^*u+h_0(1)\log\varepsilon\right). \quad (3.14)$$

(Proof of Theorem 1). We have the following explicit formula:

$$\sum_{\rho}\tilde{h}_0(\rho)=\int_0^{\infty}h_0(x)\frac{dx}{x}+\int_0^{\infty}h_0(x)dx-\sum_{m\leq 1/\mu}\Lambda(m)\left[\frac{1}{m}h_0\left(\frac{1}{m}\right)+h_o(m)\right]-(\gamma+\log\pi)h_0(1)$$
$$-\int_1^{\infty}\left[h_0(x)+\frac{1}{x}h_0\left(\frac{1}{x}\right)-\frac{2}{x^2}h_0(1)\right]\frac{xdx}{x^2-1}\quad (3.15)$$

where the sum on ρ is over all complex zeros of $\zeta(s)$. We assume that μ is not a rational number. If v is a finite place, then

$$\int'_{k_v^*}\frac{h_0\left(|u|_v^{-1}\right)}{|1-u|_v}d^*u=h_0(1)\int'_{k_v^*}\frac{1_{O_v^*}}{|1-u|_v}d^*u+\sum_{k=1}^{\infty}\frac{h_0\left(p^{-k}\right)}{p^k}\int_{|u|_v=p^k}d^*u+\sum_{k=1}^{\infty}h_o\left(p^k\right)\int_{|u|_v=p^{-k}}d^*u\,.\quad (3.16)$$

Let $A=\{u\in k_v:|u+1|_v=1\}$, and put $1_A(x)=1$, $x\in A$; $1_A(x)=0$, $x\notin A$.
Then

$$\hat{1}_A(x)=\int_A\psi_v(-xu)du=\psi_v(x)\hat{1}_{O_v^*}(x).\quad (3.17)$$

Since

$$\hat{1}_{O_v^*}=1_{O_v}-\frac{1}{p}1_{\pi_v^{-1}O_v},\quad (3.18)$$

we have

$$\hat{1}_A(x)=\psi_v(x)\left(1_{O_v}(x)-\frac{1}{p}1_{\pi_v^{-1}O_v}(x)\right).\quad (3.19)$$

Thence, the eq. (3.17) can be written also:

$$\hat{1}_A(x)=\int_A\psi_v(-xu)du=\psi_v(x)\left(1_{O_v}(x)-\frac{1}{p}1_{\pi_v^{-1}O_v}(x)\right).\quad (3.19b)$$

By definition of the principal value integral $\int'$, we have that:

$$\int'_{k_v^*}\frac{1_{O_v^*}}{|1-u|_v}d^*u=-\int_{k_v}\hat{1}_A(u)\log|u|_v du=-\int_{O_v}\psi_v(u)\log|u|_v du+\frac{1}{p}\int_{\pi_v^{-1}O_v}\psi_v(u)\log|u|_v du=$$
$$=-\frac{\log p}{p}+\left(\frac{1}{p}-1\right)\int_{O_v}\log|u|_v du=-\frac{\log p}{p}+\left(\frac{1}{p}-1\right)^2\log p\sum_{n=0}^{\infty}np^{-n}=0\,.\quad (3.20)$$

Since p is a rational prime for each finite place v of k, by the normalization (3.7) for the Haar measure on k_v^*

$$\int_{|u|_v=p^k}d^*u=\log p=\Lambda\left(p^{|k|}\right)\quad (3.21)$$

for all nonzero integers k. Therefore, by (3.11) we have

$$\sum_{m\le 1/\mu}\Lambda(m)\left[\frac{1}{m}h_0\left(\frac{1}{m}\right)+h_0(m)\right]=\sum_{v\neq\infty}\int_{k_v^*}^{'}\frac{h_0\left(|u|_v^{-1}\right)}{|1-u|_v}d^*u\,. \quad (3.22)$$

We assume that v is the infinite place of k. By definition of the principal value integral $\int^{'}$,

$$\int_{R^*}^{'}\frac{h_0\left(|u|^{-1}\right)}{|1-u|}d^*u=(\gamma+\log(2\pi))h_0(1)+\lim_{\delta\to 0}\left(\int_{|1-u|\ge\delta}\frac{h_0\left(|u|^{-1}\right)}{|1-u|}d^*u+h_0(1)\log\delta\right). \quad (3.23)$$

We have

$$\lim_{\delta\to 0}\left(\int_{|1-u|\ge\delta}\frac{h_0\left(|u|^{-1}\right)}{|1-u|}d^*u+h_0(1)\log\delta\right)=$$
$$=\lim_{\delta\to 0}\left\{\int_1^{\infty}\left(\frac{1}{u}h_0\left(\frac{1}{u}\right)+h_0(u)u^{-\delta}\right)\frac{(u+1)^{\delta-1}+(u-1)^{\delta-1}}{2}du-\frac{1}{\delta}h_0(1)\right\}. \quad (3.24)$$

Since

$$\lim_{\delta\to 0}\int_1^{\mu^{-1}}h_0(u)\left(u^{-\delta}-1\right)\frac{(u+1)^{\delta-1}+(u-1)^{\delta-1}}{2}du=\lim_{\delta\to 0}\int_1^{\mu^{-1}}h_0(u)\left(u^{-\delta}-1\right)\frac{(u-1)^{\delta-1}}{2}du=0 \quad (3.25)$$

and

$$\lim_{\delta\to 0}\left(2\int_1^{\infty}\frac{1}{u^2}\frac{(u+1)^{\delta-1}+(u-1)^{\delta-1}}{2}du-\frac{1}{\delta}\right)=\int_1^{\infty}\frac{1}{u^2(u+1)}du+\lim_{\delta\to 0}\left(\Gamma(\delta)\Gamma(2-\delta)-\frac{1}{\delta}\right)=-\log 2, \quad (3.26)$$

we have

$$\lim_{\delta\to 0}\left(\int_{|1-u|\ge\delta}\frac{h_0\left(|u|^{-1}\right)}{|1-u|}d^*u+h_0(1)\log\delta\right)=\int_1^{\infty}\left[\frac{1}{u}h_0\left(\frac{1}{u}\right)+h_0(u)-\frac{2h_0(1)}{u^2}\right]\frac{u}{u^2-1}du-h_0(1)\log 2\,. \quad (3.27)$$

Therefore,

$$\int_{R^*}^{'}\frac{h_0\left(|u|^{-1}\right)}{|1-u|}d^*u=(\gamma+\log\pi)h_0(1)+\int_1^{\infty}\left[\frac{1}{u}h_0\left(\frac{1}{u}\right)+h_0(u)-\frac{2}{u^2}h_0(1)\right]\frac{udu}{u^2-1}\,. \quad (3.28)$$

The stated identity then follows from (3.15) and (3.22).

Theorem 2.

Let h be a smooth even function of compact support in $L^2(R^*)$, and let $g(\lambda)=h(\lambda^{-1})\lambda^{-1}$. Then

$$\int_R H_\infty g(u)\cos(2\pi u)\log|u|du = -h(1)\log 2\pi - \gamma h(1)$$
$$-\lim_{\varepsilon\to 0}\left(\int_{|\lambda-1|\geq\varepsilon}\frac{h(\lambda^{-1})}{\sqrt{\lambda}}\frac{\max\{\sqrt{\lambda},1/\sqrt{\lambda}\}}{|\lambda^2-1|}d\lambda + h(1)\log\varepsilon\right). \quad (3.29)$$

Corollary 1.

Let $h(u)=h_0(|u|)$. *Then*

$$\lim_{\varepsilon\to 0}\left(\int_{|\lambda-1|\geq\varepsilon}\frac{h(\lambda^{-1})}{\sqrt{\lambda}}\frac{\max\{\sqrt{\lambda},1/\sqrt{\lambda}\}}{|\lambda^2-1|}d\lambda + h(1)\log\varepsilon\right)=$$
$$= -h_0(1)\log 2 + \int_1^\infty\left(h_0(x)+h_0^*(x)-\frac{2}{x^2}h_0(1)\right)\frac{x}{x^2-1}dx. \quad (3.30)$$

Theorem 3.

Let $h(u)=h_0(|u|)$ *and* $g(\lambda)=h(\lambda^{-1})|\lambda|^{-1}$. *Then*

$$\int_{k_v} H_v g(u)\psi_v(u)\log|u|du = -\int'_{k_v^*}\frac{h_0(|u|^{-1})}{|1-u|_v}d^*u, \quad (3.31)$$

where the principal value $\int'$ *is uniquely determined by the unique distribution on* k_v^* *which agrees with* $\frac{d^*u_v}{|1-u|_v}$ *for* $u\neq 1$ *and whose Fourier transform vanishes at 1*.

Lemma 1.

Let

$$I = R^+ \times \prod_{v\neq\infty} O_v^*. \quad (3.32)$$

Then

$$J = \bigcup_{\xi\in k^*}\xi I, \quad (3.33)$$

a disjoint union.

Let

$$\Psi(x)=\prod_v \psi_v(x_v) \quad (3.34)$$

for $x\in \mathrm{A}$, where ψ_v is given in (3.1). For $f=\prod_v f_v \in L^2(\mathrm{A})$, we define

$$Hf(\beta)=\int_A f(\alpha)\Psi(-\alpha\beta)d\alpha=\prod_v\int_{k_v} f_v(\alpha_v)\psi_v(-\alpha_v\beta_v)d\alpha_v \quad (3.35)$$

for $\beta=(\beta_v)\in \mathrm{A}$; that is

$$Hf(\beta)=\prod_v H_v f_v(\beta_v). \quad (3.36)$$

Lemma 2.

Let $f=\prod_v f_v$ *be a continuous function in* $L^1(\mathrm{A})$ *satisfying* $Hf\in L^1(\mathrm{A})$. *Then the inversion formula*

$$f(-\alpha)=HHf(\alpha) \quad (3.37)$$

holds for all $\alpha\in \mathrm{A}$, *and*

$$\|Hf\|_{L^2(\mathrm{A})}=\|f\|_{L^2(\mathrm{A})}. \quad (3.38)$$

Lemma 3.

If $f(x)$ *satisfies the conditions*:

1) $f(x)$ *is continuous in* $L^1(\mathrm{A})$,
2) $\sum_{\xi\in k} f(\alpha(x+\xi))$ *converges for all ideles* α *and adeles* x, *uniformly for* $x\in D$ *where* $D=[0,1)\times\prod_{v\neq\infty}O_v$, *and*
3) $\sum_{\xi\in k}|Hf(\alpha\xi)|$ *converges for all ideles* α, *then*

$$\sum_{\xi\in k} f(\alpha\xi)=\frac{1}{|\alpha|}\sum_{\xi\in k}Hf(\xi/\alpha). \quad (3.39)$$

The Schwartz space $S(R)$ is the space of all smooth functions f, all of whose derivatives are of rapid decay; that is

$$\frac{\partial^k f}{\partial x^k}(x)=O\left((1+|x|)^{-N}\right) \quad (3.40)$$

for all integers $k\geq 0$ and $N>0$. Let $S(\mathrm{A})$ be the Schwartz-Bruhat space on A, whose functions are finite linear combinations of functions of the form

$$f(\alpha)=\prod_v f_v(\alpha_v) \quad (3.41)$$

where

(1) f_v is in the Schwartz space $S(R)$ if v is the infinite place of k;
(2) f_v belongs to $S(k_v)$, the space of locally constant and compactly supported functions on k_v if v is finite; and
(3) $f_v=1_{O_v}$, the characteristic function of O_v, for almost all v.

Let $d^{\times}t$ be the multiplicative measure on R^* given by

$$d^{\times}t = \frac{dt}{|t|}. \quad (3.42)$$

We denote by $d^{\times}\alpha_v$ the multiplicative measure on k_v^* given by

$$d^{\times}\alpha_v = \left(1 - p^{-1}\right)^{-1} \frac{d\alpha_v}{|\alpha_v|_v}, \quad (3.43)$$

where $p^{-1} = |\pi_v|_v$. We choose the Haar measure

$$d^{\times}\alpha = \prod_v d^{\times}\alpha_v \quad (3.44)$$

on J. Then, $d^{\times}\alpha$ is also a Haar measure on C satisfying (3.3).

Lemma 4.

A function θ satisfying $\theta(0) = 0$, $H\theta(0) = 1$, $E(H\theta) \in L_0^2(C)^{\perp}$, and $E(\theta) \notin L^2(C)$ exists such that

$$\langle E(f), E(H\theta)\rangle_{L^2(C)} = f(0)\|E(H\theta)\|_{L^2(C)}^2 \quad (3.45)$$

for all $f \in S(\mathrm{A})$ with $Hf(0) = 0$.

Now, we assume that θ is given as in Lemma 4. For any element $f \in S(\mathrm{A})$, let $f_0 = f - Hf(0)\theta - f(0)H\theta$. Then $f_0 \in L_0^2(X)$ and

$$f = f_0 + Hf(0)\theta + f(0)H\theta. \quad (3.46)$$

For any $f \in S(\mathrm{A})$, we define

$$\|f\|_{L^2(X)}^2 = \|f_0\|_{L_0^2(X)}^2 + \left(|f(0)|^2 + |Hf(0)|^2\right)\|E(H\theta)\|_{L^2(C)}^2. \quad (3.47)$$

Let $L^2(X)$ be the Hilbert space that is the completion of the Schwartz-Bruhat space $S(\mathrm{A})$ for the norm given by (3.47). It follows that $L_0^2(X)$ is a subspace of $L^2(X)$, and that the orthogonal complement $L_0^2(X)^{\perp}$ of $L_0^2(X)$ in $L^2(X)$ is the subspace

$$\{a\theta + bH\theta : a, b \in C\}. \quad (3.48)$$

We define $\overline{L}^2(C)$ to be the Hilbert space that is the completion of $E(S(\mathrm{A}))$ for the norm

$$\|E(f)\|_{\overline{L}^2(C)} = \|f\|_{L^2(X)} \quad (3.49)$$

for $f \in S(\mathrm{A})$. By the following **corollary**:

If $f \in S(\mathrm{A})$ *and* $Hf(0)=0$*, then* $\|E(f)\|_{L^2(C)} = \|f\|_{L^2(X)}$, we have that $L^2(C)$ is a codimension one subspace of $\overline{L}^2(C)$. The orthogonal complement of $L^2(C)$ in $\overline{L}^2(C)$ is the subspace

$$L^2(C)^{\perp} = \{aE(\theta) : a \in C\}. \quad (3.50)$$

We define $h(u) = h_0(|u|)$ for all $u = (u_v) \in J$. There exists a real-valued function $g \in S(J)$ such that

$$h(\lambda) = \sum_{\xi \in k^*} g(\xi\lambda). \quad (3.51)$$

An operator $U(h)$ acting on the space $L^2(X)$ is defined by

$$U(h)f(x) = \int_C h(\lambda^{-1}) f(\lambda x) d^{\times}\lambda \quad (3.52)$$

for $f \in L^2(X)$, where $d^{\times}\lambda$ is given in (3.44). If $f(-\alpha) = -f(\alpha)$ for all $\alpha \in \mathrm{A}$, then $U(h)f = 0$.

Theorem 4.

E *extends to a surjective isometry from* $L^2(X)$ *to* $\overline{L}^2(C)$.

