On some mathematical connections between Fermat's Last Theorem, Modular Functions, Modular Elliptic Curves and some sector of String Theory

Michele Nardelli [1,2]

[1] Dipartimento di Scienze della Terra – Università degli Studi di Napoli "Federico II"
Largo S. Marcellino, 10 – 80138 Napoli (Italy)

[2] Dipartimento di Matematica ed Applicazioni "R. Caccioppoli"
Università degli Studi di Napoli "Federico II" – Polo delle Scienze e delle Tecnologie
Monte S. Angelo, Via Cintia (Fuorigrotta), 80126 Napoli (Italy)

Abstract

This paper is fundamentally a review, a thesis, of principal results obtained in some sectors of Number Theory and String Theory of various authoritative theoretical physicists and mathematicians.
Precisely, we have described some mathematical results regarding the Fermat's Last Theorem, the Mellin transform, the Riemann zeta function, the Ramanujan's modular equations, how primes and adeles are related to the Riemann zeta functions and the p-adic and adelic string theory.
Furthermore, we show that also the fundamental relationship concerning the Palumbo-Nardelli model (a general relationship that links bosonic string action and superstring action, i.e. bosonic and fermionic strings in all natural systems), can be related with some equations regarding the p-adic (adelic) string sector.
Thence, in conclusion, we have described some new interesting connections that are been obtained between String Theory and Number Theory, with regard the arguments above mentioned.
In the **Chapters 1** and **2**, we have described the mathematics concerning the Fermat's Last Theorem, precisely the Wiles approach in the **Chapter 1** and further mathematical aspects concerning the Fermat's Last Theorem, precisely the modular forms, the Euler products, the Shimura map and the automorphic L-functions in the **Chapter 2**. Furthermore. In this chapter, we have described also some mathematical applications of the Mellin transform, only mentioned in the Chapter 1, the zeta-function quantum field theory and the quantum L-functions.
In the **Chapter 3**, we have described how primes and adeles are related to the Riemann zeta function, precisely the Connes approach. In the **Chapter 4**, we have described the p-adic and adelic strings, precisely the open and closed p-adic strings, the adelic strings, the solitonic q-branes of p-adic string theory and the open and closed scalar zeta strings.
In the **Chapter 5**, we have described some correlations obtained between some solutions in string theory, Riemann zeta function and Palumbo-Nardelli model. Precisely, we have showed the cosmological solutions from the D3/D7 system, the classification and stability of cosmological solutions, the solution applied to ten dimensional IIB supergravity, the connections with some equations concerning the Riemann zeta function, the Palumbo-Nardelli model and the Ramanujan's identities. Furthermore, we have described the interactions between intersecting D-branes and the general action and equations of motion for a probe D3-brane moving through a type IIB supergravity background. Finally, in the **Chapter 6**, we have showed the connections between the equations of the various chapters.

Introduzione e riassunto

L'ultimo teorema di Fermat è una generalizzazione dell'equazione diofantea $a^2 + b^2 = c^2$. Già gli antichi Greci ed i Babilonesi sapevano che questa equazione ha delle soluzioni intere, come (3, 4, 5) $(3^2 + 4^2 = 5^2)$ o (5, 12, 13) $(5^2 + 12^2 = 13^2)$. Queste soluzioni sono conosciute come "terne pitagoriche" e ne esistono infinite, anche escludendo le soluzioni banali per cui a, b e c hanno un divisore in comune e quelle ancor più banali in cui almeno uno dei numeri è uguale a zero.
Secondo l'ultimo teorema di Fermat, non esistono soluzioni intere positive quando l'esponente 2 è sostituito da un numero intero maggiore. Il teorema è particolarmente noto per la sua correlazione con molti argomenti matematici che apparentemente non hanno nulla a che vedere con la Teoria dei Numeri. Inoltre, la ricerca di una dimostrazione è stata all'origine dello sviluppo di importanti aree della matematica, anche legate a moderni settori della fisica teorica, quali ad esempio la Teoria delle Stringhe.
L'ultimo teorema di Fermat può essere dimostrato per n = 4 e nel caso in cui n è un numero primo: se infatti si trova una soluzione $a^{kp} + b^{kp} = c^{kp}$, si ottiene immediatamente una soluzione $\left(a^k\right)^p + \left(b^k\right)^p = \left(c^k\right)^p$. Nel corso degli anni il teorema venne dimostrato per un numero sempre maggiore di esponenti speciali n, ma il caso generale rimaneva evasivo. Il caso n = 5 è stato dimostrato da Dirichlet e Legendre nel 1825 ed il caso n = 7 da Gabriel Lamé nel 1839. Nel 1983 G. Faltings dimostrò la congettura di Mordell, che implica che per ogni n > 2, c'è al massimo un numero finito di interi "co-primi" a, b e c con $a^n + b^n = c^n$. (In matematica, gli interi a e b si dicono "co-primi" o "primi tra loro" se e solo se essi non hanno nessun divisore comune eccetto 1 e -1, o, equivalentemente, se il loro massimo comune divisore è 1).
Utilizzando i sofisticati strumenti della geometria algebrica (in particolare curve ellittiche e forme modulari), della teoria di Galois e dell'algebra di Hecke, il matematico di Cambridge Andrew John Wiles, dell'Università di Princeton, con l'aiuto del suo primo studente, Richard Taylor, diede una dimostrazione dell'ultimo teorema di Fermat, pubblicata nel 1995 nella rivista specialistica "Annals of Mathematics".
Nel 1986, Ken Ribet aveva dimostrato la "Congettura Epsilon" di Gerhard Frey secondo la quale ogni contro-esempio $a^n + b^n = c^n$ all'ultimo teorema di Fermat avrebbe prodotto una curva ellittica definita come: $y^2 = x \cdot \left(x - a^n\right) \cdot \left(x + b^n\right)$, che fornirebbe un contro-esempio alla "Congettura di Taniyama-Shimura". Quest'ultima congettura propone un collegamento profondo fra le curve ellittiche e le forme modulari. Wiles e Taylor hanno stabilito un caso speciale della Congettura di Taniyama-Shimura sufficiente per escludere tali contro-esempi in seguito all'ultimo teorema di Fermat. In pratica, la dimostrazione che le curve ellittiche semistabili sui razionali sono modulari, rappresenta una forma ridotta della Congettura di Taniyama-Shimura che tuttavia è sufficiente per provare l'ultimo teorema di Fermat.
Le curve ellittiche sono molto importanti nella Teoria dei Numeri e ne costituiscono il maggior campo di ricerca attuale. Nel campo delle curve ellittiche, i "numeri p-adici" sono conosciuti come "numeri l-adici", a causa dei lavori di Jean-Pierre Serre. Il numero primo p è spesso riservato per l'aritmetica modulare di queste curve.
Il sistema dei numeri p-adici è stato descritto per la prima volta da Kurt Hensel nel 1897. Per ogni numero primo p, il sistema dei numeri p-adici estende l'aritmetica dei numeri razionali in modo differente rispetto l'estensione verso i numeri reali e complessi. L'uso principale di questo strumento viene fatto nella Teoria dei Numeri. L'estensione è ottenuta da un'interpretazione alternativa del concetto di valore assoluto. Il motivo della creazione dei numeri p-adici era il tentativo di introdurre il concetto e le tecniche delle "serie di potenze" nel campo della Teoria dei Numeri. Più concretamente per un dato numero primo p, il campo Q_p dei numeri p-adici è un'estensione dei numeri razionali. Se tutti i campi Q_p vengono considerati collettivamente, si

arriva al “principio locale-globale” di Helmut Hasse, il quale, a grandi linee, afferma che certe equazioni possono essere risolte nell’insieme dei numeri razionali se e solo se possono essere risolte negli insiemi dei numeri reali e dei numeri p-adici per ogni p. Il campo Q_p possiede una topologia derivata da una metrica, che è, a sua volta, derivata da una stima alternativa dei numeri razionali. Questa metrica è “completa”, nel senso che ogni serie di Cauchy converge.
Scopo del presente lavoro è quello di evidenziare le connessioni ottenute tra la matematica inerente la dimostrazione dell’ultimo teorema di Fermat ed alcuni settori della Teoria di Stringa, precisamente la supersimmetria p-adica e adelica in teoria di stringa.
I settori inerenti la dimostrazione dell’ultimo teorema di Fermat, riguardano quelle funzioni chiamate L p-adiche connesse alla funzione zeta di Riemann, quale estensione analitica al piano complesso della serie di Dirichlet. Tali funzioni sono strettamente correlate sia ai numeri primi, sia alla funzione zeta, i cui teoremi sono già stati connessi matematicamente con la teoria di stringa in alcuni precedenti lavori.
Quindi, per concludere, anche dalla matematica che riguarda l’ultimo teorema di Fermat è possibile ottenere, come vedremo nel corso del lavoro, ulteriori connessioni tra Teoria di Stringa (p-adic string theory), Numeri Primi, Funzione zeta di Riemann (numeri p-adici, funzioni L p-adiche) e Serie di Fibonacci (quindi identità e funzioni di Ramanujan), che, a loro volta, verranno correlate anche al modello Palumbo-Nardelli.

Chapter 1.

The mathematics concerning the Fermat’s Last Theorem

1.1 The Wiles approach.[1]

An elliptic curve over Q is said to be modular if it has a finite covering by a modular curve of the form $X_0(N)$. Any such elliptic curve has the property that its Hasse-Weil zeta function has an analytic continuation and satisfies a functional equation of the standard type. If an elliptic curve over Q with a given j-invariant is modular then it is easy to see that all elliptic curves with the same j-invariant are modular. A well-known conjecture which grew out of the work of Shimura and Taniyama in the 1950’s and 1960’s asserts that every elliptic curve over Q is modular.
In 1985 Frey made the remarkable observation that this conjecture should imply Fermat’s Last Theorem. The Wiles approach to the study of elliptic curves is via their associated Galois representations. Suppose that ρ_p is the representation of $Gal(\overline{Q}/Q)$ on the p-division points of an elliptic curve over Q, and suppose that ρ_3 is irreducible. The choice of 3 is critical because a crucial theorem of Langlands and Tunnell shows that if ρ_3 is irreducible then it is also modular. Thence, under the hypothesis that ρ_3 is semistable at 3, together with some milder restrictions on the ramification of ρ_3 at the other primes, every suitable lifting of ρ_3 is modular. Furthermore, Wiles has obtained that E is modular if and only if the associated 3-adic representation is modular.
The key development in the proof is a new and surprising link between two strong but distinct traditions in number theory, the relationship between Galois representations and modular forms on the one hand and the interpretation of special values of L-functions on the other.
The restriction that ρ_3 be irreducible at 3 is bypassed by means of an intriguing argument with families of elliptic curves which share a common ρ_5. Using this, we complete the proof that all semistable elliptic curves are modular. In particular, this yields to the proof of Fermat’s Last Theorem.
Now we present the methods and results in more detail.

Let f be an eigenform associated to the congruence subgroup $\Gamma_1(N)$ of $SL_2(Z)$ of weight $k \geq 2$ and character χ. Thus if T_n is the Hecke operator associated to an integer n there is an algebraic integer $c(n,f)$ such that $T_n f = c(n,f)f$ for each n. We let K_f be the number field generated over Q by the $\{c(n,f)\}$ together with the values of χ and let O_f be its ring of integers. For any prime λ of O_f let $O_{f,\lambda}$ be the completion of O_f at λ. The following theorem is due to Eichler and Shimura (for k > 2).

THEOREM 1.

For each prime $p \in Z$ and each prime $\lambda | p$ of O_f there is a continuous representation

$$\rho_{f,\lambda} : Gal(\overline{Q}/Q) \rightarrow GL_2(O_{f,\lambda}) \qquad (1)$$

which is unramified outside the primes dividing Np and such that for all primes q ∤ Np,

$$trace\ \rho_{f,\lambda}(Frob\ q) = c(q,f), \quad \det\ \rho_{f,\lambda}(Frob\ q) = \chi(q)q^{k-1}. \qquad (2)$$

We will be concerned with trying to prove results in the opposite direction, that is to say, with establishing criteria under which a λ-adic representation arises in this way from a modular form.
Assume

$$\rho_0 : Gal(\overline{Q}/Q) \rightarrow GL_2(\overline{F}_p) \qquad (3)$$

is a continuous representation with values in the algebraic closure of a finite field of characteristic p and that $\det \rho_0$ is odd. We say that ρ_0 is modular if ρ_0 and $\rho_{f,\lambda} \bmod \lambda$ are isomorphic over $\overline{F}_p$ for some f and λ and some embedding of O_f/λ in $\overline{F}_p$. Serre has conjectured that every irreducible ρ_0 of odd determinant is modular.
If O is the ring of integers of a local field (containing Q_p) we will say that

$$\rho : Gal(\overline{Q}/Q) \rightarrow GL_2(O) \qquad (4)$$

is a lifting of ρ_0 if, for a specified embedding of the residue field of O in $\overline{F}_p$, $\overline{\rho}$ and ρ_0 are isomorphic over $\overline{F}_p$. We will restrict our attention to two cases:

(I) ρ_0 is ordinary (at p) by which we mean that there is a one-dimensional subspace of $\overline{F}_p^2$, stable under a decomposition group at p and such that the action on the quotient space is unramified and distinct from the action on the subspace.

(II) ρ_0 is flat (at p), meaning that as a representation of a decomposition group at p, ρ_0 is equivalent to one that arises from a finite flat group scheme over Z_p, and $\det \rho_0$ restricted to an inertia group at p is the cyclotomic character.

CONJECTURE.

Suppose that $\rho : Gal(\overline{Q}/Q) \rightarrow GL_2(O)$ is an irreducible lifting of ρ_0 and that ρ is unramified outside of a finite set of primes. There are two cases:

(i) *Assume that ρ_0 is ordinary. Then if ρ is ordinary and $\det \rho = \varepsilon^{k-1}\chi$ for some integer $k \geq 2$ and some χ of finite order, ρ comes from a modular form.*

(ii) *Assume that ρ_0 is flat and that p is odd. Then if ρ restricted to a decomposition group at p is equivalent to a representation on a p-divisible group, again ρ comes from a modular form.*

Now we will assume that p is an odd prime, we have the following theorem:

THEOREM 2.

Suppose that ρ_0 is irreducible and satisfies either (I) or (II) above. Suppose also that

(i) *ρ_0 is absolutely irreducible when restricted to $Q\left(\sqrt{(-1)^{\frac{p-1}{2}} p}\right)$.*

(ii) *If $q \equiv -1 \mod p$ is ramified in ρ_0 then either $\rho_0|D_q$ is reducible over the algebraic closure where D_q is a decomposition group at q or $\rho_0|I_q$ is absolutely irreducible where I_q is an inertia group at q.*

Then any representation ρ as in the conjecture does indeed come from a modular form.

The only condition which really seems essential to our method is the requirement that ρ_0 is modular. The most interesting case at the moment is when p = 3 and ρ_0 can be defined over F_3. Then since $PGL_2(F_3) \cong S_4$ every such representation is modular by the theorem of Langlands and Tunnell. In particular, every representation into $GL_2(Z_3)$ whose reduction satisfies the given conditions is modular. We deduce:

THEOREM 3.

Suppose that E is an elliptic curve defined over Q and that ρ_0 is the Galois action on the 3-division points. Suppose that E has the following properties:

(i) *E has good or multiplicative reduction at 3.*

(ii) *ρ_0 is absolutely irreducible when restricted to $Q(\sqrt{-3})$.*

(iii) *For any $q \equiv -1 \mod 3$ either $\rho_0|D_q$ is reducible over the algebraic closure or $\rho_0|I_q$ is absolutely irreducible.*

Then E should be modular.

The important class of semistable curves, i.e., those with square-free conductor, satisfies (i) and (iii) but not necessarily (ii).

THEOREM 4.

Suppose that E is a semistable elliptic curve defined over Q. Then E is modular.

In 1986, Serre conjectured and Ribet proved a property of the Galois representation associated to modular forms which enabled Ribet to show that Theorem 4 implies "Fermat's Last Theorem". Furthermore, we have the following theorems:

THEOREM 5.

Suppose that $u^p + v^p + w^p = 0$ *with* $u, v, w \in Q$ *and* $p \geq 3$ *then* $uvw = 0$. *(Equivalently – there are no non-zero integers a,b,c,n with n > 2 such that* $a^n + b^n = c^n$ *.)*

THEOREM 6.

Suppose that ρ_0 *is irreducible and satisfies the hypothesis of the conjecture, including (I) above. Suppose further that*

(i) $\rho_0 = Ind_L^Q \kappa_0$ *for a character* κ_0 *of an imaginary quadratic extension L of Q which is unramified at p.*
(ii) $\det \rho_0 \big| I_p = \omega$.

Then a representation ρ as in the conjecture does indeed come from a modular form.

Wiles has worked on the Iwasawa conjecture for totally real fields and some applications of it, with the assumption that the reduction of a given ℓ-adic representation was reducible and tried to prove under this hypothesis that the representation itself would have to be modular. Thence, we write p for ℓ because of the connections with Iwasawa theory.
In the solution to the Iwasawa conjecture for totally real fields, Wiles has introduced a new technique in order to deal with the trivial zeroes.
It involved replacing the standard Iwasawa theory method of considering the fields in the cyclotomic Z_p-extension by a similar analysis based on a choice of infinitely many distinct primes $q_i \equiv 1 \mod p^{n_i}$ with $n_i \to \infty$ as $i \to \infty$. Wiles has developed further the idea of using auxiliary primes to replace the change of field that is used in Iwasawa theory.

Let p be an odd prime. Let Σ be a finite set of primes including p and let Q_Σ be the maximal extension of Q unramified outside this set and ∞. Throughout we fix an embedding of $\overline{Q}$, and so also of Q_Σ, in C. We will also fix a choice of decomposition group D_q for all primes q in Z. Suppose that k is a finite field characteristic p and that

$$\rho_0 : Gal(Q_\Sigma / Q) \to GL_2(k) \qquad (5)$$

is an irreducible representation. We will assume that ρ_0 comes with its field of definition k and that $\det \rho_0$ is odd.
We will restrict our choice of ρ_0 further by assuming that either:

(i) ρ_0 is ordinary. The restriction of ρ_0 to the decomposition group D_p has (for a suitable choice of basis) the form

$$\rho_0 \big| D_p \approx \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix} \qquad (6)$$

where χ_1 and χ_2 are homomorphisms from D_p to k^* with χ_2 unramified. Moreover we require that $\chi_1 \neq \chi_2$.

(ii) ρ_0 is flat at p but not ordinary. Then $\rho_0 \big| D_p$ is the representation associated to a finite flat group scheme over Z_p but is not ordinary in the sense of (i). We will assume also that $\det \rho_0 \big| I_p = \omega$ where I_p is an inertia group at p and ω is the Teichmuller character giving the action on p^{th} roots of unity.

Furthermore, we have the following restrictions on the deformations:

(i) (a) *Selmer deformations*. In this case we assume that ρ_0 is ordinary, with notion as above, and that the deformation has a representative $\rho : Gal(Q_\Sigma / Q) \to GL_2(A)$ with the property that (for a suitable choice of basis)

$$\rho \big| D_p \approx \begin{pmatrix} \tilde{\chi}_1 & * \\ 0 & \tilde{\chi}_2 \end{pmatrix}$$

with $\tilde{\chi}_2$ unramified, $\tilde{\chi} \equiv \chi_2 \mod m$, and $\det \rho \big| I_p = \varepsilon \omega^{-1} \chi_1 \chi_2$ where ε is the cyclotomic character, $\varepsilon : Gal(Q_\Sigma / Q) \to Z_p^*$, giving the action on all p-power roots of unity, ω is of order prime to p satisfying $\omega \equiv \varepsilon \mod p$, and χ_1 and χ_2 are the characters of (i) viewed as taking values in $k^* \mapsto A^*$.

(i) (b) *Ordinary deformations*. The same as in (i) (a) but with no condition on the determinant.

(i) (c) *Strict deformations*. This is a variant on (i) (a) which we only use when $\rho_0 \big| D_p$ is not semisimple and not flat. We also assume that $\chi_1 \chi_2^{-1} = \omega$ in this case. Then a strict deformation is an in (i) (a) except that we assume in addition that $(\tilde{\chi}_1 / \tilde{\chi}_2) \big| D_p = \varepsilon$.

(ii) *Flat (at p) deformations*. We assume that each deformations ρ to $GL_2(A)$ has the property that for any quotient A / a of finite order $\rho \big| D_p \mod a$ is the Galois representation associated to the $\overline{Q}_p$-points of a finite flat group scheme over Z_p.

In each of these four cases, as well as in the unrestricted case one can verify that Mazur's use of Schlessinger's criteria proves the existence of a universal deformation

$$\rho : Gal(Q_\Sigma / Q) \to GL_2(R) \qquad (7)$$

With regard the primes $q \neq p$ which are ramified in ρ_0, we distinguish three special cases:

(A) $\rho_0\big|D_q = \begin{pmatrix} \chi_1 & * \\ & \chi_2 \end{pmatrix}$ for a suitable choice of basis, with χ_1 and χ_2 unramified, $\chi_1\chi_2^{-1} = \omega$ and the fixed space of I_q of dimension 1,

(B) $\rho_0\big|I_q = \begin{pmatrix} \chi_q & 0 \\ 0 & 1 \end{pmatrix}$, $\chi_q \neq 1$, for a suitable choice of basis,

(C) $H^1(Q_q, W_\lambda) = 0$ where $W_\lambda = \{f \in Hom_k(U_\lambda, U_\lambda) : tracef = 0\} \cong (Sym^2 \otimes \det^{-1})\rho_0$.

Then in each case we can define a suitable deformation theory by imposing additional restrictions on those we have already considered, namely:

(A) $\rho\big|D_q = \begin{pmatrix} \psi_1 & * \\ & \psi_2 \end{pmatrix}$ for a suitable choice of basis of A^2 with ψ_1 and ψ_2 unramified and $\psi_1\psi_2^{-1} = \varepsilon$;

(B) $\rho\big|I_q = \begin{pmatrix} \chi_q & 0 \\ 0 & 1 \end{pmatrix}$ for a suitable choice of basis (χ_q of order prime to p, so the same character as above);

(C) $\det\rho\big|I_q = \det\rho_0\big|I_q$, i.e., of order prime to p.

Thus if M is a set of primes in Σ distinct from p and each satisfying one of (A), (B) or (C) for ρ_0, we will impose the corresponding restriction at each prime in M.
Thus to each set of data $D = \{\cdot, \Sigma, \mathrm{O}, \mathrm{M}\}$ where . is Se, str, ord, flat or unrestricted, we can associate a deformation theory to ρ_0 provided

$$\rho_0 : Gal(Q_\Sigma / Q) \to GL_2(k) \qquad (8)$$

is itself of type D and O is the ring of integers of a totally ramified extension of $W(k)$; ρ_0 is ordinary if . is Se or ord, strict if . is strict and flat if . is flat; ρ_0 is of type M, i.e., of type (A), (B) or (C) at each ramified primes $q \neq p$, $q \in \mathrm{M}$.

Suppose that q is a prime not dividing N. Let $\Gamma_1(N,q) = \Gamma_1(N) \cap \Gamma_0(q)$ and let $X_1(N,q) = X_1(N,q)_{/Q}$ be the corresponding curve. The two natural maps $X_1(N,q) \to X_1(N)$ induced by the maps $z \to z$ and $z \to qz$ on the upper half plane permit us to define a map $J_1(N) \times J_1(N) \to J_1(N,q)$. Using a theorem of Ihara, Ribet shows that this map is injective. Thus we can define φ by

$$0 \to J_1(N) \times J_1(N) \xrightarrow{\varphi} J_1(N,q). \qquad (9)$$

Dualizing, we define B by

$$0 \to B \xrightarrow{\psi} J_1(N,q) \xrightarrow{\hat{\varphi}} J_1(N) \times J_1(N) \to 0.$$

Let $T_1(N,q)$ be the ring of endomorphism of $J_1(N,q)$ generated by the standard Hecke operators. One can check that U_p preserves B either by an explicit calculation or by noting that B is the maximal abelian subvariety of $J_1(N,q)$ with multiplicative reduction at q. We set $J_2 = J_1(N)\times J_1(N)$. More generally, one can consider $J_H(N)$ and $J_H(N,q)$ in place of $J_1(N)$ and $J_1(N,q)$ (where $J_H(N,q)$ corresponds to $X_1(N,q)/H$) and we write $T_H(N)$ and $T_H(N,q)$ for the associated Hecke rings.
In the following lemma if m is a maximal ideal of $\mathrm{T}_1(Nq^{r-1})$ or $\mathrm{T}_1(Nq^r)$ we use $m^{(q)}$ to denote the maximal ideal of $\mathrm{T}_1^{(q)}(Nq^r,q^{r+1})$ compatible with m, the ring $\mathrm{T}_1^{(q)}(Nq^r,q^{r+1})\subset \mathrm{T}_1(Nq^r,q^{r+1})$ being the sub-ring obtained by omitting U_q from the list of generators.

LEMMA 1.

If $q\neq p$ is a prime and $r\geq 1$ then the sequence of abelian varieties

$$0\to J_1(Nq^{r-1})\xrightarrow{\xi_1} J_1(Nq^r)\times J_1(Nq^r)\xrightarrow{\xi_2} J_1(Nq^r,q^{r+1}) \quad (10)$$

where $\xi_1 = ((\pi_{1,r}\circ\pi)^*, -(\pi_{2,r}\circ\pi)^*)$ and $\xi_2 = (\pi_{4,r}^*, \pi_{3,r}^*)$ induces a corresponding sequence of p-divisible groups which becomes exact when localized at any $m^{(q)}$ for which ρ_m is irreducible.

Now, we have the following theorem:

THEOREM 7.

Assume that ρ_0 is modular and absolutely irreducible when restricted to $Q\left(\sqrt{(-1)^{\frac{p-1}{2}}p}\right)$. Assume also that ρ_0 is of type (A), (B) or (C) at each $q\neq p$ in Σ. Then the map $\varphi_D : R_D \to \mathrm{T}_D$ (remember that φ_D is an isomorphism) is an isomorphism for all D associated to ρ_0, i.e., where $D=(\cdot,\Sigma,\mathrm{O},\mathrm{M})$ with $\cdot$ =Se, str, fl or ord. In particular if $\cdot$ =Se, str or fl and f is any newform for which $\rho_{f,\lambda}$ is a deformation of ρ_0 of type D then

$$\# H_D^1(Q_\Sigma/Q, V_f) = \#(\mathrm{O}/\eta_{D,f}) < \infty \quad (11)$$

where $\eta_{D,f}$ is the invariant defined in the following equation $(\eta)=(\eta_{D,f})=(\hat{\pi}(1))$.

We assume that

$$\rho = Ind_L^Q \kappa : Gal(\overline{Q}/Q)\to GL_2(\mathrm{O}) \quad (12)$$

is the p-adic representation associated to a character $\kappa : Gal(\overline{L}/L)\to \mathrm{O}^\times$ of an imaginary quadratic field L.
Let M_∞ be the maximal abelian p-extension of $L(\nu)$ unramified outside p.

PROPOSITION 1.

