I think that it is possible the quantization of the gravity using the quantum field theory (I use the Lewis Ryder book) and the classical theory of fields (I use the Landau-Lifsits book).

It is evident, from the Lifsits book, that the Euler-Lagrange equation for the gravitational field is the Einstein field equation obtained from the Lagrangian of the gravitational field (I write the vacuum solution):

$$\begin{aligned} & G = g^{ik}\left(\Gamma^m_{il}\Gamma^l_{km} - \Gamma^l_{ik}\Gamma^m_{lm}\right) \\ & \mathcal{L} = -\frac{c^3\sqrt{-g}}{16\pi k} G \end{aligned}$$

the generalized variables, used in the Euler-Lagrange equation are:

$$\mathcal{L}\left(g^{ik}\,,\ \frac{\partial g^{lm}}{\partial x^r}\right)$$

as is evident from the variation of the action:

$$\delta S_g = -\frac{c^3}{16\pi k}\int\left\{\frac{\partial(G\sqrt{-g})}{\partial g^{ik}} - \frac{\partial}{\partial x^l}\frac{\partial(G\sqrt{-g})}{\partial\frac{\partial g^{ik}}{\partial x^l}}\right\}\delta g^{ik} d\Omega$$

so that it is simple to obtain the Hamiltonian[1] from this action if the Christoffel symbols are expressed in the variables subject to variation:

$$\Gamma^i_{kl} = -\frac{1}{2}\left(g_{pk}\frac{\partial g^{ip}}{\partial x^l} + g_{lp}\frac{\partial g^{ip}}{\partial x^k} - g^{ip}g_{kw}g_{lz}\frac{\partial g^{wz}}{\partial x^p}\right)$$

then, the Hamiltonian is:

$$\mathcal{H} = \frac{\partial g^{ij}}{\partial x^0}\frac{\partial\mathcal{L}}{\partial\frac{\partial g^{ij}}{\partial x^0}} - \mathcal{L}$$

the Lagrangian is a quadratic function in the derivative of the metric tensor, so that the Hamiltonian is just a quadratic function of the metric tensor; there are no linear terms of the time derivative of the metric tensor:

$$\mathcal{H} = \frac{1}{2}\frac{\partial^2\mathcal{L}}{\partial\frac{\partial g^{ij}}{\partial x^0}\partial\frac{\partial g^{ij}}{\partial x^0}}\frac{\partial g^{ij}}{\partial x^0}\frac{\partial g^{ij}}{\partial x^0} - \frac{1}{2}\frac{\partial^2\mathcal{L}}{\partial\frac{\partial g^{ij}}{\partial x^\mu}\partial\frac{\partial g^{kl}}{\partial x^\nu}}\frac{\partial g^{ij}}{\partial x^\mu}\frac{\partial g^{kl}}{\partial x^\nu}$$

where $\mu \in 1, 2, 3$ and so for each greek letter.

The G function is:

$$G = \frac{1}{4}\left[2g_{ac}\frac{\partial g^{ab}}{\partial x^d}\frac{\partial g^{cd}}{\partial x^b} - 2g_{ab}\frac{\partial g^{ab}}{\partial x^c}\frac{\partial g^{cd}}{\partial x^d} + g^{ik}(g_{ab}g_{cd} - g_{ad}g_{bc})\frac{\partial g^{ab}}{\partial x^i}\frac{\partial g^{cd}}{\partial x^k}\right]$$

it is simple to obtain the generalized momenta:

$$\begin{aligned} & \widetilde{\pi}_{ab} = \frac{\partial(G\sqrt{-g})}{\partial\frac{\partial g^{ab}}{\partial x^0}} = \frac{\sqrt{-g}}{4}\left(4g_{bc}\partial_a g^{c0} + 4g_{ac}\partial_b g^{c0} - 4g_{ab}\partial_c g^{c0} - 2\delta^0_b g_{cd}\partial_a g^{cd} - 2\delta^0_a g_{cd}\partial_b g^{cd} - 4g^{i0}g_{ac}g_{bd}\partial_i g^{cd} + 4g^{i0}g_{ab}g_{cd}\partial_i g^{cd}\right) = \\ & = \sqrt{-g}\left(\delta^0_a g_{bc}\partial_0 g^{c0} + \delta^0_b g_{ac}\partial_0 g^{c0} - g_{ab}\partial_0 g^{00} - \delta^0_a\delta^0_b g_{cd}\partial_0 g^{cd} - g^{00}g_{ac}g_{bd}\partial_0 g^{cd} + g^{00}g_{ab}g_{cd}\partial_0 g^{cd}\right) + \widetilde{F}_{ab} \\ & \widetilde{F}_{ab} = \frac{\sqrt{-g}}{2}\left(2\delta^\mu_a g_{bc}\partial_\mu g^{c0} + 2\delta^\mu_b g_{ac}\partial_\mu g^{c0} - 2g_{ab}\partial_\mu g^{\mu 0} - \delta^\mu_a\delta^0_b g_{cd}\partial_\mu g^{cd} - \delta^0_a\delta^\mu_b g_{cd}\partial_\mu g^{cd} - 2g^{\mu 0}g_{ac}g_{bd}\partial_\mu g^{cd} + 2g^{\mu 0}g_{ab}g_{cd}\partial_\mu g^{cd}\right) \end{aligned}$$