Let S be the subspace of $L^2(X)$ that is spanned by all functions $f \in S(\mathrm{A})$ satisfying $E(f) \in L^2(C)$. The left regular representation V of C on $L^2(C)$ is given by

$$(V(g)f)(\alpha) = f(g^{-1}\alpha) \quad (3.53)$$

for $g, \alpha \in C$ and $f \in L^2(C)$. Let $C^1 = J^1 / k^*$. Since the restriction of V to C^1 is unitary, we can decompose $L^2(C)$ as a direct sum of subspaces

$$L^2_{\chi}(C) = \{f \in L^2(C) : f(g^{-1}\alpha) = \chi(g) f(\alpha)\} \quad (3.54) \quad \text{for all } g \in C^1 \text{ and } \alpha \in C$$

for all characters χ of C^1. These subspaces correspond to projections

$$P_{\chi} = \int_{C^1} \overline{\chi}(g) V(g) d^{\times}g, \quad (3.55)$$

where $d^{\times}g$ is the restriction to C^1 of the Haar measure on C.
Let φ be an element in $L^2_{\chi}(C)$. We can write

$$\varphi(x) = \overline{\chi}(x/|x|)\varphi(|x|), \quad (3.56)$$

where $1/|x|$ is meant to be the idele $(1/|x|, 1, 1, \ldots, 1)$. If φ is orthogonal to the range of the subspace S under E, then

$$\int_C E(f)(x)\overline{\chi}\left(\frac{x}{|x|}\right)\varphi(|x|)d^\times x = 0 \quad (3.57)$$

for all $f \in S(\mathrm{A})$ satisfying $Hf(0)=0$. Let

$$f_n(t) = \frac{\sin 2n\pi t}{\pi t}. \quad (3.58)$$

Then

$$\hat{f}_n(t) = \int_{-\infty}^{\infty} f_n(x)e^{-2\pi itx}dx = \begin{cases} 1 t \in [-n,n] \\ 0 otherwise \end{cases}. \quad (3.59)$$

Let

$$\varphi_n(|x|) = \int_{-\infty}^{\infty} f_n(u)\varphi(|x|e^u)du. \quad (3.60)$$

We denote $\phi(u)=\varphi(e^u)$. Since

$$\int_{-\infty}^{\infty} \varphi(|x|e^u)e^{-2\pi iuy}du = |x|^{i2\pi y}\hat{\phi}(y), \quad (3.61)$$

by the Plancherel formula

$$\varphi_n(|x|) = \int_{-n}^{n}\hat{\phi}(y)|x|^{2\pi iy}dy. \quad (3.62)$$

Since $\varphi \in L_\chi^2(C)$, $\phi(u) \in L^2(R)$. Hence, $\hat{\phi}(y) \in L^2(R)$. It follows that

$$\varphi(|x|) - \varphi_n(|x|) = \int_{|y|>n}\hat{\phi}(y)e^{-2\pi ity}dy \quad (3.63)$$

with $|x| = e^{-t}$. Let $|E(f)(\alpha)| << |\alpha|^{-m}$ for any positive integer m as $|\alpha| \to \infty$. By Lemma 1,

$$\left|\int_C E(f)(x)\overline{\chi}\left(\frac{x}{|x|}\right)[\varphi(|x|)-\varphi_n(|x|)]d^\times x\right|^2 \le \int_C |E(f)(x)|^2 d^\times x \int_{-\infty}^{\infty}\left|\int_{|y|>n}\hat{\phi}(y)e^{-2\pi ity}dy\right|^2 dt =$$
$$= \int_C |E(f)(x)|^2 d^\times x \int_{|y|>n}\left|\hat{\phi}(y)\right|^2 dy \to 0 \quad (3.64)$$

as $n \to \infty$, where $|x| = e^{-t}$. Therefore,

$$\int_C E(f)(x)\overline{\chi}\left(\frac{x}{|x|}\right)\varphi(|x|)d^\times x = \lim_{n\to\infty}\int_C E(f)(x)\overline{\chi}\left(\frac{x}{|x|}\right)\varphi_n(|x|)d^\times x. \quad (3.65)$$

Since $|E(f)(\alpha)| << |\alpha|^{-m}$ for any positive integer m as $|\alpha| \to \infty$, we can interchange the order of integration and obtain that

$$\int_C E(f)(x)\overline{\chi}\left(\frac{x}{|x|}\right)\varphi_n(|x|)d^\times x = \int_{-n}^{n}\hat{\phi}(y)\left(\int_C E(f)(x)\overline{\chi}\left(\frac{x}{|x|}\right)|x|^{2\pi i y}d^\times x\right)dy \quad (3.66)$$

for $n=1,2,....$ By (3.65) and (3.66), we obtain that

$$\int_C E(f)(x)\overline{\chi}\left(\frac{x}{|x|}\right)\varphi(|x|)d^\times x = \int_{-\infty}^{\infty}\hat{\phi}(t)dt\int_C E(f)(x)\overline{\chi}\left(\frac{x}{|x|}\right)|x|^{2\pi i t}d^\times x . \quad (3.67)$$

Let

$$f_0(x) = f_\infty(x_\infty)\chi_\infty\left(\frac{x_\infty}{|x|}\right)\left(\prod_{unramified\,\chi_v}1_{O_v}(x_v)\right)\left(\prod_{ramified\,\chi_v}\chi_v(x_v)1_{O_v^*}(x_v)\right) \quad (3.68)$$

with $f_\infty \in S(R^+)$. If χ_v are unramified for all finite places v, we choose f_0 so that $\int_{R^+} f_\infty(x)dx = 0$. Then $f_0 \in S(\mathrm{A})$ satisfying $Hf_0(0)=0$. Using

$$\int_C E(f_0)(x)\overline{\chi}\left(\frac{x}{|x|}\right)|x|^{2\pi i t}d^\times x = \int_C E(f_1)(x)\overline{\chi}(x)|x|^{2\pi i t}d^\times x \quad (3.69)$$

where $f_1(x) = \chi_\infty(|x|)f_0(x)$, we can write

$$\int_C E(f_0)(x)\overline{\chi}\left(\frac{x}{|x|}\right)|x|^{2\pi i t}d^\times x = L\left(\overline{\chi},\frac{1}{2}+2\pi i t\right)\int_0^\infty f_\infty(u)|u|^{-1/2+2\pi i t}du \quad (3.70)$$

where $L(\overline{\chi},1/2+2\pi i t)$ is the analytic continuation of

$$L(\overline{\chi},s) = \prod_{unramified\,v}\frac{1}{1-\overline{\chi}(\pi_v)p^{-s}} \quad (3.71)$$

for $\Re s > 1$. By (3.67) and (3.70), we obtain that

$$\int_C E(f_0)(x)\overline{\chi}\left(\frac{x}{|x|}\right)\varphi(|x|)d^\times x = \int_{-\infty}^{\infty}\hat{\phi}(t)L\left(\overline{\chi},\frac{1}{2}+2\pi i t\right)dt\int_0^\infty f_\infty(u)|u|^{-1/2+2\pi i t}du . \quad (3.72)$$

By (3.57) and (3.72), we have

$$\int_{-\infty}^{\infty}\hat{\phi}(t)L\left(\overline{\chi},\frac{1}{2}+2\pi i t\right)dt\int_0^\infty f_\infty(u)|u|^{-1/2+2\pi i t}du = 0 . \quad (3.73)$$

It follows that

$$\int_{-\infty}^{\infty}\hat{\phi}(t)L\left(\overline{\chi},\frac{1}{2}+2\pi i t\right)b(t)dt = 0 \quad (3.74)$$

for all $b(t)\in L^2(R)$, which satisfy $\int_R \hat{b}(u)e^{u/2}du = 0$ if χ_v is unramified for all $v \neq \infty$. Since $L\left(\chi, \frac{1}{2}+2\pi i t\right) = 0$ for at most a discrete set of real t, the identity (3.74) implies that

$$\hat{\phi}(t) = 0 \quad (3.75)$$

for almost all real t because we can choose b so that the integrand in (3.74) is non-negative. Since

$$\varphi(|x|) = \int_{-\infty}^{\infty} \hat{\phi}(y)|x|^{2\pi i y} dy, \quad (3.76)$$

we have $\varphi(|x|) = 0$ for all $x \in C$. By (3.56), $\varphi(x) = 0$ for all $x \in C$. Therefore, the orthogonal complement of the range of S under E in $L^2_\chi(C)$ contains no non-zero element. It follows that E is a surjective isometry from S to $L^2(C)$. By (3.50), E extends to a surjective isometry from $L^2(X)$ to $\overline{L}^2(C)$.

Let $h(\lambda)$ be given as in (3.51). An operator $V(h)$ acting on the space $\overline{L}^2(C)$ is defined by

$$V(h)F(x) = \int_C h(\lambda)|\lambda|^{1/2} F(\lambda^{-1}x) d^\times\lambda \quad (3.77)$$

for $F \in \overline{L}^2(C)$. The Haar measure $d^\times\lambda$ on C is given in (3.44). If $F(-x) = -F(x)$ for all $x \in C$, then $V(h)F = 0$.

Let S_Λ be the subspace of $\overline{L}^2(C)$ given by

$$S_\Lambda = \left\{ f \in \overline{L}^2(C) : f(\alpha) = 0 \right\} \quad (3.78) \quad \text{for all } \alpha \text{ with } |\alpha| > \Lambda .$$

The corresponding orthogonal projection is also denoted by S_Λ. We denote by $S_{\Lambda,0}$ the restriction of S_Λ to the subspace $L^2_0(C)$ and the corresponding orthogonal projection.

Theorem 5.

Let S_Λ and $V(h)$ be given as in (3.78) *and* (3.77), *respectively. Then $(S_\Lambda - S_{\Lambda,0})V(h)$ is of trace class, and its trace acting on the space $\overline{L}^2(C)$ is given by*

$$trace\ \left(\{S_\Lambda - S_{\Lambda,0}\}V(h)\right) = \tilde{h}_0(1) + \tilde{h}_0(0) + o(1) \quad (3.79)$$

where $o(1)$ tends to 0 as $\Lambda \to \infty$.

If T is a bounded linear operator of trace class on a Hilbert space H, then the trace of T is also given by

$$\text{trace } (T) = \sum_{n=1}^{\infty} \langle Tf_n, f_n \rangle_H \quad (3.80)$$

where $\{f_n\}$ is an orthonormal base of H.
Let P_Λ be the orthogonal projection of $L^2(X)$ onto the subspace

$$P_\Lambda = \{f \in L^2(X) : f(x) = 0\} \quad (3.81) \quad \text{for } |x| < \Lambda^{-1}.$$

Put

$$Z_\Lambda = H^t P_\Lambda H . \quad (3.82)$$

Theorem 6.

Let h, $V(h)$, S_Λ, *and* Z_Λ *be given as in* (3.51), (3.77), (3.78), *and* (3.82) *respectively. Then* $(EZ_\Lambda E^{-1} - S_\Lambda)V(h)$ *is of trace class, and its trace acting on the space* $\overline{L}^2(C)$ *is given by the formula*

$$\text{trace } \{(EZ_\Lambda E^{-1} - S_\Lambda)V(h)\} = -\sum_v \int_{k_v^*}' \frac{h_0(u^{-1})}{|1-u|_v} d^* u \quad (3.83)$$

where the principal value $\int'$ *is uniquely determined by the unique distribution on* k_v^* *which agrees with* $\dfrac{d^* u_v}{|1-u|_v}$ *for* $u \neq 1$ *and whose Fourier transform vanishes at 1.*

We have that $Z_\Lambda U(h) - S_\Lambda U(h)$ is of trace class on $\overline{L}^2(C)$. By (3.52), (3.82), and Lemma 2, we have that

$$Z_\Lambda U(h) f(x) = \int_C h(\lambda^{-1}) f(\lambda x) d^\times \lambda - \int_{\xi \in \mathrm{A}, |\xi| < \Lambda^{-1}} \Psi(\xi x) d\xi \int_{\mathrm{A}} \Psi(-\xi u) du \int_C h(\lambda^{-1}) f(\lambda u) d^\times \lambda \quad (3.84)$$

for $f \in S(\mathrm{A})$. Hence, for $x \in C$ we have

$$EZ_\Lambda E^{-1} V(h) F(x) = \int_C F(\lambda) \sqrt{|x/\lambda|} h(x/\lambda) d^\times \lambda - \int_{\xi \in \mathrm{A}, |\xi| < \Lambda^{-1}} \Psi(\xi x) d\xi \int_{\mathrm{A}} \Psi(-\xi u) du \int_C h(\lambda^{-1}) \sqrt{|x/\lambda u|} F(\lambda u) d^\times \lambda$$
(3.85)

for all $F = E(f)$ with $f \in S(\mathrm{A})$.
We extend h to a function on A by defining $h(\lambda) = 0$ for $\lambda \notin J$. Since $f \in S(\mathrm{A})$ and $h_0 \in C_0^\infty(0, \infty)$, we can change orders of integrations to obtain that

$$-\int_{\xi \in \mathrm{A}, |\xi| < \Lambda^{-1}} \Psi(\xi x) d\xi \int_{\mathrm{A}} \Psi(-\xi u) du \int_C h(\lambda^{-1}) \sqrt{|x/\lambda u|} F(\lambda u) d^\times \lambda =$$
$$= -\int_C F(\lambda) \sqrt{|x\lambda|} \left(\int_{\xi \in \mathrm{A}, |\xi| < \Lambda^{-1}} Hh(\lambda \xi) \Psi(\xi x) d\xi \right) d^\times \lambda . \quad (3.86)$$

By (3.85), (3.78), and (3.77) we have

$$\left(EZ_\Lambda E^{-1}-S_\Lambda\right)V(h)F(x)=\int_C F(\lambda)\left\{\sqrt{|x/\lambda|}h(x/\lambda)\ell_\Lambda(x)-\sqrt{|x\lambda|}\left(\int_{\xi\in\mathbb{A},|\xi|<\Lambda^{-1}}Hh(\lambda\xi)\Psi(\xi x)d\xi\right)\right\}d^\times\lambda \tag{3.87}$$

for all $F=E(f)$ with $f\in S(\mathbb{A})$, where $\ell_\Lambda(x)=1$ if $|x|>\Lambda$ and $\ell_\Lambda(x)=0$ if $|x|\le\Lambda$. Since such elements F are dense in $\overline{L}^2(C)$, (3.87) holds for all $F\in\overline{L}^2(C)$. It follows that the trace of $\left(EZ_\Lambda E^{-1}-S_\Lambda\right)V(h)$ acting on the space $\overline{L}^2(C)$ is given by

$$\begin{aligned}\operatorname{trace}\left\{\left(EZ_\Lambda E^{-1}-S_\Lambda\right)V(h)\right\}&=\int_C\left\{h(1)\ell_\Lambda(x)-\int_{\xi\in\mathbb{A},|\xi|<\Lambda^{-1}}|x|Hh(x\xi)\Psi(x\xi)d\xi\right\}d^\times x=\\&=\int_C\left\{\int_{|u|\ge|x|\Lambda^{-1}}Hh(u)\Psi(u)du-h(1)\tau_\Lambda(x)\right\}d^\times x. \qquad (3.88)\end{aligned}$$

Let $\delta<\Lambda$ be a small positive number. We write

$$\begin{aligned}\int_C\left\{\int_{|u|\ge|x|\Lambda^{-1}}Hh(u)\Psi(u)du-h(1)\tau_\Lambda(x)\right\}d^\times x&=\left(\int_{x\in C,|x|>\delta}+\int_{x\in C,|x|\le\delta}\right)\\&\left\{\int_{|u|\ge|x|\Lambda^{-1}}Hh(u)\Psi(u)du-h(1)\tau_\Lambda(x)\right\}d^\times x. \qquad (3.89)\end{aligned}$$