There is an isomorphism

$$H^1_{unr}\left(Q_\Sigma / Q, Y^*\right) \xrightarrow{\approx} Hom\left(Gal\left(M_\infty / L(\nu)\right), (K/\mathrm{O})(\nu)\right)^{Gal(L(\nu)/L)} \quad (13)$$

where H^1_{unr} *denotes the subgroup of classes which are Selmer at p and unramified everywhere else.*

Now we write $H^1_{str}\left(Q_\Sigma / Q, Y^*_n\right)$ (where $Y^*_n = Y^*_{\lambda^n}$ and similarly for Y_n) for the subgroup of $H^1_{unr}\left(Q_\Sigma / Q, Y^*_n\right) = \left\{\alpha \in H^1_{unr}\left(Q_\Sigma / Q, Y^*_n\right) : \alpha_p = 0 in H^1\left(Q_p, Y^*_n / \left(Y^*_n\right)^0\right)\right\}$ where $\left(Y^*_n\right)^0$ is the first step in the filtration under D_p, thus equal to $\left(Y_n / Y^0_n\right)^*$ or equivalently to $\left(Y^*\right)^0_{\lambda^n}$ where $\left(Y^*\right)^0$ is the divisible submodule of Y^* on which the action of I_p is via ε^2. It follows from an examination of the action I_p on Y_λ that

$$H^1_{str}\left(Q_\Sigma / Q, Y_n\right) = H^1_{unr}\left(Q_\Sigma / Q, Y_n\right). \quad (14)$$

In the case of Y^* we will use the inequality

$$\# H^1_{str}\left(Q_\Sigma / Q, Y^*\right) \leq \# H^1_{unr}\left(Q_\Sigma / Q, Y^*\right). \quad (15)$$

Furthermore, for n sufficiently large the map

$$H^1_{str}\left(Q_\Sigma / Q, Y^*_n\right) \to H^1_{str}\left(Q_\Sigma / Q, Y^*\right) \quad (16)$$

is injective.
The above map is then injective whenever the connecting homomorphism

$$H^0\left(L_{p^*}, (K/\mathrm{O})(\nu)\right) \to H^1\left(L_{p^*}, (K/\mathrm{O})(\nu)_{\lambda^n}\right)$$

is injective, which holds for sufficiently large n. Furthermore, we have

$$\frac{\# H^1_{str}\left(Q_\Sigma / Q, Y_n\right)}{\# H^1_{str}\left(Q_\Sigma / Q, Y^*_n\right)} = \# H^0\left(Q_p, \left(Y^0_n\right)^*\right) \frac{\# H^0\left(Q, Y_n\right)}{\# H^0\left(Q, Y^*_n\right)}. \quad (17)$$

Thence, setting $t = \inf_q \#\left(\mathrm{O}/\left(1-\nu(q)\right)\right)$ if $\nu \bmod \lambda = 1$ or $t = 1$ if $\nu \bmod \lambda \neq 1$ (17b), we get

$$\# H^1_{Se}\left(Q_\Sigma / Q, Y\right) \leq \frac{1}{t} \cdot \prod_{\in\Sigma} \ell_q \cdot \# Hom\left(Gal\left(M_\infty / L(\nu)\right), (K/\mathrm{O})(\nu)\right)^{Gal(L(\nu)/L)} \quad (18)$$

where $\ell_q = \# H^0\left(Q_q, Y^*\right)$ for $q \neq p$, $\ell_p = \lim_{n\to\infty} \# H^0\left(Q_p, \left(Y^0_n\right)^*\right)$. This follows from Proposition 1, (14)-(17) and the elementary estimate

$$\#\left(H^1_{Se}\left(Q_\Sigma / Q, Y\right) / H^1_{unr}\left(Q_\Sigma / Q, Y\right)\right) \leq \prod_{q\in\Sigma-\{p\}} \ell_q, \quad (19)$$

which follows from the fact that $\# H^1\left(Q_q^{unr}, Y\right)^{Gal\left(Q_q^{unr}/Q_q\right)} = \ell_q$. (Remember that ℓ is the ℓ-adic representation).

Let w_f denote the number of roots of unity ζ of L such that $\zeta \equiv 1 \bmod f$ (f an integral ideal of O_L). We choose an f prime to p such that $w_f = 1$. Then there is a grossencharacter φ of L satisfying $\varphi((\alpha)) = \alpha$ for $\alpha \equiv 1 \bmod f$. According to Weil, after fixing an embedding $\overline{Q} \mapsto \overline{Q}_p$ we can associate a p-adic character φ_p to φ. We choose an embedding corresponding to a prime above p and then we find $\varphi_p = \kappa \cdot \chi$ for some χ of finite order and conductor prime to p.

The grossencharacter φ (or more precisely $\varphi \circ N_{F/L}$) is associated to a (unique) elliptic curve E defined over $F = L(f)$, the ray class field of conductor f, with complex multiplication by O_L and isomorphic over C to C/O_L. We may even fix a Weierstrass model of E over O_F which has good reduction at all primes above p. For each prime B of F above p we have a formal group $\hat{E}_B$, and this is a relative Lubin-Tate group with respect to F_B over L_p. We let $\lambda = \lambda_{\hat{E}_B}$ be the logarithm of this formal group.

Let U_∞ be the product of the principal local units at the primes above p of $L(fp^\infty)$; i.e.,

$$U_\infty = \prod_{B|p} U_{\infty,B} \quad \text{where} \quad U_{\infty,B} = \lim_{\leftarrow} U_n, B.$$

To an element $u = \lim_{\leftarrow} u_n \in U_\infty$ we can associate a power series $f_{u,B}(T) \in O_B[T]^\times$ where O_B is the ring of integers of F_B. For B we will choose the prime above p corresponding to our chosen embedding $\overline{Q} \mapsto \overline{Q}_p$. This power series satisfies $u_{n,B} = (f_{u,B})(\omega_n)$ for all $n > 0, n \equiv 0(d)$ where $d = \lfloor F_B : L_p \rfloor$ and $\{\omega_n\}$ is chosen as an inverse system of π^n division points of $\hat{E}_B$. We define a homomorphism $\delta_k : U_\infty \to O_B$ by

$$\delta_k(u) := \delta_{k,B}(u) = \left(\frac{1}{\lambda'_{\hat{E}_B}(T)} \frac{d}{dT} \right)^k \log f_{u,B}(T)\Big|_{T=0}. \quad (20)$$

Then

$$\delta_k(u^\tau) = \theta(\tau)^k \delta_k(u) \quad (21) \quad \text{for} \quad \tau \in Gal(\overline{F}/F)$$

where θ denotes the action on $E[p^\infty]$. Now $\theta = \varphi_p$ on $Gal(\overline{F}/F)$. We want a homomorphism on u_∞ with a transformation property corresponding to ν on all of $Gal(\overline{L}/L)$. We observe that $\nu = \varphi_p^2$ on $Gal(\overline{F}/F)$.

Let S be a set of coset representatives for $Gal(\overline{L}/L)/Gal(\overline{L}/F)$ and define

$$\Phi_2(u) = \sum_{\sigma \in S} \nu^{-1}(\sigma) \delta_2(u^\sigma) \in O_B[\nu]. \quad (22)$$

Each term is independent of the choice of coset representative by (17b) and it is easily checked that

$$\Phi_2(u^\sigma) = \nu(\sigma)\Phi_2(u).$$

It takes integral values in $O_B[\nu]$. Let $U_\infty(\nu)$ denote the product of the groups of local principal units at the primes above p of the field $L(\nu)$. Then Φ_2 factors through $U_\infty(\nu)$ and thus defines a continuous homomorphism

$$\Phi_2 : U_\infty(\nu) \to C_p.$$

Let C_∞ be the group of projective limits of elliptic units in $L(\nu)$. Then we have a crucial theorem of Rubin:

THEOREM 8.

There is an equality of characteristic ideals as $\Lambda = Z_p[[Gal(L(\nu)/L)]]$*-modules*:

$$char \wedge (Gal(M_\infty / L(\nu))) = char \wedge (U_\infty(\nu)/\overline{C}_\infty).$$

Let $\nu_0 = \nu \bmod \lambda$. For any $Z_p[Gal(L(\nu_0)/L)]$-module X we write $X^{(\nu_0)}$ for the maximal quotient of $X \underset{Z_p}{\otimes} O$ on which the action of $Gal(L(\nu_0)/L)$ is via the Teichmuller lift of ν_0. Since $Gal(L(\nu)/L)$ decomposes into a direct product of a pro-p group and a group of order prime to p,

$$Gal(L(\nu)/L) \cong Gal(L(\nu)/L(\nu_0)) \times Gal(L(\nu_0)/L),$$

we can also consider any $Z_p[[Gal(L(\nu)/L)]]$-module also as a $Z_p[Gal(L(\nu_0)/L)]$-module. In particular $X^{(\nu_0)}$ is a module over $Z_p[Gal(L(\nu_0)/L)]^{(\nu_0)} \cong O$. Also $\Lambda^{(\nu_0)} \cong O[[T]]$.
Now according to results of Iwasawa, $U_\infty(\nu)^{(\nu_0)}$ is a free $\Lambda^{(\nu_0)}$-module of rank one. We extend Φ_2 O-linearly to $U_\infty(\nu) \otimes_{Z_p} O$ and it then factors through $U_\infty(\nu)^{(\nu_0)}$. Suppose that u is a generator of $U_\infty(\nu)^{(\nu_0)}$ and β an element of $\overline{C}_\infty^{(\nu_0)}$. Then $f(\gamma - 1)u = \beta$ for some $f(T) \in O[[T]]$ and γ a topological generator of $Gal(L(\nu)/L(\nu_0))$. Computing Φ_2 on both u and β gives

$$f(\nu(\gamma) - 1) = \phi_2(\beta)/\Phi_2(u). \quad (23)$$

We have that ν can be interpreted as the grossencharacter whose associated p-adic character , via the chosen embedding $\overline{Q} \mapsto \overline{Q}_p$, is ν, and $\overline{\nu}$ is the complex conjugate of ν.
Furthermore, we can compute $\Phi_2(u)$ by choosing a special local unit and showing that $\Phi_2(u)$ is a p-adic unit.
Now, if we have that

$$\# H^1_{Se}(Q_\Sigma / Q, Y) \leq \#(O/\Omega^{-2} L_{f_0}(2, \overline{\nu})) \cdot \prod_{q \in \Sigma} \ell_q,$$

and

$$\#(O/h_L) \cdot \prod_{q \in \Sigma - \{p\}} \ell_q, \quad (24)$$

where $\ell_q = \# H^0(Q_q, ((K/O)(\psi) \oplus K/O)^*)$ and h_L is the class number of O_L, combining these we obtain the following relation:

$$\# H^1_{Se}(Q_\Sigma / Q, V) \leq \#\left(O/\Omega^{-2} L_{f_0}(2, \bar{v})\right) \#(O/h_L) \cdot \prod_{q \in \Sigma} \ell_q, \quad (25)$$

where $\ell_q = \# H^0(Q_q, V^*)$ (for $q \neq p$), $\ell_p = \# H^0\left(Q_p, (Y^0)^*\right)$. (Also here, we remember that ℓ is p-adic).

Let ρ_0 be an irreducible representation as in (5). Suppose that f is a newform of weight 2 and level N, λ a prime of O_f above p and $\rho_{f,\lambda}$ a deformation of ρ_0. Let m be the kernel of the homomorphism $T_1(N) \to O_f / \lambda$ arising from f.

We now give an explicit formula for η developed by Hida by interpreting $\langle , \rangle$ in terms of the cup product pairing on the cohomology of $X_1(N)$, and then in terms of the Petersson inner product of f with itself. Let

$$(,) : H^1(X_1(N), O_f) \times H^1(X_1(N), O_f) \to O_f \quad (26)$$

be the cup product pairing with O_f as coefficients. Let p_f be the minimal prime of $T_1(N) \otimes O_f$ associated to f, and let

$$L_f = H^1(X_1(N), O_f)[p_f].$$

If $f = \sum a_n q^n$ let $f^\rho = \sum \bar{a}_n q^n$. Then f^ρ is again a newform and we define L_{f^ρ} by replacing f by f^ρ in the definition of L_f. Then the pairing $(,)$ induces another by restriction

$$(,) : L_f \times L_{f^\rho} \to O_f. \quad (27)$$

Replacing O by the localization of O_f at p (if necessary) we can assume that L_f and L_{f^ρ} are free of rank 2 and direct summands as O_f-modules of the respective cohomology groups. Let δ_1, δ_2 be a basis of L_f. Then also $\bar{\delta}_1, \bar{\delta}_2$ is a basis of $L_{f^\rho} = \bar{L}_f$. Here complex conjugation acts on $H^1(X_1(N), O_f)$ via its action on O_f. We can then verify that

$$(\delta, \bar{\delta}) := \det(\delta_i, \bar{\delta}_j)$$

is an element of O_f whose image in $O_{f,\lambda}$ is given by $\pi(\eta^2)$ (unit).

To give a more useful expression for $(\delta, \bar{\delta})$ we observe that f and $\bar{f}^\rho$ can be viewed as elements of $H^1(X_1(N), C) \cong H^1_{DR}(X_1(N), C)$ via $f \mapsto f(z)dz$, $\bar{f}^\rho \mapsto \bar{f}^\rho d\bar{z}$. Then $\{f, \bar{f}^\rho\}$ form a basis for $L_f \otimes_{O_f} C$. Similarly $\{\bar{f}, f^\rho\}$ form a basis for $L_{f^\rho} \otimes_{O_f} C$. Define the vectors $\omega_1 = (f, \bar{f}^\rho)$, $\omega_2 = (\bar{f}, f^\rho)$ and write $\omega_1 = C\delta$ and $\omega_2 = \overline{C}\,\bar{\delta}$ with $C \in M_2(C)$. Then writing $f_1 = f, f_2 = \bar{f}^\rho$ we set

$$(\omega, \varpi) := \det((f_i, \bar{f}_j)) = (\delta, \bar{\delta}) \det(C\overline{C}).$$

Now (ω,ϖ) is given explicitly in terms of the (non-normalized) Petersson inner product $\langle , \rangle$:
$(\omega,\varpi) = -4\langle f, f\rangle^2$ where $\langle f, f\rangle = \int_{\Im/\Gamma_1(N)} f\bar{f}dxdy$. Hence, we have the following equation:

$$(\omega,\varpi) = -4\left(\int_{\Im/\Gamma_1(N)} f\bar{f}dxdy\right)^2. \quad (28)$$

To compute $\det(C)$ we consider integrals over classes in $H_1(X_1(N), O_f)$. By Poincaré duality there exist classes c_1, c_2 in $H_1(X_1(N), O_f)$ such that $\det\left(\int_{c_j} \delta_i\right)$ is a unit in O_f . Hence $\det C$ generates the same O_f -module as is generated by $\left\{\det\left(\int_{c_j} f_i\right)\right\}$ for all such choices of classes (c_1, c_2) and with $\{f_1, f_2\} = \{f, \bar{f}^\rho\}$. Letting u_f be a generator of the O_f -module $\left\{\det\left(\int_{c_j} f_i\right)\right\}$ we have the following formula of Hida:

$$\pi(\eta^2) = \langle f, f\rangle^2 / u_f \bar{u}_f \times (\text{unit in } O_{f,\lambda}).$$

Now, we choose a (primitive) grossencharacter φ on L together with an embedding $\overline{Q} \mapsto \overline{Q}_p$ corresponding to the prime p above p such that the induced p-adic character φ_p has the properties:

(i) $\varphi_p \bmod \overline{p} = \kappa_0$ ($\overline{p} =$ maximal ideal of $\overline{Q}_p$).
(ii) φ_p factors through an abelian extension isomorphic to $Z_p \oplus T$ with T of finite order prime to p.
(iii) $\varphi((\alpha)) = \alpha$ for $\alpha \equiv 1(f)$ for some integral ideal f prime to p.

Let $p_0 = \ker \psi_f : T_1(N) \to O_f$ and let $A_f = J_1(N)/ p_0 J_1(N)$ be the abelian variety associated to f by Shimura. Over F^+ there is an isogeny $A_{f/F^+} \approx (E_{/F^+})^d$ where $d = [O_f : Z]$.
We have that the p-adic Galois representation associated to the Tate modules on each side are equivalent to $(Ind_F^{F^+} \varphi_0) \otimes_{Z_p} K_{f,p}$ where $K_{f,p} = O_f \otimes Q_p$ and where $\varphi_p : Gal(\overline{F}/F) \to Z_p^\times$ is the p-adic character associated to φ and restricted to F .We now give an expression for $\langle f_\varphi, f_\varphi\rangle$ in terms of the L-function of φ. We note that $L_N(2,\overline{\nu}) = L_N(2,\nu) = L_N(2, \varphi^2 \overline{\hat{\chi}})$ and remember that ν is the p-adic character, and $\overline{\nu}$ is the complex conjugate of ν, we have that:

$$\langle f_\varphi, f_\varphi\rangle = \frac{1}{16\pi^3} N^2 \left\{\prod_{\substack{q|N \\ q \notin S_\varphi}} \left(1 - \frac{1}{q}\right)\right\} L_N(2, \varphi^2 \overline{\hat{\chi}}) L_N(1, \psi), \quad (29)$$

where χ *is the character of* f_φ *and* $\hat{\chi}$ *its restriction to* L*;* ψ *is the quadratic character associated to* L*;* $L_N(\ )$ *denotes that the Euler factors for primes dividing* N *have been removed;* S_φ *is the set of primes* $q|N$ *such that* $q = qq'$ *with* $q \mid \text{cond } \varphi$ *and* q, q' *primes of* L*, not necessarily distinct.*

THEOREM 9.

Suppose that ρ_0 *as in* (5) *is an irreducible representation of odd determinant such that* $\rho_0 = Ind_L^Q \kappa_0$ *for a character* κ_0 *of an imaginary quadratic extension* L *of* Q *which is unramified at p. Assume also that:*

(i) $\det \rho_0\big|_{I_p} = \omega$;

(ii) ρ_0 *is ordinary.*

Then for every $D = (\cdot, \Sigma, \mathrm{O}, \phi)$ *such that* ρ_0 *is of type* D *with* $\cdot =$ *Se or ord,*

$$R_D \cong \mathrm{T}_D$$

and T_D *is a complete intersection.*

COROLLARY.

For any ρ_0 *as in the theorem suppose that*

$$\rho : Gal(\overline{Q}/Q) \to GL_2(\mathrm{O})$$

is a continuous representation with values in the ring of integers of a local field, unramified outside a finite set of primes, satisfying $\overline{\rho} \cong \rho_0$ *when viewed as representations to* $GL_2(\overline{F}_p)$. *Suppose further that:*

(i) $\rho\big|_{D_p}$ *is ordinary;*

(ii) $\det \rho\big|_{I_p} = \chi \varepsilon^{k-1}$ *with* χ *of finite order,* $k \geq 2$.

Then ρ *is associated to a modular form of weight* k.

THEOREM 10. (Langlands-Tunnell)

Suppose that $\rho : Gal(\overline{Q}/Q) \to GL_2(C)$ *is a continuous irreducible representation whose image is finite and solvable. Suppose further that* $\det \rho$ *is odd. Then there exists a weight one newform* f *such that* $L(s, f) = L(s, \rho)$ *up to finitely many Euler factors.*

Suppose then that

$$\rho_0 : Gal(\overline{Q}/Q) \to GL_2(F_3)$$

is an irreducible representation of odd determinant. This representation is modular in the sense that over $\overline{F}_3$, $\rho_0 \approx \rho_{g,\mu} \bmod \mu$ for some pair (g, μ) with g some newform of weight 2. There exists a representation

$$i : GL_2(F_3) \mapsto GL_2\left(\mathrm{Z}\left[\sqrt{-2}\right]\right) \subset GL_2(C).$$

By composing i with an automorphism of $GL_2(F_3)$ if necessary we can assume that i induces the identity on reduction $\bmod\left(1+\sqrt{-2}\right)$. So if we consider $i \circ \rho_0 : Gal(\overline{Q}/Q) \to GL_2(C)$ we obtain an irreducible representation which is easily seen to be odd and whose image is solvable.

Now pick a modular form E of weight one such that $E \equiv 1(3)$. For example, we can take $E = 6E_{1,\chi}$ where $E_{1,\chi}$ is the Eisenstein series with Mellin transform given by $\zeta(s)\zeta(s,\chi)$ for χ the quadratic

character associated to $Q(\sqrt{-3})$. Then $fE \equiv f \bmod 3$ and using the Deligne-Serre lemma we can find an eigenform g' of weight 2 with the same eigenvalues as f modulo a prime μ' above $(1+\sqrt{-2})$. There is a newform g of weight 2 which has the same eigenvalues as g' for almost all T_l's, and we replace (g',μ') by (g,μ) for some prime μ above $(1+\sqrt{-2})$. Then the pair (g,μ) satisfies our requirements for a suitable choice of μ (compatible with μ').
We can apply this to an elliptic curve E defined over Q, and we have the following fundamental theorems:

THEOREM 11.

All semistable elliptic curves over Q are modular.

THEOREM 12.

Suppose that E is an elliptic curve defined over Q with the following properties:
(i) *E has good or multiplicative reduction at 3, 5,*
(ii) *For p = 3, 5 and for any prime $q \equiv -1 \bmod p$ either $\bar{\rho}_{E,p}|D_q$ is reducible over $\bar{F}_p$ or $\bar{\rho}_{E,p}|I_q$ is irreducible over $\bar{F}_p$.*
Then E is modular.

Chapter 2.

Further mathematical aspects concerning the Fermat's Last Theorem

2.1 <u>On the modular forms, Euler products, Shimura map and automorphic L-functions.</u>

A. Modular forms[2]

We know that there is a direct relation with elliptic curves, via the concept of *modularity* of elliptic curves over Q.
Let E be an elliptic curve over Q, given by some Weierstrass equation. Such a Weierstrass equation can be chosen to have its coefficients in Z. A Weierstrass equation for E with coefficients in Z is called *minimal* if its *discriminant* is minimal among all Weierstrass equations for E with coefficients in Z; this discriminant then only depends on E and will be denoted discr(E). Thence, E has a Weierstrass minimal model over Z, that will be denoted by E_Z.
For each prime number p, we let E_{F_p} denote the curve over F_p given by reducing a minimal Weierstrass equation modulo p; it is the fibre of E_Z over F_p. The curve E_{F_p} is smooth if and only if p does not divide discr(E).
The possible singular fibres have exactly one singular point: an ordinary double point with rational tangents, or with conjugate tangents, or an ordinary cusp. The three types of reduction are called

split multiplicative, non-split multiplicative and additive, respectively, after the type of group law that one gets on the complement of the singular point. For each p we then get an integer a_p by requiring the following identity:

$$p+1-a_p = \# E(F_p). \quad (1)$$

This means that for all p, a_p is the trace of F_p on the degree one étale cohomology of $E_{\overline{F}_p}$, with coefficients in F_l, or in Z/l^nZ or in the l-adic numbers Z_l. For p not dividing discr(E) we know that $|a_p| \leq 2p^{1/2}$. If E_{F_p} is multiplicative, then $a_p = 1$ or -1 in the split and non-split case. If E_{F_p} is additive, then $a_p = 0$. We also define, for each p an element $\varepsilon(p)$ in $\{0,1\}$ by setting $\varepsilon(p)=1$ for p not dividing discr(E). The Hasse-Weil L-function of E is then defined as:

$$L_E(s) = \prod_p L_{E,p}(s), \quad L_{E,p}(s) = \left(1-a_p p^{-s} + \varepsilon(p) p p^{-2s}\right)^{-1}, \quad (2)$$

for s in C with $R(s) > 3/2$. We note that for all p and for all $l \neq p$ we have the identity:

$$1-a_p t + \varepsilon(p) t^2 = \det\left(1-tF_p^*, H^1\left(E_{\overline{F},et}, Q_l\right)\right). \quad (3)$$

We use étale cohomology with coefficients in Q_l, the field of l-adic numbers, and not in F_l.
The function L_E was conjectured to have a holomorphic continuation over all of C, and to satisfy a certain precisely given functional equation relating the values at s and $2-s$. In that functional equation appears a certain positive integer N_E called the conductor of E, composed of the primes p dividing discr(E) with exponents that depend on the behaviour of E at p, i.e., on E_{Z_p}. This conjecture on continuation and functional equation was proved for semistable E (i.e., E such that there is no p where E has additive reduction) by Wiles and Taylor-Wiles, and in the general case by Breuil, Conrad, Diamond and Taylor. In fact, the continuation and functional equation are direct consequences of the modularity of E that was proved by Wiles, Taylor-Wiles, etc.
The weak Birch and Swinnerton-Dyer conjecture says that the dimension of the Q-vector space $Q \otimes E(Q)$ is equal to the order of vanishing of L_E at 1. Anyway, the function L_E gives us integers a_n for all $n \geq 1$ as follows:

$$L_E(s) = \sum_{n\geq 1} a_n n^{-s}, \quad \text{for } R(s) > 3/2. \quad (4)$$

From these a_n one can then consider the following function:

$$f_E : H = \{\tau \in C \mid \Im(\tau) > 0\} \to C, \quad \tau \mapsto \sum_{n\geq 1} a_n e^{2\pi i n \tau}. \quad (5)$$

Equivalently, we have:

$$f_E = \sum_{n\geq 1} a_n q^n, \quad \text{with } q : H \to C, \quad \tau \mapsto e^{2\pi i \tau}. \quad (6)$$

A more conceptual way to state the relation between L_E and f_E is to say that L_E is obtained, up to elementary factors, as the *Mellin transform* of f_E:

$$\int_0^\infty f_E(it)t^s \frac{dt}{t} = (2\pi)^{-s}\Gamma(s)L_E(s), \quad \text{for } \; R(s) > 3/2. \quad (7)$$

Hence, we can finally state what the modularity of E means:
f_E is a modular form of weight two for the congruence subgroup $\Gamma_0(N_E)$ of $SL_2(Z)$.
The last statement means that f_E has an enormous amount of symmetry.
A typical example of a modular form of weight higher than two is the *discriminant* modular form, usually denoted Δ. One way to view Δ is as the holomorphic function on the upper half plane H given by:

$$\Delta = q\prod_{n\geq 1}(1-q^n)^{24}, \quad (8)$$

where q is the function from H to C given by $z \mapsto \exp(2\pi i z)$. The coefficients in the power series expansion:

$$\Delta = \sum_{n\geq 1}\tau(n)q^n \quad (9)$$

define the famous *Ramanujan τ-function*.
To say that Δ is a modular form of weight 12 for the group $SL_2(Z)$ means that for all elements $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $SL_2(Z)$ the following identity holds for all z in H:

$$\Delta\left(\frac{az+b}{cz+d}\right) = (cz+d)^{12}\Delta(z), \quad (10)$$

which is equivalent to saying that the multi-differential form $\Delta(z)(dz)^{\otimes 6}$ is invariant under the action of $SL_2(Z)$. As $SL_2(Z)$ is generated by the elements $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, it suffices to check the identity in (10) for these two elements. The fact that Δ is q times a power series in q means that Δ is a *cusp form*: it vanishes at "$q = 0$". It is a fact that Δ is the first example of a non-zero cusp form for $SL_2(Z)$: there is no non-zero cusp form for $SL_2(Z)$ of weight smaller than 12, i.e., there are no non-zero holomorphic functions on H satisfying (10) with the exponent 12 replaced by a smaller integer, whose Laurent series expansion in q is q times a power series. Moreover, the C-vector space of such functions of weight 12 is one-dimensional, and hence Δ is a basis of it.
The one-dimensionality of this space has as a consequence that Δ is an eigenform for certain operators on this space, called *Hecke operators*, that arise from the action on H of $GL_2(Q)^+$, the subgroup of $GL_2(Q)$ of elements whose determinant is positive. This fact explains that the coefficients $\tau(n)$ satisfy certain relations which are summarised by the following identity of Dirichlet series:

$$L_\Delta(s) := \sum_{n\geq 1}\tau(n)n^{-s} = \prod_p\left(1-\tau(p)p^{-s}+p^{11}p^{-2s}\right)^{-1}. \quad (11)$$

These relations:

$$\tau(mn) = \tau(m)\tau(n) \qquad \text{if } m \text{ and } n \text{ are relatively prime;}$$

$$\tau(p^n) = \tau(p^{n-1})\tau(p) - p^{11}\tau(p^{n-2}) \quad \text{if } p \text{ is prime and } n \geq 2$$

were conjectured by Ramanujan, and proved by Mordell. Using these identities, $\tau(n)$ can be expressed in terms of the $\tau(p)$ for p dividing n. As L_Δ is the Mellin transform of Δ, L_Δ is holomorphic on C, and satisfies the functional equation (Hecke):

$$(2\pi)^{-(12-s)}\Gamma(12-s)L_\Delta(12-s) = (2\pi)^{-s}\Gamma(s)L_\Delta(s). \quad (12)$$

The famous *Ramanujan conjecture* states that for all primes p one has the inequality:

$$|\tau(p)| < 2p^{11/2}, \quad (13)$$

or, equivalently, that the complex roots of the polynomial $x^2 - \tau(p)x + p^{11}$ are complex conjugates of each other, and hence are of absolute value $p^{11/2}$.