[1] $-\frac{c^3}{16\pi k}$ is a scale factor that is a constant in the calculations, the terms without the scale factors have the tilde grapheme

the Hamiltonian density $\widetilde{\mathcal{H}}\sqrt{-g}$ in the space-time is:

$$\widetilde{\mathcal{H}}\sqrt{-g} = \sqrt{-g}\left[2g_{\mu\mu}\frac{\partial g^{\mu 0}}{\partial x^0}\frac{\partial g^{\mu 0}}{\partial x^0} + g^{00}(g_{ab}g_{ab} - g_{aa}g_{bb})\frac{\partial g^{ab}}{\partial x^0}\frac{\partial g^{ab}}{\partial x^0}\right] +$$
$$-\frac{\sqrt{-g}}{4}\left[2g_{ab}\frac{\partial g^{a\nu}}{\partial x^\mu}\frac{\partial g^{b\mu}}{\partial x^\nu} - 2g_{ab}\frac{\partial g^{ab}}{\partial x^\mu}\frac{\partial g^{\mu\nu}}{\partial x^\nu} + g^{\mu\nu}(g_{ab}g_{cd} - g_{ad}g_{bc})\frac{\partial g^{ab}}{\partial x^\mu}\frac{\partial g^{cd}}{\partial x^\nu}\right]$$

The Hamiltonian (complete with the scale factors) in the quadrimensional point is:

$$\boxed{\mathcal{H} = \frac{c^3}{16\pi k}\left\{\left[g^{00}(g_{aa}g_{bb} - g_{ab}g_{ab})\frac{\partial g^{ab}}{\partial x^0}\frac{\partial g^{ab}}{\partial x^0} - 2g_{\mu\mu}\frac{\partial g^{\mu 0}}{\partial x^0}\frac{\partial g^{\mu 0}}{\partial x^0}\right] + \frac{1}{4}\left[2g_{ab}\frac{\partial g^{a\nu}}{\partial x^\mu}\frac{\partial g^{b\mu}}{\partial x^\nu} - 2g_{ab}\frac{\partial g^{ab}}{\partial x^\mu}\frac{\partial g^{\mu\nu}}{\partial x^\nu} + g^{\mu\nu}(g_{ab}g_{cd} - g_{ad}g_{bc})\frac{\partial g^{ab}}{\partial x^\mu}\frac{\partial g^{cd}}{\partial x^\nu}\right]\right\}}$$

this is a classical Hamiltonian, and it is not possible to obtain the generalized momenta from the time derivative of the metric tensor.

If the metric tensor is a constant in time, then:

- it is possible to add (arbitrary null) terms to the first part of the Hamiltonian so to obtain an invertible function between momenta and time derivative of the metric tensor
- it is possible to add a time-dependent arbitrary stress-tensor, containing the null time-derivative of the metric tensor
- it is possible that the relativistic invariant used in the action of the gravitational field is something slightly different from $G\sqrt{-g}$

so that the new momenta, and new Hamiltonian are[2]:

$$\widetilde{\pi}_{ab} = \frac{\partial(G\sqrt{-g})}{\partial\frac{\partial g^{ab}}{\partial x^0}} = -\sqrt{-g}\; g^{00}g_{ac}g_{bd}\partial_0 g^{cd} + \widetilde{F}_{ab} = \sqrt{-g}\; g^{00}\partial_0 g_{ab} + \widetilde{F}_{ab}$$
$$\partial_0 g_{ab} = \frac{\widetilde{\pi}_{ab} - \widetilde{F}_{ab}}{g^{00}\sqrt{-g}}$$
$$\partial_0 g^{ab} = -\frac{g^{ac}g^{bd}}{g^{00}\sqrt{-g}}(\widetilde{\pi}_{cd} - \widetilde{F}_{cd})$$
$$\widetilde{\mathcal{H}}\sqrt{-g} = -\sqrt{-g}g^{00}g_{aa}g_{bb}\frac{\partial g^{ab}}{\partial x^0}\frac{\partial g^{ab}}{\partial x^0} - \frac{\sqrt{-g}}{4}\left[2g_{ab}\frac{\partial g^{a\nu}}{\partial x^\mu}\frac{\partial g^{b\mu}}{\partial x^\nu} - 2g_{ab}\frac{\partial g^{ab}}{\partial x^\mu}\frac{\partial g^{\mu\nu}}{\partial x^\nu} + g^{\mu\nu}(g_{ab}g_{cd} - g_{ad}g_{bc})\frac{\partial g^{ab}}{\partial x^\mu}\frac{\partial g^{cd}}{\partial x^\nu}\right]$$
$$\widetilde{\mathcal{H}} = -g^{00}g_{aa}g_{bb}g^{ac}g^{bd}g^{ae}g^{bf}(-F_{cd} + \pi_{cd})(-F_{ef} + \pi_{ef}) - \frac{1}{4}\left[2g_{ab}\frac{\partial g^{a\nu}}{\partial x^\mu}\frac{\partial g^{b\mu}}{\partial x^\nu} - 2g_{ab}\frac{\partial g^{ab}}{\partial x^\mu}\frac{\partial g^{\mu\nu}}{\partial x^\nu} + g^{\mu\nu}(g_{ab}g_{cd} - g_{ad}g_{bc})\frac{\partial g^{ab}}{\partial x^\mu}\frac{\partial g^{cd}}{\partial x^\nu}\right]$$