Since

$$\begin{aligned}&\int_{x\in C,|x|\le\delta}\left\{\int_{|u|\ge|x|\Lambda^{-1}}Hh(u)\Psi(u)du-h(1)\tau_\Lambda(x)\right\}d^\times x=-\int_{x\in C,|x|\le\delta}\left\{\int_{|u|<|x|\Lambda^{-1}}Hh(u)\Psi(u)du\right\}d^\times x=\\&=-\int_{|u|<\delta\Lambda^{-1}}Hh(u)\Psi(u)\log\frac{\delta}{|u|\Lambda}du \qquad (3.90)\end{aligned}$$

and

$$\begin{aligned}&\int_{x\in C,|x|>\delta}\left\{\int_{|u|\ge|x|\Lambda^{-1}}Hh(u)\Psi(u)du-h(1)\tau_\Lambda(x)\right\}d^\times x=\int_{|u|>\delta\Lambda^{-1}}Hh(u)\Psi(u)\log\frac{|u|\Lambda}{\delta}du-h(1)\log\frac{\Lambda}{\delta}=\\&=\int_{\mathbb{A}}Hh(u)\Psi(u)\log|u|du-\int_{|u|\le\delta\Lambda^{-1}}\Psi(u)Hh(u)\log\frac{|u|\Lambda}{\delta}du, \qquad (3.91)\end{aligned}$$

if we notice that $\log\frac{|u|\Lambda}{\delta}=0$ for $|u|=\delta\Lambda^{-1}$ then

$$\int_C\left\{\int_{|u|\ge|x|\Lambda^{-1}}Hh(u)\Psi(u)du-h(1)\tau_\Lambda(x)\right\}d^\times x=\int_{\mathbb{A}}Hh(u)\Psi(u)\log|u|du. \qquad (3.92)$$

By (3.88) and (3.92),

$$\operatorname{trace}\left\{\left(EZ_\Lambda E^{-1}-S_\Lambda\right)V(h)\right\}=\int_{\mathbb{A}}Hh(u)\Psi(u)\log|u|du. \qquad (3.93)$$

Let $g(\lambda)=h(\lambda^{-1})|\lambda|^{-1}$, and let

$$g_v(\lambda_v)=h((1,\ldots,1,\lambda_v^{-1},1,\ldots))|\lambda_v|_v^{-1}. \qquad (3.94)$$

By Fourier inversion formula, Theorem 2, and Theorem 3 we get

$$\int_{\mathrm{A}} Hg(u)\Psi(u)\log|u|du = \sum_{v\neq\infty}\int_{k_v} H_v g_v(u_v)\psi_v(u_v)\log|u_v|_v du_v + \int_R H_\infty g_\infty(u)\psi_\infty(u)\log|u|_\infty du =$$

$$= -h_0(1)\log 2\pi - \gamma h_0(1) - \sum_{v\neq\infty}\int_{k_v^*}' \frac{h_0(u^{-1})}{|1-u|_v} d^*u - \lim_{\varepsilon\to 0}\left(\int_{|\lambda-1|\geq\varepsilon} \frac{h_0(\lambda^{-1})}{\sqrt{\lambda}} \frac{\max\{\sqrt{\lambda},1/\sqrt{\lambda}\}}{|\lambda^2-1|} d\lambda + h_0(1)\log\varepsilon\right) \tag{3.95}$$

where the principal value $\int'$ is uniquely determined by the unique distribution on k_v^* which agrees with $\frac{d^*u_v}{|1-u|_v}$ for $u\neq 1$ and whose Fourier transform vanishes at 1. Since

$$\int_{R^*}' \frac{h_0(|u|^{-1})}{|1-u|} d^*u = (\gamma + \log(2\pi))h_0(1) + \lim_{\varepsilon\to 0}\left(\int_{|1-u|\geq\varepsilon} \frac{h_0(|u|^{-1})}{|1-u|} d^*u + h_0(1)\log\varepsilon\right), \quad (3.96)$$

by Corollary 1 and the proof of Theorem 1 we have

$$\int_{\mathrm{A}} Hg(u)\Psi(u)\log|u|du = -\sum_v \int_{k_v^*}' \frac{h_0(u^{-1})}{|1-u|_v} d^*u. \quad (3.97)$$

Therefore,

$$\mathrm{trace}\{(EZ_\Lambda E^{-1} - S_\Lambda)V(h)\} = -\sum_v \int_{k_v^*}' \frac{h_0(u^{-1})}{|1-u|_v} d^*u. \quad (3.98)$$

3.1 On some equations concerning the Selberg trace formula. [6]

Let $N>1$ be an integer, which is not square free. Denote by $\Gamma_0(N)$ the Hecke congruence subgroup of level N. Now let $\boldsymbol{a}$ be a cusp of $\Gamma_0(N)$. We denote its stabilizer by $\Gamma_{\boldsymbol{a}}$. An element $\sigma_{\boldsymbol{a}}$ in $PSL_2(R)$ exists such that $\sigma_{\boldsymbol{a}}\infty = \boldsymbol{a}$ and $\sigma_{\boldsymbol{a}}^{-1}\Gamma_{\boldsymbol{a}}\sigma_{\boldsymbol{a}} = \Gamma_\infty$. If ψ is an eigenfunction of the Laplacian associated with a positive discrete eigenvalue λ, then it has a Fourier expansion

$$\psi(\sigma_{\boldsymbol{a}} z) = \sqrt{y}\sum_{m\neq 0}\rho_{\boldsymbol{a}}(m)K_{i\kappa}(2\pi|m|y)e^{2m\pi i x} \quad (3.99)$$

at every cusp $\boldsymbol{a}$ of $\Gamma_0(N)$, where $\kappa = \sqrt{\lambda - 1/4}$ and $K_\nu(y)$ is given by the formula

$$K_\nu(y) = \frac{2^\nu\Gamma\left(\nu+\frac{1}{2}\right)}{y^\nu\sqrt{\pi}}\int_0^\infty \frac{\cos(yt)}{(1+t^2)^{\nu+\frac{1}{2}}} dt. \quad (3.100)$$

We have also the following trace formula:

$$d(n)h\left(-\frac{i}{2}\right)+\sqrt{n}\sum_{j=1}^{\infty}h(\kappa_j)tr_{\lambda_j}T_n=\int_D\{K(z,z)-H(z,z)\}dz\,. \quad (3.100b)$$

Every cusp of $\Gamma_0(N)$ is equivalent to one of the following inequivalent cusps

$$\frac{u}{w} \text{ with } u,w>0,\ (u,w)=1,\ w|N\,. \quad (3.101)$$

By the Riemann-Lebesgue theorem, we have also that

$$\lim_{Y\to\infty}\int_{-\infty}^{\infty}h(r)\left(\frac{a}{d}\right)^{ir}\frac{\varphi_{jj}\left(\frac{1}{2}-ir\right)Y^{2ir}}{ir}dr=\lim_{Y\to\infty}\int_{-\infty}^{\infty}h(r)\mathcal{R}\left(\varphi_{jj}\left(\frac{1}{2}-ir\right)\right)\frac{\sin\{r\ln(aY^2/d)\}}{r}dr=\pi h(0)\varphi_{jj}\left(\frac{1}{2}\right).$$
$$(3.102)$$

Furthermore, we have that:

$$\lim_{Y\to\infty}\left(c(\infty)_Y+\sum_{\{P\},\Gamma_P=\{1_2\}}c(P)_Y\right)=\sqrt{n}\delta_n\nu(N)g(0)\ln\frac{\sqrt{n}}{2}+\frac{\sqrt{n}}{4}h(0)\left\{\delta_n\nu(N)+\sum_{j=1}^{\nu(N)}\sum_{\substack{ad=n,d>0,a\neq d\\ a_j=\frac{u_j}{w_j},\left(w_j,\frac{N}{w_j}\right)|(a-d)}}\varphi_{jj}\left(\frac{1}{2}\right)\right\}+$$
$$+\frac{\sqrt{n}}{2}\sum_{w|N,w>0}\sum_{ad=n,d>0,a\neq d}\sum_b\frac{\ln\left\{(a-d)^2wN\left[\frac{C^2}{\ell^2},N\right]/C^2\right\}}{|a-d|}g\left(\ln\frac{a}{d}\right)-\frac{\delta_n\nu(N)\sqrt{n}}{2\pi}\int_{-\infty}^{\infty}h(r)\frac{\Gamma'}{\Gamma}(1+ir)dr+$$
$$+\sum_{j=1}^{\nu(N)}\sum_{\substack{ad=n,d>0\\ a_j=\frac{u_j}{w_j},\left(w_j,\frac{N}{w_j}\right)|a-d}}\frac{\sqrt{n}}{4\pi}\int_{-\infty}^{\infty}h(r)\left(\frac{a}{d}\right)^{ir}\sum_{i=1}^{\nu(N)}\varphi_{ij}'\left(\frac{1}{2}+ir\right)\varphi_{ij}\left(\frac{1}{2}-ir\right)dr+\sum_{\substack{ad=n,d>0,a\neq d\\ a_j=\frac{u_j}{w_j},\left(w_j,\frac{N}{w_j}\right)|(a-d)}}\frac{1}{2}|a-d|\int_1^{\infty}k\left(\frac{(a-d)^2}{n}t\right)\frac{\ln t}{\sqrt{t-1}}dt$$
$$-\frac{1}{8}h(0)\sum_{j=1}^{\nu(N)}\sum_{a,d,b}^{*}\left(\frac{A'}{D'}\right)^{1/2}\varphi_{jj'}\left(\frac{1}{2}\right) \quad (3.103)$$

with $C=(a-d)w-bwN$, where $\ell=(C,bN)$ and the summation on b is taken over all numbers b such that $C,bN/w\in Z,N|C$ and $0\leq b<|a-d|$. Note that there are exactly $|a-d|$ number of such numbers b. Denote by $c(\infty)$ the right side of the identity (3.103). We conclude that the trace formula (3.100b) can be written as

$$d(n)h\left(-\frac{i}{2}\right)+\sqrt{n}\sum_{j=1}^{\infty}h(\kappa_j)tr_jT_n=c(I)+\sum_{\{R\}}c(R)+\sum_{\{P\},\Gamma_P\neq\{1_2\}}c(P)+c(\infty) \quad (3.104)$$

for $\operatorname{Re}s>1$, where the summations on the right side of the identity are taken over the conjugacy classes.
Now, we have

$$g^{(4)}(\log u) = A(s)u^{\frac{1}{2}-s} + O_s\left(u^{-\frac{1}{2}}\right), \quad (3.105)$$

where $A(s)$ is an analytic function of s for $\operatorname{Re} s > 0$ and, for every complex number s with $\operatorname{Re} s > 0$, there exists a finite constant $B(s)$ depending only on s such that

$$\left|O_s\left(u^{-\frac{1}{2}}\right)\right| \le B(s)u^{-\frac{1}{2}}. \quad (3.106)$$

Moreover, for every fixed value of u, the term $O_s\left(u^{-\frac{1}{2}}\right)$ also represents an analytic function of s for $\operatorname{Re} s > 0$. Since

$$h(r) = \frac{1}{r^4}\int_0^\infty g^{(4)}(\ln u)u^{ir-1}du \qquad (3.107)$$

for non-zero r, we have

$$h(r) = \frac{A(s)}{r^4}\left(\frac{1}{s-\frac{1}{2}-ir} + \frac{1}{s-\frac{1}{2}+ir}\right) + O_s\left(r^{-4}\right) \quad (3.108)$$

for $\operatorname{Re} s > 1$ and for non-zero r with $|\operatorname{Im} r| < \frac{1}{2} - \varepsilon$. By analytic continuation, we obtain that

$$h(r) = \frac{2A(s)\left(s-\frac{1}{2}\right)}{r^4\left[\left(s-\frac{1}{2}\right)^2 + r^2\right]} + O_s\left(r^{-4}\right) \quad (3.109)$$

for $\operatorname{Re} s > 0$ and for non-zero r with $|\operatorname{Im} r| < \frac{1}{2} - \varepsilon$. It follows that the left side of (3.104) is an analytic function of s for $\operatorname{Re} s > 0$ except for having simple poles at $s = 1, \frac{1}{2} \pm i\kappa_j$, $j = 1,2,\ldots$. Then the right side of (3.104) can be interpreted as an analytic function of s in the same region by analytic continuation. Since $k(t) = (1 + t/4)^{-s}$, we have that $c(R)$ is analytic for $\operatorname{Re} s > 0$ except for a simple pole at $s = 1/2$. There are only a finite number of elliptic conjugacy classes $\{R\}$. The term $c(I)$ is a constant. Since $g(0) = 2\sqrt{\pi}\,\Gamma\left(s-\frac{1}{2}\right)\Gamma(s)^{-1}$,

$$h(0) = 2\sqrt{\pi}\,4^s\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\int_1^\infty\left(u + \frac{1}{u} + 2\right)^{\frac{1}{2}-s}\frac{du}{u} \quad (3.110)$$

and

$$g\left(\ln\frac{a}{d}\right)=2\sqrt{\pi}4^{s-\frac{1}{2}}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\left(\frac{a}{d}+\frac{d}{a}+2\right)^{\frac{1}{2}-s}, \quad (3.111)$$

the sum of first three terms and the last two terms on the right side of the identity (3.103) is analytic for $\operatorname{Re}s>0$ except for a pole at $s=1/2$. By Stirling's formula the identity

$$\frac{\Gamma'(z)}{\Gamma(z)}=\ln z+O(1) \quad (3.112)$$

holds uniformly when $|\arg z|\le\pi-\delta$ for a small positive number δ. It follows from (3.102) and (3.112) that the fourth term on the right side of the identity (3.103) is analytic for $\operatorname{Re}s>0$ except for a possible pole at $s=1/2$. We know that, by theorem of Kubota, that each Eisenstein series $E_i(z,s)$ has a meromorphic continuation to the whole s-plane, and the identity

$$\sum_{i=1}^{\nu(N)}\varphi_{ij}\left(\frac{1}{2}+ir\right)\varphi_{ij}\left(\frac{1}{2}-ir\right)=1 \quad (3.113)$$

hold for all real r and for $i=1,2,...,\nu(N)$. It follows that functions $\varphi_{ij}(s)$, $i,j=1,2,...,\nu(N)$, are analytic on the line $\operatorname{Re}s=1/2$. Let Y be a fixed large positive number. By the Maass-Selberg relation, we have

$$\operatorname{Re}\sum_{i=1}^{\nu(N)}\varphi_{ij}'\left(\frac{1}{2}+ir\right)\varphi_{ij}\left(\frac{1}{2}-ir\right)=2\ln Y+\frac{\varphi_{jj}(1/2-ir)Y^{2ir}-\varphi_{jj}(1/2+ir)Y^{-2ir}}{2ir}+$$
$$-\int_{D_Y}E_j\left(z,\frac{1}{2}+ir\right)E_j\left(z,\frac{1}{2}-ir\right)dz+o(1). \quad (3.114)$$