B. Euler products[3]

We know that the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^s}, \quad (14)$$

converges for $R(s) > 1$ and gives rise by analytic continuation to a meromorphic function $\zeta(s)$ in C. For $R(s) > 1$ $\zeta(s)$ admits the absolutely convergent infinite product expansion

$$\prod_p \frac{1}{1-p^{-s}}, \quad (15)$$

taken over the set of primes. This "Euler product" may be regarded as an analytic formulation of the principle of unique factorization in the ring Z of integers. It is, as well, the product taken over all the non-Archimedean completions of the rational field Q (which completions Q_p are indexed by the set of primes) of the "Mellin transform" in Q_p

$$\xi_p(s) = \frac{1}{1-p^{-s}}, \quad (16)$$

(where the Mellin transform is, more or less, Fourier transform on the multiplicative group. Classically, the Mellin transform φ of f is given formally by $\varphi(s) = \int_0^\infty f(x)x^s(dx/x)$. (17)) of the canonical "Gaussian density" $\Phi_p(x) =$ 1 if $x \in$ closure of Z in Q_p; 0 otherwise, which Gaussian density is equal to its own Fourier transform. For the Archimedean completion $Q_\infty = R$ of the rational field Q one forms the classical Mellin transform

$$\xi_\infty(s) = \pi^{-(s/2)}\Gamma(s/2) \quad (18)$$

of the classical Gaussian density

$$\Phi_\infty(x) = e^{-\pi x^2}, \quad (19)$$

(which also is equal to its own Fourier transform). Then the function

$$\xi(s) = \xi_\infty(s)\zeta(s) = \prod_{p \le \infty} \xi_p(s) \quad (20)$$

is meromorphic in C, and satisfies the functional equation

$$\xi(1-s) = \xi(s). \quad (21)$$

The connection of Riemann's ζ-function with the subject of modular forms begins with the observation that $\zeta(2s)$ is essentially the Mellin transform of $\theta_I(x) = \theta(ix) - 1$, where θ, which is a modular form of weight 1/2 and level 8, is defined in the upper-half plane H by the formula

$$\theta(\tau) = \sum_{m \in Z} \exp(\pi i \tau m^2). \quad (22)$$

In fact, one of the classical proofs of the functional equation (21) is given by applying the Poisson summation formula to the function $x \mapsto \exp(\pi i \tau x^2)$, while observing that the substitution $s \mapsto (1/2) - s$ for $\zeta(2s)$ corresponds in the upper-half plane to the substitution $\tau \mapsto -1/\tau$ for the theta series. If f is a cuspform for a congruence group Γ containing

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad (23)$$

and so, consequently, $f(\tau+1) = f(\tau)$, then one has the following Fourier expansion

$$f(\tau) = \sum_{m=1}^{\infty} c_m e^{2\pi i m \tau}. \quad (24)$$

The Mellin transform $\varphi(s)$ of f_I leads to the Dirichlet series

$$\varphi(s) = \sum_{m=1}^{\infty} c_m m^{-s}, \quad (25)$$

which may be seen to have a positive abscissa of convergence.
For the "modular group" $\Gamma(1)$ the Dirichlet series associated to every cuspform of weight w admits an analytic continuation with functional equation under the substitution $s \mapsto w - s$. Since $\Gamma(1)$ is generated by the two matrices T and

$$W = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (26)$$

and since the functional equation of a modular form f relative to T is reflected in the formation of the Fourier series (24), the condition that an absolutely convergent series (24) is a modular form for $\Gamma(1)$ is the functional equation for a modular form relative solely to W.

Observing that the formula

$$ds^2 = \frac{dx^2 + dy^2}{y^2} \quad \text{for} \quad \tau = x + iy \in H\,, \quad (27)$$

gives a (the hyperbolic) $SL_2(R)$-invariant metric in H with associated invariant measure

$$d\mu = \frac{dxdy}{y^2}\,, \quad (28)$$

one introduces the Petersson (Hermitian) inner product in the space of cuspform of weight w for Γ with the definition:

$$\langle f, g\rangle = \int_{H/\Gamma} f(\tau)\overline{g}(\tau)\Im(\tau)^w d\mu(\tau)\,. \quad (29) \text{ (see also page 13 eq. (28))}$$

(Integration over the quotient H/Γ makes sense since the integrand $f(\tau)\overline{g}(\tau)y^w$ (30) is Γ-invariant).

For the modular group $\Gamma(1)$ the n^{th} Hecke operator $T(n) = T_w(n)$ is the linear endomorphism of the space of cuspforms of weight w arising from the following considerations. Let S_n be the set of 2×2 matrices in Z with determinant n. For

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S_n \quad (31)$$

and for a function f in H one defines

$$(M \cdot_w f)(\tau) = \det(M)^{w-1}(c\tau + d)^{-w} f(\tau)\,, \quad (32)$$

and then, observing that $\Gamma(1)$ under $\cdot_w$ acts trivially on the modular forms of weight w, one may define the Hecke operator $T_w(n)$ by

$$T_w(n)(f) = \sum_{M\in S_n/\Gamma(1)} (M \cdot_w f)(\tau)\,, \quad (33)$$

where the quotient $S_n/\Gamma(1)$ refers to the action of $\Gamma(1)$ by left multiplication on the set S_n. One finds for m,n coprime that

$$T(mn) = T(m)T(n)\,, \quad (34)$$

and furthermore one has

$$T\left(p^{e+1}\right) = T\left(p^e\right)T(p) - p^{w-1}T\left(p^{e-1}\right). \quad (35)$$

Consequently, the operators $T(n)$ commute with each other, and, therefore, generate a commutative algebra of endomorphisms of the space of cusp forms of weight w for $\Gamma(1)$. It is not difficult to see that the Hecke operators are self-adjoint for the Petersson inner product on the space of cuspforms. Consequently, the space of cuspforms of weight w admits a basis of simultaneous eigenforms for the Hecke algebra. A "Hecke eigencuspform" is said to be *normalized* if its Fourier coefficient $c_1 = 1$. If f is a normalized Hecke eigencuspform, then:

(i) The Fourier coefficient c_m of f is the eigenvalue of f for $T(m)$.

(ii) The Fourier coefficients $c(m) = c_m$ of f satisfy

$c(mn) = c(m)c(n)$ for m,n coprime, and

$c(p^{e+1}) = c(p^e)c(p) - p^{w-1}c(p^{e-1})$ for p prime.

Consequently, the Dirichlet series associated with a simultaneous Hecke eigencuspform of level 1 and weight w admits an Euler product

$$\varphi(s) = \prod_p \frac{1}{1 - c_p p^{-s} + p^{w-1-2s}}. \quad (36)$$

For example, when f is the unique normalized cuspform Δ of level 1 and weight 12, one has

$$\varphi(s) = \prod_p \frac{1}{1 - \tau(p)p^{-s} + p^{11-2s}}, \quad (37)$$

(in fact, if $w = 12$, then $w - 1 - 2s = 11 - 2s$)

where $c_p = \tau(p)$ is the function τ of Ramanujan.

C. Shimura map[3]

Shimura showed for a given W_N-compatible Hecke eigencuspform f of weight 2 for the group $\Gamma_0(N)$ with rational Fourier coefficients how to construct an elliptic curve E_f defined over Q such that the Dirichlet series $\varphi(s)$ associated with f is the same as the L-function $L(E_f, s)$.

Let Γ be a congruence subgroup of $SL_2(Z)$, and let $X(\Gamma)$ denote the compact Riemann surface H^*/Γ. The inclusion of Γ in $\Gamma(1)$ induces a "branched covering"

$$X(\Gamma) \to X(1) \cong P^1. \quad (38)$$

One may use the elementary Riemann-Hurwitz formula from combinatorial topology to determine the Euler number, and consequently the genus, of $X(\Gamma)$. The genus is the dimension of the space of cuspforms of weight 2. Even when the genus is zero one obtains embeddings of $X(\Gamma)$ in projective spaces P^r through holomorphic maps

$$\tau \mapsto (f_0(\tau), f_1(\tau), \ldots, f_r(\tau)), \quad (39)$$

where $f_0, f_1, \ldots, f_r$ is a basis of the space of modular forms of weight w with w sufficiently large.

Using the corresponding projective embedding one finds a *model* for $X_0(N) = X(\Gamma_0(N))$ over Q, i.e., an algebraic curve defined over Q in projective space that is isomorphic as a compact Riemann surface to $X_0(N)$.

Associated with any "complete non-singular" algebraic curve X of genus g is a complex torus, the "Jacobian" $J(X)$ of X, that is the quotient of g-dimensional complex vector space C^g by the lattice Ω generated by the "period matrix", which is the $g \times 2g$ matrix in C obtained by integrating each of the g members ω_i of a basis of the space of holomorphic differentials over each of the $2g$ loops in X representing the members of a homology basis in dimension 1. Furthermore, if one picks a base point z_0 in X, then for any z in X, the path integral from z_0 to z of each of the g holomorphic differentials is well-defined modulo the periods of the differential. One obtains a holomorphic map $X \to J(X)$ from the formula

$$z \mapsto \left(\int_{z_0}^{z} \omega_1, \ldots, \int_{z_0}^{z} \omega_g \right) \bmod \Omega . \quad (40)$$

This map is universal for pointed holomorphic maps from X to complex tori. Furthermore, the Jacobian $J(X)$ is an algebraic variety that admits definition over any field of definition for X and z_0, and the universal map also admits definition over any such field. The complex tori that admit embeddings in projective space are the abelian group objects in the category of projective varieties. They are called *abelian varieties*. Every abelian variety is isogenous to the product of "simple" abelian varieties: abelian varieties having no abelian subvarieties. Shimura showed that one of the simple isogeny factors of $J(X_0(N))$ is an elliptic curve E_f defined over Q characterized by the fact that its one-dimensional space of holomorphic differentials induces on $X_0(N)$, via the composition of the universal map with projection on E_f, the one-dimensional space of differentials on $X_0(N)$ determined by the cuspform f.

He showed further that $L(E_f, s)$ is the Dirichlet series $\varphi(s)$ with Euler product given by f. An elliptic curve E defined over Q is said to be *modular* if it is isogenous to E_f for some W_N-compatible Hecke eigencuspform of weight 2 for $\Gamma_0(N)$. Equivalently E is modular if and only if $L(E, s)$ is the Dirichlet series given by such a cuspform. The Shimura-Taniyama-Weil Conjecture states that every elliptic curve defined over Q is modular. Shimura showed that this conjecture is true in the special case where the Z-module rank of the ring of endomorphisms of E is grater than one. In this case the point τ of the upper-half plane corresponding to $E(C)$ is a quadratic imaginary number, and $L(E, s)$ is a number-theoretic L-function associated with the corresponding imaginary quadratic number field.

D. Automorphic L-functions[4]

Talking about zeta functions in general one inevitably is led to start with the Riemann zeta function $\zeta(s)$. It is defined as a *Dirichlet series*:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} , \quad (41)$$

which converges for each complex number s of real part greater than one. In the same region it possesses a representation as a *Mellin integral*:

$$\zeta(s)=\frac{1}{\Gamma(s)}\int_0^\infty \frac{1}{e^t-1}t^s\frac{dt}{t}. \quad (42)$$

Let f be a cusp form of weight $2k$ for some natural number k, i.e., the function f is holomorphic on the upper half plane H in C, and has a certain invariance property under the action of the modular group $SL_2(Z)$ on H. Then f admits a Fourier expansion

$$f(z)=\sum_{n=1}^\infty a_n e^{2\pi izn}. \quad (43)$$

Define its L-function for $\mathrm{Re}(s)>1$ by

$$L(f,s)=\sum_{n=1}^\infty \frac{a_n}{n^s}. \quad (44)$$

The easily established integral representation

$$\hat{L}(f,s)=(2\pi)^{-s}\Gamma(s)L(f,s)=\int_0^\infty f(it)t^s\frac{dt}{t}, \quad (45)$$

implies that $L(f,s)$ extends to an entire function satisfying the functional equation $\hat{L}(f,s)=(-1)^k\hat{L}(f,2k-s)$. With $\Lambda(f,s)=\hat{L}(f,2ks)$ this becomes

$$\Lambda(f,s)=(-1)^k\Lambda(f,1-s). \quad (46)$$

This construction can be extended to cusp forms for suitable subgroups of the modular group. These L-functions look like purely analytical objects. Thus it was particularly daring of A. Weil, G. Shimura, and Y. Taniyama in 1955 to propose the conjecture that the zeta function of any elliptic curve over Q coincides with a $\Lambda(f,s)$ for a suitable cusp form f. This conjecture was proved in part by A. Wiles and R. Taylor providing a proof of Fermat's Last Theorem as a consequence.
The upper half plane is a homogeneous space of the group $SL_2(R)$, and so cusp forms may be viewed as functions on this group, in particular, they are vectors in the natural unitary representation of $SL_2(R)$ on the space

$$L^2(SL_2(Z)\backslash SL_2(R)). \quad (47)$$

Going even further one can extend this quotient space to the quotient of the adele group $GL_2(A)$ modulo its discrete subgroup $GL_2(Q)$, so cusp forms become vectors in

$$L^2\left(GL_2(Q)\backslash GL_2(A)^1\right), \quad (48)$$

where $GL_2(A)^1$ denotes the set of all matrices in $GL_2(A)$ whose determinant has absolute value one. Now GL_2 can be replaced by GL_n for $n \in N$ and one can imitate the methods of Tate's thesis (the case n = 1) to arrive at a much more general definition of an automorphic L-function: this is an Euler product $L(\pi, s)$ attached to an automorphic representation π of $GL_n(A)^1$, i.e., an irreducible subrepresentation π of $L^2(GL_n(Q) \setminus GL_n(A)^1)$. As in the GL_1-case it has an integral representation as a Mellin transform and it extends to a meromorphic representation, which is entire if π is cuspidal and $n > 1$. Furthermore it satisfies a functional equation

$$L(\pi, s) = \varepsilon(\pi, s) L(\tilde{\pi}, 1 - s), \quad (49)$$

where $\tilde{\pi}$ is the contragredient representation and $\varepsilon(\pi, s)$ is a constant multiplied by an exponential. We conclude remember that extending the Weil-Shimura-Taniyama conjecture, R.P. Langlands conjectured in the 1960s that any motivic L-function coincides with $L(\pi, s)$ for some cuspidal π.

2.2 <u>On some mathematical applications of the Mellin transform</u>.[5]

Harmonic sums are sums of the form

$$G(x) = \sum_k \lambda_k g(\mu_k x), \quad (50)$$

where the λ_k are the *amplitudes*, the μ_k are the *frequencies* and $g(x)$ is the *base function*. We consider harmonic sums because we wish to evaluate $G(x)$ at a set of particular points $x_0, x_1, \ldots$ or at all $x \in R$.

Definition of the harmonic sum and computation of the appropriate Mellin transform.

Now, let $\lambda_k = 1/k$, $\mu_k = 1/k$ and $g(x) = x/(1+x) = 1/(1+1/x)$; and we consider the harmonic sum

$$h(x) = \sum_x \lambda_k g(\mu_k x) = \sum \frac{1}{k} \frac{x/k}{1 + x/k} = \sum \left(\frac{1}{k} - \frac{1}{x+k} \right). \quad (51)$$

This sum is of interest because

$$h(n) = \sum \left(\frac{1}{k} - \frac{1}{n+k} \right) = \sum \frac{1}{k} - \sum_{k=n+1} \frac{1}{k} = \sum_{k=1}^{n} \frac{1}{k} = H_n, \quad (52)$$

the nth harmonic number.
The principal operation in the evaluation of harmonic sums is the computation of the Mellin transform of the base function $g(y)$ and the computation of the Dirichlet generating function $\Lambda(s)$. We first compute the transform of the base function. We have $\mathrm{M}[1/(1+x); s] = \pi / \sin(\pi s)$ and hence

$$\mathrm{M}\left[\frac{x}{1+x}; s \right] = -\frac{\pi}{\sin(\pi s)}. \quad (53)$$

Now we compute the Dirichlet generating function $\Lambda(s)$. We have

$$\Lambda(s) = \sum \frac{1}{k} k^s = \sum \frac{1}{k^{1-s}} = \zeta(1-s). \quad (54)$$

We conclude that the Mellin transform of $h(x)$ is

$$-\frac{\pi}{\sin(\pi s)} \zeta(1-s). \quad (55)$$

Inversion of the map.

Now, by Mellin inversion we obtain:

$$\mathrm{M}^{-1}\left[-\frac{\pi}{\sin(\pi s)} \zeta(1-s); x\right] = h(x). \quad (56)$$

This is equivalent to the inversion integral

$$\int_{c-i\infty}^{c+i\infty} \left(-\frac{\pi}{\sin(\pi s)} \zeta(1-s)\right) x^{-s} ds = h(x). \quad (57)$$

This integral representation permits the computation of $h(x)$, because the integral can evaluated by the Cauchy Residue theorem, i.e., it is a sum of residues of $h^*(s)x^{-s}$.

Computation of the poles of the transform function and the corresponding terms in the asymptotic expansion.

We use the fact that

$$h(x) \approx -\sum_{\varsigma \in Sing(h^*(s)x^{-s}) \cap H} \operatorname{Re} s\left(h^*(s)x^{-s}; s = \varsigma\right), \quad (58)$$

where H is the right half-plane, chosen for an expansion at infinity. We must compute the set of poles $Sing\left(h^*(s)x^{-s}\right) \cap H$ and map them back to the terms of the expansion of $h(x)$. The poles of $h^*(s)$ in the right half-plane are at $s = 0$, where we have a double pole and

$$h^*(s) = \frac{1}{s^2} - \frac{\gamma}{s} + \ldots \quad (59)$$

and at $s = k, k \in Z^+$, where we have

$$h^*(s) = -(-1)^k \frac{\zeta(1-k)}{s-k} + \ldots . \quad (60)$$

These poles map back to $-\log x - \gamma$ (61) and $-\frac{1}{2x}$ for $k = 1$, $-\frac{(-1)^k B_k}{k} \frac{1}{x^k}$ for $k \geq 2$. (62)

We conclude that Harmonic numbers satisfy the asymptotic expansion

$$H_n \approx \log n + \gamma + \frac{1}{2n} + \sum_{k\geq 2} \frac{(-1)^k B_k}{k} \frac{1}{n^k}. \quad (63)$$

This expansion is exact; it converges for $n \geq 1$.
The Mellin transform maps the space of functions that are integrable along the positive real line to that of complex functions that are analytic on a vertical strip of the complex plane. This strip may in many cases be extended to a larger domain. The map is given by the fundamental formula:

$$\mathrm{M}[f(x);s] = f^*(s) = \int_0^{+\infty} f(x) x^{s-1} dx. \quad (64)$$

The Mellin-Perron formula is a specific instance of generalized Mellin summation. The traditional proof uses the "discontinuous factor" described by the following lemma:

$$\phi(y) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{y^s}{s(s+1)\ldots(s+m)} ds = \frac{1}{m!}\left(1 - \frac{1}{y}\right)^m \text{ if } 1 \leq y;$$

$$\phi(y) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{y^s}{s(s+1)\ldots(s+m)} ds = 0 \text{ if } 0 < y \leq 1, \quad (65)$$

where $y \in R^+, m \in Z^+$ and $c \geq 1$.
The above equality for the discontinuous factor $\phi(y)$ is easily verified with the Cauchy residue theorem.
Hence, there are two cases.

Case 1. $1 \leq y$.

The term $\dfrac{y^s}{s(s+1)\ldots(s+m)}$ (66) is meromorphic with residues

$$\frac{y^{-k}}{(-k)(-k+1)\ldots(-k+k-1)(-k+k+1)\ldots(-k+m)} = y^{-k} \frac{(-1)^k}{k!(m-k)!} \quad (67)$$

where $0 \leq k \leq m$. Therefore the sum of these residues is

$$\sum_{k=0}^{m} y^{-k} \frac{(-1)^k}{k!(m-k)!} = \frac{1}{m!} \sum_{k=0}^{m} \binom{m}{k} \left(-\frac{1}{y}\right)^k 1^{m-k} = \frac{1}{m!}\left(1 - \frac{1}{y}\right)^m. \quad (68)$$

Now consider the left contour. The integral along the vertical segment at c in the right-half plane approaches

$$\int_{c-i\infty}^{c+i\infty} \frac{y^s}{s(s+1)\ldots(s+m)} ds \quad (69)$$

as T goes to infinity. Along the two horizontal segments from $-T \pm iT$ to $c \pm iT$, the integrand is bounded by $\frac{y^{\sigma}}{T^m}$ and because the term $\frac{1}{1+\sigma}\frac{y^{1+\sigma}}{T^m}$ with $\sigma = -T$, $\sigma = c$ vanishes as T goes to infinity (recall that $1 \le y$), the contribution from these two segments is zero. The integrand is bounded by $\frac{y^{-T}}{T(T-1)\ldots(T-m)}$ on the vertical segment in the left half-plane; hence the integral is bounded by $\frac{2y^{-T}}{(T-1)\ldots(T-m)}$ and its contribution is zero also.

Case 2. $0 < y \le 1$.

Consider the contour in the right half-plane. Along the horizontal segments we may use the same bound as in the first case, with $\sigma = c$ and $\sigma = T$; hence these integrals vanish $(0 < y \le 1)$. The integrand is bounded by $\frac{y^{T}}{T(T+1)\ldots(T+m)}$ on the vertical segment in the right half-plane; its contribution is zero because $0 < y \le 1$.

The principal feature of the "discontinuous factor" is that it can be used to evaluate finite sums. Suppose we have a finite sum over the indices k from 1 to $n-1$. Evidently $\phi(y)$ is non-zero if $1/y$ lies in $(0,1)$ and zero otherwise. We need only find a map such that the set $\{1,\ldots n-1\}$ maps to a subrange of $(0,1)$ and $\{n, n+1\ldots\}$ to a subrange of $[1,\infty)$. Clearly $1/y = k/n$ is such a map. We obtain

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{k^s}\frac{n^s}{s(s+1)\ldots(s+m)}ds = \frac{1}{m!}\left(1-\frac{k}{n}\right)^m \quad \text{if } k < n$$

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{k^s}\frac{n^s}{s(s+1)\ldots(s+m)}ds = 0 \quad \text{if } n \le k \qquad (70)$$

By a formal argument we finally have

$$\frac{1}{m!}\sum_{k=1}^{n-1}\lambda_k\left(1-\frac{k}{n}\right)^m = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\sum\frac{\lambda_k}{k^s}\right)\frac{n^s}{s(s+1)\ldots(s+m)}ds. \qquad (71)$$

This is the **Mellin-Perron formula**.

The Mellin-transform view adds two additional perspectives. One, that the Mellin-Perron formula is a specific instance of harmonic sum formulas, and hence, two, that its evaluation corresponds to Mellin inversion.

We wish to evaluate the harmonic sum $\sum_{1\le k<n}\lambda_k\left(1-\frac{k}{n}\right)^m$ (72) where $m, n \in Z^+$. This is equivalent to $\sum_1^{\infty}\lambda_k g\left(\frac{k}{n}\right)$ (73) where $g(x) = (1-x)^m$ if $0 < x \le 1$; $g(x) = 0$ otherwise. It is no difficult to see that $\mathrm{M}[g(x);s] = \frac{1}{s(s+1)\ldots(s+m)}$. (74)

Evidently the sum $\sum_1^{\infty}\lambda_k g\left(\frac{k}{n}\right)$ is a harmonic sum $G(x)$ of the form $\sum_1^{\infty}\lambda_k g(kx)$ with amplitudes λ_k, frequencies $\mu_k = k$ and evaluated at $x = 1/n$. Therefore the transform function $G^*(s)$ is

$$\Lambda(s)\frac{1}{s(s+1)\ldots(s+m)}. \quad (75)$$

By Mellin inversion we thus have

$$G(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Lambda(s)\frac{x^{-s}}{s(s+1)\ldots(s+m)}ds \quad (76)$$

and in particular

$$G\left(\frac{1}{n}\right)=\sum_{1\le k<n}\lambda_k\left(1-\frac{k}{n}\right)^m=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Lambda(s)\frac{n^s}{s(s+1)\ldots(s+m)}ds. \quad (77)$$

This is the Mellin-Perron formula.

The Mellin transform: Definitions, Theorems and Lemmas

Definition 2.2.1

The open strip of complex numbers $\langle\alpha,\beta\rangle$ is the set $\{s=\sigma+it|\alpha<\sigma<\beta\}$.

Definition 2.2.2

Let $f(x)$ be locally Lebesgue integrable over $(0,+\infty)$. The Mellin transform of $f(x)$ is defined by

$$\mathrm{M}[f(x);s]=f^*(s)=\int_0^{+\infty}f(x)x^{s-1}dx. \quad (78)$$

The fundamental strip is the largest open strip where the integral converges.

Lemma 2.2.1

The conditions $f(x)_{x\to 0^+}\in \mathrm{O}(x^u), \quad f(x)_{x\to+\infty}\in\mathrm{O}(x^v), \quad (79)$

when $u>v$, guarantee that $f^*(x)$ exists in the strip $\langle -u,-v\rangle$.

Definition 2.2.3

Let $H_0(x)=1$ if $x\in[0,1]$; $H_0(x)=0$ if $x>1$ (80) be defined on $[0,+\infty)$ and let

$$H_m(x)=(1-x)^m H_0(x) \text{ when } m\in Z^+. \quad (81)$$

Note that $H_0(x)$ has a discontinuity at $x=1$; we have $\lim_{x\to 1-}H_0(x)=1$ and $\lim_{x\to 1+}H_0(x)=0$. Note also that $\lim_{x\to 1-}H_m(x)=\lim_{x\to 1+}H_m(x)=0$ when $m\in Z^+$; $H_m(x)$ is continuous at $x=1$.

Lemma 2.2.2

The Mellin transform $H_m^*(x)$ of $H_m(x)$, where $m \in N$, exists in $\langle 0,+\infty \rangle$ and is given by

$$H_m^*(x) = \frac{m!}{s(s+1)...(s+m)}. \quad (82)$$

We have $H_m(x)_{x \to 0+} \in O(1)$ and $H_m(x)_{x \to +\infty} \in O(x^{-b})$ for any $b > 0$ and for $m \in N$, hence $H_m^*(x)$ exists in $\langle 0,+\infty \rangle$. Note that

$$H_0^*(x) = \int_0^1 x^{s-1} dx = \frac{1}{s}\left[x^s\right]_0^1 = \frac{1}{s}. \quad (83)$$

We also have

$$H_m^*(s) = \int_0^1 H_m(x) x^{s-1} dx = \int_0^1 H_{m-1}(x) x^{s-1} dx - \int_0^1 H_{m-1}(x) x^s dx =$$
$$= H_{m-1}^*(x) - \int_0^1 \frac{(1-x)^m}{m} s x^{s-1} dx = H_{m-1}^*(x) - \frac{s}{m} H_m^*(x). \quad (84)$$

This gives $$H_m^*(x) = \frac{m}{s+m} H_{m-1}^*(x) \quad (85)$$

Now, we will be concerned with the linearity and the rescaling property of the Mellin transform.

Theorem 2.2.1

Let $\mathrm{K} \subset Z$ be a finite set of integers; let $\mu_k, \lambda_k \in R^+$. Let the fundamental strip of $\mathrm{M}[f(x);s]$ be $\langle \alpha, \beta \rangle$. We have

$$\mathrm{M}\left[\sum_k \lambda_k f(\mu_k x); s\right] = \left(\sum_k \frac{\lambda_k}{\mu_k^s}\right) \mathrm{M}[f(x);s], \quad (86)$$

where $s \in \langle \alpha, \beta \rangle$.
Let $y = \mu_k x$ and $dy = \mu_k dx$. Note that

$$\int_0^\infty \left(\sum_k \lambda_k f(\mu_k x)\right) x^{s-1} dx = \sum_k \lambda_k \int_0^\infty f(\mu_k x) x^{s-1} dx = \sum_k \lambda_k \int_0^\infty f(y) y^{s-1} \frac{dy}{\mu_k^s} = \left(\sum_k \frac{\lambda_k}{\mu_k^s}\right) f^*(s). \quad (87)$$

We were able to exchange the integral with the summation because K is finite. It can be shown that this operation extends to infinite K as long as $\sum_k \lambda_k / \mu_k^s$ converges absolutely. The extended property holds in the intersection of the half-plane of convergence of $\sum_k \lambda_k / \mu_k^s$ and the fundamental strip $\langle \alpha, \beta \rangle$ of $f(x)$.