then it is possible now the quantization of the Scharzschild metric, or the black hole diagonal metric:

$$\widehat{\mathcal{H}} = \frac{c^3}{16\pi k}g^{00}g_{aa}g_{bb}g^{ac}g^{bd}g^{ae}g^{bf}\left(F_{cd} + i\hbar\frac{\partial}{\partial g^{cd}}\right)\left(F_{ef} + i\hbar\frac{\partial}{\partial g^{ef}}\right) + \frac{c^3}{64\pi k}\left[2g_{ab}\frac{\partial g^{a\nu}}{\partial x^\mu}\frac{\partial g^{b\mu}}{\partial x^\nu} - 2g_{ab}\frac{\partial g^{ab}}{\partial x^\mu}\frac{\partial g^{\mu\nu}}{\partial x^\nu} + g^{\mu\nu}(g_{ab}g_{cd} - g_{ad}g_{bc})\frac{\partial g^{ab}}{\partial x^\mu}\frac{\partial g^{cd}}{\partial x^\nu}\right]$$

the Scharzschild metric is:

$$ds^2 = \left(1 - \frac{r_g}{r}\right)c^2dt^2 - r^2(\sin^2\theta d\phi^2 + d\theta^2) - \frac{dr^2}{1 - \frac{r_g}{r}}$$
$$g^{00} = \left(1 - \frac{r_g}{r}\right)^{-1} \;,\; \partial_1 g^{00} = -\left(1 - \frac{r_g}{r}\right)^{-2}\frac{r_g}{r^2}$$
$$g^{11} = g^{rr} = -\left(1 - \frac{r_g}{r}\right) \;,\; \partial_1 g^{11} = -\frac{r_g}{r^2}$$
$$g^{22} = g^{\theta\theta} = -r^{-2} \;,\; \partial_1 g^{22} = \frac{2}{r^3}$$
$$g^{33} = g^{\phi\phi} = -r^2\sin^{-2}\theta \;,\; \partial_1 g^{33} = r^{-3}\sin^{-2}\theta \;,\; \partial_2 g^{33} = 2r^{-2}\sin^{-2}\theta\cot\theta$$
$$g_{aa} = \frac{1}{g^{aa}}$$

[2]this is obtained adding quadratic terms $\partial_0 g^{ab}\partial_0 g^{cd}$, that eliminate some terms in the Lagrangian, and in the momenta

then:

$$F_{aa} = 0$$

$$i\hbar \frac{d}{dt}\Psi = -\frac{\hbar^2 c^3}{16\pi k} g^{00} g^{aa} g^{aa} \frac{\partial^2}{\partial g^{aa} \partial g^{aa}} + \frac{c^3}{64\pi k}\left[2 g_{\mu\mu} \partial_\mu g^{\mu\mu} \partial_\mu g^{\mu\mu} - 2 g_{aa} \partial_\mu g^{aa} \partial_\mu g^{\mu\mu} + g^{\mu\mu} g_{aa} g_{bb} \partial_\mu g^{aa} \partial_\mu g^{bb} - g^{\mu\mu} g_{aa} g_{aa} \partial_\mu g^{aa} \partial_\mu g^{aa}\right] \Psi$$

$$\boxed{i\hbar \frac{d}{dt}\Psi\left(g^{mm}, t\right) = \frac{c^3}{16\pi k}\left\{-\hbar^2 g^{00} g^{aa} g^{aa} \frac{\partial^2}{\partial g^{aa} \partial g^{aa}} - \frac{g^{11}}{2}\left[\frac{2r_g}{r^5}\left(1 - \frac{r_g}{r}\right)^{-2} \frac{1}{g^{00} g^{22}} + \frac{2r_g \sin^{-2}\theta}{r^5}\left(1 - \frac{r_g}{r}\right)^{-2} \frac{1}{g^{00} g^{33}} - \frac{4\sin^{-2}\theta}{r^6} \frac{1}{g^{22} g^{33}}\right]\right\} \Psi\left(g^{mm}, t\right)}$$

this is the wave function of a massless particle moving in the vacuum in a stationary gravitational field; this is also the differential equation for gravitational field fluctuations with massless bosons (for example gravitons).