Let $\boldsymbol{a}_i=u_i/w_i$ be a cusp given in (3.101), and let $\eta=1/2+ir$. We have that

$$E_i(z,\eta)=\delta_{\boldsymbol{a}_i\infty}y^{\eta}+\frac{\sqrt{\pi}\Gamma\left(\eta-\frac{1}{2}\right)}{\Gamma(\eta)}\varphi_{\boldsymbol{a}_i\infty,0}(\eta)y^{1-\eta}+\frac{2\pi^{\eta}\sqrt{y}}{\Gamma(\eta)}\sum_{m\neq0}|m|^{\eta-\frac{1}{2}}K_{\eta-\frac{1}{2}}(2|m|\pi y)\varphi_{\boldsymbol{a}_i\infty,m}(\eta)e(mx)$$
(3.115)

where

$$\varphi_{\boldsymbol{a}_i\infty,0}(\eta)=\frac{\varphi(w_i)}{\varphi((w_i,N/w_i))}\left(\frac{(w_i,N/w_i)}{w_iN}\right)^{\eta}\prod_{p|N}\frac{1}{1-p^{-2\eta}}\prod_{p\left|\frac{N}{w_i}\right.}\left(1-p^{1-2\eta}\right)\frac{\zeta 2\eta-1}{\zeta(2\eta)} \quad (3.116)$$

and

$$\varphi_{\boldsymbol{a}_i\infty,m}(\eta)=\left(\frac{(w_i,N/w_i)}{w_iN}\right)^{\eta}\sum_{(c,N/w_i)=1}\frac{1}{c^{2\eta}}\sum_{\substack{d\bmod(cw_i),(d,cw_i)=1\\ cd\equiv u_i\bmod(w_i,N/w_i)}}e\left(-\frac{md}{cw_i}\right). \quad (3.117)$$

It follows from the functional identity of the Riemann zeta-function that

$$\frac{\Gamma\left(\eta-\frac{1}{2}\right)}{\Gamma(\eta)}\varphi_{a_i\infty,0}(\eta)=\pi^{2\eta-\frac{3}{2}}\frac{\Gamma(1-\eta)}{\Gamma(\eta)}\frac{\zeta(2-2\eta)}{\zeta(2\eta)}\frac{\varphi(w_i)}{\varphi((w_i,N/w_i))}\times\left(\frac{(w_i,N/w_i)}{w_iN}\right)^{\eta}\prod_{p|N}\frac{1}{1-p^{-2\eta}}\prod_{p\left|\frac{N}{w_i}\right.}\left(1-p^{1-2\eta}\right)$$

(3.118)

By using Stirling's formula

$$|\Gamma(\sigma+it)|\approx\sqrt{2\pi}e^{-\pi|t|/2}|t|^{\sigma-1/2} \quad (3.119)$$

for any fixed real value of σ as $t\to\infty$, we obtain that

$$\frac{\Gamma\left(\eta-\frac{1}{2}\right)}{\Gamma(\eta)}\varphi_{a_i\infty,0}(\eta)y^{1-\eta}<<\sqrt{y}\ln^2(|r|+1). \quad (3.120)$$

By (3.100) and by partial integration, we find that

$$\frac{K_{\eta-1/2}(2|m|\pi y)}{\Gamma(\eta)}=\frac{2^{\eta-1/2}}{(2|m|\pi y)^{\eta-1/2}\sqrt{\pi}}\int_0^{\infty}\frac{\cos(2|m|\pi yt)}{(1+t^2)^{\eta}}dt<<\frac{1+|r|^3}{|m|^3y^3}. \quad (3.121)$$

Then, it follows that the fifth term on the right side of the identity (3.103) is analytic for $\operatorname{Re}s>0$ except for a possible pole at $s=1/2$.

4. On some equations concerning p-adic strings, p-adic and adelic zeta functions, zeta strings and zeta nonlocal scalar fields. [7] [8] [9] [10] [11] [12]

Like in the ordinary string theory, the starting point of p-adic strings is a construction of the corresponding scattering amplitudes. Recall that the ordinary crossing symmetric Veneziano amplitude can be presented in the following forms:

$$A_{\infty}(a,b)=g^2\int_R|x|_{\infty}^{a-1}|1-x|_{\infty}^{b-1}dx=g^2\left[\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}+\frac{\Gamma(b)\Gamma(c)}{\Gamma(b+c)}+\frac{\Gamma(c)\Gamma(a)}{\Gamma(c+a)}\right]=g^2\frac{\zeta(1-a)}{\zeta(a)}\frac{\zeta(1-b)}{\zeta(b)}\frac{\zeta(1-c)}{\zeta(c)}=$$

$$=g^2\int DX\exp\left(-\frac{i}{2\pi}\int d^2\sigma\partial^{\alpha}X_{\mu}\partial_{\alpha}X^{\mu}\right)\prod_{j=1}^{4}\int d^2\sigma_j\exp\left(ik_{\mu}^{(j)}X^{\mu}\right), \quad (4.1-4.4)$$

where $\hbar=1$, $T=1/\pi$, and $a=-\alpha(s)=-1-\frac{s}{2}$, $b=-\alpha(t)$, $c=-\alpha(u)$ with the condition $s+t+u=-8$, i.e. $a+b+c=1$.

The p-adic generalization of the above expression

$$A_{\infty}(a,b)=g^2\int_R|x|_{\infty}^{a-1}|1-x|_{\infty}^{b-1}dx,$$

is:

$$A_p(a,b) = g_p^2 \int_{Q_p} |x|_p^{a-1} |1-x|_p^{b-1} dx \,, \quad (4.5)$$

where $|\ldots|_p$ denotes p-adic absolute value. In this case only string world-sheet parameter x is treated as p-adic variable, and all other quantities have their usual (real) valuation.
Now, we remember that the Gauss integrals satisfy adelic product formula

$$\int_R \chi_\infty (ax^2 + bx) d_\infty x \prod_{p \in P} \int_{Q_p} \chi_p (ax^2 + bx) d_p x = 1 \,, \quad a \in Q^\times \,, \quad b \in Q \,, \quad (4.6)$$

what follows from

$$\int_{Q_v} \chi_v (ax^2 + bx) d_v x = \lambda_v(a) |2a|_v^{-\frac{1}{2}} \chi_v \left(-\frac{b^2}{4a} \right), \quad v = \infty, 2, \ldots, p \ldots . \quad (4.7)$$

These Gauss integrals apply in evaluation of the Feynman path integrals

$$K_v(x'', t''; x', t') = \int_{x',t'}^{x'',t''} \chi_v \left(-\frac{1}{h} \int_{t'}^{t''} L(\dot{q}, q, t) dt \right) D_v q \,, \quad (4.8)$$

for kernels $K_v(x'', t''; x', t')$ of the evolution operator in adelic quantum mechanics for quadratic Lagrangians. In the case of Lagrangian

$$L(\dot{q}, q) = \frac{1}{2} \left(-\frac{\dot{q}^2}{4} - \lambda q + 1 \right),$$

for the de Sitter cosmological model one obtains

$$K_\infty(x'', T; x', 0) \prod_{p \in P} K_p(x'', T; x', 0) = 1 \,, \quad x'', x', \lambda \in Q \,, \ T \in Q^\times \,, \quad (4.9)$$

where

$$K_v(x'', T; x', 0) = \lambda_v(-8T) |4T|_v^{-\frac{1}{2}} \chi_v \left(-\frac{\lambda^2 T^3}{24} + [\lambda(x''+x') - 2] \frac{T}{4} + \frac{(x''-x')^2}{8T} \right). \quad (4.10)$$

Also here we have the number 24 that correspond to the Ramanujan function that has 24 “modes”, i.e., the physical vibrations of a bosonic string. Hence, we obtain the following mathematical connection:

$$K_v(x'', T; x', 0) = \lambda_v(-8T) |4T|_v^{-\frac{1}{2}} \chi_v \left(-\frac{\lambda^2 T^3}{24} + [\lambda(x''+x') - 2] \frac{T}{4} + \frac{(x''-x')^2}{8T} \right) \Rightarrow$$

$$\Rightarrow \frac{4\left[anti\log \frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi^2}{4}w'} \phi_{w'}(itw')}\right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (4.10b)$$

The adelic wave function for the simplest ground state has the form

$$\psi_A(x) = \psi_\infty(x) \prod_{p \in P} \Omega\left(|x|_p\right) = \begin{cases} \psi_\infty(x), x \in Z \\ 0, x \in Q \setminus Z \end{cases}, \quad (4.11)$$

where $\Omega\left(|x|_p\right) = 1$ if $|x|_p \le 1$ and $\Omega\left(|x|_p\right) = 0$ if $|x|_p > 1$. Since this wave function is non-zero only in integer points it can be interpreted as discreteness of the space due to p-adic effects in adelic approach. The Gel'fand-Graev-Tate gamma and beta functions are:

$$\Gamma_\infty(a) = \int_R |x|_\infty^{a-1} \chi_\infty(x) d_\infty x = \frac{\zeta(1-a)}{\zeta(a)}, \quad \Gamma_p(a) = \int_{Q_p} |x|_p^{a-1} \chi_p(x) d_p x = \frac{1-p^{a-1}}{1-p^{-a}}, \quad (4.12)$$

$$B_\infty(a,b) = \int_R |x|_\infty^{a-1} |1-x|_\infty^{b-1} d_\infty x = \Gamma_\infty(a)\Gamma_\infty(b)\Gamma_\infty(c), \quad (4.13)$$

$$B_p(a,b) = \int_{Q_p} |x|_p^{a-1} |1-x|_p^{b-1} d_p x = \Gamma_p(a)\Gamma_p(b)\Gamma_p(c), \quad (4.14)$$

where $a,b,c \in C$ with condition $a+b+c=1$ and $\zeta(a)$ is the Riemann zeta function. With a regularization of the product of p-adic gamma functions one has adelic products:

$$\Gamma_\infty(u) \prod_{p \in P} \Gamma_p(u) = 1, \quad B_\infty(a,b) \prod_{p \in P} B_p(a,b) = 1, \quad u \ne 0,1, \quad u = a,b,c, \quad (4.15)$$

where $a+b+c=1$. We note that $B_\infty(a,b)$ and $B_p(a,b)$ are the crossing symmetric standard and p-adic Veneziano amplitudes for scattering of two open tachyon strings. Introducing real, p-adic and adelic zeta functions as

$$\zeta_\infty(a) = \int_R \exp\left(-\pi x^2\right) |x|_\infty^{a-1} d_\infty x = \pi^{-\frac{a}{2}} \Gamma\left(\frac{a}{2}\right), \quad (4.16)$$

$$\zeta_p(a) = \frac{1}{1-p^{-1}} \int_{Q_p} \Omega\left(|x|_p\right) |x|_p^{a-1} d_p x = \frac{1}{1-p^{-a}}, \quad \mathrm{Re}\, a > 1, \quad (4.17)$$

$$\zeta_A(a) = \zeta_\infty(a) \prod_{p \in P} \zeta_p(a) = \zeta_\infty(a)\zeta(a), \quad (4.18)$$

one obtains

$$\zeta_A(1-a) = \zeta_A(a), \quad (4.19)$$

where $\zeta_A(a)$ can be called adelic zeta function. We have also that

$$\zeta_A(a) = \zeta_\infty(a)\prod_{p\in P}\zeta_p(a) = \zeta_\infty(a)\zeta(a) = \int_R \exp\left(-\pi x^2\right)|x|_\infty^{a-1}d_\infty x \cdot \frac{1}{1-p^{-1}}\int_{Q_p}\Omega\left(|x|_p\right)|x|_p^{a-1}d_p x . \quad (4.19b)$$

Let us note that $\exp\left(-\pi x^2\right)$ and $\Omega\left(|x|_p\right)$ are analogous functions in real and p-adic cases. Adelic harmonic oscillator has connection with the Riemann zeta function. The simplest vacuum state of the adelic harmonic oscillator is the following Schwartz-Bruhat function:

$$\psi_A(x) = 2^{\frac{1}{4}}e^{-\pi x_\infty^2}\prod_{p\in P}\Omega\left(|x_p|_p\right), \quad (4.20)$$

whose the Fourier transform

$$\psi_A(k) = \int \chi_A(kx)\psi_A(x) = 2^{\frac{1}{4}}e^{-\pi k_\infty^2}\prod_{p\in P}\Omega\left(|k_p|_p\right) \quad (4.21)$$

has the same form as $\psi_A(x)$. The Mellin transform of $\psi_A(x)$ is

$$\Phi_A(a) = \int \psi_A(x)|x|^a d_A^\times x = \int_R \psi_\infty(x)|x|^{a-1}d_\infty x \prod_{p\in P}\frac{1}{1-p^{-1}}\int_{Q_p}\Omega\left(|x|_p\right)|x|^{a-1}d_p x = \sqrt{2}\Gamma\left(\frac{a}{2}\right)\pi^{-\frac{a}{2}}\zeta(a) \quad (4.22)$$

and the same for $\psi_A(k)$. Then according to the Tate formula one obtains (4.19).
The exact tree-level Lagrangian for effective scalar field φ which describes open p-adic string tachyon is

$$\mathcal{L}_p = \frac{1}{g^2}\frac{p^2}{p-1}\left[-\frac{1}{2}\varphi p^{-\frac{\Box}{2}}\varphi + \frac{1}{p+1}\varphi^{p+1}\right], \quad (4.23)$$

where p is any prime number, $\Box = -\partial_t^2 + \nabla^2$ is the D-dimensional d'Alambertian and we adopt metric with signature $(-+...+)$. Now, we want to show a model which incorporates the p-adic string Lagrangians in a restricted adelic way. Let us take the following Lagrangian

$$L = \sum_{n\ge 1} C_n \mathcal{L}_n = \sum_{n\ge 1}\frac{n-1}{n^2}\mathcal{L}_n = \frac{1}{g^2}\left[-\frac{1}{2}\phi\sum_{n\ge 1} n^{-\frac{\Box}{2}}\phi + \sum_{n\ge 1}\frac{1}{n+1}\phi^{n+1}\right]. \quad (4.24)$$

Recall that the Riemann zeta function is defined as

$$\zeta(s) = \sum_{n\ge 1}\frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}, \quad s = \sigma + i\tau, \quad \sigma > 1. \quad (4.25)$$

Employing usual expansion for the logarithmic function and definition (4.25) we can rewrite (4.24) in the form

$$L = -\frac{1}{g^2}\left[\frac{1}{2}\phi\zeta\left(\frac{\Box}{2}\right)\phi + \phi + \ln(1-\phi)\right], \quad (4.26)$$

where $|\phi|<1$. $\zeta\left(\frac{\Box}{2}\right)$ acts as pseudodifferential operator in the following way:

$$\zeta\left(\frac{\Box}{2}\right)\phi(x)=\frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk\,, \quad -k^2=k_0^2-\vec{k}^2>2+\varepsilon\,, \quad (4.27)$$

where $\tilde{\phi}(k)=\int e^{(-ikx)}\phi(x)dx$ is the Fourier transform of $\phi(x)$.

Dynamics of this field ϕ is encoded in the (pseudo)differential form of the Riemann zeta function. **When the d'Alambertian is an argument of the Riemann zeta function we shall call such string a "zeta string".** Consequently, the above ϕ is an open scalar zeta string. The equation of motion for the zeta string ϕ is

$$\zeta\left(\frac{\Box}{2}\right)\phi=\frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon}e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk=\frac{\phi}{1-\phi} \quad (4.28)$$

which has an evident solution $\phi=0$.

For the case of time dependent spatially homogeneous solutions, we have the following equation of motion

$$\zeta\left(\frac{-\partial_t^2}{2}\right)\phi(t)=\frac{1}{(2\pi)}\int_{|k_0|>\sqrt{2}+\varepsilon}e^{-ik_0t}\zeta\left(\frac{k_0^2}{2}\right)\tilde{\phi}(k_0)dk_0=\frac{\phi(t)}{1-\phi(t)}. \quad (4.29)$$

With regard the open and closed scalar zeta strings, the equations of motion are

$$\zeta\left(\frac{\Box}{2}\right)\phi=\frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk=\sum_{n\geq1}\theta^{\frac{n(n-1)}{2}}\phi^n\,, \quad (4.30)$$

$$\zeta\left(\frac{\Box}{4}\right)\theta=\frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{4}\right)\tilde{\theta}(k)dk=\sum_{n\geq1}\left[\theta^{n^2}+\frac{n(n-1)}{2(n+1)}\theta^{\frac{n(n-1)}{2}-1}\left(\phi^{n+1}-1\right)\right], \quad (4.31)$$

and one can easily see trivial solution $\phi=\theta=0$.