Definition 2.2.4

1. (Lebesgue integration)

Let $f(x)$ be integrable with fundamental strip $\langle\alpha,\beta\rangle$. If $c\in(\alpha,\beta)$ and $f^*(c+it)$ is integrable, then

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}f^*(s)x^{-s}ds=f(x)\quad (88)$$

almost everywhere. If $f(x)$ is continuous, the equality holds everywhere on $(0,+\infty)$.

2. (Riemann integration.)

Let $f(x)$ be locally integrable with fundamental strip $\langle\alpha,\beta\rangle$ and be of bounded variation in a neighbourhood of x_0. Then

$$\lim_{T\to\infty}\frac{1}{2\pi i}\int_{c-iT}^{c+iT}f^*(s)x^{-s}ds\Big|_{x_0}=\frac{f(x_0^+)+f(x_0^-)}{2}\quad (89)$$

for $c\in(\alpha,\beta)$. Of course if $\lim_{x\to x_0^+}f(x)=\lim_{x\to x_0^-}f(x)$ then

$$\frac{f(x_0^+)+f(x_0^-)}{2}=f(x_0).\quad (90)$$

Theorem 2.2.2 (Mellin-Perron formula)

Let $c\in R^+$ lie in the half-plane of absolute convergence of $\sum_k \lambda_k/k^s$. Then we have

$$\frac{1}{m!}\sum_{1\le k<n}\lambda_k\left(1-\frac{k}{n}\right)^m=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\sum_{k\ge1}\frac{\lambda_k}{k^s}\right)n^s\frac{ds}{s(s+1)\ldots(s+m)}\quad (91)$$

for $m\in Z^+$. We have

$$\sum_{1\le k<n}\lambda_k+\frac{\lambda_n}{2}=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\sum_{k\ge1}\frac{\lambda_k}{k^s}\right)n^s\frac{ds}{s}\quad (92)$$

when $m=0$.
This theorem is a straightforward application of Mellin inversion.

Proof.

Let $F(x)=\sum_k \lambda_k f(\mu_k x)$ and use the rescaling property to obtain

$$\mathrm{M}[F(x);s]=F^*(s)=\left(\sum_k\frac{\lambda_k}{\mu_k^s}\right)f^*(s).\quad (93)$$

Consider Riemann-integrable $f(x)$ and apply the Mellin inversion formula

$$\sum_k \lambda_k \frac{f(\mu_k x^+)+f(\mu_k x^-)}{2} = \lim_{T\to\infty}\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\left(\sum_k \frac{\lambda_k}{\mu_k^s}\right) f^*(s)x^{-s}ds . \quad (94)$$

Let $f(x)=H_m(x)$, $m\in N$ and let $\mu_k = k$. Recall that the fundamental strip of $H_m(x)$ is $\langle 0,\infty\rangle$; let $x=1/n$. This gives

$$\sum_k \lambda_k \frac{f(\mu_k x^+)+f(\mu_k x^-)}{2} = \sum_k \lambda_k \frac{H_m\left(\frac{k}{n^-}\right)+H_m\left(\frac{k}{n^+}\right)}{2} =$$

$$= \sum_{1\le k<n} \lambda_k \frac{\left(1-\frac{k}{n^-}\right)^m + \left(1-\frac{k}{n^+}\right)^m}{2} + \lambda_n \frac{H_m(1^+)+H_m(1^-)}{2} = \sum_{1\le k<n} \lambda_k \left(1-\frac{k}{n}\right)^m + \lambda_n \frac{H_m(1^+)+H_m(1^-)}{2} . \quad (95)$$

Note that

$$\lambda_n \frac{H_m(1^+)+H_m(1^-)}{2} = \lambda_n/2 \text{ if } m=0; \quad \lambda_n \frac{H_m(1^+)+H_m(1^-)}{2} = 0 \text{ if } m\in Z^+ . \quad (96)$$

Continuing the substitution, we have

$$\lim_{T\to\infty}\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\left(\sum_k \frac{\lambda_k}{\mu_k^s}\right) f^*(s)x^{-s}ds = \lim_{T\to\infty}\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\left(\sum_k \frac{\lambda_k}{k^s}\right)\frac{m!}{s(s+1)\dots(s+m)} n^s ds =$$

$$= \frac{m!}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\sum_k \frac{\lambda_k}{k^s}\right) n^s \frac{ds}{s(s+1)\dots(s+m)} . \quad (97)$$

This concludes the proof. Because the fundamental strip of $H_m(x)$ is $\langle 0,\infty\rangle$, the choice of $c>0$ is determined by the half-plane of convergence of $\sum_k \lambda_k / k^s$ only.

Now we presents two Mellin-Perron formulae for the generalized ζ-function. We apply the Mellin inversion theorem to $F(x)=\sum_k \lambda_k f(\mu_k x)$ with $x=r/n$, $r,n\in Z^+$, $\mu_k = k+a$, $\lambda_k = 1$, $a\in R$, $a\in(0,1]$, $f(x)=H_1(x)=(1-x)H_0(x)$. As we require $\mu_k \in R^+$ we take $k\in N$. We have

$$F(x)=\sum_{k\in N}\left(1-(k+a)\frac{r}{n}\right)H_0\left((k+a)\frac{r}{n}\right) \quad (98)$$

and

$$F^*(s)=\left(\sum_{k\in N}\frac{1}{(k+a)^s}\right)f^*(s)=\frac{\zeta(s,a)}{s(s+1)} \quad (99)$$

where $\sigma>1$. We need to evaluate $F(x)$. $H_0(x)$ vanishes outside of $[0,1)$, hence we require $0\le(k+a)r/n<1$ or $k<n/r-a$. Let $N(u)=\{v<u \mid v\in N\}$ where $u\in R^+$. We have

$$F(x)=\sum_{k\in N(n/r-a)}\left(1-(k+a)\frac{r}{n}\right). \quad (100)$$

With these settings the Mellin inversion formula yields the following theorem.

Theorem 2.2.3

Let $c>1$.

$$\sum_{k\in N(n/r-a)}\left(1-(k+a)\frac{r}{n}\right)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{r^s}\zeta(s,a)\frac{n^s}{s(s+1)}ds. \quad (101)$$

This theorem has several useful corollaries. The first of these is obtained by setting $r=1$. Let $\alpha\in(-1,0)$.

Corollary 2.2.3

Let $n\in N$.

$$\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\zeta(s,a)\frac{n^s}{s(s+1)}ds=0. \quad (102)$$

Let $c=1$. The set of poles of $\zeta(s,a)n^s/(s(s+1))$ in $\langle\alpha,c\rangle$ is $\{1,0\}$. We apply the shifting lemma with $\Phi(s)=n^s$ and $T_j=j$. Because $|n^s|=n^\sigma$ we can take $M=n^c$.

$$\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\zeta(s,a)\frac{n^s}{s(s+1)}ds=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\zeta(s,a)\frac{n^s}{s(s+1)}$$
$$-\operatorname{Res}\left(\zeta(s,a)\frac{n^s}{s(s+1)};s=1\right)-\operatorname{Res}\left(\zeta(s,a)\frac{n^s}{s(s+1)};s=0\right)$$
$$=\sum_{0\le k<n}\left(1-(k+a)\frac{1}{n}\right)-\frac{n}{2}-\zeta(0,a)=n-n\frac{a}{n}-\frac{1}{n}\frac{1}{2}(n-1)n-\frac{n}{2}-\zeta(0,a)=\frac{1}{2}-a-\zeta(0,a)=0. \quad (103)$$

The second corollary results from taking $r=4$.

Corollary 2.2.4

Let $n\in N$.

$$\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{1}{4^s}\zeta(s,a)\frac{n^s}{s(s+1)}ds. \quad (104)$$

We let $c=1$ as before and consider the poles of $\zeta(s,a)n^s/(4^s s(s+1))$ in $\langle\alpha,c\rangle$, which are at 1 and 0. We apply the shifting lemma with $\Phi(s)=(n/4)^s$, $T_j=j$ and take $M=(n/4)^c$.

$$\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\zeta(s,a)\frac{n^s}{4^s s(s+1)}ds=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\zeta(s,a)\frac{n^s}{4^s s(s+1)}$$
$$-\operatorname{Res}\left(\zeta(s,a)\frac{n^s}{4^s s(s+1)};s=1\right)-\operatorname{Res}\left(\zeta(s,a)\frac{n^s}{4^s s(s+1)};s=0\right)$$
$$=\sum_{k\in N(n/4-a)}\left(1-(k+a)\frac{4}{n}\right)-\frac{n}{8}-\zeta(0,a)=\varepsilon(n,a)-\frac{n}{8}-\zeta(0,a). \quad (105)$$

Suppose $n = 4m + m_1$ where $m_1 \in \{0,1,2,3\}$. We have $n/4 - a = [n/4] + m_1/4 - a$. If $m_1/4 < a$, the sum over $N(n/4-a)$ ranges from 0 to $[n/4]-1$. If $m_1/4 \geq a$ the sum includes $[n/4]$. We have two cases:

$$\varepsilon(n,a) = \left[\frac{n}{4}\right] - a\frac{4}{n}\left[\frac{n}{4}\right] - \frac{2}{n}\left(\left[\frac{n}{4}\right]-1\right)\left[\frac{n}{4}\right] \qquad \text{if } \frac{m_1}{4} < a$$

$$\varepsilon(n,a) = \left[\frac{n}{4}\right] + 1 - a\frac{4}{n}\left(\left[\frac{n}{4}\right]+1\right) - \frac{2}{n}\left(\left[\frac{n}{4}\right]+1\right)\left[\frac{n}{4}\right] \qquad \text{if } \frac{m_1}{4} \geq a .$$

We note that $[n/4] = (n - m_1)/4$ and $[n/4]4/n = 1 - m_1/n$. Hence the two terms evaluate to

$$\frac{1}{8}n + \frac{1}{2} - a + \frac{1}{n}\left(am_1 - \frac{1}{2}m_1 - \frac{1}{8}m_1^2\right) \quad \text{and} \quad \frac{1}{8}n + \frac{1}{2} - a + \frac{1}{n}\left(a(m_1 - 4) + \frac{1}{2}m_1 - \frac{1}{8}m_1^2\right).$$

We conclude that

$$\frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} \zeta(s,a)\frac{n^s}{4^s s(s+1)}ds = \varepsilon(n,a) - \frac{1}{8}n - \frac{1}{2} + a . \quad (106)$$

2.3 The zeta-function quantum field theory and the quantum L-functions.[6]

The Riemann zeta-function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}, \quad s = \sigma + i\tau, \quad \sigma > 1 \quad (107)$$

and there is an Euler adelic representation

$$\zeta(s) = \prod_p \left(1 - p^{-s}\right)^{-1} . \quad (108)$$

Now, we have the Riemann ξ-function

$$\xi(s) = \frac{s(s-1)}{2}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) \quad (109)$$

which is an entire function. The zeros of the ξ-function are the same as the nontrivial zeros of the ζ-function. There is the functional equation

$$\xi(s) = \xi(1-s) \quad (110)$$

and the Hadamard representation for the ξ-function

$$\xi(s) = \frac{1}{2}e^{as}\prod_\rho\left(1 - \frac{s}{\rho}\right)e^{s/\rho} . \quad (111)$$

Here ρ are nontrivial zeros of the zeta-function and

$$a=-\frac{1}{2}\gamma-1+\frac{1}{2}\log 4\pi \quad (112)$$

where γ is Euler's constant.
If $F(\tau)$ is a function of a real variable τ then we define a pseudo-differential operator $F(\Box)$ by using the Fourier transform

$$F(\Box)\phi(x)=\int e^{ixk}F(k^2)\tilde{\phi}(k)dk\,. \quad (113)$$

Here $\Box$ is the d'Alambertian operator

$$\Box=-\frac{\partial^2}{\partial x_0^2}+\frac{\partial^2}{\partial x_1^2}+\ldots+\frac{\partial^2}{\partial x_{d-1}^2}, \quad (114)$$

$\phi(x)$ is a function from $x\in R^d$, $\tilde{\phi}(k)$ is the Fourier transform and $k^2=k_0^2-k_1^2-\ldots-k_{d-1}^2$.
We assume that the integral (113) converges.
One can introduce a natural field theory related with the real valued function $F(\tau)=\xi\left(\frac{1}{2}+i\tau\right)$ defined by means of the zeta-function. We consider the following Lagrangian

$$L=\phi\xi(1/2+i\,\Box)\phi, \quad (115)$$

the integral

$$\xi(1/2+i\,\Box)\phi\,(x)=\int e^{ixk}\xi(1/2+i\,\Box)\tilde{\phi}(k)dk \quad (116)$$

converges if $\phi(x)$ is a decreasing function since $\xi\left(\frac{1}{2}+i\tau\right)$ is bounded.
The operator $\xi(1/2+i\,\Box)$ (or $\zeta(1/2+i\,\Box)$) is the first quantization the Riemann zeta-function.
From the Hadamard representation (111) we get

$$\xi\left(\frac{1}{2}+i\tau\right)=\frac{C}{2}\prod_{n=1}^{\infty}\left(1-\frac{\tau^2}{m_n^4}\right). \quad (117)$$

It is possible to write the formula (117) in the form

$$\xi\left(\frac{1}{2}+i\tau\right)=\frac{C}{2}\prod_{\varepsilon,n}\left(1+\frac{\tau}{\varepsilon m_n^2}\right) \quad (118)$$

where $\varepsilon=\pm 1$ and a regularization is assumed.
To quantize the zeta-function classical field $\phi(x)$ which satisfies the equation in the Minkowski space

$$F(\Box)\phi(x)=0 \quad (119)$$

where $F(\Box) = \xi(1/2 + i\Box)$ we can try to interpret $\phi(x)$ as an operator valued distribution in a Hilbert space H which satisfies the equation (119). We suppose that there is a representation of the Poincare group and an invariant vacuum vector $|0\rangle$ in H. Then the Wightman function

$$W(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle$$

is a solution of the equation

$$F(\Box)W(x) = 0. \quad (120)$$

By using (118) we can write the formal Kallen-Lehmann representation

$$W(x) = \sum_{\varepsilon n} \int e^{ixk} f_{\varepsilon n}(k)\delta(k^2 + \varepsilon m_n^2)dk. \quad (121)$$

One introduces also another useful function

$$Z(\tau) = \pi^{-i\tau/2} \frac{\Gamma\left(\frac{1}{4} + \frac{i\tau}{2}\right)}{\left|\Gamma\left(\frac{1}{4} + \frac{i\tau}{2}\right)\right|} \zeta\left(\frac{1}{2} + i\tau\right) = e^{i\vartheta(\tau)}\zeta\left(\frac{1}{2} + i\tau\right). \quad (122)$$

Here $\Gamma(z)$ is the gamma function. The function $Z(\tau)$ is called the Riemann-Siegel (or Hardy) function. It is known that $Z(\tau)$ is real for real τ and there is a bound

$$Z(\tau) = O(|\tau|^{\varepsilon}), \quad \varepsilon > 0. \quad (123)$$

One can introduce a natural field theory related with the real valued functions $Z(\tau)$ defined by means of the zeta-function by considering the following Lagrangian

$$L = \phi Z(\Box)\phi.$$

The integral (113) converges if $\phi(x)$ is a decreasing function since there is the bound (123). Thence, we have the following connection:

$$\pi^{-i\tau/2} \frac{\Gamma\left(\frac{1}{4} + \frac{i\tau}{2}\right)}{\left|\Gamma\left(\frac{1}{4} + \frac{i\tau}{2}\right)\right|} \zeta\left(\frac{1}{2} + i\tau\right) = e^{i\vartheta(\tau)}\zeta\left(\frac{1}{2} + i\tau\right) = O\left(|\tau|^{\varepsilon}\right) \Rightarrow$$

$$\Rightarrow F(\Box)\phi(x) = \int e^{ixk} F(k^2)\tilde{\phi}(k)dk, \quad \varepsilon > 0. \quad (124)$$

For any character to modulus q one defines the corresponding Dirichlet L-function by setting

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \quad (\sigma > 1). \quad (125)$$

If χ is primitive then $L(s,\chi)$ has an analytic continuation to the whole complex plane. The zeros lie in the critical strip and symmetrically distributed about the critical line $\sigma = 1/2$.
If we quantize the L-function by considering the pseudo-differential operator

$$L(\sigma + i\,\square, \chi) \quad (126)$$

then we can try to avoid the appearance of tachyons and/or ghosts by choosing an appropriate character χ.
The Taniyama-Weil conjecture relates elliptic curves and modular forms. It asserts that if E is an elliptic curve over Q, then there exists a weight-two cusp form f which can be expressed as the Fourier series

$$f(z) = \sum a_n e^{2\pi n z} \quad (127)$$

with the coefficients a_n depending on the curve E. Such a series is a modular form if and only if its Mellin transformation, i.e. the Dirichlet L-series

$$L(s,f) = \sum a_n n^{-s} \quad (128)$$

has a holomorphic extension to the full s-plane and satisfies a functional equation. For the elliptic curve E we obtain the L-series $L(s,E)$. The Taniyama-Weil conjecture was proved by Wiles and Taylor for semistable elliptic curves and it implies Fermat's Last Theorem.
Quantization of the L-functions can be performed similarly to the quantization of the Riemann zeta-function discussed above by considering the corresponding pseudo-differential operator $L(\sigma + i\,\square)$.

Chapter 3.

How primes and adeles are related to the Riemann zeta function[7]

A. Connes has reduced the Riemann hypothesis for L-function on a global field k to the validity of a trace formula for the action of the idele class group on the noncommutative space quotient of the adeles of k by the multiplicative group of k.
Connes has devised a Hermitian operator whose eigenvalues are the Riemann zeros on the critical line. Connes gets a discrete spectrum by making the operator act on an abstract space where the primes appearing in the Euler product for the Riemann zeta function are built in; the space is constructed from collections of p-adic numbers (adeles) and the associated units (ideles).

Hence, the geometric framework involves the space X of Adele classes, where two adeles which belong to the same orbit of the action of $GL_1(k)$ (k a global field), are considered equivalent. The group $C_k = GL_1(A)/GL_1(k)$ of Idele classes (which is the class field theory counterpart of the Galois group) acts by multiplication on X.

We have a trace formula (Theorem 3) for the action of the multiplicative group K^* of a local field K on the Hilbert space $L^2(K)$, and (Theorem 4) a trace formula for the action of the multiplicative group C_s of Idele classes associated to a finite set S of places of a global field k, on the Hilbert space of square integrable functions $L^2(X_S)$, where X_S is the quotient of $\prod_{v\in S} k_v$ by the action of the group O_S^* of S-units of k. The validity of the trace formula for any finite set of places follows from Theorem 4, but in the global case is left open and shown (Theorem 5) to be equivalent to the validity of the Riemann Hypothesis for all L functions with Grossencharakter.

H. Montgomery has proved (assuming RH) a weakening of the following conjecture (with $\alpha, \beta > 0$),

$$Card\{(i,j); i,j \in 1,...,M; x_i - x_j \in [\alpha, \beta]\} \approx M \int_\alpha^\beta \left(1 - \left(\frac{\sin(\pi u)}{\pi u}\right)^2\right) du \qquad (1)$$

This law, i.e. the equation (1), is precisely the same as the correlation between eigenvalues of hermitian matrices of the gaussian unitary ensemble. Moreover, numerical tests due to A. Odlyzko have confirmed with great precision the behaviour (1) as well as the analogous behaviour for more than two zeros. N. Katz and P. Sarnak has proved an analogue of the Montgomery-Odlyzko law for zeta and L-functions of function fields over curves.

It is thus an excellent motivation to try and find a natural pair (H, D) where naturality should mean for instance that one should not even have to define the zeta function, let alone its analytic continuation, in order to obtain the pair (in order for instance to avoid the joke of defining H as the ℓ^2 space built on the zeros of zeta).

Theorem 1.

Let K be a local field with basic character α. Let $h \in S(K^*)$ have compact support. Then $R_\Lambda U(h)$ is a trace class operator and when $\Lambda \to \infty$, one has

$$Trace(R_\Lambda U(h)) = 2h(1)\log'\Lambda + \int' \frac{h(u^{-1})}{|1-u|} d^*u + o(1) \qquad (2)$$

where $2\log'\Lambda = \int_{\lambda\in K^*, |\lambda|\in[\Lambda^{-1},\Lambda]} d^*\lambda$, and the principal value $\int'$ is uniquely determined by the pairing with the unique distribution on K which agrees with $\frac{du}{|1-u|}$ for $u \neq 1$ and whose Fourier transform vanishes at 1.

Proof.

We normalize the additive Haar measure to be the selfdual one on K. Let the constant $\rho > 0$ be determined by the equality,

$$\int_{1\le|\lambda|\le\Lambda}\frac{d\lambda}{|\lambda|}\approx\rho\log\Lambda \quad \text{when} \quad \Lambda\to\infty\,, \quad (3)$$

so that $d^*\lambda=\rho^{-1}\frac{d\lambda}{|\lambda|}$. Let L be the unique distribution, extension of $\rho^{-1}\frac{du}{|1-u|}$ whose Fourier transform vanishes at 1, $\hat{L}(1)=0$. One then has by definition,

$$\int' \frac{h(u^{-1})}{|1-u|}d^*u=\left\langle L,\frac{h(u^{-1})}{|u|}\right\rangle, \quad (4)$$

where $\frac{h(u^{-1})}{|u|}=0$ for u^{-1} outside the support of h. Let $T=U(h)$. We can write the Schwartz kernel of T as,

$$k(x,y)=\int h(\lambda^{-1})\delta(y-\lambda x)d^*\lambda\,. \quad (5)$$

Given any such kernel k we introduce its symbol,

$$\sigma(x,\xi)=\int k(x,x+u)\alpha(u\xi)du \quad (6)$$

as its partial Fourier transform. The Schwartz kernel $r_\Lambda^t(x,y)$ of the transpose R_Λ^t is given by,

$$r_\Lambda^t(x,y)=\rho_\Lambda(x)(\hat{\rho}_\Lambda)(x-y)\,. \quad (7)$$

Thus, the symbol σ_Λ of R_Λ^t is simply,

$$\sigma_\Lambda(x,\xi)=\rho_\Lambda(x)\rho_\Lambda(\xi)\,. \quad (8)$$

The operator R_Λ is of trace class and one has,

$$\text{Trace}\,(R_\Lambda T)=\int k(x,y)r_\Lambda^t(x,y)dxdy\,. \quad (9)$$

Using the Parseval formula we thus get,

$$\text{Trace}\,(R_\Lambda T)=\int_{|x|\le\Lambda,|\xi|\le\Lambda}\sigma(x,\xi)dxd\xi\,. \quad (10)$$

Now the symbol σ of T is given by,

$$\sigma(x,\xi)=\int h(\lambda^{-1})\left(\int\delta(x+u-\lambda x)\alpha(u\xi)du\right)d^*\lambda\,. \quad (11)$$

One has,

$$\int\delta(x+u-\lambda x)\alpha(u\xi)du=\alpha((\lambda-1)x\xi)\,, \quad (12)$$

thus (11) gives,

$$\sigma(x,\xi)=\rho^{-1}\int_K g(\lambda)\alpha(\lambda x\xi)d\lambda \quad (13)$$

where,

$$g(\lambda)=h\left((\lambda+1)^{-1}\right)|\lambda+1|^{-1}. \quad (14)$$

Since h is smooth with compact support on K^* the function g belongs to $C_c^\infty(K)$. Thus $\sigma(x,\xi)=\rho^{-1}\hat{g}(x\xi)$ and

$$\mathrm{Trace}\,(R_\Lambda T)=\rho^{-1}\int_{|x|\le\Lambda,|\xi|\le\Lambda}\hat{g}(x\xi)dxd\xi\,. \quad (15)$$

With $u=x\xi$ one has $dxd\xi=du\dfrac{dx}{|x|}$ and, for $|u|\le\Lambda^2$,

$$\rho^{-1}\int_{\frac{|u|}{\Lambda}\le|x|\le\Lambda}\frac{dx}{|x|}=2\log'\Lambda-\log|u|\,. \quad (16)$$

Thus we can rewrite (15) as,

$$\mathrm{Trace}\,(R_\Lambda T)=\int_{|u|\le\Lambda^2}\hat{g}(u)\left(2\log'\Lambda-\log|u|\right)du\,. \quad (17)$$

Since $g\in C_c^\infty(K)$ one has,

$$\int_{|u|\ge\Lambda^2}|\hat{g}(u)|du=O\left(\Lambda^{-N}\right) \qquad \forall N \quad (18)$$

and similarly for $|\hat{g}(u)\log|u||$. Thus

$$\mathrm{Trace}\,(R_\Lambda T)=2g(0)\log'\Lambda-\int\hat{g}(u)\log|u|du+o(1). \quad (19)$$

Now for any local field K and basic character α, if we take for the Haar measure da the selfdual one, the Fourier transform of the distribution $\varphi(u)=-\log|u|$ is given outside 0 by

$$\hat{\varphi}(a)=\rho^{-1}\frac{1}{|a|}, \quad (20)$$

with ρ determined by (3). To see this one lets P be the distribution on K given by,

$$P(f)=\lim_{\substack{\varepsilon\to 0\\ \varepsilon\in Mod(K)}}\left(\int_{|x|\ge\varepsilon}f(x)d^*x+f(0)\log\varepsilon\right). \quad (21)$$

One has $P(f_a)=P(f)-\log|a|f(0)$ which is enough to show that the function $\hat{P}(x)$ is equal to $-\log|x|+\mathrm{cst},$ and $\hat{\varphi}$ differs from P by a multiple of δ_0. Thus the Parseval formula gives, with the convention of Theorem 3,

$$-\int \hat{g}(u)\log|u|du = \frac{1}{\rho}\int' g(a)\frac{da}{|a|}. \quad (22)$$

Replacing a by $\lambda - 1$ and applying (14) gives the desired result.

Now, let k be a global field and S a finite set of places of k containing all infinite places. The group O_S^* of S-units is defined as the subgroup of k^*, $O_S^* = \{q \in k^*, |q_v| = 1, v \notin S\}$. It is co-compact in J_S^1 where, $J_S = \prod_{v\in S} k_v^*$ and, $J_S^1 = \{j \in J_S, |j| = 1\}$. Thus the quotient group $C_S = J_S / O_S^*$ plays the same role as C_k, and acts on the quotient X_S of $A_S = \prod_{v\in S} k_v$ by O_S^*.

Theorem 2.

Let A_S be as above, with basic character $\alpha = \prod \alpha_v$. Let $h \in S(C_S)$ have compact support. Then when $\Lambda \to \infty$, one has

$$Trace(R_\Lambda U(h)) = 2h(1)\log'\Lambda + \sum_{v\in S}\int'_{k_v^*}\frac{h(u^{-1})}{|1-u|}d^*u + o(1) \quad (23)$$

where $2\log'\Lambda = \int_{\lambda\in C_S, |\lambda|\in[\Lambda^{-1},\Lambda]} d^*\lambda$, *each* k_v^* *is embedded in* C_S *by the map* $u \to (1,1,\dots,u,\dots,1)$ *and the principal value* $\int'$ *is uniquely determined by the pairing with the unique distribution on* k_v *which agrees with* $\frac{du}{|1-u|}$ *for* $u \neq 1$ *and whose Fourier transform relative to* α_v *vanishes at 1.*

Proof.

We normalize the additive Haar measure dx to be the selfdual one on the abelian group A_S. Let the constant $\rho > 0$ be determined by the equality,

$$\int_{\lambda\in D, 1\le|\lambda|\le\Lambda}\frac{d\lambda}{|\lambda|} \approx \rho\log\Lambda \quad \text{when} \quad \Lambda\to\infty,$$

so that $d^*\lambda = \rho^{-1}\frac{d\lambda}{|\lambda|}$. We let f be a smooth compactly supported function on J_S such that

$$\sum_{q\in O_S^*} f(qg) = h(g) \quad \forall g \in C_S. \quad (24)$$

The existence of such an f follows from the discreteness of O_S^* in J_S. We then have the equality $U(f) = U(h)$, where

$$U(f) = \int f(\lambda)U(\lambda)d^*\lambda. \quad (25)$$

Now, for an operator T, acting on functions on A_S, which commutes with the action of O_S^* and is represented by an integral kernel,

$$T(\xi) = \int k(x,y)\xi(y)dy, \quad (26)$$

the trace of its action on $L^2(X_S)$ is given by,

$$Tr(T) = \sum_{q \in O_S^*} \int_D k(x,qx)dx, \quad (27)$$

where D is a fundamental domain for the action of O_S^* on the subset J_S of A_S, whose complement is negligible. Let $T = U(f)$. We can write the Schwartz kernel of T as,

$$k(x,y) = \int f(\lambda^{-1})\delta(y - \lambda x)d^*\lambda, \quad (28)$$

by construction one has,

$$k(qx,qy) = k(x,y) \qquad q \in O_S^*. \quad (29)$$

For any $q \in O_S^*$, we shall evaluate the integral,

$$I_q = \int_{x \in D} k(qx,y) r_\Lambda^t(x,y)dydx \quad (30)$$

where the Schwartz kernel $r_\Lambda^t(x,y)$ for the transpose R_Λ^t is given by,

$$r_\Lambda^t(x,y) = \rho_\Lambda(x)(\hat{\rho}_\Lambda)(x-y). \quad (31)$$

To evaluate the above integral, we let $y = x + a$ and perform a Fourier transform in a. For the Fourier transform in a of $r_\Lambda^t(x,x+a)$, one gets,

$$\sigma_\Lambda(x,\xi) = \rho_\Lambda(x)\rho_\Lambda(\xi). \quad (32)$$

For the Fourier transform in a of $k(qx,x+a)$, one gets,

$$\sigma(x,\xi) = \int f(\lambda^{-1})\left(\int \delta(x + a - \lambda qx)\alpha(a\xi)da\right)d^*\lambda. \quad (33)$$

One has,

$$\int \delta(x + a - \lambda qx)\alpha(a\xi)da = \alpha((\lambda q - 1)x\xi), \quad (34)$$

thus (33) gives,

$$\sigma(x,\xi) = \rho^{-1}\int_{A_S} g_q(u)\alpha(ux\xi)du \quad (35)$$

where,

$$g_q(u) = f\left(q(u+1)^{-1}\right)|u+1|^{-1}. \quad (36)$$

Since f is smooth with compact support on A_S^* the function g_q belongs to $C_c^\infty(A_S)$.
Thus $\sigma(x,\xi)=\rho^{-1}\hat{g}_q(x\xi)$ and, using the Parseval formula we get,

$$I_q=\int_{x\in D,|x|\leq\Lambda,|\xi|\leq\Lambda}\sigma(x,\xi)dxd\xi\,. \quad (37)$$

This gives,

$$I_q=\rho^{-1}\int_{x\in D,|x|\leq\Lambda,|\xi|\leq\Lambda}\hat{g}_q(x\xi)dxd\xi\,. \quad (38)$$

With $u=x\xi$ one has $dxd\xi=du\dfrac{dx}{|x|}$ and, for $|u|\leq\Lambda^2$,

$$\rho^{-1}\int_{x\in D,\frac{|u|}{\Lambda}\leq|x|\leq\Lambda}\frac{dx}{|x|}=2\log'\Lambda-\log|u|\,. \quad (39)$$

Thus we can rewrite (38) as,

$$\text{Trace}\,(R_\Lambda T)=\sum_{q\in O_S^*}\int_{|u|\leq\Lambda^2}\hat{g}_q(u)(2\log'\Lambda-\log|u|)du\,. \quad (40)$$

Now $\log|u|=\sum_{v\in S}\log|u_v|$, and we shall first prove that,

$$\sum_{q\in O_S^*}\int\hat{g}_q(u)du=h(1), \quad (41)$$

while for any $v\in S$,

$$\sum_{q\in O_S^*}\int\hat{g}_q(u)(-\log|u_v|)du=\int'_{k_v^*}\frac{h(u^{-1})}{|1-u|}d^*u\,. \quad (42)$$

In fact all the sums in q will have only finitely many non zero terms. It will then remain to control the error term, namely to show that,

$$\sum_{q\in O_S^*}\int\hat{g}_q(u)(\log|u|-2\log'\Lambda)^+du=0(\Lambda^{-N}), \quad (43)$$

for any N, where we used the notation $x^+=0$ if $x\leq 0$ and $x^+=x$ if $x>0$.
Now recall that for (36), $g_q(u)=f\left(q(u+1)^{-1}\right)|u+1|^{-1}$, so that $\int\hat{g}_q(u)du=g_q(0)=f(q)$. Since f has compact support in A_S^*, the intersection of O_S^* with the support of f is finite and by (24) we get the equality (41). To prove (42), we consider the natural projection pr_v from $\prod_{l\in S}k_l^*$ to $\prod_{l\neq v}k_l^*$. The image $pr_v(O_S^*)$ is still a discrete subgroup of $\prod_{l\neq v}k_l^*$, thus there are only finitely many $q\in O_S^*$ such that k_v^* meets the support of f_q, where $f_q(a)=f(qa)$ for all a.

For each $q \in O_S^*$ one has,

$$\int \hat{g}_q(u)(-\log|u_v|)du = \int_{k_v^*}' \frac{f_q(u^{-1})}{|1-u|} d^*u\,, \quad (44)$$

and this vanishes except for finitely many $q's$, so that by (24) we get the equality (42).

Theorem 3.

Let k be a global field of positive characteristic and Q_Λ be the orthogonal projection on the subspace of $L^2(X)$ spanned by the $f \in S(A)$ such that $f(x)$ and $\hat{f}(x)$ vanish for $|x| > \Lambda$. Let $h \in S(C_k)$ have compact support. Then the following conditions are equivalent,

a) *When $\Lambda \to \infty$, one has*

$$\text{Trace}\,(Q_\Lambda U(h)) = 2h(1)\log'\Lambda + \sum_v \int_{k_v^*}' \frac{h(u^{-1})}{|1-u|} d^*u + o(1)\,. \quad (45)$$

b) *All L functions with Grossencharakter on k satisfy the Riemann Hypothesis.*

To prove that (a) implies (b), we shall prove (assuming (a)) the positivity of the Weil distribution,

$$\Delta = \log|d^{-1}|\delta_1 + D - \sum_v D_v\,. \quad (46)$$

We have that for $\delta = 0$, the map E,

$$E(f)(g) = |g|^{1/2} \sum_{q \in k^*} f(qg) \qquad \forall g \in C_k\,, \quad (47)$$

defines a surjective isometry from $L^2(X)_0$ to $L^2(C_k)$ such that,

$$EU(a) = |a|^{1/2} V(a) E\,, \quad (48)$$

where the left regular representation V of C_k on $L^2(C_k)$ is given by,

$$(V(a)\xi)(g) = \xi(a^{-1}g) \qquad \forall g, a \in C_k\,. \quad (49)$$

Let S_Λ be the subspace of $L^2(C_k)$ given by,

$$S_\Lambda = \{\xi \in L^2(C_k); \xi(g) = 0, \forall g, |g| \notin [\Lambda^{-1}, \Lambda]\}. \quad (50)$$

We shall denote by the same letter the corresponding orthogonal projection.

Let $B_{\Lambda,0}$ be the subspace of $L^2(X)_0$ spanned by the $f \in S(A)_0$ such that $f(x)$ and $\hat{f}(x)$ vanish for $|x| > \Lambda$ and $Q_{\Lambda,0}$ be the corresponding orthogonal projection. Let $f \in S(A)_0$ be such that $f(x)$ and $\hat{f}(x)$ vanish for $|x| > \Lambda$, then $E(f)(g)$ vanishes for $|g| > \Lambda$, and the equality

$$E(f)(g) = E(\hat{f})\left(\frac{1}{g}\right) \quad f \in S(A)_0, \quad (51)$$

shows that $E(f)(g)$ vanishes for $|g| < \Lambda^{-1}$.

This shows that $E(B_{\Lambda,0}) \subset S_\Lambda$, so that if we let $Q'_{\Lambda,0} = EQ_{\Lambda,0}E^{-1}$, we get the inequality,

$$Q'_{\Lambda,0} \leq S_\Lambda \quad (52)$$

and for any Λ the following distribution on C_k is of positive type,

$$\Delta_\Lambda(f) = \text{Trace}\left((S_\Lambda - Q'_{\Lambda,0})V(f)\right), \quad (53)$$

i.e. one has,

$$\Delta_\Lambda(f * f^*) \geq 0, \quad (54)$$

where $f^*(g) = \bar{f}(g^{-1})$ for all $g \in C_k$.

Let then $f(g) = |g|^{-1/2} h(g^{-1})$, so that by (48) one has $EU(h) = V(\tilde{f})E$ where $\tilde{f}(g) = f(g^{-1})$ for all $g \in C_k$. Then, we have:

$$\sum_v D_v(f) - \log|d^{-1}| = \sum_v \int'_{k_v^*} \frac{h(u^{-1})}{|1-u|} d^*u . \quad (55)$$

One has $\text{Trace}(S_\Lambda V(f)) = 2f(1)\log'\Lambda$, thus using (a) we see that the limit of Δ_Λ when $\Lambda \to \infty$ is the Weil distribution Δ. The term D in the latter comes from the nuance between the subspaces B_Λ and $B_{\Lambda,0}$. This shows using (53), that the distribution Δ is of positive type so that (b) holds.

Let us now show that (b) implies (a). We shall compute from the zeros of L-functions and independently of any hypothesis the limit of the distributions Δ_Λ when $\Lambda \to \infty$.

We choose an isomorphism

$$C_k \cong C_{k,1} \times N . \quad (56)$$

where $N = \text{range } |\ | \subset R_+^*$, $N \cong Z$ is the subgroup $q^Z \subset R_+^*$. For $\rho \in C$ we let $d\mu_\rho(z)$ be the harmonic measure of ρ with respect to the line $iR \subset C$. It is a probability measure on the line iR and coincides with the Dirac mass at ρ when ρ is on the line.

The implication (b) $\Rightarrow$ (a) follows immediately from the explicit formulas and the following lemma,

Lemma 1.

The limit of the distributions Δ_Λ when $\Lambda \to \infty$ is given by,

$$\Delta_\infty(f) = \sum_{\substack{L\left(\tilde{\chi},\frac{1}{2}+\rho\right)=0 \\ \rho\in B/N^\perp}} N\left(\tilde{\chi},\frac{1}{2}+\rho\right)\int_{z\in iR}\hat{f}(\tilde{\chi},z)d\mu_\rho(z) \quad (57)$$

where B *is the open strip* $B=\left\{\rho\in C;\mathrm{Re}(\rho)\in\left]\frac{-1}{2},\frac{1}{2}\right[\right\}$, $N\left(\tilde{\chi},\frac{1}{2}+\rho\right)$ *is the multiplicity of the zero,* $d\mu_\rho(z)$ *is the harmonic measure of* ρ *with respect to the line* $iR\subset C$, *and the Fourier transform* $\hat{f}$ *of* f *is defined by*

$$\hat{f}(\tilde{\chi},\rho)=\int_{C_k} f(u)\tilde{\chi}(u)|u|^\rho d^*u. \quad (58)$$

Let us first recall the Weil explicit formulas. One lets k be a global field. One identifies the quotient $C_k/C_{k,1}$ with the range of the module,

$$N=\{|g|;g\in C_k\}\subset R_+^*. \quad (59)$$

One endows N with its normalized Haar measure d^*x. Given a function F on N such that, for some $b>\frac{1}{2}$,

$$|F(\nu)|=0(\nu^b) \quad \nu\to 0, \quad |F(\nu)|=0(\nu^{-b}), \quad \nu\to\infty, \quad (60)$$

one lets,

$$\Phi(s)=\int_N F(\nu)\nu^{1/2-s}d^*\nu. \quad (61)$$

Given a Grossencharakter χ, i.e. a character of C_k and any ρ in the strip $0<\mathrm{Re}(\rho)<1$, one lets $N(\chi,\rho)$ be the order of $L(\chi,s)$ at $s=\rho$. One lets,

$$S(\chi,F)=\sum_\rho N(\chi,\rho)\Phi(\rho) \quad (62)$$

where the sum takes place over ρ's in the above open strip. One then defines a distribution Δ on C_k by,

$$\Delta=\log\left|d^{-1}\right|\delta_1+D-\sum_v D_v, \quad (63)$$

where δ_1 is the Dirac mass at $1\in C_k$, where d is a differential idele of k so that $|d|^{-1}$ is up to sign the discriminant of k when char $(k)=0$ and is q^{2g-2} when k is a function field over a curve of genus g with coefficients in the finite field F_q. The distribution D is given by,

$$D(f)=\int_{C_k} f(w)\left(|w|^{1/2}+|w|^{-1/2}\right)d^*w \quad (64)$$

where the Haar measure $d^* w$ is normalized. The distributions D_v are parametrized by the places v of k and are obtained as follows. For each v one considers the natural proper homomorphism,

$$k_v^* \to C_k\,, \quad x \to \text{class of } (1,\ldots,x,1\ldots) \quad (65)$$

of the multiplicative group of the local field k_v in the idele class group C_k. One then has,

$$D_v(f) = Pfw \int_{k_v^*} \frac{f(u)}{|1-u|} |u|^{1/2} d^* u \quad (66)$$

where the Haar measure $d^* u$ is normalized, and where the Weil Principal value Pfw of the integral is obtained as follows, for a local field $K = k_v$,

$$Pfw \int_{k_v^*} 1_{R_v^*} \frac{1}{|1-u|} d^* u = 0\,, \quad (67)$$

if the local field k_v is non Archimedean, and otherwise:

$$Pfw \int_{k_v^*} \varphi(u) d^* u = PF_0 \int_{R_+^*} \psi(v) d^* v\,, \quad (68)$$

where $\psi(v) = \int_{|u|=v} \varphi(u) d_v u$ is obtained by integrating φ over the fibers, while

$$PF_0 \int \psi(v) d^* v = 2\log(2\pi)c + \lim_{t\to\infty}\left(\int \left(1 - f_0^{2t}\right)\psi(v) d^* v - 2c\log t\right), \quad (69)$$

where one assumes that $\psi - c f_1^{-1}$ is integrable on R_+^*, and $f_0(v) = \inf\left(v^{1/2}, v^{-1/2}\right) \quad \forall v \in R_+^*$, $f_1 = f_0^{-1} - f_0$. The Weil explicit formula is then,

Theorem 4.

With the above notations one has $S(\chi, F) = \Delta\left(F(|w|)\chi(w)\right)$.

Let K be non Archimedean, furthermore, let α be a character of K such that,

$$\alpha / R = 1\,, \quad \alpha / \pi^{-1} R \neq 1\,. \quad (70)$$

Then, for the Fourier transform given by,

$$(Ff)(x) = \int f(y)\alpha(y) dy\,, \quad (71)$$

with dy the selfdual Haar measure, one has

$$F(1_R) = 1_R\,. \quad (72)$$

Lemma 2.

With the above choice of α one has

$$\int' \frac{h(u^{-1})}{|1-u|} d^*u = Pfw \int \frac{h(u^{-1})}{|1-u|} d^*u \qquad (73)$$

with the notations of theorem 1.

By construction the two sides can only differ by a multiple of $h(1)$. Let us recall from theorem 1 that the left hand side is given by

$$\left\langle L, \frac{h(u^{-1})}{|u|} \right\rangle, \qquad (74)$$

where L is the unique extension of $\rho^{-1} \dfrac{du}{|1-u|}$ whose Fourier transform vanishes at 1, $\hat{L}(1) = 0$. Thus from (67) we just need to check that (74) vanishes for $h = 1_{R^*}$, i.e. that

$$\langle L, 1_{R^*} \rangle = 0. \qquad (75)$$

Equivalently, if we let $Y = \{y \in K; |y-1| = 1\}$ we just need to show, using Parseval, that,

$$\langle \log|u|, \hat{1}_Y \rangle = 0. \qquad (76)$$

One has $\hat{1}_Y(x) = \int_Y \alpha(xy)dy = \alpha(x)\hat{1}_{R^*}(x)$, and $1_{R^*} = 1_R - 1_P$, $\hat{1}_{R^*} = 1_R - |\pi| 1_{\pi^{-1}R}$, thus, with $q^{-1} = |\pi|$,

$$\hat{1}_Y(x) = \alpha(x)\left(1_R - \frac{1}{q} 1_{\pi^{-1}R}\right)(x). \qquad (77)$$

We now need to compute $\int \log|x| \hat{1}_Y(x)dx = A + B$,

$$A = -\frac{1}{q} \int_{\pi^{-1}R^*} \alpha(x)(\log q)dx, \quad B = \left(1 - \frac{1}{q}\right) \int_R \log|x| dx. \qquad (78)$$

Let us show that $A + B = 0$. One has $\int_R dx = 1$, and

$$A = -\int_{R^*} \alpha(\pi^{-1}y)(\log q)dy = -\log q \left(\int_R \alpha(\pi^{-1}y)dy - \int_P dy\right) = \frac{1}{q} \log q, \qquad (79)$$

since $\int_R \alpha(\pi^{-1}y)dy = 0$ as $\alpha / \pi^{-1}R \neq 1$.

To compute B, note that $\int_{\pi^n R^*} dy = q^{-n}\left(1 - \frac{1}{q}\right)$ so that

$$B=\left(1-\frac{1}{q}\right)\int_R \log|x|dx=\left(1-\frac{1}{q}\right)^2\sum_{n=0}^{\infty}(-n\log q)q^{-n}=-q^{-1}\log q=-\frac{1}{q}\log q, \quad (80)$$

and $A+B=0$.

Let us now treat the case of Archimedean fields. We take $K=R$ first, and we normalize the Fourier transform as,

$$(Ff)(x)=\int f(y)e^{-2\pi ixy}dy \quad (81)$$

so that the Haar measure dx is selfdual.

With the notations of (68) one has,

$$Pfw\int_{R^*} f_0^3(|u|)\frac{|u|^{1/2}}{|1-u|}d^*u=\log\pi+\gamma \quad (82)$$

where γ is Euler's constant, $\gamma=-\Gamma'(1)$. Indeed integrating over the fibers gives $f_0^4\times\left(1-f_0^4\right)^{-1}$, and one gets,

$$PF_0\int_{R_+^*} f_0^4\times\left(1-f_0^4\right)^{-1}d^*u=\left[\log(2\pi)+\lim_{t\to\infty}\left(\int_{R_+^*}\left(1-f_0^{2t}\right)f_0^4\left(1-f_0^4\right)^{-1}d^*u-\log t\right)\right]=\log 2\pi+\gamma-\log 2. \quad (83)$$

Now let $\varphi(u)=-\log|u|$, it is a tempered distribution on R and one has,

$$\left\langle\varphi,e^{-\pi u^2}\right\rangle=\frac{1}{2}\log\pi+\frac{\gamma}{2}+\log 2, \quad (84)$$

as one obtains from $\frac{\partial}{\partial s}\int|u|^{-s}e^{-\pi u^2}du=\frac{\partial}{\partial s}\left[\pi^{\frac{s-1}{2}}\Gamma\left(\frac{1-s}{2}\right)\right]$ evaluated at $s=0$, using $\frac{\Gamma'\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}=-\gamma-2\log 2$. Thus by the Parseval formula one has,

$$\left\langle\hat{\varphi},e^{-\pi x^2}\right\rangle=\frac{1}{2}\log\pi+\frac{\gamma}{2}+\log 2, \quad (85)$$

which gives, for any test function f,

$$\left\langle\hat{\varphi},f\right\rangle=\lim_{\varepsilon\to 0}\left[\int_{|x|\geq\varepsilon}f(x)d^*x+(\log\varepsilon)f(0)\right]+\lambda f(0) \quad (86)$$

where $\lambda=\log(2\pi)+\gamma$. In order to get (86) one uses the equality,

$$\lim_{\varepsilon\to 0}\left[\int_{|x|\geq\varepsilon} f(x)d^*x+(\log\varepsilon)f(0)\right]=\lim_{\varepsilon\to 0}\left[\int f(x)|x|^{\varepsilon}d^*x-\frac{1}{\varepsilon}f(0)\right], \quad (87)$$

which holds since both sides vanish for $f(x)=1$ if $|x|\leq 1$, $f(x)=0$ otherwise. Thus from (86) one gets,

$$\int_R' f(u)\frac{1}{|1-u|}d^*u=\lambda f(1)+\lim_{\varepsilon\to 0}\left[\int_{|1-u|\geq\varepsilon}\frac{f(u)}{|1-u|}d^*u+(\log\varepsilon)f(1)\right]. \quad (88)$$

Taking $f(u)=|u|^{1/2} f_0^3(|u|)$, the right hand side of (88) gives $\lambda-\log 2=\log\pi+\gamma$, thus we conclude using (82) that for any test function f,

$$\int_R' f(u)\frac{1}{|1-u|}d^*u=Pfw\int_R f(u)\frac{1}{|1-u|}d^*u. \quad (89)$$

Let us finally consider the case $K=C$. We choose the basic character α as

$$\alpha(z)=\exp 2\pi i(z+\bar{z}), \quad (90)$$

the selfdual Haar measure is $dzd\bar{z}=|dz\wedge d\bar{z}|$, and the function $f(z)=\exp-2\pi|z|^2$ is selfdual. The normalized multiplicative Haar measure is

$$d^*z=\frac{|dz\wedge d\bar{z}|}{2\pi|z|^2}. \quad (91)$$

Let us compute the Fourier transform of the distribution

$$\varphi(z)=-\log|z|_C=-2\log|z|. \quad (92)$$

One has

$$\left\langle\varphi,\exp-2\pi|z|^2\right\rangle=\log 2\pi+\gamma, \quad (93)$$

as is seen using $\frac{\partial}{\partial\varepsilon}\left(\int e^{-2\pi|z|^2}|z|^{-2\varepsilon}|dz\wedge d\bar{z}|\right)=\frac{\partial}{\partial\varepsilon}\left[(2\pi)^{\varepsilon}\Gamma(1-\varepsilon)\right].$

Thus $\left\langle\hat{\varphi},\exp-2\pi|u|^2\right\rangle=\log 2\pi+\gamma$ and one gets,

$$\left\langle\hat{\varphi},f\right\rangle=\lim_{\varepsilon\to 0}\left[\int_{|u|_C\geq\varepsilon} f(u)d^*u+\log\varepsilon f(0)\right]+\lambda' f(0) \quad (94)$$

where $\lambda'=2(\log 2\pi+\gamma)$.

To see this one uses the analogue of (87) for $K=C$, to compute the right hand side of (94) for $f(z)=\exp-2\pi|z|^2$. Thus, for any test function f, one has,

$$\int_C' f(u)\frac{1}{|1-u|_C}d^*u = \lambda' f(1) + \lim_{\varepsilon\to 0}\left[\int_{|1-u|_C\geq\varepsilon}\frac{f(u)}{|1-u|_C}d^*u + (\log\varepsilon)f(1)\right]. \quad (95)$$

Let us compare it with Pfw. When one integrates over the fibers of $C^* \xrightarrow{|\ |_C} R_+^*$ the function $|1-z|_C^{-1}$ one gets,

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{\left|1-e^{i\theta}z\right|^2}d\theta = \frac{1}{1-|z|^2} \text{ if } |z|<1, \text{ and } \frac{1}{|z|^2-1} \text{ if } |z|>1. \quad (96)$$

Thus for any test function f on R_+^* one has by (68)

$$Pfw\int f(|u|_C)\frac{1}{|1-u|_C}d^*u = PF_0\int f(\nu)\frac{1}{|1-\nu|}d^*\nu \quad (97)$$

with the notations of (69). With $f_2(\nu) = \nu^{\frac{1}{2}}f_0(\nu)$ we thus get, using (69),

$$Pfw\int f_2(|u|_C)\frac{1}{|1-u|_C}d^*u = PF_0\int f_0 f_1^{-1}d^*\nu = 2(\log 2\pi + \gamma). \quad (98)$$

We shall now show that,

$$\lim_{\varepsilon\to 0}\left[\int_{|1-u|_C\geq\varepsilon}\frac{f_2(|u|_C)}{|1-u|_C}d^*u + \log\varepsilon\right] = 0, \quad (99)$$

it will then follow that, using (95),

$$\int_C' f(u)\frac{1}{|1-u|_C}d^*u = Pfw\int f(u)\frac{1}{|1-u|_C}d^*u. \quad (100)$$

To prove (99) it is enough to investigate the integral,

$$\int_{|z|\leq 1, |1-z|\geq\varepsilon}\left[(1-z)(1-\bar{z})\right]^{-1}|dz\wedge d\bar{z}| = j(\varepsilon) \quad (101)$$

and show that $j(\varepsilon) = \alpha\log\varepsilon + o(1)$ for $\varepsilon\to 0$. A similar statement then holds for

$$\int_{|z|\leq 1, |1-z^{-1}|\geq\varepsilon}\left[(1-z)(1-\bar{z})\right]^{-1}|dz\wedge d\bar{z}|.$$

One has $j(\varepsilon) = \int_D |dZ\wedge d\bar{Z}|$, where $Z = \log(1-z)$ and the domain D is contained in the rectangle,

$$\left\{Z = (x+iy); \log\varepsilon\leq x\leq\log 2, -\frac{\pi}{2}\leq y\leq\pi/2\right\} = R_\varepsilon \quad (102)$$

and bounded by the curve $x = \log(2\cos y)$ which comes from the equation of the circle $|z| = 1$ in polar coordinates centred at $z = 1$. One thus gets,

$$j(\varepsilon) = 4\int_{\log\varepsilon}^{\log 2} Arc\cos(e^x/2)dx, \quad (103)$$

when $\varepsilon \to 0$ one has $j(\varepsilon) \approx 2\pi\log(1/\varepsilon)$, which is the area of the following rectangle (in the measure $|dz \wedge d\bar{z}|$),

$$\{Z = (x+iy); \log\varepsilon \le x \le 0, -\pi/2 \le y \le \pi/2\}. \quad (104)$$

One has $|R_\varepsilon| - 2\pi\log 2 = 2\pi\log(1/\varepsilon)$. When $\varepsilon \to 0$ the area of $R_\varepsilon \setminus D$ converges to

$$4\int_{-\infty}^{\log 2} Arc\sin(e^x/2)dx = -4\int_0^{\pi/2}\log(\sin u)du = 2\pi\log 2, \quad (105)$$

so that $j(\varepsilon) = 2\pi\log(1/\varepsilon) + o(1)$ when $\varepsilon \to 0$.

Thus we can assert that with the above choice of basic characters for local fields one has, for any test function f,

$$\int_K' f(u)\frac{1}{|1-u|}d^*u = Pfw\int f(u)\frac{1}{|1-u|}d^*u. \quad (106)$$

Now, we have the following

Lemma 3.

Let K be a local field, α_0 a normalized character as above and α, $\alpha(x) = \alpha_0(\lambda x)$ an arbitrary character of K.

Hence, we obtain that:

$$\int_K' f(u)\frac{1}{|1-u|}d^*u = \log|\lambda| f(1) + Pfw\int f(u)\frac{1}{|1-u|}d^*u. \quad (107)$$

Chapter 4.