The exact tree-level Lagrangian of effective scalar field φ, which describes open p-adic string tachyon, is:

$$\mathcal{L}_p=\frac{m_p^D}{g_p^2}\frac{p^2}{p-1}\left[-\frac{1}{2}\varphi p^{-\frac{\Box}{2m_p^2}}\varphi+\frac{1}{p+1}\varphi^{p+1}\right], \quad (4.32)$$

where p is any prime number, $\Box=-\partial_t^2+\nabla^2$ is the D-dimensional d'Alambertian and we adopt metric with signature $(-+...+)$, as above. Now, we want to introduce a model which incorporates all the above string Lagrangians (4.32) with p replaced by $n\in N$. Thence, we take the sum of all Lagrangians $\mathcal{L}_n$ in the form

$$L=\sum_{n=1}^{+\infty}C_n\mathcal{L}_n=\sum_{n=1}^{+\infty}C_n\frac{m_n^D}{g_n^2}\frac{n^2}{n-1}\left[-\frac{1}{2}\phi n^{-\frac{\Box}{2m_n^2}}\phi+\frac{1}{n+1}\phi^{n+1}\right],\quad (4.33)$$

whose explicit realization depends on particular choice of coefficients C_n, masses m_n and coupling constants g_n.

Now, we consider the following case

$$C_n=\frac{n-1}{n^{2+h}},\quad (4.34)$$

where h is a real number. The corresponding Lagrangian reads

$$L_h=\frac{m^D}{g^2}\left[-\frac{1}{2}\phi\sum_{n=1}^{+\infty}n^{-\frac{\Box}{2m^2}-h}\phi+\sum_{n=1}^{+\infty}\frac{n^{-h}}{n+1}\phi^{n+1}\right]\quad (4.35)$$

and it depends on parameter h. According to the Euler product formula one can write

$$\sum_{n=1}^{+\infty}n^{-\frac{\Box}{2m^2}-h}=\prod_p\frac{1}{1-p^{-\frac{\Box}{2m^2}-h}}.\quad (4.36)$$

Recall that standard definition of the Riemann zeta function is

$$\zeta(s)=\sum_{n=1}^{+\infty}\frac{1}{n^s}=\prod_p\frac{1}{1-p^{-s}},\quad s=\sigma+i\tau,\quad \sigma>1,\quad (4.37)$$

which has analytic continuation to the entire complex s plane, excluding the point $s=1$, where it has a simple pole with residue 1. Employing definition (4.37) we can rewrite (4.35) in the form

$$L_h=\frac{m^D}{g^2}\left[-\frac{1}{2}\phi\zeta\left(\frac{\Box}{2m^2}+h\right)\phi+\sum_{n=1}^{+\infty}\frac{n^{-h}}{n+1}\phi^{n+1}\right].\quad (4.38)$$

Here $\zeta\left(\frac{\Box}{2m^2}+h\right)$ acts as a pseudodifferential operator

$$\zeta\left(\frac{\Box}{2m^2}+h\right)\phi(x)=\frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2m^2}+h\right)\tilde{\phi}(k)dk,\quad (4.39)$$

where $\tilde{\phi}(k)=\int e^{(-ikx)}\phi(x)dx$ is the Fourier transform of $\phi(x)$.

We consider Lagrangian (4.38) with analytic continuations of the zeta function and the power series $\sum\frac{n^{-h}}{n+1}\phi^{n+1}$, i.e.

$$L_h=\frac{m^D}{g^2}\left[-\frac{1}{2}\phi\zeta\left(\frac{\Box}{2m^2}+h\right)\phi+AC\sum_{n=1}^{+\infty}\frac{n^{-h}}{n+1}\phi^{n+1}\right],\quad (4.40)$$

where AC denotes analytic continuation.
Potential of the above zeta scalar field (4.40) is equal to $-L_h$ at $\Box = 0$, i.e.

$$V_h(\phi) = \frac{m^D}{g^2}\left(\frac{\phi^2}{2}\zeta(h) - AC\sum_{n=1}^{+\infty}\frac{n^{-h}}{n+1}\phi^{n+1}\right), \quad (4.41)$$

where $h \neq 1$ since $\zeta(1) = \infty$. The term with ζ -function vanishes at $h = -2, -4, -6, \ldots$. The equation of motion in differential and integral form is

$$\zeta\left(\frac{\Box}{2m^2} + h\right)\phi = AC\sum_{n=1}^{+\infty} n^{-h}\varphi^n , \quad (4.42)$$

$$\frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2} + h\right)\tilde{\phi}(k)dk = AC\sum_{n=1}^{+\infty} n^{-h}\phi^n , \quad (4.43)$$

respectively.
Now, we consider five values of h, which seem to be the most interesting, regarding the Lagrangian (4.40): $h = 0$, $h = \pm 1$, and $h = \pm 2$. For $h = -2$, the corresponding equation of motion now read:

$$\zeta\left(\frac{\Box}{2m^2} - 2\right)\phi = \frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2} - 2\right)\tilde{\phi}(k)dk = \frac{\phi(\phi+1)}{(1-\phi)^3}. \quad (4.44)$$

This equation has two trivial solutions: $\phi(x) = 0$ and $\phi(x) = -1$. Solution $\phi(x) = -1$ can be also shown taking $\tilde{\phi}(k) = -\delta(k)(2\pi)^D$ and $\zeta(-2) = 0$ in (4.44).
For $h = -1$, the corresponding equation of motion is:

$$\zeta\left(\frac{\Box}{2m^2} - 1\right)\phi = \frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2} - 1\right)\tilde{\phi}(k)dk = \frac{\phi}{(1-\phi)^2}. \quad (4.45)$$

where $\zeta(-1) = -\frac{1}{12}$.
The equation of motion (4.45) has a constant trivial solution only for $\phi(x) = 0$.
For $h = 0$, the equation of motion is

$$\zeta\left(\frac{\Box}{2m^2}\right)\phi = \frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2}\right)\tilde{\phi}(k)dk = \frac{\phi}{1-\phi}. \quad (4.46)$$

It has two solutions: $\phi = 0$ and $\phi = 3$. The solution $\phi = 3$ follows from the Taylor expansion of the Riemann zeta function operator

$$\zeta\left(\frac{\Box}{2m^2}\right) = \zeta(0) + \sum_{n\geq 1}\frac{\zeta^{(n)}(0)}{n!}\left(\frac{\Box}{2m^2}\right)^n , \quad (4.47)$$

as well as from $\tilde{\phi}(k) = (2\pi)^D 3\delta(k)$.
For $h = 1$, the equation of motion is:

$$\frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2}+1\right)\tilde{\phi}(k)dk=-\frac{1}{2}\ln(1-\phi)^2\,, \quad (4.48)$$

where $\zeta(1)=\infty$ gives $V_1(\phi)=\infty$.
In conclusion, for $h=2$, we have the following equation of motion:

$$\frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2}+2\right)\tilde{\phi}(k)dk=-\int_0^{\phi}\frac{\ln(1-w)^2}{2w}dw\,. \quad (4.49)$$

Since holds equality

$$-\int_0^1\frac{\ln(1-w)}{w}dw=\sum_{n=1}^{\infty}\frac{1}{n^2}=\zeta(2)$$

one has trivial solution $\phi=1$ in (4.49).

Now, we want to analyze the following case: $C_n=\frac{n^2-1}{n^2}$. In this case, from the Lagrangian (4.33), we obtain:

$$L=\frac{m^D}{g^2}\left[-\frac{1}{2}\phi\left\{\zeta\left(\frac{\Box}{2m^2}-1\right)+\zeta\left(\frac{\Box}{2m^2}\right)\right\}\phi+\frac{\phi^2}{1-\phi}\right]. \quad (4.50)$$

The corresponding potential is:

$$V(\phi)=-\frac{m^D}{g}\frac{31-7\phi}{24(1-\phi)}\phi^2\,. \quad (4.51)$$

The equation of motion is:

$$\left[\zeta\left(\frac{\Box}{2m^2}-1\right)+\zeta\left(\frac{\Box}{2m^2}\right)\right]\phi=\frac{\phi\left[(\phi-1)^2+1\right]}{(\phi-1)^2}. \quad (4.52)$$

Its weak field approximation is:

$$\left[\zeta\left(\frac{\Box}{2m^2}-1\right)+\zeta\left(\frac{\Box}{2m^2}\right)-2\right]\phi=0\,, \quad (4.53)$$

which implies condition on the mass spectrum

$$\zeta\left(\frac{M^2}{2m^2}-1\right)+\zeta\left(\frac{M^2}{2m^2}\right)=2\,. \quad (4.54)$$

From (4.54) it follows one solution for $M^2>0$ at $M^2\approx 2.79m^2$ and many tachyon solutions when $M^2<-38m^2$.
With regard the extension by ordinary Lagrangian, we have the Lagrangian, potential, equation of motion and mass spectrum condition that, when $C_n=\frac{n^2-1}{n^2}$, are:

$$L=\frac{m^D}{g^2}\left[\frac{\phi}{2}\left\{\frac{\Box}{m^2}-\zeta\left(\frac{\Box}{2m^2}-1\right)-\zeta\left(\frac{\Box}{2m^2}\right)-1\right\}\phi+\frac{\phi^2}{2}\ln\phi^2+\frac{\phi^2}{1-\phi}\right], \quad (4.55)$$

$$V(\phi)=\frac{m^D}{g^2}\frac{\phi^2}{2}\left[\zeta(-1)+\zeta(0)+1-\ln\phi^2-\frac{1}{1-\phi}\right], \quad (4.56)$$

$$\left[\zeta\left(\frac{\Box}{2m^2}-1\right)+\zeta\left(\frac{\Box}{2m^2}\right)-\frac{\Box}{m^2}+1\right]\phi=\phi\ln\phi^2+\phi+\frac{2\phi-\phi^2}{(1-\phi)^2}, \quad (4.57)$$

$$\zeta\left(\frac{M^2}{2m^2}-1\right)+\zeta\left(\frac{M^2}{2m^2}\right)=\frac{M^2}{m^2}. \quad (4.58)$$

In addition to many tachyon solutions, equation (4.58) has two solutions with positive mass: $M^2\approx 2.67m^2$ and $M^2\approx 4.66m^2$.

Now, we describe the case of $C_n=\mu(n)\frac{n-1}{n^2}$. Here $\mu(n)$ is the Mobius function, which is defined for all positive integers and has values 1, 0, – 1 depending on factorization of n into prime numbers p. It is defined as follows:

$$\mu(n)=\begin{cases}0, & n=p^2m\\ (-1)^k, & n=p_1p_2\ldots p_k,\, p_i\neq p_j\\ 1, & n=1,(k=0).\end{cases} \quad (4.59)$$

The corresponding Lagrangian is

$$L_\mu=C_0\mathcal{L}_0+\frac{m^D}{g^2}\left[-\frac{1}{2}\phi\sum_{n=1}^{+\infty}\frac{\mu(n)}{n^{\frac{\Box}{2m^2}}}\phi+\sum_{n=1}^{+\infty}\frac{\mu(n)}{n+1}\phi^{n+1}\right] \quad (4.60)$$

Recall that the inverse Riemann zeta function can be defined by

$$\frac{1}{\zeta(s)}=\sum_{n=1}^{+\infty}\frac{\mu(n)}{n^s}, \quad s=\sigma+it, \quad \sigma>1. \quad (4.61)$$

Now (4.60) can be rewritten as

$$L_\mu=C_0\mathcal{L}_0+\frac{m^D}{g^2}\left[-\frac{1}{2}\phi\frac{1}{\zeta\left(\frac{\Box}{2m^2}\right)}\phi+\int_0^\infty\mathcal{M}(\phi)d\phi\right], \quad (4.62)$$

where $\mathcal{M}(\phi)=\sum_{n=1}^{+\infty}\mu(n)\phi^n=\phi-\phi^2-\phi^3-\phi^5+\phi^6-\phi^7+\phi^{10}-\phi^{11}-\ldots$ The corresponding potential, equation of motion and mass spectrum formula, respectively, are:

$$V_\mu(\phi) = -L_\mu(\Box = 0) = \frac{m^D}{g^2}\left[\frac{C_0}{2}\phi^2(1-\ln\phi^2) - \phi^2 - \int_0^\phi \mathcal{M}(\phi)d\phi\right], \quad (4.63)$$

$$\frac{1}{\zeta\left(\frac{\Box}{2m^2}\right)}\phi - \mathcal{M}(\phi) - C_0\frac{\Box}{m^2}\phi - 2C_0\phi\ln\phi = 0, \quad (4.64)$$

$$\frac{1}{\zeta\left(\frac{M^2}{2m^2}\right)} - C_0\frac{M^2}{m^2} + 2C_0 - 1 = 0, \quad |\phi| << 1, \quad (4.65)$$

where usual relativistic kinematic relation $k^2 = -k_0^2 + \vec{k}^2 = -M^2$ is used.
Now, we take the pure numbers concerning the eqs. (4.54) and (4.58). They are: 2.79, 2.67 and 4.66. We note that all the numbers are related with $\Phi = \frac{\sqrt{5}+1}{2}$, thence with the aurea ratio, by the following expressions:

$$2{,}79 \cong (\Phi)^{15/7}; \qquad 2{,}67 \cong (\Phi)^{13/7} + (\Phi)^{-21/7}; \qquad 4{,}66 \cong (\Phi)^{22/7} + (\Phi)^{-30/7}. \quad (4.66)$$

4.1 On some equations concerning a general class of cosmological models driven by a non-local scalar field inspired by string field theories and p-adic cosmology. [13] [14]

In this sub-section we consider a model of gravity coupling with a non-local scalar field which induced by strings field theory

$$S = \int d^4x\sqrt{-g}\left[\frac{M_p^2}{2}R + \frac{M_s^4}{g_4}\left(\frac{1}{2}\phi F\left(-\frac{\Box_g}{M_s^2}\right)\phi - \Lambda'\right)\right], \quad (4.67)$$

where g is the metric, $\Box_g = \frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu$, M_p is a mass Planck, M_s is a characteristic string scale related with the string tension α', $M_s = 1/\sqrt{\alpha'}$, ϕ is a dimensionless scalar field (tachyon or dilaton), g_4 is a dimensionless four dimensional effective coupling constant related with the ten dimensional string coupling constant g_0 and the compactification scale. $\Lambda = \frac{M_s^4}{g_4}\Lambda'$ is an effective four dimensional cosmological constant. The form of the function F is inspired by a nonlocal action appeared in string field theories. In particular cases

$$F(z) = -\xi^2 z + 1 - ce^{-2z}, \quad (4.68)$$

ξ is a real parameter and c is a positive constant. Using dimensional space-time variables and after a rescaling we can rewrite (4.67) for F given by (4.68) as follows

$$S = \int d^4x\sqrt{-g}\left(\frac{m_p^2}{2}R + \frac{\xi^2}{2}\phi\Box_g\phi + \frac{1}{2}(\phi^2 - c\Phi^2) - \Lambda'\right), \quad (4.69)$$

where $\Phi = e^{\Box_g}\phi$ and $m_p^2 = g_4 M_p^2 / M_s^2$. Generally speaking the string scale does not coincide with the Planck mass. This gives a possibility to get a realistic value of Λ. The form of the term $(e^{\Box_g}\phi)^2$ is analogous to the form of the interaction in the action for the string field tachyon in non-flat background, which is a generalization of the SFT (String Field Theory) tachyon interaction term in a flat background. This type of models does appear in SFT and in the p-adic string models. The case of the open Cubic Superstring Field Theory (CSSFT) tachyon corresponds to $\xi^2 = -1/\left[4\ln\left(\frac{4}{3\sqrt{3}}\right)\right] \approx 0.9556$ and $c = 3$. We consider in detail action (4.69) at $c = 1$, which is invariant under translation $\phi \to \phi + const$.