On p-adic and adelic strings

4.1 Open and closed p-adic strings.[8]

Let us now discuss the question of the construction of a dynamical theory for open and closed p-adic strings. It was proposed (Volovich, 1987) to consider p-adic generalization of the Veneziano string amplitude in two ways, according to two equivalent representations

$$A(a,b)=\int_0^1 |x|^{a-1}|1-x|^{b-1}dx=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \quad (1)$$

The first way corresponds to an interpretation of the amplitude A(a, b) as a convolution of two characters and the second one to the p-adic interpolation of the gamma function. Using the first approach a complex-valued string amplitude over a finite Galois field has been constructed.
Consideration of string amplitudes as a convolution of characters is a very general concept applicable to characters on number fields, groups and algebras.
Now, we have the string amplitudes of the following form

$$A(\gamma_a,\gamma_b)=\int_K \gamma_a(x)\gamma_b(1-x)dx, \quad (2)$$

where K is a field F, i.e., K = F, $\gamma_a(x)$ is a multiplicative character on K, and dx is a measure on K. Note that the range of integration in (2) is over the entire field F, and hence this p-adic generalization is rather one of the Virasoro-Shapiro amplitude

$$A=\int_C |z|^{a-1}|1-z|^{b-1}dz, \quad (3)$$

than of the Veneziano amplitude (1), where the integration is over the unit segment on the real axis. The equation (3) is just a particular case of (2) for K = C and $\gamma_a(z)=|z|^{a-1}$. The ordinary Veneziano amplitude can be rewritten in the following way

$$A=\int_R |x|^{a-1}|1-x|^{b-1}\theta_{[0,1]}(x)dx, \quad (4)$$

where $\theta(x)$ is the characteristic function of the segment $[0,1]$. In particular, it can be written in terms of the Heaviside function $\theta_{[0,1]}(x)=\theta(x)\theta(1-x)$. Hence, in order to have a generalization of the expression (4) on an arbitrary field F one should have on F an analogue of the Heaviside function or the function sign x.
We have a generalization of the amplitude (4), in the case of an arbitrary locally compact disconnected field F, in the following form

$$A_{F,\tau}^{open}(\gamma_a,\gamma_b)=\int_F |x|^{a-1}|1-x|^{b-1}\theta_{\tau[0,1]}dx \quad (5)$$

where $\theta_{\tau[0,1]}(x)$ is a p-adic generalization of the characteristic function of the segment $[0,1]$ on F related to a quadratic extension $F(\sqrt{\tau})$. In particular one can take the function $\theta_{\tau[0,1]}(x)$ in the form $\theta_\tau(x)\theta_\tau(1-x)$ where $\theta_\tau(x)$ is a p-adic analogue of the Heaviside function.
In the ordinary case there is an important relation between amplitudes of the open and the closed strings. This relation give a connection on the tree level as follows

$$A_{tree}^{closed}(s,t,u)=\sin\left(\frac{\pi t}{8}\right)A_{tree}^{open}\left(\frac{s}{4},\frac{t}{4}\right)A_{tree}^{open}\left(\frac{t}{4},\frac{u}{4}\right), \quad (6)$$

where s, t, u are the Mandelstam variables.

Let F in eq. (5) be a non-discrete totally disconnected and locally compact field and define also the generalized Heaviside function in the form

$$\theta_\tau(\omega) = \frac{1 + sign_\tau \omega}{2} \quad (7)$$

which is an analogue of the ordinary one.
Now we will consider the amplitude (5) with the characteristic function in one of the following forms:

$$\theta_{\tau[0,1]}(x) = \theta_\tau(x)\theta_\tau(1-x), \quad (8.1)$$

$$\theta_{\tau[0,1]}(x) = \frac{1}{2}\left(1 + Sign_\tau x \cdot Sign_\tau(1-x)\right), \quad (8.2)$$

$$\theta_{\tau[0,1]}(x) = \frac{1}{2}\left(Sign_\tau x + Sign_\tau(1-x)\right), \quad (8.3)$$

$$\theta_{\tau[0,1]}(x) = \frac{1}{2}\left(Sign_\tau x - Sign_\tau(-1)\cdot Sign_\tau(1-x)\right), \quad \tau = \varepsilon, \quad (8.4)$$

$$\theta_{\tau[0,1]}(x) = \frac{1}{2}\left(1 - Sign_\tau(-1)\cdot Sign_\tau x \cdot Sign_\tau(1-x)\right). \quad (8.5)$$

The corresponding amplitudes (5) can be calculated with the help of the general formula

$$B(\pi_1, \pi_2) = \frac{\Gamma(\pi_1)\Gamma(\pi_2)}{\Gamma(\pi_1\pi_2)}, \quad (9)$$

which connects the beta function

$$B(\pi_1, \pi_2) = \int_F \pi_1(x)|x|^{-1}\pi_2(1-x)|1-x|^{-1}dx, \quad (10)$$

where $\pi(x)$ is a multiplicative character with the gamma function defined by an additive character χ

$$\Gamma(\pi) = \int_F \chi(x)\pi(x)|x|^{-1}dx. \quad (11)$$

Consider now the string amplitudes, constructed over the p-adic fields Q_p and their quadratic extension $Q_p\left(\sqrt{\tau}\right)$, from the point of view of the product formulae (6) which relates amplitudes of closed and open strings in a very simple form. With regard the case $\tau = \varepsilon$, the closed string amplitude defined on the quadratically extended field $K = Q_p\left(\sqrt{\varepsilon}\right)$, has the form

$$A^{closed}_{Q_p\left(\sqrt{\varepsilon}\right)}(a,b,c) = \int_{Q_p\left(\sqrt{\varepsilon}\right)} |x|^{a-1}|1-x|^{b-1}dx = \frac{1-q^{a-1}}{1-q^{-a}}\cdot\frac{1-q^{b-1}}{1-q^{-b}}\cdot\frac{1-q^{c-1}}{1-q^{-c}}, \quad (12)$$

where $q = p^2$. There are no such formulae as simple as (7) for the above constructed open string amplitudes. However, there exists a formula in the following form

$$A^{closed}_{Q_p(\sqrt{\varepsilon})}(a,b,c) = A^{open,total}_{Q_p}(a,b,c)\tilde{A}_{Q_p}(a,b,c), \quad (13)$$

where

$$A^{open,total}_{Q_p}(a,b,c) = \int_{Q_p} |x|^{a-1}|1-x|^{b-1}dx = \frac{1-p^{a-1}}{1-p^{-a}} \cdot \frac{1-p^{b-1}}{1-p^{-b}} \cdot \frac{1-p^{c-1}}{1-p^{-c}} \quad (14)$$

is a p-adic analogue of the totally crossing symmetric Veneziano amplitude.
Furthermore, the p-adic generalization of the N-point tree amplitude for vector particles in the bosonic case, can be proposed in the following form

$$A(\zeta_1 k_1,\dots,\zeta_n k_n) = g^{n-2} \int_{(Q_p)^{n-3}} \theta_{[0,y_{n-1},\dots,y_{3,1}]}(y) F(\zeta,k,y) \cdot \prod_{3\le i<j\le n-1} |y_i - y_j|_p^{k_i k_j} \prod_{3\le i\le n-1} \left(|y_i|_p^{k_n k_i} |1-y_i|^{k_2 k_i} dy_i \right), \quad (15)$$

where $\theta_{[0,y_{n-1},\dots,y_{3,1}]}(y)$ is a p-adic generalization of the characteristic function of the simplex $0 \le y_{n-1} \le \dots \le y_4 \le y_3 \le 1$ and $F(\zeta,k,y)$ is the part of $\exp \sum_{i=j} \left\{ \frac{1}{2}(\zeta_i \zeta_j)/(y_i - y_j)^2 - k_i k_j / y_i - y_j \right\}$ that is multilinear in all the polarization vectors ζ_i.

4.2 On adelic strings.[9]

The set of all adeles A may be given in the form

$$A = \bigcup_S \mathrm{A}(S), \quad \mathrm{A}(S) = R \times \prod_{p\in S} Q_p \times \prod_{p\notin S} Z_p . \quad (16)$$

A has the structure of a topological ring.
We recall that quantum amplitudes defined by means of path integral may be symbolically presented as

$$A(K) = \int A(X)\chi\left(-\frac{1}{h}S[X]\right)DX , \quad (17)$$

where K and X denote classical momenta and configuration space, respectively. $\chi(a)$ is an additive character, $S[X]$ is a classical action and h is the Planck constant.
Now we consider simple p-adic and adelic bosonic string amplitudes based on the functional integral (17). The scattering of two real bosonic strings in 26-dimensional space-time at the tree level can be described in terms of the path integral in 2-dimensional quantum field theory formalism as follows:

$$A_\infty(k_1,\dots,k_4) = g_\infty^2 \int DX \exp\left(\frac{2\pi i}{h}S_0[X]\right) \times \prod_{j=1}^{4} \int d^2\sigma_j \exp\left[\frac{2\pi i}{h} k_\mu^{(j)} X^\mu(\sigma_j,\tau_j)\right], \quad (18)$$

where $DX = DX^0(\sigma,\tau)DX^1(\sigma,\tau)\dots DX^{25}(\sigma,\tau)$, $d^2\sigma_j = d\sigma_j d\tau_j$ and

$$S_0[X] = -\frac{T}{2}\int d^2\sigma \partial_\alpha X^\mu \partial^\alpha X_\mu \quad (19)$$

with $\alpha = 0,1$ and $\mu = 0,1,...,25$. Using the usual procedure one can obtain the crossing symmetric Veneziano amplitude

$$A_\infty(k_1,...,k_4) = g_\infty^2 \int_R |x|_\infty^{k_1k_2} |1-x|_\infty^{k_2k_3} dx \quad (20)$$

and similarly the Virasoro-Shapiro one for closed bosonic strings.
As p-adic Veneziano amplitude, it was postuled p-adic analogue of (20), i.e.

$$A_p(k_1,...,k_4) = g_p^2 \int_{Q_p} |x|_p^{k_1k_2} |1-x|_p^{k_2k_3} dx, \quad (21)$$

where only the string world sheet (parametrized by x) is p-adic. Expressions (20) and (21) are Gel'fand-Graev beta functions on R and Q_p, respectively.
Now we take p-adic analogue of (18), i.e.

$$A_p(k_1,...,k_4) = g_p^2 \int DX \chi_p\left(-\frac{1}{h} S_0[X]\right) \times \prod_{j=1}^{4} \int d^2\sigma_j \chi_p\left(-\frac{1}{h} k_\mu^{(j)} X^\mu(\sigma_j,\tau_j)\right), \quad (22)$$

to be p-adic string amplitude, where $\chi_p(u) = \exp(2\pi i\{u\}_p)$ is p-adic additive character and $\{u\}_p$ is the fractional part of $u \in Q_p$. In (22), all space-time coordinates X_μ, momenta k_i and world sheet (σ,τ) are p-adic.
Evaluation of (22), in analogous way to the real case, leads to

$$A_p(k_1,...,k_4) = g_p^2 \prod_{j=1}^{4} \int d^2\sigma_j \times \chi_p\left[\frac{\sqrt{-1}}{2hT} \sum_{i<j} k_i k_j \log\left((\sigma_i-\sigma_j)^2 + (\tau_i-\tau_j)^2\right)\right]. \quad (23)$$

Adelic string amplitude is product of real and all p-adic amplitudes, i.e.

$$A_A(k_1,...,k_4) = A_\infty(k_1,...,k_4) \prod_p A_p(k_1,...,k_4). \quad (24)$$

In the case of the Veneziano amplitude and $(\sigma_i,\tau_j) \in \mathrm{A}(S) \times \mathrm{A}(S)$, where $\mathrm{A}(S)$ is defined in (16), we have

$$A_A(k_1,...,k_4) = g_\infty^2 \int_R |x|_\infty^{k_1k_2} |1-x|_\infty^{k_2k_3} dx \times \prod_{p \in S} g_p^2 \prod_{j=1}^{4} \int d^2\sigma_j \times \prod_{p \notin S} g_p^2. \quad (25)$$

There is the sense to take adelic coupling constant as

$$g_A^2 = |g|_\infty^2 \prod_p |g|_p^2 = 1, \quad 0 \neq g \in Q. \quad (26)$$

Hence, it follows that p-adic effects in the adelic Veneziano amplitude induce discreteness of string momenta and contribute to an effective coupling constant in the form

$$g_{ef}^2 = g_A^2 \prod_{p \in S} \prod_{j=1}^{4} \int d^2\sigma_j \geq 1. \quad (26b)$$

4.3 Solitonic q-branes of p-adic string theory.[10]

Now we consider the expressions for various amplitudes in ordinary bosonic open string theory, written as integrals over the boundary of the world sheet which is the real line R. Now replace the integrals over R by integrals over the p-adic field Q_p with appropriate measure, and the norms of the functions in the integrand by the p-adic norms. Using p-adic analysis, it is possible to compute N tachyon amplitudes at tree-level for all $N \geq 3$.
This leads to an exact action for the open string tachyon in d dimensional p-adic string theory. This action is:

$$S = \int d^d x L = \frac{1}{g^2}\frac{p^2}{p-1}\int d^d x\left[-\frac{1}{2}\phi p^{-\frac{1}{2}\Box}\phi + \frac{1}{p+1}\phi^{p+1}\right], \quad (27)$$

where $\Box$ denotes the d dimensional Laplacian, ϕ is the tachyon field, g is the open string coupling constant, and p is an arbitrary prime number.
The equation of motion derived from this action is,

$$p^{-\frac{1}{2}\Box}\phi = \phi^p. \quad (28)$$

The following configuration

$$\phi(x) = f\left(x^{q+1}\right)f\left(x^{q+2}\right)...f\left(x^{d-1}\right) \equiv F^{(d-q-1)}\left(x^{q+1},...,x^{d-1}\right), \quad (29)$$

with

$$f(\eta) \equiv p^{\frac{1}{2(p-1)}}\exp\left(-\frac{1}{2}\frac{p-1}{p\ln p}\eta^2\right), \quad (30)$$

describes a soliton solution with energy density localised around the hyperplane $x^{q+1} = ... = x^{d-1} = 0$.
This follows from the identity:

$$p^{-\frac{1}{2}\partial_\eta^2} f(\eta) = (f(\eta))^p. \quad (31)$$

We shall call (29), with f as in (30), the solitonic q-brane solution. Let us denote by $x_\perp = \left(x^{q+1},...,x^{d-1}\right)$ the coordinates transverse to the brane and by $x_\parallel = \left(x^0,...,x^q\right)$ those tangential to it. The energy density per unit q-volume of this brane, which can be identified as its tension T_q, is given by

$$T_q = -\int d^{d-q-1}x_\perp L\left(\phi = F^{(d-q-1)}(x_\perp)\right) = \frac{1}{2g_q^2}\frac{p^2}{p+1} \quad (32)$$

where

$$g_q = g\left[\frac{p^2-1}{2\pi p^{2p/(2p-1)}\ln p}\right]^{(d-q-1)/4}. \quad (33)$$

Hence, we obtain the following equation

$$T_q = -\int d^{d-q-1}x_\perp L\left(F^{(d-q-1)}(x_\perp)\right) = \frac{1}{2\left\{g\left[\frac{p^2-1}{2\pi p^{2p/(2p-1)}\ln p}\right]^{(d-q-1)/4}\right\}^2} \frac{p^2}{p+1}. \quad (33b)$$

Let us now consider a configuration of the type

$$\phi(x) = F^{(d-q-1)}(x_\perp)\psi(x_\parallel), \quad (34)$$

with $F^{(d-q-1)}(x_\perp)$ as defined in (29), (30). For $\psi = 1$ this describes the solitonic q-brane. Fluctuations of ψ around 1 denote fluctuations of ϕ localised on the soliton; thus $\psi(x_\parallel)$ can be regarded as one of the fields on its world-volume. We shall call this the tachyon field on the solitonic q-brane world-volume. Substituting (34) into (28) and using (31) we get

$$p^{-\frac{1}{2}\Box_\parallel}\psi = \psi^p, \quad (35)$$

where $\Box_\parallel$ denotes the (q+1) dimensional Laplacian involving the world-volume coordinates $x_\parallel$ of the q-brane. The action involving ψ can be obtained by substituting (34) into (27):

$$S_q(\psi) = S\left(\phi = F^{(d-q-1)}(x_\perp)\psi(x_\parallel)\right) = \frac{1}{g_q^2}\frac{p^2}{p-1}\int d^{q+1}x_\parallel\left[-\frac{1}{2}\psi p^{-\frac{1}{2}\Box_\parallel}\psi + \frac{1}{p+1}\psi^{p+1}\right], \quad (36)$$

where g_q has been defined in eq.(33).

In conclusion, we shall now show the world-volume action on the Dirichlet q-brane. Let us consider the situation where we start with the action (27) with g replaced by another coupling constant $\bar{g}$, and compactify (d – q – 1) directions on circles of radii $1/\sqrt{2}$. Let u^i denote the compact coordinates and z^μ the non-compacts ones, and consider an expansion of the field ϕ of the form:

$$\phi(x) = \tilde{\psi}(z) + \sqrt{\frac{C}{p}}\sum_{i=1}^{d-q-1}\xi^i(z)\left(\sqrt{2}\cos\left(\sqrt{2}u^i\right)\right) + \ldots. \quad (37)$$

Substituting this into (27), with g replaced by $\bar{g}$, we get the action:

$$\frac{1}{\bar{g}^2}\frac{p^2}{p-1}\left(\frac{2\pi}{\sqrt{2}}\right)^{d-q-1}\int d^{q+1}z\left[-\frac{1}{2}\tilde{\psi}p^{-\frac{1}{2}\Box_z}\tilde{\psi} + \frac{1}{p+1}\tilde{\psi}^{p+1} - C\left\{\frac{1}{2}\tilde{\xi}^i p^{-\frac{1}{2}\Box_z}\tilde{\xi}^i - \frac{1}{2}\tilde{\psi}^{p-1}\tilde{\xi}^i\tilde{\xi}^i\right\} + O\left(\tilde{\xi}^3\right) + \ldots\right].$$

(38)

4.4 Open and closed scalar zeta strings.[11]

The exact tree-level Lagrangian for effective scalar field φ which describes open p-adic string tachyon is

$$L_p = \frac{1}{g^2}\frac{p^2}{p-1}\left[-\frac{1}{2}\varphi p^{-\frac{\Box}{2}}\varphi + \frac{1}{p+1}\varphi^{p+1}\right], \quad (39)$$

where p is any prime number, $\Box = -\partial_t^2 + \nabla^2$ is the D-dimensional d'Alembertian and we adopt metric with signature (– + … +).

Now we want to show a model which incorporates the p-adic string Lagrangians in a restricted adelic way. The eq. (39) take the form:

$$L = \sum_{n\geq 1} C_n L_n = \sum_{n\geq 1}\frac{n-1}{n^2}L_n = \frac{1}{g^2}\left[-\frac{1}{2}\phi\sum_{n\geq 1} n^{-\frac{\Box}{2}}\phi + \sum_{n\geq 1}\frac{1}{n+1}\phi^{n+1}\right]. \quad (39b)$$

Recall that the Riemann zeta function is defined as

$$\zeta(s) = \sum_{n\geq 1}\frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}, \quad s = \sigma + i\tau, \quad \sigma > 1. \quad (40)$$

Employing usual expansion for the logarithmic function and definition (40) we can rewrite (39b) in the form

$$L = -\frac{1}{g^2}\left[1/2\phi\zeta\,(\Box/2)\,\phi + \phi + \ln(1-\phi)\ \right], \quad (41)$$

where $|\phi| < 1$. $\zeta\,(\Box/2)$ acts as pseudo-differential operator in the following way:

$$\zeta\,(\Box/2)\,\phi(x) = \frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk, \quad -k^2 = k_0^2 - \vec{k}^2 > 2+\varepsilon, \quad (42)$$

where $\tilde{\phi}(k) = \int e^{(-ikx)}\phi(x)dx$ is the Fourier transform of $\phi(x)$.

Dynamics of this field ϕ is encoded in the (pseudo)differential form of the Riemann zeta function. When the d'Alembertian is an argument of the Riemann zeta function we shall call such string a *zeta string*. Consequently, the above ϕ is an open scalar zeta string. The equation of motion for the zeta string ϕ is

$$\zeta\,(\Box/2)\,\phi = \frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \frac{\phi}{1-\phi} \quad (43)$$

which has an evident solution $\phi = 0$.

For the case of time dependent spatially homogeneous solutions, we have the following equation of motion

$$\zeta\left(\frac{-\partial_t^2}{2}\right)\phi(t) = \frac{1}{(2\pi)}\int_{|k_0|>\sqrt{2}+\varepsilon} e^{-ik_0 t}\zeta\left(\frac{k_0^2}{2}\right)\tilde{\phi}(k_0)dk_0 = \frac{\phi(t)}{1-\phi(t)}. \quad (44)$$

Finally, with regard the open and closed scalar zeta strings, the equations of motion are

$$\zeta\,(\Box/2)\,\phi = \frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \sum_{n\geq 1}\theta^{\frac{n(n-1)}{2}}\phi^n\,, \qquad (45)$$

$$\zeta\,(\Box/4)\,\theta = \frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{4}\right)\tilde{\theta}(k)dk = \sum_{n\geq 1}\left[\theta^{n^2} + \frac{n(n-1)}{2(n+1)}\theta^{\frac{n(n-1)}{2}-1}\left(\phi^{n+1}-1\right)\right], \quad (46)$$

and one can easily see trivial solution $\phi = \theta = 0$.

Chapter 5

On some correlations obtained between some solutions in string theory, Riemann zeta function and Palumbo-Nardelli Model.

With regard the paper: "Brane Inflation, Solitons and Cosmological Solutions:I", that dealt various cosmological solutions for a D3/D7 system directly from M-theory with fluxes and M2-branes, and the paper: "General brane geometries from scalar potentials: gauged supergravities and accelerating universes", that dealt time-dependent configurations describing accelerating universes, we have obtained interesting connections between some equations concerning cosmological solutions, some equations concerning the Riemann zeta function and the relationship of Palumbo-Nardelli model.

5.1 Cosmological solutions from the D3/D7 system.[14]

The full action in M-theory will consist of three pieces: a bulk term, S_{bulk}, a quantum correction term, $S_{quantum}$, and a membrane source term, S_{M2}. The action is then given as the sum of these three pieces:

$$S = S_{bulk} + S_{quantum} + S_{M2}. \quad (1)$$

The individual pieces are:

$$S_{bulk} = \frac{1}{2\kappa^2}\int d^{11}x\sqrt{-g}\left[R - \frac{1}{48}G^2\right] - \frac{1}{12\kappa^2}\int C\wedge G\wedge G\,, \quad (2)$$

where we have defined G = dC, with C being the usual three form of M-theory, and $\kappa^2 \equiv 8\pi G_N^{(11)}$. This is the bosonic part of the classical eleven-dimensional supergravity action. The leading quantum correction to the action can be written as:

$$S_{quantum} = b_1 T_2 \int d^{11}x\sqrt{-g}\left[J_0 - \frac{1}{2}E_8\right] - T_2\int C\wedge X_8\,. \quad (3)$$

The coefficient T_2 is the membrane tension. For our case, $T_2 = \left(\frac{2\pi^2}{\kappa^2}\right)^{1/3}$, and b_1 is a constant number given explicitly as $b_1 = (2\pi)^{-4}3^{-2}2^{-13}$. The M2 brane action is given by:

$$S_{M2} = -\frac{T_2}{2}\int d^3\sigma\sqrt{-\gamma}\left[\gamma^{\mu\nu}\partial_\mu X^M \partial_\nu X^N g_{MN} - 1 + \frac{1}{3}\varepsilon^{\mu\nu\rho}\partial_\mu X^M\partial_\nu X^N\partial_\rho X^P C_{MNP}\right], \quad (4)$$

where X^M are the embedding coordinates of the membrane. The world-volume metric $\gamma_{\mu\nu}, \mu,\nu = 0,1,2$ is simply the pull-back of g_{MN}, the space-time metric. The motion of this M2 brane is obviously influenced by the background G-fluxes.

5.2 Classification and stability of cosmological solutions.[14]

The metric that we get in type IIB is of the following generic form:

$$ds^2 = \frac{f_1}{t^\alpha}\left(-dt^2 + dx_1^2 + dx_2^2\right) + \frac{f_2}{t^\beta}dx_3^2 + \frac{f_3}{t^\gamma}g_{mn}dy^m dy^n \quad (5)$$

where $f_i = f_i(y)$ are some functions of the fourfold coordinates and α,β and γ could be positive or negative number. For arbitrary $f_i(y)$ and arbitrary powers of t , the type IIB metric can in general come from an M-theory metric of the form

$$ds^2 = e^{2A}\eta_{\mu\nu}dx^\mu dx^\nu + e^{2B}g_{mn}dy^m dy^n + e^{2C}|dz|^2, \quad (6)$$

with three different warp factors A, B and C , given by:

$$A = \frac{1}{2}\log\frac{f_1 f_2^{\frac{1}{3}}}{t^{\alpha+\frac{\beta}{3}}} + \frac{1}{3}\log\frac{\tau_2}{|\tau|^2}, \quad B = \frac{1}{2}\log\frac{f_3 f_2^{\frac{1}{3}}}{t^{\gamma+\frac{\beta}{3}}} + \frac{1}{3}\log\frac{\tau_2}{|\tau|^2}, \quad C = -\frac{1}{3}\left[\log\frac{f_2}{t^\beta} + \log\frac{\tau_2^2}{|\tau|}\right]. \quad (7)$$

To see what the possible choices are for such a background, we need to find the difference B – C . This is given by:

$$B - C = \frac{1}{2}\log\frac{f_2 f_3}{t^{\gamma+\beta}} + \log\frac{\tau_2}{|\tau|}. \quad (8)$$

Since the space and time dependent parts of (8) can be isolated, (8) can only vanish if

$$f_2 = f_3^{-1}\cdot\frac{|\tau|}{\tau_2}, \qquad \gamma + \beta = 0, \quad (9)$$

with α and $f_1(y)$ remaining completely arbitrary.
We now study the following interesting case, where $\alpha = \beta = 2$, $\gamma = 0$ $f_1 = f_2$. The internal six manifold is time independent. This example would correspond to an exact de-Sitter background, and therefore this would be an accelerating universe with the three warp factors given by:

$$A = \frac{2}{3}\log\frac{f_1}{t^2}, \quad B = \frac{1}{2}\left[\log f_3 + \frac{1}{3}\log\frac{f_1}{t^2}\right], \quad C = -\frac{1}{3}\log\frac{f_1}{t^2}. \quad (10)$$

We see that the internal fourfold has time dependent warp factors although the type IIB six dimensional space is completely time independent. Such a background has the advantage that the four dimensional dynamics that would depend on the internal space will now become time independent.

This case, assumes that the time-dependence has a peculiar form, namely the 6D internal manifold of the IIB theory is assumed constant, and the non-compact directions correspond to a 4D de-Sitter space. Using (10), the corresponding 11D metric in the M-theory picture, can then, in principle, be inserted in the equations of motion that follow from (1). Hence, for the Palumbo-Nardelli model, we have the following connection:

$$-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]=$$
$$=\int_0^\infty\frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_V\left(\left|F_2\right|^2\right)\right]\Rightarrow$$
$$\Rightarrow\frac{1}{2\kappa^2}\int d^{11}x\sqrt{-g}\left[R-\frac{1}{48}G^2\right]-\frac{1}{12\kappa^2}\int C\wedge G\wedge C \quad (11),$$

where the third term is the bosonic part of the classical eleven-dimensional super-gravity action.