We take the metric in the form

$$ds^2 = -dt^2 + a^2(t)\left(dx_1^2 + dx_2^2 + dx_3^2\right) \quad (4.70)$$

and get the following equation of motion for the space homogeneous scalar field ϕ:

$$F(-\mathcal{D})\phi = 0, \quad (4.71)$$

where

$$\mathcal{D} \equiv -\partial_t^2 - 3H(t)\partial_t, \quad H = \frac{\dot{a}}{a} \quad \text{and} \quad \dot{a} \equiv \partial_t a. \quad (4.72)$$

The Friedmann equations have the following form

$$3H^2 = \frac{1}{m_p^2}\varepsilon, \quad 3H^2 + 2\dot{H} = -\frac{1}{m_p^2}\mathcal{P}, \quad (4.73)$$

where the energy and the pressure are obtained from the action (4.67) using standard formula

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}, \quad T_{\mu\nu} = diag\{\varepsilon, \mathcal{P}, \mathcal{P}, \mathcal{P}\}. \quad (4.74)$$

For the case of F given by (4.68) the energy and the pressure have additional nonlocal terms ε_{nl1} and ε_{nl2}. We have the following equations:

$$\varepsilon = \frac{\xi^2}{2}(\partial\phi)^2 - \frac{1}{2}\left(\phi^2 - c\left(e^{\mathcal{D}}\phi\right)^2\right) - c\int_0^1\left(\partial e^{(1+\rho)\mathcal{D}}\phi\right)\left(\partial e^{(1-\rho)\mathcal{D}}\phi\right)d\rho + c\int_0^1\left(e^{(1+\rho)\mathcal{D}}\phi\right)\left(-\mathcal{D}e^{(1-\rho)\mathcal{D}}\phi\right)d\rho + \Lambda',$$
$$\mathcal{P} = \frac{\xi^2}{2}(\partial\phi)^2 + \frac{1}{2}\left(\phi^2 - c\left(e^{\mathcal{D}}\phi\right)^2\right) - c\int_0^1\left(\partial e^{(1+\rho)\mathcal{D}}\phi\right)\left(\partial e^{(1-\rho)\mathcal{D}}\phi\right)d\rho - c\int_0^1\left(e^{(1+\rho)\mathcal{D}}\phi\right)\left(-\mathcal{D}e^{(1-\rho)\mathcal{D}}\phi\right)d\rho + \Lambda'. \quad (4.75)$$

Nonlocal term ε_{nl1} plays a role of an extra potential term and ε_{nl2} a role of an extra kinetic term.

We use the Weierstrass product representation for the function F in (4.67),

$$F(z) = e^{f(z)} \prod_n \left(1 - \frac{z}{\alpha_n^2}\right), \quad (4.76)$$

where α_n^2 are complex numbers, and represent the flat analog of (4.67) as

$$S_{flat} = \frac{1}{2}\int d^4x \phi F(-\Box)\phi \approx \frac{1}{2}\sum \left[\varepsilon_n \psi_n e^{f(-\Box)}\left(\Box + \alpha_n^2\right)\psi_n + c.c.\right], \quad (4.77)$$

where $\Box$ is the d'Alambertian in the flat space-time (here ε_n is not the ε of the equations (4.73)-(4.75)).
Now, we use a representation of nonlocal dynamics given by action (4.67) in terms of local fields

$$\int d^4x \sqrt{-g}\left(\frac{m_p^2}{2}R + \frac{1}{2}\sum \left[\varepsilon_n \psi_n e^{f(-\Box_g)}\left(\Box_g + \alpha_n^2\right)\psi_n + \Lambda' + c.c.\right]\right). \quad (4.78)$$

We perform a deformation of this model by several steps. First, we consider an approximation to the model (4.78) in the form

$$\int d^4x \sqrt{-g}\left(\frac{m_p^2}{2}R + \sum \left[\frac{\varepsilon_n}{2}\psi_n e^{f(\alpha_n^2)}\left(\Box_g + \alpha_n^2\right)\psi_n + \Lambda' + c.c.\right]\right). \quad (4.78b)$$

Second, we restrict a number of local fields and, third, we add potentials of the order $1/m_p^2$ in which Λ' is also included:

$$\int d^4x \sqrt{-g}\left(\frac{m_p^2}{2}R + \sum \left[\frac{\varepsilon_n}{2}\psi_n e^{f(\alpha_n^2)}\left(\Box_g + \alpha_n^2\right)\psi_n + c.c.\right] - \mathcal{V}(\psi_1,\ldots,\psi_n)\right), \quad (4.79)$$

such that solutions of the field equations in the non-flat case are the same as the flat case. Finally, we find the corresponding scale factor $a(t)$ and study cosmological properties of approximated solutions to our model. In the flat case the action (4.67) has the following form:

$$S_{flat} = \frac{1}{2}\int d^4x \phi F(-\Box)\phi . \quad (4.80)$$

Equation of motion on the space-homogeneous configurations (4.71) is reduced to the following linear equation:

$$F(\partial^2)\phi = 0 . \quad (4.81)$$

A plane wave

$$\phi = e^{\alpha t} \quad (4.82)$$

is a solution of (4.81) if α is a root of the characteristic equation

$$F(\alpha^2) = 0 . \quad (4.83)$$

For a case of F given by (4.68) equation (4.81) has the following form

$$-\xi^2\partial^2\phi+\phi-ce^{-2\partial^2}\phi=0\,. \quad (4.84)$$

This equation has an infinite number of derivatives and can be treated as a pseudodifferential as well as an integral equation. The corresponding characteristic equation:

$$F\left(\alpha^2\right)\equiv-\xi^2\alpha^2+1-ce^{-2\alpha^2}=0 \quad (4.85)$$

has the following solutions

$$\alpha_n=\pm\frac{1}{2\xi}\sqrt{4+2\xi^2W_n\left(-\frac{2ce^{-2/\xi^2}}{\xi^2}\right)}, \quad n=0,\pm1,\pm2,... \quad (4.86)$$

where W_n is the n-s branch of the Lambert function satisfying a relation $W(z)e^{W(z)}=z$. The Lambert function is a multivalued function, so eq. (4.85) has an infinite number of roots. Parameters ξ and c are real, therefore if α_n is a root of (4.85), then the adjoined number α_n^* is a root as well. Note that if α_n is a root of (4.85), then $-\alpha_n$ is a root too. In other words, equation (4.85) has quadruples of complex roots

$$\alpha_{n,\pm\pm}=\pm\mathrm{Re}(\alpha_n)\pm i\,\mathrm{Im}(\alpha_n)\,. \quad (4.87)$$

If $\alpha^2=\alpha_0^2$ is a multiple root, then at this point $F\left(\alpha_0^2\right)=0$ and $F'\left(\alpha_0^2\right)=0$. These equations give that

$$\alpha_0^2=\frac{1}{\xi^2}-\frac{1}{2}, \quad (4.88)$$

hence α_0^2 is a real number and all multiple roots of $F\left(\alpha_0^2\right)=0$ are either real or pure imaginary. The multiple roots exist if and only if

$$c=\frac{\xi^2}{2e}e^{2/\xi^2}\,. \quad (4.89)$$

Real roots for any ξ and c, except $\xi^2=0$ and $c=\infty$, are no more then double degenerated, because $F''\left(\alpha_0^2\right)\neq0$. Summing up we note that according as the values of parameters c and ξ^2 there exist the following types of the general real solution of (4.84):

- If $c\neq\frac{\xi^2}{2e}e^{2/\xi^2}$ and $c\neq1$ then the general real solution is

$$\phi=\sum_n R_n e^{m_n t}+\sum_n\left(C_n e^{\alpha_n t}+C_n^* e^{\alpha_n^* t}\right), \quad (4.90)$$

where R_n and C_n are arbitrary real and complex numbers respectively.

- If $c=\frac{\xi^2}{2e}e^{2/\xi^2}>1$, then to get the general real solution one has to add to (4.90)

$$\phi_0 = \tilde{R}_1 t e^{m_0 t} + \tilde{R}_2 t e^{-m_0 t}, \qquad m_0 = \sqrt{\frac{1}{\xi^2} - \frac{1}{2}} \quad \text{if} \quad \xi^2 < 2, \quad (4.91)$$

$$\phi_0 = \tilde{C}_1 t e^{i\alpha_0 t} + \tilde{C}_1^* t e^{-i\alpha_0 t}, \quad \alpha_0 = i\sqrt{\frac{1}{2} - \frac{1}{\xi^2}} \quad \text{if} \quad \xi^2 > 2. \quad (4.92)$$

- If $c = 1$ then to get the general real solution one has to add to (4.90)

$$\phi_0 = C_1 t + C_0, \quad \text{if} \quad \xi^2 \neq 2, \quad (4.93)$$

$$\phi_0 = C_3 t^3 + C_2 t^2 + C_1 t + C_0, \quad \text{if} \quad \xi^2 = 2. \quad (4.94)$$

Now we consider a special values of ξ^2 and c, which have been obtain in the SFT inspired cosmological model. From the action for the tachyon in the CSSFT the following equation has been obtained:

$$\left(-\xi_0^2 \tilde{\alpha}^2 + 1\right) = 3e^{-\tilde{\alpha}^2/4}, \quad (4.95)$$

where

$$\xi_0^2 = -\frac{1}{4\ln\left(\frac{4}{3\sqrt{3}}\right)} \approx 0.9556. \quad (4.96)$$

Substituting $\tilde{\alpha} = 2\sqrt{2}\alpha$, we obtain eq. (4.85) with $\xi^2 = 8\xi_0^2$ and $c = 3$. *Note that the pure number 8 is related to the physical vibrations of the superstrings, thence with the Ramanujan modular equation (eq. (2.18)) and that 3 is a Fibonacci's number*. From (4.89) it is follows that all roots are simple. We obtain that $\xi_{\min}^2 > \xi^2 > \xi_{\max}^2$, so there exist neither real roots not pure imaginary roots.
Equation (4.84) has the conserved energy which is defined by the formula that is a flat analog of (4.75). The energy density is as follows:

$$E = E_k + E_p + E_{nl1} + E_{nl2}, \quad (4.97)$$

where

$$E_k = \frac{\xi^2}{2}(\partial\phi)^2, \qquad E_p = -\frac{1}{2}\phi^2 + \frac{c}{2}\Phi^2, \quad (4.98)$$

$$E_{nl1} = c\int_0^1 \left(e^{-\rho\partial^2}\Phi\right)\left(\partial^2\left(e^{\rho\partial^2}\Phi\right)\right)d\rho, \qquad E_{nl2} = -c\int_0^1 \left(\partial\left(e^{-\rho\partial^2}\Phi\right)\right)\left(\partial\left(e^{\rho\partial^2}\Phi\right)\right)d\rho. \quad (4.99)$$

For the pressure

$$P = E_k - E_p - E_{nl1} + E_{nl2}, \quad (4.100)$$

we have the following explicit form

$$P = \frac{\xi^2}{2}(\partial\phi)^2 + \frac{1}{2}\phi^2 - \frac{c}{2}\Phi^2 - c\int_0^1 \left\{\left(e^{-\rho\partial^2}\Phi\right)\left(\partial^2\left(e^{\rho\partial^2}\Phi\right)\right) - \left(\partial\left(e^{-\rho\partial^2}\Phi\right)\right)\left(\partial\left(e^{\rho\partial^2}\Phi\right)\right)\right\}d\rho. \quad (4.101)$$

Let us calculate the energy density and pressure for the following solution

$$\phi = \sum_{n=1}^{N} C_n e^{\alpha_n t}\,, \quad (4.102)$$

where N is a natural number, C_n are some constant and α_n are solutions to eq. (4.85). For $N = 1$ and

$$\phi = Ce^{\alpha t} \quad (4.103)$$

we obtain

$$E\left(Ce^{\alpha t}\right) = 0\,, \quad (4.104) \qquad P\left(Ce^{\alpha t}\right) = C^2 p_\alpha e^{2\alpha t}\,. \quad (4.105)$$

We denote the energy density and pressure of function $\phi(t)$ as the functionals $E(\phi)$ and $P(\phi)$, respectively, and use the following notation

$$p_\alpha \equiv \alpha^2\left(\xi^2 - 2 + 2\xi^2\alpha^2\right). \quad (4.106)$$

For $N = 2$ and

$$\phi = C_1 e^{\alpha_1 t} + C_2 e^{\alpha_2 t}\,, \quad (4.107)$$

where α_1 and α_2 are different roots of (4.85) we have

$$E\left(C_1 e^{\alpha_1 t} + C_2 e^{\alpha_2 t}\right) = -2C_1C_2 p_{\alpha_1}\,, \quad \text{at} \quad \alpha_2 = -\alpha_1\,, \quad (4.108)$$

and

$$E\left(C_1 e^{\alpha_1 t} + C_2 e^{\alpha_2 t}\right) = 0\,, \quad \text{at} \quad \alpha_2 \neq -\alpha_1\,. \quad (4.109)$$

The pressure $P(\phi)$ for solution (4.107) is

$$P\left(C_1 e^{-\alpha t} + C_2 e^{\alpha t}\right) = \left(C_1^2 e^{-2\alpha t} + C_2^2 e^{2\alpha t}\right) p_\alpha\,. \quad (4.110)$$

In the general case we have

$$E\left(\sum_{n=1}^{N} C_n e^{\alpha_n t}\right) = -2\sum_{n=1}^{N}\sum_{k=n+1}^{N} C_n C_k\, p_{\alpha_n} \delta_{\alpha_n, -\alpha_k}\,, \quad (4.111)$$

where

$$\begin{aligned} \delta_{\alpha_n, -\alpha_k} &= 1\,, \qquad \alpha_n = -\alpha_k\,, \\ \delta_{\alpha_n, -\alpha_k} &= 0\,, \qquad \alpha_n \neq -\alpha_k\,. \quad (4.112) \end{aligned}$$

From formula (4.111) we see that the energy density is a sum of the crossing terms. At the same time the pressure is a sum of "individual" pressures and has no crossing term. In the case of an arbitrary finite number of summands the pressure is as follows:

$$P\left(\sum_{n=1}^{N} C_n e^{\alpha_n t}\right) = \sum_{n=1}^{N} C_n^2 P\left(e^{\alpha_n t}\right) = \sum_{n=1}^{N} C_n^2\, p_{\alpha_n} e^{2\alpha_n t}\,. \quad (4.113)$$

If the parameters ξ^2 and c are such that the characteristic equation (4.85) have double roots, then eq. (4.84) has the following solution

$$\phi_0(t) = B_1 e^{\alpha_0 t} t + C_1 e^{\alpha_0 t} + B_2 e^{-\alpha_0 t} t + C_2 e^{-\alpha_0 t}, \quad (4.114)$$

where B_1, B_2, C_1 and C_2 are constants, $\alpha_0 \neq 0$ is defined by (4.88). Using formulas (4.97) and (4.100) and substituting

$$\alpha_0 = \frac{\sqrt{4-2\xi^2}}{2\xi}, \quad (4.115)$$

we obtain

$$E(\phi_0) = -\frac{(\xi^2-2)}{3\xi^2}\left[3C_1 B_2 \sqrt{4-2\xi^2}\,\xi - 3C_2 B_1 \sqrt{4-2\xi^2}\,\xi + 8B_1 B_2 (2\xi^2-1)\right]. \quad (4.116)$$

The pressure is as follows

$$P(\phi_0) = -\frac{(\xi^2-2)}{3\xi^2}\left(B_2 e^{-\frac{t\sqrt{4-2\xi^2}}{\xi}}\left(B_2\left(8\xi^2 - 3t\sqrt{4-2\xi^2}\,\xi - 4\right) - 3C_2\xi\sqrt{4-2\xi^2}\right) + \right.$$
$$\left. + B_1 e^{\frac{t\sqrt{4-2\xi^2}}{\xi}}\left(3C_1\sqrt{4-2\xi^2}\,\xi + B_1\left(8\xi^2 + 3t\sqrt{4-2\xi^2}\,\xi - 4\right)\right)\right). \quad (4.117)$$

We can present $F(-\Box)$ in the action (4.67) as the Ostrogradski Representation. To this purpose let us construct the Weierstrass product for the function $F(z)$ of a complex variable z. Let us recall that a complex function $R(z)$ such that its logarithmic derivative $R'(z)/R(z)$ is a meromorphic function regular in the point $z=0$, has simple poles and satisfies $|R'(z)/R(z)| < C, z \in \Gamma_n, n = 1,2,...$, can be presented as

$$R(z) = R(0) e^{\frac{R'(0)}{R(0)} z} \prod \left(1 - \frac{z}{z_k}\right) e^{z/z_k}. \quad (4.118)$$

$\Gamma_n, n = 1,2,...$ is a set of special closed contours Γ_n such that the point $z=0$ is in all Γ_n, Γ_n is in Γ_{n+1}, and $S_n / d_n \leq C$, where S_n is a length of the contour Γ_n, and d_n is its distance from zero. In the case of a more week requirement $|R'(z)/R(z)| < M|z|^p\, z \in \Gamma_n, n = 1,2...$ we have

$$R(z) = e^{f(z)} \prod \left(1 - \frac{z}{z_k}\right) e^{Q_k(z)}, \qquad Q_k(z) = \sum_{l=1}^{p} \frac{1}{l}\left(\frac{z}{z_k}\right)^l, \quad (4.119)$$

where $f(z)$ is an entire function. In the case $R = F$ given by (4.68) the Weierstrass product can be written in the form

$$F(\alpha^2) = e^{f(\alpha^2)} \prod_n (\alpha^2 - \alpha_n^2). \quad (4.120)$$

The function $f(z)$ in our case is

$$f(z) = A + \beta z, \quad (4.121)$$

where constants A and β are determined by ξ and c. It is convenient to pick out real roots in (4.120) and combine the complex conjugated roots:

$$F(\alpha^2) = e^{A+\beta\alpha^2} \prod (\alpha^2 - m_k^2) \prod (\alpha^2 - \alpha_n^2)(\alpha^2 - \alpha_n^{*2}), \quad (4.122)$$

where m_k denote real roots. In the case of simple roots the Lagrangian up to a total derivative can be presented as a sum of an infinite number of fields

$$\mathcal{L} = \frac{1}{2}\phi F(\partial^2)\phi \approx \frac{1}{2}\sum \left[\varepsilon_n \psi_n e^{f(\partial^2)}(-\partial^2 + \alpha_n^2)\psi_n + c.c.\right], \quad (4.123)$$

where $\approx$ means equivalence up to a total derivative, ε_n are constants. It is the Ostrogradski representation. Note that for complex roots ψ_n are complex.
According to a general procedure of construction of the energy and pressure we write a generalization of (4.123) to a non-flat case

$$\mathcal{L}_g = \sum \mathcal{L}_g(\psi_n), \qquad \mathcal{L}_g(\psi_n) = \frac{\varepsilon_n}{2}\sqrt{-g}\,\psi_n e^{f(-\Box_g)}(\Box_g + \alpha_n^2)\psi_n, \quad (4.124)$$

and find

$$E_\psi = \sum_n E_n, \qquad E_n = \frac{\varepsilon_n}{2}(\dot{\psi}_n^2 - \alpha_n^2\psi_n^2)e^{f(\alpha_n^2)}, \quad (4.125)$$

$$P_\psi = \sum_n P_n, \qquad P_n = \frac{\varepsilon_n}{2}(\dot{\psi}_n^2 + \alpha_n^2\psi_n^2)e^{f(\alpha_n^2)}. \quad (4.126)$$

The E. O. M. for ψ_n is

$$(\partial^2 - \alpha_n^2)\psi_n = 0 \quad (4.127)$$

and its solutions are

$$\psi_n = A_n e^{\alpha_n t} + B_n e^{-\alpha_n t}. \quad (4.128)$$

For solutions (4.128) we obtain

$$E_\psi = 2\sum_n \alpha_n^2 \varepsilon_n A_n B_n e^{\beta\alpha_n^2}, \quad (4.129)$$

$$P_\psi = \sum_n \varepsilon_n \alpha_n^2 \left(A_n^2 e^{2\alpha_n t} + B_n^2 e^{-2\alpha_n t}\right)e^{\beta\alpha_n^2}. \quad (4.130)$$

On the other hand according to (4.111) and (4.113) we have

$$E = -2\sum_n A_n B_n \alpha_n^2 (\xi^2 - 2 + 2\xi^2\alpha_n^2), \quad (4.131)$$

$$P = \sum_n \left(A_n^2 e^{-2\alpha_n t} + B_n^2 e^{2\alpha_n t}\right)\alpha_n^2(\xi^2 - 2 + 2\xi^2\alpha_n^2). \quad (4.132)$$

Comparing (4.129), (4.130) and (4.131), (4.132) and using equation (4.85) we obtain that

$$E = E_{\psi}, \qquad P = P_{\psi} \quad (4.133)$$

if and only if

$$\varepsilon_n = -\left(2ce^{-2\alpha_n^2} + \xi^2\right)e^{-\beta\alpha_n^2}, \quad (4.134)$$

that is in accordance with general formula for ε_n. Note that we consider only simple roots α_n.
Now we consider the nonlocal model (4.67) in the Friedmann Universe. To consider the dynamics in such a system we need to solve nonlinear Friedmann equations (4.73), which represent hopelessly complicated problem. From (4.73) we obtain the following nonlinear integral equation in $H(t)$:

$$\dot{H} = -\frac{1}{2m_p^2}(\mathcal{P} + \mathcal{E}) = -\frac{1}{m_p^2}\left(\frac{\xi^2}{2}(\partial\phi)^2 - c\int_0^1 \left(\partial e^{(1+\rho)\mathcal{D}}\phi\right)\left(\partial e^{(1-\rho)\mathcal{D}}\phi\right)d\rho\right), \quad (4.135)$$

where $\mathcal{D} \equiv -\partial_t^2 - 3H(t)\partial_t$.
We choose a special solution of eq. (4.84) and find the corresponding Ostrogradski approximation in the flat space-time. After we deform the obtained approximate model to the case of the Friedmann Universe, assuming that exact solutions in the Friedmann metric are coincide with exact solutions in the flat space-time. Our starting point is the Lagrangian (4.123). The corresponding action in the non-flat space-time is as follows:

$$S_{Ostr.} = \int d^4x\sqrt{-g}\left(\frac{m_p^2}{2}R + \sum_n\left[\frac{\varepsilon_n}{2}\psi_n e^{\beta\alpha_n^2}\left(\Box_g + \alpha_n^2\right)\psi_n + c.c.\right] - \mathcal{V}(\psi_1,\dots,\psi_n)\right). \quad (4.136)$$

In the fields ϕ_n depend only on time and the metric is a spatially flat Friedmann metric, then we have the following equation for ψ_n

$$\varepsilon_n\left(\mathcal{D} + \alpha_n^2\right)\psi_n - e^{-\beta\alpha_n^2}\mathcal{V}'_{\psi_n} = 0 \Leftrightarrow \left(2ce^{-2\alpha_n^2} + \xi^2\right)\left(\mathcal{D} + \alpha_n^2\right)\psi_n + \mathcal{V}'_{\psi_n} = 0, \quad (4.137)$$

where $\mathcal{V}'_{\psi_n}$ is a derivative of $\mathcal{V}$ on ψ_n. Note that form of $\mathcal{V}(\psi_1,\dots,\psi_n)$ depends on choose of special solutions $\psi_1,\dots,\psi_n$. The energy and the pressure density in the Friedmann metric have the form

$$\mathcal{E}_{\text{mod}} = E_{\psi} + \mathcal{V}, \quad (4.138) \qquad \mathcal{P}_{\text{mod}} = P_{\psi} - \mathcal{V}, \quad (4.139)$$

where E_{ψ} and P_{ψ} are given by formulas (4.125) and (4.126) respectively. This means that the extra term $\mathcal{V}$ play a role of a potential term. The Friedmann equations of motion are:

$$3H^2 = \frac{1}{m_p^2}\left(E_{\psi} + \mathcal{V}\right), \qquad 3H^2 + 2\dot{H} = -\frac{1}{m_p^2}\left(P_{\psi} - \mathcal{V}\right). \quad (4.140)$$

Therefore

$$\dot{H} = -\frac{1}{2m_p^2}\left(P_\psi + E_\psi\right). \quad (4.141)$$

We choose such υ that ψ_k in the non-flat case are the same as in the flat case. Using (4.129) and (4.130) we get

$$\dot{H} = \frac{1}{2m_p^2}\sum_n \alpha_n^2 \varepsilon_n e^{\beta\alpha_n^2}\left(A_n^2 e^{2\alpha_n t} + 2A_n B_n + B_n^2 e^{-2\alpha_n t}\right). \quad (4.142)$$

Using (4.128) we can rewrite (4.142) as follows

$$\dot{H} = \frac{1}{2m_p^2}\sum_n \varepsilon_n e^{\beta\alpha_n^2}\dot{\phi}_n^2 = -\frac{1}{2m_p^2}\sum_n \left(2ce^{-2\alpha_n^2} + \xi^2\right)\dot{\phi}_n^2. \quad (4.143)$$

Substituting values of ε_n (formula 4.134) and using formulas (4.111) and (4.113), we obtain that

$$\dot{H} = \frac{1}{2m_p^2}\left(\frac{\xi^2}{2}(\partial\phi)^2 - c\int_0^1 \left(\partial\left(e^{-\rho\partial^2}\Phi\right)\right)\left(\partial\left(e^{\rho\partial^2}\Phi\right)\right)d\rho\right) = \frac{1}{2m_p^2}\left(E(\phi) + P(\phi)\right), \quad (4.144)$$

where

$$\phi(t) = \sum_n \phi_n(t) = \sum_n \left(A_n e^{\alpha_n t} + B_n e^{-\alpha_n t}\right). \quad (4.145)$$

Therefore,

$$H(t) = \frac{1}{2m_p^2}\left(\sum_n 2A_n B_n p_{\alpha_n} t - \sum_n\left(-\frac{A_n^2}{2\alpha_n}e^{-2\alpha_n t} + \frac{B_n^2}{2\alpha_n}e^{2\alpha_n t}\right)p_{\alpha_n}\right) + H_0, \quad (4.146)$$

where H_0 is an integration constant and we assume the sum goes over the complex conjugated roots. It is convenient to rewrite (4.143) as follows

$$\dot{H} = -\frac{1}{2m_p^2}\sum_n \frac{p_{\alpha_n}}{\alpha_n^2}\dot{\phi}_n^2 = -\frac{1}{2m_p^2}\left(\left(\xi^2 - 2\right)\sum_n \dot{\phi}_n^2 + 2\xi^2\sum_n \alpha_n^2\dot{\phi}_n^2\right). \quad (4.147)$$

Thus to obtain the crossing of cosmological constant barrier one should consider the case $\xi^2 < 2$ and the field $\phi(t)$, which consists of at least two modes. It is easy to see that $H(t)$ has no singular point at finite time. For some values of parameters we obtain bouncing solutions, which satisfy the conditions $H(0) = 0$ and $\dot{H}(0) > 0$.
An action for the tachyon in the CSSFT in the flat background when fields up to zero mass are taken into account is found to be

$$S_{SFT} = \frac{1}{g_0^2\alpha'^2}\int dx\left(u^2(x) - \frac{\alpha'}{2}\eta^{\mu\nu}\partial_\mu\phi(x)\partial_\nu\phi(x) + \frac{1}{4}\phi^2(x) + \frac{e^{2\lambda}}{3}\tilde{\phi}^2(x)\tilde{u}(x)\right) \quad (4.148)$$

where $\phi(x)$ is the tachyon field, $u(x)$ is an auxiliary field,

$$\tilde{\phi} = e^{\alpha'\lambda\Box}\phi , \quad (4.149)$$

and $\lambda = -\log\frac{4}{3\sqrt{3}} \approx 0.2616$. η is the flat Minkowskian metric, $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$. An auxiliary field $u(x)$ can be integrated out to yield

$$S_{tach} = \frac{1}{g_0^2}\int dx\left(-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi(x)\partial_\nu\phi(x) + \frac{1}{4}\phi^2(x) - \frac{e^{4\lambda}}{36}\left(\tilde{\phi}^2(x)\right)^2(x)\right). \quad (4.150)$$

A reasonable assumption that u has no the tilde simplifies the last term in this action. Namely, under this assumption and a rescaling $x \to 2\sqrt{2\lambda}x, \phi \to \frac{3}{\sqrt{2}}e^{-2\lambda}\phi$, and $g_0 \to 12\lambda e^{-2\lambda}g_0$ the action for the tachyon becomes

$$S_{tach,approx} = \frac{1}{g_0^2}\int dx\left(-\frac{\xi^2}{2}\eta^{\mu\nu}\partial_\mu\phi(x)\partial_\nu\phi(x) + \frac{1}{2}\phi^2(x) - \frac{1}{4}\left(e^{\frac{1}{8}\Box}\phi\right)^4(x)\right) \quad (4.151)$$

where $\xi^2 = \frac{1}{4\lambda} \approx 0.9556$.