5.3 Solution applied to ten dimensional IIB supergravity (uplifted 10-dimensional solution).[14]

This solution can be oxidized on a three sphere S^3 to give a solution to ten dimensional IIB supergravity. This 10D theory contains a graviton, a scalar field, and the NSNS 3-form among other fields, and has a ten dimensional action given by

$$S_{10}=\int d^{10}x\sqrt{|g|}\left[\frac{1}{4}R-\frac{1}{2}(\partial\phi)^2-\frac{1}{12}e^{-2\phi}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right].\quad (12)$$

We have a ten dimensional configuration given by

$$ds_{10}^2=\left(\frac{2}{r}\right)^{3/4}\left[-h(r)dt^2+r^2dx_{0,5}^2+\frac{r^2}{h(r)}dr^2\right]+\left(\frac{r}{2}\right)^{5/4}\left[d\theta^2+d\psi^2+d\varphi^2+\left(d\psi+\cos\theta d\varphi-\frac{Q}{5r^5}dt\right)^2\right]$$

$$\phi=-\frac{5}{4}\log\frac{r}{2},$$

$$H_3=-\frac{Q}{r^6}dr\wedge dt\wedge\left(d\psi+\cos\theta d\varphi\right)-\frac{g}{\sqrt{2}}\sin\theta d\theta\wedge d\varphi\wedge d\psi.\quad (13)$$

This uplifted 10-dimensional solution describes NS-5 branes intersecting with fundamental strings in the time direction.
Now we make the manipulation of the angular variables of the three sphere simpler by introducing the following left-invariant 1-forms of SU(2):

$$\sigma_1=\cos\psi d\theta+\sin\psi\sin\theta d\varphi,\quad \sigma_2=\sin\psi d\theta-\cos\psi\sin\theta d\varphi,\quad \sigma_3=d\psi+\cos\theta d\varphi,\quad (14)$$

and

$$h_3=\sigma_3-\frac{Q}{5}\frac{1}{r^5}dt.\quad (15)$$

Next, we perform the following change of variables

$$\frac{r}{2}=\rho^{\frac{4}{5}},\quad t=\frac{5}{32}\tilde{t},\quad dx_4=\frac{1}{2\sqrt{2}}d\tilde{x}_4,\quad dx_5=\frac{1}{2}dZ,\quad g=\sqrt{2}\tilde{g},\quad Q=\sqrt{2}2^7\tilde{Q},\quad \sigma_i=\frac{1}{\tilde{g}}\tilde{\sigma}_i.\ (16)$$

It is straightforward to check that the 10-dimensional solution (13) becomes, after these changes

$$d\tilde{s}_{10}^2=\frac{1}{2}\rho^{-1}\left[d\tilde{s}_6^2\right]+\frac{\rho}{\tilde{g}^2}\left[\tilde{\sigma}_1^2+\tilde{\sigma}_2^2+\left(\tilde{\sigma}_3-\frac{\tilde{g}\tilde{Q}}{4\sqrt{2}}\frac{1}{\rho^4}d\tilde{t}\right)^2\right]+\rho dZ^2,$$

$$\phi=-\ln\rho,$$

$$H_3=-\frac{1}{\tilde{g}^2}\tilde{\sigma}_1\wedge\tilde{\sigma}_2\wedge\tilde{h}_3+\frac{\tilde{Q}}{\sqrt{2}\tilde{g}\rho^5}d\tilde{t}\wedge d\rho\wedge\tilde{h}_3,\quad (17)$$

where we define

$$d\tilde{s}_6^2=-\tilde{h}(\rho)d\tilde{t}^2+\frac{\rho^2}{\tilde{h}(\rho)}d\rho^2+\rho^2 d\tilde{x}_{0,4}^2\quad (18)$$

and, after re-scaling M,

$$\tilde{h}=-\frac{2\tilde{M}}{\rho^2}+\frac{\tilde{g}^2}{32}\rho^2+\frac{\tilde{Q}^2}{8}\frac{1}{\rho^6}.\quad (19)$$

We now transform the solution from the Einstein to the string frame. This leads to

$$d\tilde{s}_{10}^2=\frac{1}{2}\rho^{-2}\left[d\tilde{s}_6^2\right]+\frac{1}{\tilde{g}^2}\left[\tilde{\sigma}_1^2+\tilde{\sigma}_2^2+\left(\tilde{\sigma}_3-\frac{\tilde{g}\tilde{Q}}{4\sqrt{2}}\frac{1}{\rho^4}d\tilde{t}\right)^2\right]+dZ^2,$$

$$\bar{\phi}=-2\ln\rho,$$

$$\bar{H}_3=H_3.\qquad (20)$$

We have a solution to 10-dimensional IIB supergravity with a nontrivial NSNS field. If we perform an S-duality transformation to this solution we again obtain a solution to type-IIB theory but with a nontrivial RR 3-form, F_3. The S-duality transformation acts only on the metric and on the dilaton, leaving invariant the three form. In this way we are led to the following configuration, which is S-dual to the one derived above

$$d\tilde{s}_{10}^2=\frac{1}{2}\left[d\tilde{s}_6^2\right]+\frac{\rho^2}{\tilde{g}^2}\left[\tilde{\sigma}_1^2+\tilde{\sigma}_2^2+\left(\tilde{\sigma}_3-\frac{\tilde{g}\tilde{Q}}{4\sqrt{2}}\frac{1}{\rho^4}d\tilde{t}\right)^2\right]+\rho^2 dZ^2,$$

$$\bar{\phi}=2\ln\rho,$$

$$F_3 = H_3 . \qquad (21)$$

With regard the T-duality, in the string frame we have

$$d\bar{s}_{10}^2 = \frac{1}{2}\left[ds_6^2\right] + \frac{r^2}{g^2}\left[\sigma_1^2 + \sigma_2^2 + \left(\sigma_3 - \frac{gQ}{4\sqrt{2}}\frac{1}{r^4}dt\right)^2\right] + r^{-2}dZ^2 . \quad (22)$$

This gives a solution to IIA supergravity with excited RR 4-form, C_4. We proceed by performing a T-duality transformation, leading to a solution of IIB theory with nontrivial RR 3-form, C_3. The complete solution then becomes

$$d\bar{s}_{10}^2 = \frac{1}{2}\left[ds_6^2\right] + \frac{r^2}{g^2}\left[\sigma_1^2 + \sigma_2^2 + \left(\sigma_3 - \frac{gQ}{4\sqrt{2}}\frac{1}{r^4}dt\right)^2\right] + r^2 dZ^2 ,$$

$$\bar{\phi} = 2\ln r$$

$$C_3 = -\frac{1}{g^2}\sigma_1 \wedge \sigma_2 \wedge h_3 - \frac{Q}{\sqrt{2}g}\frac{1}{r^5}dt \wedge dr \wedge h_3 . \quad (23)$$

We are led in this way to precisely the same 10D solution as we found earlier [see formula (21)].
With regard the Palumbo-Nardelli model, we have the following connection with the eq. (12):

$$-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] =$$
$$= \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(\left|F_2\right|^2\right)\right] \to$$
$$\to \int d^{10}x\sqrt{|g|}\left[\frac{1}{4}R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{12}e^{-2\phi}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right] . \quad (24)$$

5.4 Connections with some equations concerning the Riemann zeta function.[14]

We have obtained interesting connections between some cosmological solutions of a D3/D7 system, some solutions concerning ten dimensional IIB supergravity and some equations concerning the Riemann zeta function, specifying the Goldston-Montgomery theorem.
In the chapter “Goldbach’s numbers in short intervals” of Languasco’s paper “The Goldbach’s conjecture”, is described the Goldston-Montgomery theorem.

THEOREM 1

Assume the Riemann hypothesis. We have the following implications: (1) If $0 < B_1 \le B_2 \le 1$ and $F(X,T) \approx \frac{1}{2\pi}T\log T$ uniformly for $\frac{X^{B_1}}{\log^3 X} \le T \le X^{B_2}\log^3 X$, then

$$\int_1^X (\psi(1+\delta)x) - \psi(x) - \delta(x)^2 \, dx \approx \frac{1}{2}\delta X^2 \log\frac{1}{\delta}, \quad (25)$$

uniformly for $\frac{1}{X^{B_2}} \leq \delta \leq \frac{1}{X^{B_1}}$.

(2) If $1 < A_1 \leq A_2 < \infty$ and $\int_1^X (\psi((1+\delta)x) - \psi(x) - \delta x)^2 \, dx \approx \frac{1}{2}\delta X^2 \log\frac{1}{\delta}$ uniformly for $\frac{1}{X^{1/A_1}\log^3 X} \leq T \leq \frac{1}{X^{1/A_2}}\log^3 X$, then $F(X,T) \approx \frac{1}{2\pi} T \log T$ uniformly for

$$T^{A_1} \leq X \leq T^{A_2}.$$

Now, for show this theorem, we must to obtain some preliminary results .

Preliminaries Lemma. (Goldston-Montgomery)

Lemma 1.

We have $f(y) \geq 0 \quad \forall y \in R$ and let $I(Y) = \int_{-\infty}^{+\infty} e^{-2|y|} f(Y+y)\,dy = 1 + \varepsilon(Y)$. If R(y) is a Riemann-integrable function, we have:

$$\int_a^b R(y) f(Y+y)\,dy = \left(\int_a^b R(y)\,dy\right)(1 + \varepsilon'(y)).$$

Furthermore, fixed R, $|\varepsilon'(Y)|$ is little if $|\varepsilon(y)|$ is uniformly small for $Y + a - 1 \leq y \leq Y + b + 1$.

Lemma 2.

Let $f(t) \geq 0$ a continuous function defined on $[0,+\infty)$ such that $f(t) << \log^2(t+2)$.
If

$$J(T) = \int_0^T f(t)\,dt = (1 + \varepsilon(T)) T \log T,$$

then

$$\int_0^\infty \left(\frac{\sin ku}{u}\right)^2 f(u)\,du = \left(\frac{\pi}{2} + \varepsilon'(k)\right) k \log\frac{1}{k},$$

with $|\varepsilon'(k)|$ small for $k \to 0^+$ if $|\varepsilon(T)|$ is uniformly small for

$$\frac{1}{k \log^2 k} \leq T \leq \frac{1}{k}\log^2 k.$$

Lemma 3.

Let $f(t) \geq 0$ a continuous function defined on $[0,+\infty)$ such that $f(t) << \log^2(t+2)$. If

$$I(k) = \int_0^{\infty} \left(\frac{\sin ku}{u}\right)^2 f(u)du = \left(\frac{\pi}{2} + \varepsilon'(k)\right) k \log\frac{1}{k}, \quad (26)$$ then

$$J(T) = \int_0^T f(t)dt = (1+\varepsilon')T\log T \quad , \quad (27)$$

with $|\varepsilon'|$ small if $|\varepsilon(k)| \leq \varepsilon$ uniformly for $\frac{1}{T\log T} \leq k \leq \frac{1}{T}\log^2 T$.

Lemma 4.

Let $F(X,T) := \sum_{0<\gamma,\gamma'<T} \frac{4X^{i}(\gamma-\gamma')}{4+(\gamma-\gamma')^2}$. Then (i) $F(X,T) \geq 0$; (ii) $F(X,T) = F(1/X,T)$; (iii) If

The Riemann hypothesis is preserved, then we have

$$F(X,T) = T\left(\frac{1}{X^2}\log^2 T + \log X\right)\left(\frac{1}{2\pi} + O\left(\sqrt{\frac{\log\log T}{\log T}}\right)\right)$$

uniformly for $1 \leq X \leq T$.

Lemma 5.

Let $\delta \in (0,1]$ and $a(s) = \frac{(1+\delta)^s - 1}{s}$. If $c(\gamma) \leq 1 \quad \forall y$ we have that

$$\int_{-\infty}^{+\infty} |a(it)|^2 \left|\sum_{\gamma} \frac{c(\gamma)}{1+(t-\gamma)^2}\right|^2 dt = \int_{-\infty}^{+\infty} \left|\sum_{|\gamma|\leq Z} a(1/2+i\gamma)\frac{c(\gamma)}{1+(t-\gamma)^2}\right|^2 dt + O\left(\delta^2 \log^3 \frac{2}{\delta}\right) + O\left(\frac{1}{Z}\log^3 Z\right)$$

for $Z > \frac{1}{\delta}$.

For to show the Theorem 1, there are two parts. We go to prove (1).
We define

$$J(X,T) = 4\int_0^T \left|\sum_{\gamma} \frac{X^{i\gamma}}{1+(t-\gamma)^2}\right|^2 dt.$$

Montgomery has proved that $J(X,T) = 2\pi F(X,T) + O(\log^3 T)$ and thence the hypothesis $F(X,T) \approx \frac{1}{2\pi}T\log T$ is equal to $J(X,T) = (1+o(1))T\log T$. Putting $k = \frac{1}{2}\log(1+\delta)$, we have

$$|a(it)|^2 = 4\left(\frac{\sin kt}{t}\right)^2.$$

For the Lemma 2, we obtain that

$$\int_0^{\infty} |a(it)|^2 \left|\sum_{\gamma} \frac{X^{i\gamma}}{1+(t-\gamma)^2}\right|^2 dt = \left(\frac{\pi}{2}+o(1)\right)k\log\frac{1}{k} = \left(\frac{\pi}{4}+o(1)\right)\delta\log\frac{1}{\delta}$$

for

$$\frac{1}{\delta\log^2\frac{1}{\delta}} \le T \le \frac{3}{\delta}\log^2\frac{1}{\delta}.$$

For the Lemma 5 and the parity of the integrand, we have that

$$\int_{-\infty}^{+\infty} \left|\sum_{|\gamma|\le Z} a(\rho)\frac{X^{i\gamma}}{1+(t-\gamma)^2}\right|^2 dt = \left(\frac{\pi}{2}+o(1)\right)\delta\log\frac{1}{\delta} \qquad \text{(a)}$$

if $Z \ge \frac{1}{\delta}\log^3\frac{1}{\delta}$.

From the $S(t) = \sum_{|\gamma|\le Z} a(\rho)\frac{X^{i\gamma}}{1+(t-\gamma)^2}$ we note that the Fourier's transformed verify that

$$\hat{S}(u) = \pi\sum_{|\gamma|\le Z} a(\rho)X^{i\gamma}e(-\gamma u)e^{-2\pi|u|}.$$

From the Plancherel identity, we have that

$$\int_{-\infty}^{+\infty} \left|\sum_{|\gamma|\le Z} a(\rho)X^{i\gamma}e(-\gamma u)\right|^2 e^{-4\pi|u|}du = \left(\frac{2}{\pi}+o(1)\right)\delta\log\frac{1}{\delta}.$$

For the substitution $Y = \log X$, $-2\pi u = y$ we obtain

$$\int_{-\infty}^{+\infty} \left|\sum_{|\gamma|\le Z} a(\rho)e^{i\gamma(Y+y)}\right|^2 e^{-2|y|}dy = (1+o(1))\delta\log\frac{1}{\delta}. \qquad \text{(b)}$$

Using the Lemma 1 with $R(y) = e^{2y}$ if $0 \le y \le \log 2$ and $R(y) = 0$ otherwise, and putting $x = e^{Y+y}$ we have that

$$\int_X^{2X} \left|\sum_{|\gamma|\le Z} a(\rho)x^{\rho}\right|^2 dx = \left(\frac{3}{2}+o(1)\right)\delta X^2\log\frac{1}{\delta}.$$

Substituting X with $X2^{-j}$, summarizing on j, $1 \le j \le K$, and using the explicit formula for $\psi(x)$ with $Z = X\log^3 X$ we obtain

$$\int_{X2^{-K}}^{X} (\psi((1+\delta)x)-\psi(x)-\delta x)^2\,dx = \frac{1}{2}\left(1-2^{-2K}+o(1)\right)\delta X^2 \log\frac{1}{\delta}.$$

Furthermore, we put $K = [\log\log X]$ and we utilize, for the interval $1 \le x \le X2^{-K}$, the estimate of Lemma 4 (placing $X2^{-K}$ for X). Thus, we obtain (1).
Now, we prove (2).
We fix an real number X_1. Making an integration for parts between X_1 and $X_2 = X_1 \log^{2/3} X_1$ we obtain, remembering that for hypothesis we have

$$\int_{1}^{X} (\psi((1+\delta)x)-\psi(x)-\delta x)^2\,dx \approx \frac{1}{2}\delta X^2 \log\frac{1}{\delta},$$

that $$\int_{X_1}^{X_2} (\psi((1+\delta)x)-\psi(x)-\delta x)^2 x^{-4}\,dx = \left(\frac{1}{2}+o(1)\right)\delta X_1^{-2} \log\frac{1}{\delta}. \quad \text{(c)}$$

Utilizing the estimate, valid under the Riemann hypothesis

$$\int_{1}^{X} (\psi((1+\delta)x)-\psi(x)-\delta x)^2\,dx << \delta X^2 \log^2\frac{2}{\delta},$$

we obtain analogously as before that

$$\int_{X_2}^{\infty} (\psi((1+\delta)x)-\psi(x)-\delta x)^2 x^{-4}\,dx << \delta X_2^{-2} \log^2\frac{1}{\delta} = o\left(\delta X_1^{-2}\log\frac{1}{\delta}\right). \quad \text{(d)}$$

Now, summarizing (c) and (d) and multiplying the sum for X_1^2 we obtain

$$\int_{1}^{\infty} \min\left(\frac{x^2}{X_1^2},\frac{X_1^2}{x^2}\right)(\psi((1+\delta)x)-\psi(x)-\delta x)^2 x^{-2}\,dx = (1+o(1))\delta\log\frac{1}{\delta}.$$

Putting $X_1 = X$, $Y = \log X$, $x = e^{Y+y}$ and using the explicit formula for $\psi(x)$ with $Z = X\log^3 X$, we obtain the equation (b).
Now, we take the equation (10) and precisely $A = \frac{2}{3}\log\frac{f_1}{t^2}$. We note that from the equation (27) for $\varepsilon' = -\frac{2}{3}$ and T = 2, we have $J(T) = \int_0^T f(t)dt = (1+\varepsilon')T\log T = \frac{2}{3}\log 2$. This result is related to $A = \frac{2}{3}\log\frac{f_1}{t^2}$ putting $\frac{f_1}{t^2} = 2$, hence with the Lemma 3 of Goldston-Montgomery theorem. Then, we have the following interesting relation

$$A = \frac{2}{3}\log\frac{f_1}{t^2} \Rightarrow \int_0^T f(t)dt = (1+\varepsilon')T\log T, \quad (28)$$

hence the connection between the cosmological solution and the equation related to the Riemann zeta function.

Now, we take the equations (13) e (21) and precisely $\phi = -\frac{5}{4}\log\frac{r}{2}$ and $\bar{\phi} = 2\ln\rho$. We note that from the equation (27) for $\varepsilon' = \frac{3}{2}$ and T = 1/2 , we have $J(T) = \int_0^T f(t)dt = (1+\varepsilon')T\log T = \frac{5}{4}\log\frac{1}{2}$.

Furthermore, for $\varepsilon' = 3$ and T = 1/2 , we have $J(T) = \int_0^T f(t)dt = (1+\varepsilon')T\log T = 2\log\frac{1}{2}$.

These results are related to $\phi = -\frac{5}{4}\log\frac{r}{2}$ putting r = 1 and to $\bar{\phi} = 2\ln\rho$ putting $\rho = 1/2$, hence with the Lemma 3 of Goldston-Montgomery theorem. Then, we have the following interesting relations:

$$\phi = -\frac{5}{4}\log\frac{r}{2} \Rightarrow -\int_0^T f(t)dt = -[(1+\varepsilon')T\log T] \text{, (29a)} \qquad \bar{\phi} = 2\ln\rho \Rightarrow \int_0^T f(t)dt = (1+\varepsilon')T\log T, \Rightarrow$$

$$\Rightarrow \int d^{10}x\sqrt{|g|}\left[\frac{1}{4}R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{12}e^{-2\phi}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right] \quad (29b)$$

hence the connection between the 10-dimensional solutions (12) and some equations related to the Riemann zeta function.

From this the possible connection between cosmological solutions concerning string theory and some mathematical sectors concerning the zeta function, whose the Goldston-Montgomery Theorem and the related Goldbach's Conjecture.

5.5 The P-N Model (Palumbo-Nardelli model) and the Ramanujan identities.[15]

Palumbo (2001) ha proposed a simple model of the birth and of the evolution of the Universe. Palumbo and Nardelli (2005) have compared this model with the theory of the strings, and translated it in terms of the latter obtaining:

$$-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] =$$
$$= \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(\left|F_2\right|^2\right)\right], \qquad (30)$$

A general relationship that links bosonic and fermionic strings acting in all natural systems.

It is well-known that the series of Fibonacci's numbers exhibits a fractal character, where the forms repeat their similarity starting from the reduction factor $1/\phi$ = 0,618033 = $\frac{\sqrt{5}-1}{2}$ (Peitgen et al. 1986). Such a factor appears also in the famous fractal Ramanujan identity (Hardy 1927):

$$0,618033 = 1/\phi = \frac{\sqrt{5}-1}{2} = R(q) + \frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}, \qquad (31)$$

and

$$\pi = 2\Phi - \frac{3}{20}\left[R(q) + \frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right], \qquad (32)$$

where

$$\Phi = \frac{\sqrt{5}+1}{2}.$$

Furthermore, we remember that π arises also from the following identity:

$$\pi = \frac{12}{\sqrt{130}}\log\left[\frac{(2+\sqrt{5})(3+\sqrt{13})}{\sqrt{2}}\right], \ (32a) \ \text{and} \ \pi = \frac{24}{\sqrt{142}}\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]. \ (32b)$$

The introduction of (31) and (32) in (30) provides:

$$-\int d^{26}x\sqrt{g}\left[-\frac{R}{16G}\cdot\frac{1}{2\Phi - \frac{3}{20}\left(R(q) + \frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f\left(-t^{1/5}\right)}\frac{dt}{t^{4/5}}\right)}\right)} - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi) + \right.$$

$$\left. -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\ \right] = \int_0^\infty \frac{R}{\kappa_{11}^2}\cdot 2\Phi - \frac{3}{20}\left[R(q) + \frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right].$$

$$\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{11}^2}{2\Phi - \frac{3}{20}\left(R(q) + \frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right)2Rg_{10}^2} Tr_\nu \right.$$

$$\left. \left(\left|F_2\right|^2\right)\ \right], \qquad (33)$$

which is the translation of (30) in the terms of the Theory of the Numbers, specifically the possible connection between the Ramanujan identity and the relationship concerning the Palumbo-Nardelli model.

5.6 Interactions between intersecting D-branes.[12]

Let us consider two Dp-branes in type II string theory, intersecting at n angles inside the ten-dimensional space.
The interaction between the branes can be computed from the exchange of massless closed string modes. This can be computed from the one-loop vacuum amplitude for the open strings stretched between the two Dp-branes, that is given by

$$\mathrm{A} = 2\int \frac{dt}{2t} Tre^{-tH}\ , \quad (34)$$

where H is the open string Hamiltonian. For two Dp-branes making n angles in ten dimensions this amplitudes can be computed to give

$$\mathrm{A} = V_p \int_0^\infty \frac{dt}{t} \exp^{-\frac{tY^2}{2\pi^2\alpha'}} \left(8\pi^2\alpha' t\right)^{-\frac{p-3}{2}} \left(-iL\eta(it)^{-3}(8\pi\alpha' t)^{-1/2}\right)^{4-n} \left(Z_{NS} - Z_R\right), \quad (35)$$

with

$$Z_{NS} = \left(\Theta_3(0|it)\right)^{4-n} \prod_{j=1}^{n} \frac{\Theta_3\left(i\Delta\theta_j t|it\right)}{\Theta_1\left(i\Delta\theta_j t|it\right)} - \left(\Theta_4(0|it)\right)^{4-n} \prod_{j=1}^{n} \frac{\Theta_4\left(i\Delta\theta_j t|it\right)}{\Theta_1\left(i\Delta\theta_j t|it\right)},$$

$$Z_R = \left(\Theta_2(0|it)\right)^{4-n} \prod_{j=1}^{n} \frac{\Theta_2\left(i\Delta\theta_j t|it\right)}{\Theta_1\left(i\Delta\theta_j t|it\right)}, \quad (36)$$

being the contributions coming from the NS and R sectors. Thence, the eq. (35) can be rewritten also

$$\mathrm{A} = V_p \int_0^\infty \frac{dt}{t} \exp^{-\frac{tY^2}{2\pi^2\alpha'}} \left(8\pi^2\alpha' t\right)^{-\frac{p-3}{2}} \left(-iL\eta(it)^{-3}(8\pi\alpha' t)^{-1/2}\right)^{4-n}$$

$$\left[\ \left(\Theta_3(0|it)\right)^{4-n} \prod_{j=1}^{n} \frac{\Theta_3\left(i\Delta\theta_j t|it\right)}{\Theta_1\left(i\Delta\theta_j t|it\right)} - \left(\Theta_4(0|it)\right)^{4-n} \prod_{j=1}^{n} \frac{\Theta_4\left(i\Delta\theta_j t|it\right)}{\Theta_1\left(i\Delta\theta_j t|it\right)}\ \right] - \left[\ \left(\Theta_2(0|it)\right)^{4-n} \prod_{j=1}^{n} \frac{\Theta_2\left(i\Delta\theta_j t|it\right)}{\Theta_1\left(i\Delta\theta_j t|it\right)}\ \right]. \quad (36b)$$

Also in (36) Θ_i are the usual Jacobi functions and η is the Dedekin function. Furthermore, in (35) by Y we mean the distance between both branes, $Y = \sqrt{\Sigma_k Y_k^2}$ where k labels the dimensions in which the branes are separated and Y_k the distance between both branes along the k direction.
Now we take the small t limit of (35), that is, the large distance limit $(Y >> l_s)$. This is the right limit that takes into account the contributions coming from the massless closed strings exchanged between the branes.
Using the well known modular properties of the Θ and η functions we obtain, in the $t \to 0$ limit, that the amplitude is just given by

$$\mathrm{A}\left(Y, \Delta\theta_j\right) = \frac{V_p L^{4-n} F\left(\Delta\theta_j\right)}{2^{p-2}\left(2\pi^2\alpha'\right)^{(p+1-n)/2}} \int t^{-\frac{p+n-5}{2}} \exp^{-\frac{tY^2}{2\pi^2\alpha'}} dt\ , \quad (37)$$

where the function F contains the dependence on the relative angles between the branes, and is extracted from the small t limit of (36). The exact form of this function is given by

$$F(\Delta\theta_j) = \frac{(4-n) + \Sigma_{j+1}^{n} \cos 2\Delta\theta_j - 4\Pi_{j=1}^{n} \cos\Delta\theta_j}{2\Pi_{j=1}^{n} \sin\Delta\theta_j}. \quad (38)$$

Hence, the eq. (37) can be rewritten also

$$\mathrm{A}(Y,\Delta\theta_j) = \frac{V_p L^{4-n}}{2^{p-2}(2\pi^2\alpha')^{(p+1-n)/2}} \frac{(4-n) + \Sigma_{j+1}^{n} \cos 2\Delta\theta_j - 4\Pi_{j=1}^{n} \cos\Delta\theta_j}{2\Pi_{j=1}^{n} \sin\Delta\theta_j} \int t^{-\frac{p+n-5}{2}} \exp^{-\frac{tY^2}{2\pi^2\alpha'}} dt. \quad (38b)$$

The interaction potential between the branes can then be calculated by performing the integral (37). This integral is just given in terms of the Euler Γ-function, so the potential has the following form

$$V(Y,\Delta\theta_j) = -\frac{V_p L^{4-n} F(\Delta\theta_j)}{2^{p-2}(2\pi^2\alpha')^{p-3}} \Gamma\left(\frac{7-p-n}{2}\right) Y^{(p+n-7)}. \quad (39)$$

Note that for $p+n=7$ this expression is not valid as $\Gamma(0)$ is not a well defined function. In fact in that case the integral (37) is divergent, so we need to introduce a lower cutoff to perform it. If we denote by Λ_c the cutoff, the integral becomes

$$V(Y,\Delta\theta_j) = \frac{V_p L^{p-3} F(\Delta\theta_j)}{(4\pi^2\alpha')^{p-3}} \ln\frac{Y}{\Lambda_c}. \quad (40)$$

When dealing with compact spaces the expression (37) is modified in the following way

$$\mathrm{A}(Y,\Delta\theta_j) = \frac{V_p L^{4-n} F(\Delta\theta_j)}{2^{p-2}(2\pi^2\alpha')^{(p+1-n)/2}} \sum_{\omega_k \in Z} \int_0^\infty t^{-\frac{p+n-5}{2}} \exp^{-\frac{t\Sigma_k (Y_k + 2\pi\omega_k R)^2}{2\pi^2\alpha'}} dt, \quad (41)$$

where ω_k represents the winding modes of the strings on the directions transverse to the branes. That means that the summation over k in (41) has only one term in the D6-brane case and it will be $Y_9 = |x_9^{(1)} - x_9^{(2)}|$. In the D5-brane case we will have two terms: $Y_8 = |x_8^{(1)} - x_8^{(2)}|$ and $Y_9 = |x_9^{(1)} - x_9^{(2)}|$. Also in both cases we will denote $Y = \sqrt{\Sigma_k Y_k^2}$. Nevertheless, if the distance between the branes is small compared with the compactification radii $(Y << (2\pi R))$, the winding modes would be too massive and then will not contribute to the low energy regime. That is, it will cost a lot of energy to the strings to wind around the compact space. If we translate this assumption to (41), the dominant mode will be the zero mode, and the potential can be written as in (39), (40), taking into account that we focus on the case where the number of angles is $n=2$. In this case the potential, when normalised over the non-compact directions, for branes of different dimensions is just given by

$$V_{Dp}(Y,\Delta\theta_j) = -\frac{(2\pi R)^{(p-5)} F(\Delta\theta_j)}{2^{p-2}(2\pi^2\alpha')^{p-3}} \Gamma\left(\frac{5-p}{2}\right) Y^{(p-5)}, \quad (42) \qquad V_{D5}(Y,\Delta\theta_j) = \frac{F(\Delta\theta_j)}{(4\pi^2\alpha')^2} \ln\frac{Y}{\Lambda_c}, \quad (43)$$

where the $(2\pi R)^{p-5}$ factor arises from the dimensions in which the branes become parallel on the compact dimensions. Furthermore, remember that R denotes the radius of the torus.
Now we note that the eq. (37) can be rewritten substituting to π the corresponding Ramanujan's identity (32). Hence, we obtain

$$A(Y,\Delta\theta_j)=\frac{V_p L^{4-n} F(\Delta\theta_j)}{2^{p-2}(2\alpha')^{(p+1-n)/2}\left\{2\Phi-\frac{3}{20}\left[R(q)+\frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q\frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right]\right\}}\int t^{-\frac{p+n-5}{2}}\exp^{-\frac{tY^2}{2\pi^2\alpha'}}dt$$

(43a)

With regard the eq. (40), we note that can be related with the expression (29b) concerning the lemma 3 of Goldston-Montgomery Theorem and with the Palumbo-Nardelli Model. Hence, we can write the following interesting connections:

$$\frac{V_p L^{p-3} F(\Delta\theta_j)}{(4\pi^2\alpha')^{p-3}}\ln\frac{Y}{\Lambda_c}=2\ln\rho\Rightarrow\int_0^T f(t)dt=(1+\varepsilon')T\log T\,,\Rightarrow$$

$$\Rightarrow\int d^{10}x\sqrt{|g|}\left[\frac{1}{4}R-\frac{1}{2}(\partial\phi)^2-\frac{1}{12}e^{-2\phi}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right]\Rightarrow$$

$$\Rightarrow\int_0^\infty\frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(\left|F_2\right|^2\right)\right]=$$

$$-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr(G_{\mu\nu}G_{\rho\sigma})f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]. \quad (43b)$$

5.7 General action and equations of motion for a probe D3-brane moving through a type IIB supergravity background.[44]

Now we will show the general action and equations of motion for a probe D3-brane moving through a type IIB supergravity background describing a configuration of branes and fluxes.
We start by specifying the ansatz for the background fields that we consider, and the form of the brane action. We are interested in compactifications of type IIB theory, in which the metric takes the following general form (in the Einstein frame)

$$ds^2=h^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu+h^{1/2}g_{mn}dy^m dy^n\,. \quad (44)$$

We now embed a probe D3-brane in this background, with its four infinite dimensions parallel to the four large dimensions of the background solution. The motion of such a brane is described by the sum of the Dirac-Born-Infeld (DBI) action and the Wess-Zumino (WZ) action. The DBI action is given, in the string frame, by

$$S_{DBI} = -T_3 g_s^{-1} \int d^4\xi e^{-\phi} \sqrt{-\det(\gamma_{ab} + F_{ab})}, \quad (45)$$

where $F_{ab} = B_{ab} + 2\pi\alpha' f_{ab}$, with B_2 the pullback of the 2-form field to the brane and f_2 the world-volume gauge field. $\gamma_{ab} = g_{MN}\partial_a x^M \partial_b x^N$, is the pullback of the ten-dimensional metric g_{MN} in the string frame. Finally $\alpha' = \ell_s^2$ is the string scale and ξ^a are the brane world-volume coordinates.
The WZ part is given by

$$S_{WZ} = qT_3 \int_W C_4, \quad (46)$$

where W is the world-volume of the brane and $q = 1$ for a probe D3-brane and $q = -1$ for a probe anti-brane. We are interested in exploring the effect of angular momentum on the motion of the brane, and therefore assume that there are no gauge fields living in the world-volume of the probe brane, $f_{ab} = 0$. For convenience we take the static gauge, that is, we use the non-compact coordinates as our brane coordinates: $\xi^a = x^{\mu=a}$. Since, in addition, we are interested in cosmological solutions for branes, we consider the case where the perpendicular positions of the brane, y^m, depend only on time. Thus

$$\gamma_{00} = g_{00} + g_{mn}\dot{y}^m \dot{y}^n h^{1/2} = -h^{-1/2}(1 - hv^2) \quad (47)$$

and $B_{ab} = 0$. Hence

$$S_{DBI} = -T_3 g_s^{-1} \int d^4x e^{-3\phi} \sqrt{1 - hv^2}, \quad (48)$$

in the Einstein frame. Thence, summing the DBI and WZ actions, we have the total action for the probe brane

$$S = -T_3 g_s^{-1} \int d^4x h^{-1} \left[e^{-3\phi}\sqrt{1 - hv^2} - q \right]. \quad (49)$$

This action is valid for arbitrarily high velocities. Furthermore, this equation correspond to the Born-Infeld action for the D-brane embedded in the 10-dimensional space of type IIB theory.
The functions appearing in the following equations

$$h(\eta) = \frac{27\pi\alpha'^2}{4\eta^4}\left[g_s N + \frac{3(g_s M)^2}{2\pi}\left(\ln\frac{\eta}{\tilde{\eta}} + \frac{1}{4} \right) \right] = \frac{c}{\eta^4}(1 + b\ln\eta), \quad (50)$$

$$N_{eff} = N + \frac{3(g_s M)^2}{2\pi}\ln\frac{\eta}{\eta_0}, \quad (51)$$

are the solutions of the equations of motion for the IIB theory in 10-dimensions, defining the background. Thence, putting eqs. (50) and (51) in (49), we can determine the trajectory of the brane in ten dimensions.
Here, $\eta = \tilde{\eta}$ determines the UV scale at which the KT throat joins to the Calabi-Yau space. This solution has a naked singularity at the point where $h(\eta_0) = 0$, located at $\eta_0 = \tilde{\eta} e^{-1/b}$. In this configuration, the supergravity approximation is valid when $g_s M, g_s N >> 1$: in this limit the curvatures are small, and we keep $g_s < 1$.
We note that also the eqs. (50) and (51), can be related with the expression (29b) and with the relationship concerning the Palumbo-Nardelli Model. Hence, we obtain the following connections:

$$h(\eta)-\frac{c}{\eta^4}=\frac{c}{\eta^4}b\ln\eta \Rightarrow 2\ln\rho \Rightarrow \int_0^T f(t)dt=(1+\varepsilon')T\log T ,\Rightarrow$$

$$\Rightarrow \int d^{10}x\sqrt{|g|}\left[\frac{1}{4}R-\frac{1}{2}(\partial\phi)^2-\frac{1}{12}e^{-2\phi}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right]\Rightarrow$$

$$\Rightarrow \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(\left|F_2\right|^2\right)\right]=$$
$$-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right], \quad (51a)$$

$$\frac{2\pi N+3(g_s M)^2}{2\pi}\ln\frac{\eta}{\eta_0}\Rightarrow 2\ln\rho \Rightarrow \int_0^T f(t)dt=(1+\varepsilon')T\log T ,\Rightarrow$$

$$\Rightarrow \int d^{10}x\sqrt{|g|}\left[\frac{1}{4}R-\frac{1}{2}(\partial\phi)^2-\frac{1}{12}e^{-2\phi}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right]\Rightarrow$$

$$\Rightarrow \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(\left|F_2\right|^2\right)\right]=$$
$$-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]. \quad (51b)$$

Furthermore, the eq. (49) is also related with the relationship concerning the Palumbo-Nardelli Model applied to the D-branes. Hence, we have:

$$-\mu_{25}\int d^{26}\xi Tr\left\{e^{-\Phi}\left[-\det\left(G_{ab}+B_{ab}+2\pi\alpha' F_{ab}\right)\right]^{1/2}\right\}=\int_0^\infty -\frac{1}{(2\pi\alpha')^2 g_{YM}^2}\int d^{10}xTr\left\{\left[-\det\left(\eta_{\mu\nu}+2\pi\alpha' F_{\mu\nu}\right)\right]^{1/2}\right\}$$
$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]=\int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}$$
$$\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{k_{10}^2}{g_{10}^2}Tr_\nu\left(\left|F_2\right|^2\right)\right]\Rightarrow -T_3 g_s^{-1}\int d^4x h^{-1}\left[e^{-3\phi}\sqrt{1-hv^2}-q\right]. \quad (52)$$

Chapter 6

Connections.

Now we take the eq. (20) of **Chapter 1**. We note that can be related with the Godston-Montgomery equation, the ten dimensional action (12) and the relationship of Palumbo-Nardelli model (30) of **Chapter 5**, hence we have the following connection:

$$\delta_k(u):=\delta_{k,\mathrm{B}}(u)=\left(\frac{1}{\lambda'_{\hat{E}_\mathrm{B}}(\mathrm{T})}\frac{d}{d\mathrm{T}}\right)^k \log f_{u,\mathrm{B}}(\mathrm{T})\Big|_{\mathrm{T}=0} \Rightarrow 2\ln\rho \Rightarrow \int_0^T f(t)dt=(1+\varepsilon')T\log T \;,\Rightarrow$$

$$\Rightarrow \int d^{10}x\sqrt{|g|}\left[\frac{1}{4}R-\frac{1}{2}(\partial\phi)^2-\frac{1}{12}e^{-2\phi}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right]\Rightarrow$$

$$\Rightarrow \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(\left|F_2\right|^2\right)\right]=$$

$$-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]. \quad (1)$$

Now we take the eq. (29) of **Chapter 1**. We note that can be related with the equation regarding the Palumbo-Nardelli model and with the Ramanujan's identity concerning π. Hence, we have the following connections:

$$\langle f_\varphi,f_\varphi\rangle=\frac{1}{16\pi^3}N^2\left\{\prod_{\substack{q|N\\ q\notin S_\varphi}}\left(1-\frac{1}{q}\right)\right\}L_N\left(2,\varphi^2\bar{\chi}\right)L_N(1,\psi)\Rightarrow$$

$$\Rightarrow\int_0^\infty \pi^2\langle f_\varphi,f_\varphi\rangle\cdot\frac{1}{N^2\left\{\prod_{q|N}\left(1-\frac{1}{q}\right)\right\}L_N\left(2,\varphi^2\bar{\chi}\right)L_N(1,\psi)G_N}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\cdot$$

$$\cdot\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(\left|F_2\right|^2\right)\right]=$$

$$=-\int d^{26}x\sqrt{g}\left[\left(-R\cdot\pi^2\langle f_\varphi,f_\varphi\rangle\cdot\frac{1}{N^2\left\{\prod_{q|N}\left(1-\frac{1}{q}\right)\right\}L_N\left(2,\varphi^2\bar{\chi}\right)L_N(1,\psi)G}\right)-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)+\right.$$

$$\left.-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right], \quad (2)$$

$$\left\langle f_\varphi, f_\varphi \right\rangle = \frac{1}{16\pi^3} N^2 \left\{ \prod_{\substack{q|N \\ q \notin S_\varphi}} \left(1 - \frac{1}{q}\right) \right\} L_N\left(2, \varphi^2 \bar{\hat{\chi}}\right) L_N(1, \psi) \Rightarrow$$

$$\Rightarrow \frac{1}{16\left\{2\Phi - \frac{3}{20}\left[R(q) + \frac{\sqrt{5}}{1 + \frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right]\right\}^3} N^2 \left\{ \prod_{\substack{q|N \\ q \notin S_\varphi}} \left(1 - \frac{1}{q}\right) \right\} L_N\left(2, \varphi^2 \bar{\hat{\chi}}\right) L_N(1, \psi).$$

(3)

Now we take the eqs. (8) and (9) and (11) of the **Chapter 2**. We note that can be related with the Ramanujan's modular equation (32b) and the Ramanujan's identity concerning π (32). Thence, we have the following connection:

$$\Delta = \sum_{n \geq 1} \tau(n) q^n = q \prod_{n \geq 1} \left(1 - q^n\right)^{24} \Rightarrow \frac{\pi\sqrt{142}}{\ln\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} \Rightarrow$$

$$\Rightarrow 2\Phi - \frac{3}{20}\left[R(q) + \frac{\sqrt{5}}{1 + \frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right] \cdot \frac{\sqrt{142}}{\ln\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (4)$$

Also for the eqs. (11) and (37), we obtain of the similar connections:

$$L_\Delta(s) := \sum_{n \geq 1} \tau(n) n^{-s} = \prod_p \left(1 - \tau(p) p^{-s} + p^{11} p^{-2s}\right)^{-1} \Rightarrow$$

$$\Rightarrow 2\Phi - \frac{3}{20}\left[R(q) + \frac{\sqrt{5}}{1 + \frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right] \cdot \frac{\sqrt{142}}{\ln\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}, \quad (5)$$

$$\varphi(s)=\prod_p \frac{1}{1-\tau(p)p^{-s}+p^{11-2s}} \Rightarrow$$

$$\Rightarrow 2\Phi-\frac{3}{20}\left[R(q)+\frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right]\cdot\frac{\sqrt{142}}{\ln\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (6)$$

Also with regard the eqs. (101) and (106) of **Chapter 2**, we note that can be related with the Ramanujan's identity concerning π. Thence, we have the following connections:

$$\sum_{k\in N(n/r-a)}\left(1-(k+a)\frac{r}{n}\right)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{r^s}\zeta(s,a)\frac{n^s}{s(s+1)}ds \Rightarrow$$

$$\Rightarrow \frac{1}{2\left\{2\Phi-\frac{3}{20}\left[R(q)+\frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right]\right\}i}\int_{c-i\infty}^{c+i\infty}\frac{1}{r^s}\zeta(s,a)\frac{n^s}{s(s+1)}ds, \quad (7)$$

$$\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\zeta(s,a)\frac{n^s}{4^s s(s+1)}ds=\varepsilon(n,a)-\frac{1}{8}n-\frac{1}{2}+a \Rightarrow$$

$$\Rightarrow \frac{1}{2\left\{2\Phi-\frac{3}{20}\left[R(q)+\frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right]\right\}i}\int_{c-i\infty}^{c+i\infty}\frac{1}{r^s}\zeta(s,a)\frac{n^s}{4^s s(s+1)}ds=$$

$$=\varepsilon(n,a)-\frac{1}{8}n-\frac{1}{2}+a. \quad (8)$$

Now we take the eqs. (79), (82), (83), (98) and (105) of **Chapter 3**. We note that can be related with the Goldston-Montgomery equation (29b) and with the Palumbo-Nardelli relationship (30) of chapter 5. Hence, we obtain the following connections:

$$A = -\int_{R^*} \alpha(\pi^{-1}y)(\log q)dy = -\log q\left(\int_R \alpha(\pi^{-1}y)dy - \int_P dy\right) = \frac{1}{q}\log q \Rightarrow$$

$$\Rightarrow 2\ln\rho \Rightarrow \int_0^T f(t)dt = (1+\varepsilon')T\log T \Rightarrow$$

$$\Rightarrow \int d^{10}x\sqrt{|g|}\left[\frac{1}{4}R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{12}e^{-2\phi}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right] \Rightarrow$$

$$\Rightarrow \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(\left|F_2\right|^2\right)\right] =$$

$$-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr(G_{\mu\nu}G_{\rho\sigma})f(\phi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right], \quad (9)$$

$$Pfw\int_{R^*} f_0^3(|u|)\frac{|u|^{1/2}}{|1-u|}d^*u = \log\pi + \gamma \Rightarrow 2\ln\rho \Rightarrow \int_0^T f(t)dt = (1+\varepsilon')T\log T, \Rightarrow$$

$$\Rightarrow \int d^{10}x\sqrt{|g|}\left[\frac{1}{4}R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{12}e^{-2\phi}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right] \Rightarrow$$

$$\Rightarrow \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(\left|F_2\right|^2\right)\right] =$$

$$-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr(G_{\mu\nu}G_{\rho\sigma})f(\phi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right], \quad (10)$$

$$PF_0\int_{R_+^*} f_0^4 \times (1-f_0^4)^{-1}d^*u = \left[\log(2\pi) + \lim_{t\to\infty}\left(\int_{R_+^*}(1-f_0^{2t})f_0^4(1-f_0^4)^{-1}d^*u - \log t\right)\right] = \log 2\pi + \gamma - \log 2$$

$$\Rightarrow 2\ln\rho \Rightarrow \int_0^T f(t)dt = (1+\varepsilon')T\log T, \Rightarrow$$

$$\Rightarrow \int d^{10}x\sqrt{|g|}\left[\frac{1}{4}R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{12}e^{-2\phi}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right] \Rightarrow$$

$$\Rightarrow \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(\left|F_2\right|^2\right)\right] =$$

$$-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr(G_{\mu\nu}G_{\rho\sigma})f(\phi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right], \quad (11)$$

$$Pfw\int f_2\left(|u|_C\right)\frac{1}{|1-u|_C}d^*u = PF_0\int f_0 f_1^{-1}d^*\nu = 2(\log 2\pi + \gamma) \Rightarrow$$

$$\Rightarrow 2\ln\rho \Rightarrow \int_0^T f(t)dt = (1+\varepsilon')T\log T\ , \Rightarrow$$

$$\Rightarrow \int d^{10}x\sqrt{|g|}\left[\frac{1}{4}R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{12}e^{-2\phi}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right] \Rightarrow$$

$$\Rightarrow \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(|F_2|^2\right)\right] =$$

$$-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right], \quad (12)$$

$$4\int_{-\infty}^{\log 2} Arc\sin\left(e^x/2\right)dx = -4\int_0^{\pi/2}\log(\sin u)du = 2\pi\log 2 \Rightarrow$$

$$\Rightarrow 2\ln\rho \Rightarrow \int_0^T f(t)dt = (1+\varepsilon')T\log T\ , \Rightarrow$$

$$\Rightarrow \int d^{10}x\sqrt{|g|}\left[\frac{1}{4}R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{12}e^{-2\phi}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right] \Rightarrow$$

$$\Rightarrow \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(|F_2|^2\right)\right] =$$

$$-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]. \quad (13)$$

Now, we take the eqs. (15), (22), (25) and (27) of **Chapter 4**. We note that can be related with the Palumbo-Nardelli relationship. Thence, we have the following connections:

$$A(\zeta_1 k_1,\ldots,\zeta_n k_n) = g^{n-2}\int_{(Q_p)^{n-3}}\theta_{[0,y_{n-1},\ldots,y_{3,1}]}(y)F(\zeta,k,y)\ \cdot\prod_{3\le i<j\le n-1}\left|y_i - y_j\right|_p^{k_i k_j}\prod_{3\le i\le n-1}\left(\left|y_i\right|_p^{k_n k_i}\left|1-y_i\right|^{k_2 k_i}dy_i\right) \Rightarrow$$

$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] =$$

$$= \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(|F_2|^2\right)\right], \quad (14)$$

$$A_p(k_1,\ldots,k_4) = g_p^2\int DX\chi_p\left(-\frac{1}{h}S_0[X]\right)\times\prod_{j=1}^4\int d^2\sigma_j\chi_p\left(-\frac{1}{h}k_\mu^{(j)}X^\mu\left(\sigma_j,\tau_j\right)\right) \Rightarrow$$

$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] =$$

$$= \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(|F_2|^2\right)\right], \quad (15)$$

$$A_A(k_1,...,k_4)=g_\infty^2\int_R |x|_\infty^{k_1k_2}|1-x|_\infty^{k_2k_3}dx\times\prod_{p\in S}g_p^2\prod_{j=1}^4\int d^2\sigma_j\times\prod_{p\notin S}g_p^2\Rightarrow$$
$$\Rightarrow-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr(G_{\mu\nu}G_{\rho\sigma})f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]=$$
$$=\int_0^\infty\frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(\left|F_2\right|^2\right)\right],\quad(16)$$

$$S=\int d^dxL=\frac{1}{g^2}\frac{p^2}{p-1}\int d^dx\left[-\frac{1}{2}\phi p^{-\frac{1}{2}[]}\phi+\frac{1}{p+1}\phi^{p+1}\right]\Rightarrow$$
$$\Rightarrow\int_0^\infty\frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(\left|F_2\right|^2\right)\right]=$$
$$-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr(G_{\mu\nu}G_{\rho\sigma})f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right].\quad(17)$$

While, if we take the eqs. (18), (33b), (38), (43) and (46) of **Chapter 4**, we note that can be related with the Ramanujan's identity concerning π and with Palumbo-Nardelli model. Then, we obtain the following connections:

$$A_\infty(k_1,...,k_4)=g_\infty^2\int DX\exp\left(\frac{2\pi i}{h}S_0[X]\right)\times\prod_{j=1}^4\int d^2\sigma_j\exp\left[\frac{2\pi i}{h}k_\mu^{(j)}X^\mu(\sigma_j,\tau_j)\right]\Rightarrow$$
$$\Rightarrow-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr(G_{\mu\nu}G_{\rho\sigma})f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]=$$
$$=\int_0^\infty\frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(\left|F_2\right|^2\right)\right]\Rightarrow$$
$$\Rightarrow g_\infty^2\int DX\exp\left\{2\left\{2\Phi-\frac{3}{20}\left[R(q)+\frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q\frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right]\right\}i\frac{1}{h}S_0X\right\}\times$$
$$\times\prod_{j=1}^4\int d^2\sigma_j\exp\left\{2\left\{2\Phi-\frac{3}{20}\left[R(q)+\frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q\frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right]\right\}\frac{i}{h}k_\mu^{(j)}X^\mu(\sigma_j,\tau_j)\right\},$$
$$(18)$$

$$T_q=-\int d^{d-q-1}x_\perp L\left(F^{(d-q-1)}(x_\perp)\right)=\frac{1}{2\left\{g\left[\frac{p^2-1}{2\pi p^{2p/(2p-1)}\ln p}\right]^{(d-q-1)/4}\right\}^2}\frac{p^2}{p+1}\Rightarrow$$
$$\Rightarrow-\int d^{d-q-1}x_\perp L\left(F^{(d-q-1)}(x_\perp)\right)=$$

$$
=\frac{1}{2\left\{g\left[\frac{p^{2}-1}{2\left(2\Phi-\frac{3}{20}\left(R(q)+\frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp(\frac{1}{\sqrt{5}}\int_{0}^{q}\frac{f^{5}(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}})}\right)\right)p^{2p/(2p-1)}\ln p}\right]^{(d-q-1)/4}\right\}^{2}}\frac{p^{2}}{p+1}
$$

$\Rightarrow$

$$
-\mu_{25}\int d^{26}\xi Tr\left\{e^{-\Phi}\left[-\det\left(G_{ab}+B_{ab}+2\pi\alpha' F_{ab}\right)\right]^{1/2}\right\}=\int_{0}^{\infty}-\frac{1}{(2\pi\alpha')^{2}g_{YM}^{2}}\int d^{10}xTr\left\{\left[-\det\left(\eta_{\mu\nu}+2\pi\alpha' F_{\mu\nu}\right)\right]^{1/2}\right\}
$$

$$
\Rightarrow-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi\right]=\int_{0}^{\infty}\frac{1}{2\kappa_{10}^{2}}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\cdot
$$

$$
\cdot\left[R+4\partial_{\mu}\Phi\partial^{\mu}\Phi-\frac{1}{2}\left|\tilde{H}_{3}\right|^{2}-\frac{k_{10}^{2}}{g_{10}^{2}}Tr_{\nu}\left(\left|F_{2}\right|^{2}\right)\right],\quad (19)
$$

$$
\frac{1}{g^{2}}\frac{p^{2}}{p-1}\left(\frac{2\pi}{\sqrt{2}}\right)^{d-q-1}\int d^{q+1}z\left[-\frac{1}{2}\tilde{\psi}p^{-\frac{1}{2}\Box_{z}}\tilde{\psi}+\frac{1}{p+1}\tilde{\psi}^{p+1}-C\left\{\frac{1}{2}\tilde{\xi}^{i}p^{-\frac{1}{2}\Box_{z}}\tilde{\xi}^{i}-\frac{1}{2}\tilde{\psi}^{p-1}\tilde{\xi}^{i}\tilde{\xi}^{i}\right\}+O\left(\tilde{\xi}^{3}\right)+\ldots\right]\Rightarrow
$$

$$
\Rightarrow\frac{1}{g^{2}}\frac{p^{2}}{p-1}\left\{\frac{2\left\{2\Phi-\frac{3}{20}\left[R(q)+\frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_{0}^{q}\frac{f^{5}(-t)}{f\left(-t^{1/5}\right)}\frac{dt}{t^{4/5}}\right)}\right]\right\}}{\sqrt{2}}\right\}^{d-q-1}.
$$

$$
\cdot\int d^{q+1}z\left[-\frac{1}{2}\tilde{\psi}p^{-\frac{1}{2}\Box_{z}}\tilde{\psi}+\frac{1}{p+1}\tilde{\psi}^{p+1}-C\left\{\frac{1}{2}\tilde{\xi}^{i}p^{-\frac{1}{2}\Box_{z}}\tilde{\xi}^{i}-\frac{1}{2}\tilde{\psi}^{p-1}\tilde{\xi}^{i}\tilde{\xi}^{i}\right\}+O\left(\tilde{\xi}^{3}\right)+\ldots\right]\Rightarrow
$$

$$
\Rightarrow-\mu_{25}\int d^{26}\xi Tr\left\{e^{-\Phi}\left[-\det\left(G_{ab}+B_{ab}+2\pi\alpha' F_{ab}\right)\right]^{1/2}\right\}=\int_{0}^{\infty}-\frac{1}{(2\pi\alpha')^{2}g_{YM}^{2}}\int d^{10}xTr\left\{\left[-\det\left(\eta_{\mu\nu}+2\pi\alpha' F_{\mu\nu}\right)\right]^{1/2}\right\}
$$

$$
\Rightarrow-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi\right]=\int_{0}^{\infty}\frac{1}{2\kappa_{10}^{2}}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\cdot
$$

$$
\cdot\left[R+4\partial_{\mu}\Phi\partial^{\mu}\Phi-\frac{1}{2}\left|\tilde{H}_{3}\right|^{2}-\frac{k_{10}^{2}}{g_{10}^{2}}Tr_{\nu}\left(\left|F_{2}\right|^{2}\right)\right],\quad (20)
$$

$$\zeta\,(\Box/2)\,\phi = \frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \frac{\phi}{1-\phi} \Rightarrow$$

$$\Rightarrow \frac{1}{\left\{2\left\{2\Phi - \frac{3}{20}\left[R(q) + \frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right]\right\}\right\}^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \frac{\phi}{1-\phi}$$

$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] =$$

$$= \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(\left|F_2\right|^2\right)\right], \quad (21)$$

$$\zeta\,(\Box/4)\,\theta = \frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{4}\right)\tilde{\theta}(k)dk = \sum_{n\geq 1}\left[\theta^{n^2} + \frac{n(n-1)}{2(n+1)}\theta^{\frac{n(n-1)}{2}-1}\left(\phi^{n+1}-1\right)\right] \Rightarrow$$

$$\Rightarrow \frac{1}{\left\{2\left\{2\Phi - \frac{3}{20}\left[R(q) + \frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right]\right\}\right\}^D}$$

$$\int e^{ixk}\zeta\left(-\frac{k^2}{4}\right)\tilde{\theta}(k)dk = \sum_{n\geq 1}\left[\theta^{n^2} + \frac{n(n-1)}{2(n+1)}\theta^{\frac{n(n-1)}{2}-1}\left(\phi^{n+1}-1\right)\right] \Rightarrow$$

$$\Rightarrow \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(\left|F_2\right|^2\right)\right] =$$

$$-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]. \quad (22)$$

Furthermore, we can see easily that the equations described in the **Chapter 5** and **6** can be connected also among them.

Conclusion

Hence, in conclusion, also for some mathematical sectors concerning the Fermat's Last Theorem, can be obtained interesting and new connections with other sectors of Number Theory and String Theory, principally the p-adic and adelic numbers, the Ramanujan's modular equations, some formulae related to the Riemann zeta functions and p-adic and adelic strings.

Furthermore, also the fundamental relationship concerning the Palumbo-Nardelli model, a general relationship that links bosonic string action and superstring action (i.e. bosonic and fermionic strings acting in all natural systems), can be related with some equations regarding the p-adic (adelic) string sector.

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