With regard the cosmological scenarios with our universe to be considered as a D3-brane embedded in 10-dimensional space-time, the dynamics of this brane is given by the following covariant version of action (4.151) in a non-flat space

$$S = \int dx\sqrt{-g}\left(\frac{R}{2\kappa^2} + \frac{1}{g_0^2}\left(-\frac{\xi^2}{2}g^{\mu\nu}\partial_\mu\phi(x)\partial_\nu\phi(x) + \frac{1}{2}\phi^2(x) - \frac{1}{4}\Phi^4(x) - \Lambda\right)\right) \quad (4.152)$$

where

$$\Phi = e^{\frac{1}{8}\Box_g}\phi , \qquad \Box_g = \frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}\,g^{\mu\nu}\partial_\nu . \quad (4.153)$$

Here g is the metric, κ is a gravitational coupling constant and we choose such units that it is dimensionless, Λ is a constant. We focus on the four dimensional universe with the spatially flat FRW metric which can be written as

$$g_{\mu\nu} = diag\left(-1, a^2, a^2, a^2\right) \quad (4.154)$$

with $a = a(t)$ being a space homogeneous scale factor. In this particular case $\Box$ is expressed through a as

$$\Box_g = -\partial_t^2 - 3H\partial_t + \frac{1}{a^2}\partial_{x_i}^2 \quad (4.155)$$

where $H \equiv \dot{a}/a$ is the Hubble parameter and the dot denotes the time derivative.
The Friedmann equations for the space homogeneous tachyon field have the form

$$3H^2 = \frac{\kappa^2}{g_0^2}\left[\frac{\xi^2}{2}\dot{\phi}^2 - \frac{1}{2}\phi^2 + \frac{1}{4}\Phi^4 - \frac{1}{8}\left(\int_0^1 ds\left(e^{\frac{1}{8}s\mathcal{D}}\Phi^3\right)\mathcal{D}e^{-\frac{1}{8}s\mathcal{D}}\Phi + \int_0^1 ds\left(\partial_t e^{\frac{1}{8}s\mathcal{D}}\Phi^3\right)\partial_t e^{-\frac{1}{8}s\mathcal{D}}\Phi\right) + \Lambda\right], \quad (4.156)$$

$$\dot{H} = \frac{\kappa^2}{g_0^2}\left(-\frac{\xi^2}{2}\dot{\phi}^2 + \frac{1}{8}\int_0^1 ds\left(\partial_t e^{\frac{1}{8}s\mathcal{D}}\Phi^3\right)\partial_t e^{-\frac{1}{8}s\mathcal{D}}\Phi\right), \quad (4.157)$$

with

$$\Phi = e^{\frac{1}{8}\mathcal{D}}\phi, \qquad \mathcal{D} = -\partial_t^2 - 3H(t)\partial_t. \quad (4.158)$$

The equation of motion for the tachyon is

$$\left(\xi^2\mathcal{D} + 1\right)e^{-\frac{1}{4}\mathcal{D}}\Phi = \Phi^3. \quad (4.159)$$

The latter equation is in fact the continuity equation for the cosmic fluid.
The equation of motion for space homogeneous configurations of the tachyon field is found to be

$$\left(-\xi^2\partial_t^2 + 1\right)e^{\frac{1}{4}\partial_t^2}\Phi(t) = \Phi(t)^3, \quad (4.160)$$

where $\Phi = e^{-\frac{1}{8}\partial_t^2}\phi$.
The tachyon field starts from the origin, rolls down to the minimum of the tachyon potential and eventually stops in the minimum. The minima are located at $\Phi_0 = \pm 1$. For $\xi^2 \neq 0$ and $\xi^2 < \xi_{cr} \approx$ 1.38 there are damping fluctuations near the minimum. Let us note that in our case $\xi^2 < \xi_{cr}^2$. To analyze the late time behaviour one can linearize equation (4.160) as $\Phi = \Phi_0 - \delta\Phi$ keeping only liner in $\delta\Phi$ terms. A substitution yields the following equation for $\delta\Phi$

$$\left(-\xi^2\partial_t^2 + 1\right)e^{\frac{1}{4}\partial_t^2}\delta\Phi = 3\delta\Phi. \quad (4.161)$$

The most general real vanishing solution to equation (4.161) is

$$\delta\Phi(t) = \sum_{k\geq 0}\left(A_k e^{-m_k t} + A_k^* e^{-m_k^* t}\right). \quad (4.162)$$

The main contribution in (4.162) is given by $k = 0$ and can be represented as

$$\delta\phi(t) = Ce^{-rt}\sin(\nu t + \varphi) \text{ where } r \approx 1.1365, \; \nu \approx 1.7051 \quad (4.163)$$

Note that for $\xi^2 = 0$ (this case corresponds to a p-adic string with p = 3) equation (4.161) is simplified drastically and we have

$$m_k^2 = 4(\log 3 + 2\pi k i) \quad (4.164)$$

where again different branches may be considered. The principal branch is $k = 0$ and it corresponds to the rolling solution.
Now we describe the mathematical connection with some formulae concerning the aurea ratio and some numerical results of this chapter. We have that

For $\Phi = \frac{\sqrt{5}+1}{2}$, that is the value of aurea ratio, we obtain:

$$0.9556 \cong (\Phi)^{7/7} - (\Phi)^{-6/7} = 1.6180339 - 0.662014858 = 0.9560;$$
$$0.2616 \cong (\Phi)^{-24/7} + (\Phi)^{-38/7} = 0.192075047 + 0.073366139 = 0.2654;$$
$$1.38 \cong (\Phi)^{4/7} + (\Phi)^{-39/7} = 1.316501956 + 0.06849207 = 1.3849;$$
$$1.1365 \cong (\Phi)^{1/7} + (\Phi)^{-40/7} = 1.071162542 + 0.063941808 = 1.1351;$$
$$1.7051 \cong (\Phi)^{6/7} + (\Phi)^{-24/7} = 1.510540115 + 0.192075047 = 1.7026.$$

5. Mathematical connections

In this section we want to show some interesting mathematical connections that we have obtained between various equations regarding the **Sections 1, 3 and 4**.
We have the following mathematical connections between eqs. (1.33), (1.87b) and (1.127) of **Section 1** and various equations of **Section 4**. Indeed, we obtain that

$$\int D\Phi e^{iTr(Z\Phi)} \sum_{k=1}^{\infty} \left(\pi^2 k^4 e^{2Tr\Phi} - \frac{3}{2}\pi k^2 e^{Tr\Phi} \right) e^{-\pi k^2 Tre^{\Phi}} \Rightarrow \frac{1}{(2\pi)^D} \int_{k_0^2 - \vec{k}^2 > 2+\varepsilon} e^{ixk} \zeta\left(-\frac{k^2}{2} \right) \tilde{\phi}(k) dk = \frac{\phi}{1-\phi}$$
$$\Rightarrow \frac{1}{(2\pi)} \int_{|k_0| > \sqrt{2}+\varepsilon} e^{-ik_0 t} \zeta\left(\frac{k_0^2}{2} \right) \tilde{\phi}(k_0) dk_0 = \frac{\phi(t)}{1-\phi(t)} \Rightarrow \frac{1}{(2\pi)^D} \int_{R^D} e^{ixk} \zeta\left(-\frac{k^2}{2m^2} + h \right) \tilde{\phi}(k) dk = AC \sum_{n=1}^{+\infty} n^{-h} \phi^n$$
$$\Rightarrow \int d^4 x \sqrt{-g} \left(\frac{m_p^2}{2} R + \frac{\xi^2}{2} \phi \Box_g \phi + \frac{1}{2}\left(\phi^2 - c\Phi^2 \right) - \Lambda' \right); \quad (5.1)$$

$$\sum_{n=0}^{\infty} 4 \int_1^{\infty} d\ell \left(\ell^{-1/4} \sum_{q=1}^{\infty} \left(q^4 \pi^2 \ell - \frac{3}{2} q^2 \pi \right) \ell^{1/2} e^{-q^2 \pi \ell} \left(\frac{1}{2} \log \ell \right)^{2n} \right) \frac{(-1)^n}{(2n)!} z^{2n} \Rightarrow$$
$$\frac{1}{(2\pi)^D} \int_{k_0^2 - \vec{k}^2 > 2+\varepsilon} e^{ixk} \zeta\left(-\frac{k^2}{2} \right) \tilde{\phi}(k) dk = \frac{\phi}{1-\phi} \Rightarrow$$
$$\Rightarrow \frac{1}{(2\pi)} \int_{|k_0| > \sqrt{2}+\varepsilon} e^{-ik_0 t} \zeta\left(\frac{k_0^2}{2} \right) \tilde{\phi}(k_0) dk_0 = \frac{\phi(t)}{1-\phi(t)} \Rightarrow \frac{1}{(2\pi)^D} \int_{R^D} e^{ixk} \zeta\left(-\frac{k^2}{2m^2} + h \right) \tilde{\phi}(k) dk = AC \sum_{n=1}^{+\infty} n^{-h} \phi^n$$
$$\Rightarrow \int d^4 x \sqrt{-g} \left(\frac{m_p^2}{2} R + \frac{\xi^2}{2} \phi \Box_g \phi + \frac{1}{2}\left(\phi^2 - c\Phi^2 \right) - \Lambda' \right); \quad (5.2)$$

$$\sum_{n=-\infty}^{\infty}\int_{-\infty}^{+\infty}\frac{1}{(2\pi)^p}dk_1...dk_p\left[\frac{1}{\rho^2}\left(k_1^2+...+k_p^2+\frac{4\pi^2n^2}{\beta^2t^2}\right)^2+\kappa\left(\lambda_1\left(k_1^2+...+k_p^2\right)+\frac{4\pi^2\lambda_0 n^2}{\beta^2t^2}\right)\right]^{-s} =$$

$$=\frac{V_p\rho^{2s}}{2(2\pi)^p}\sum_{n=-\infty}^{\infty}\int_0^{\infty}dkk^{p/2-1}(k+c_+)^{-s}(k+c_-)^{-s} \Rightarrow \frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon}e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk=\frac{\phi}{1-\phi}$$

$$\Rightarrow \frac{1}{(2\pi)}\int_{|k_0|>\sqrt{2}+\varepsilon}e^{-ik_0t}\zeta\left(\frac{k_0^2}{2}\right)\tilde{\phi}(k_0)dk_0=\frac{\phi(t)}{1-\phi(t)} \Rightarrow \frac{1}{(2\pi)^D}\int_{R^D}e^{ixk}\zeta\left(-\frac{k^2}{2m^2}+h\right)\tilde{\phi}(k)dk=AC\sum_{n=1}^{+\infty}n^{-h}\phi^n$$

$$\Rightarrow \int d^4x\sqrt{-g}\left(\frac{m_p^2}{2}R+\frac{\xi^2}{2}\phi\Box_g\phi+\frac{1}{2}\left(\phi^2-c\Phi^2\right)-\Lambda'\right). \quad (5.3)$$

In conclusion, with regard the very interesting mathematical connections between eqs. (3.72), (3.87), (3.95) and (3.121) of **Section 3** and some equations of **Section 4**, we obtain that

$$\int_C E(f_0)(x)\overline{\chi}\left(\frac{x}{|x|}\right)\varphi(|x|)d^\times x=\int_{-\infty}^{\infty}\hat{\phi}(t)L\left(\overline{\chi},\frac{1}{2}+2\pi it\right)dt\int_0^{\infty}f_\infty(u)|u|^{-1/2+2\pi it}du \Rightarrow$$

$$\Rightarrow \frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon}e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk=\frac{\phi}{1-\phi} \Rightarrow$$

$$\Rightarrow \frac{1}{(2\pi)}\int_{|k_0|>\sqrt{2}+\varepsilon}e^{-ik_0t}\zeta\left(\frac{k_0^2}{2}\right)\tilde{\phi}(k_0)dk_0=\frac{\phi(t)}{1-\phi(t)} \Rightarrow \frac{1}{(2\pi)^D}\int_{R^D}e^{ixk}\zeta\left(-\frac{k^2}{2m^2}+h\right)\tilde{\phi}(k)dk=AC\sum_{n=1}^{+\infty}n^{-h}\phi^n$$

$$\Rightarrow \int d^4x\sqrt{-g}\left(\frac{m_p^2}{2}R+\frac{\xi^2}{2}\phi\Box_g\phi+\frac{1}{2}\left(\phi^2-c\Phi^2\right)-\Lambda'\right); \quad (5.4)$$

$$\left(EZ_\Lambda E^{-1}-S_\Lambda\right)V(h)F(x)=\int_C F(\lambda)\left\{\sqrt{|x/\lambda|}h(x/\lambda)\ell_\Lambda(x)-\sqrt{|x\lambda|}\left(\int_{\xi\in A,|\xi|<\Lambda^{-1}}Hh(\lambda\xi)\Psi(\xi x)d\xi\right)\right\}d^\times\lambda \Rightarrow$$

$$\Rightarrow \frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon}e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk=\frac{\phi}{1-\phi} \Rightarrow$$

$$\Rightarrow \frac{1}{(2\pi)}\int_{|k_0|>\sqrt{2}+\varepsilon}e^{-ik_0t}\zeta\left(\frac{k_0^2}{2}\right)\tilde{\phi}(k_0)dk_0=\frac{\phi(t)}{1-\phi(t)} \Rightarrow \frac{1}{(2\pi)^D}\int_{R^D}e^{ixk}\zeta\left(-\frac{k^2}{2m^2}+h\right)\tilde{\phi}(k)dk=AC\sum_{n=1}^{+\infty}n^{-h}\phi^n$$

$$\Rightarrow \int d^4x\sqrt{-g}\left(\frac{m_p^2}{2}R+\frac{\xi^2}{2}\phi\Box_g\phi+\frac{1}{2}\left(\phi^2-c\Phi^2\right)-\Lambda'\right); \quad (5.5)$$

$$\int_{A} Hg(u)\Psi(u)\log|u|du = \sum_{v\neq\infty}\int_{k_v} H_v g_v(u_v)\psi_v(u_v)\log|u_v|_v du_v + \int_R H_\infty g_\infty(u)\psi_\infty(u)\log|u|_\infty du =$$

$$= -h_0(1)\log 2\pi - \gamma h_0(1) - \sum_{v\neq\infty}\int'_{k_v^*}\frac{h_0(u^{-1})}{|1-u|_v}d^*u - \lim_{\varepsilon\to 0}\left(\int_{|\lambda-1|\geq\varepsilon}\frac{h_0(\lambda^{-1})}{\sqrt{\lambda}}\frac{\max\{\sqrt{\lambda},1/\sqrt{\lambda}\}}{|\lambda^2-1|}d\lambda + h_0(1)\log\varepsilon\right) \Rightarrow$$

$$\Rightarrow \frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \frac{\phi}{1-\phi} \Rightarrow$$

$$\Rightarrow \frac{1}{(2\pi)}\int_{|k_0|>\sqrt{2}+\varepsilon} e^{-ik_0t}\zeta\left(\frac{k_0^2}{2}\right)\tilde{\phi}(k_0)dk_0 = \frac{\phi(t)}{1-\phi(t)} \Rightarrow \frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2}+h\right)\tilde{\phi}(k)dk = AC\sum_{n=1}^{+\infty} n^{-h}\phi^n$$

$$\Rightarrow \int d^4x\sqrt{-g}\left(\frac{m_p^2}{2}R + \frac{\xi^2}{2}\phi\Box_g\phi + \frac{1}{2}(\phi^2 - c\Phi^2) - \Lambda'\right); \quad (5.6)$$

$$\frac{K_{\eta-1/2}(2|m|\pi y)}{\Gamma(\eta)} = \frac{2^{\eta-1/2}}{(2|m|\pi y)^{\eta-1/2}\sqrt{\pi}}\int_0^\infty \frac{\cos(2|m|\pi yt)}{(1+t^2)^\eta}dt << \frac{1+|r|^3}{|m|^3 y^3} \Rightarrow$$

$$\Rightarrow \frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \frac{\phi}{1-\phi} \Rightarrow$$

$$\Rightarrow \frac{1}{(2\pi)}\int_{|k_0|>\sqrt{2}+\varepsilon} e^{-ik_0t}\zeta\left(\frac{k_0^2}{2}\right)\tilde{\phi}(k_0)dk_0 = \frac{\phi(t)}{1-\phi(t)} \Rightarrow \frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2}+h\right)\tilde{\phi}(k)dk = AC\sum_{n=1}^{+\infty} n^{-h}\phi^n$$

$$\Rightarrow \int d^4x\sqrt{-g}\left(\frac{m_p^2}{2}R + \frac{\xi^2}{2}\phi\Box_g\phi + \frac{1}{2}(\phi^2 - c\Phi^2) - \Lambda'\right); \quad (5.7)$$

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