Proof of the Beale's Conjecture

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***Abstract*—This article solves the Conjecture using the Addition Method that I proposed**

***Index Terms*—algorithm**

I. Definition of the Beale's Conjecture

If:

$$A^n + B^m = C^l, \tag{1}$$

where $A, B, C, n, m, l \in \mathbb{N}^*$ and $n, m, l > 2$;
then A, B, C have a common prime divisor.

$\mathbb{N}^*$ be natural numbers without zero.

II. Algorithm for Proof of the Beale's Conjecture

A. Proof for the option of even A, B, C

If A, B, C are even numbers, then their common divisor is 2 and the Beale's Conjecture for (II-A) is true.

B. Detailing the initial conditions

Let:

$$y \quad \text{be a natural odd number.} \tag{2}$$

Then (1) for the remaining options can be represented as follows:

$$y_1^n + y_2^m = (2^k y_3)^l, \tag{3}$$

where $k \in \mathbb{N}^*$,
or:

$$y_1^n + (2^k y_2)^m = y_3^l, \tag{4}$$

where $k \in \mathbb{N}^*$.

C. Proof of the converse for the option of the multiplicity of both terms

Let:

$$(y_o \geq 3) \quad \text{be a odd prime.} \tag{5}$$

Then (3) and (4) can be represented as follows:

$$(y_o y_1)^n + (y_o y_2)^m = (2^k y_3)^l, \tag{6}$$

where $\frac{y_3}{y_o} \notin \mathbb{N}^*$,
or:

$$(y_o y_1)^n + (2^k y_o y_2)^m = y_3^l, \tag{7}$$

where $\frac{y_3}{y_o} \notin \mathbb{N}^*$.

Let's transform (6) and (7):

$$y_o(y_o^{n-1} y_1^n + y_o^{m-1} y_2^m) = (2^k y_3)^l, \tag{8}$$

where $\frac{y_3}{y_o} \notin \mathbb{N}^*$,
or:

$$y_o(y_o^{n-1} y_1^n + y_o^{m-1} (2^k y_2)^m) = y_3^l,, \tag{9}$$

where $\frac{y_3}{y_o} \notin \mathbb{N}^*$.

The left side of (8) and (9) will be a multiple of y_o, while the right side will not. I.e.:

$$(y_o y_1)^n + (y_o y_2)^m \neq (2^k y_3)^l, \tag{10}$$

where $\frac{y_3}{y_o} \notin \mathbb{N}^*$,
or:

$$(y_o y_1)^n + (2^k y_o y_2)^m \neq y_3^l, \tag{11}$$

where $\frac{y_3}{y_o} \notin \mathbb{N}^*$.

The Beale's Conjecture for (II-C) is true.

D. Proof of the converse for the option of multiplicity of one term and sum

Expressions (3) and (4) can be represented as follows:

$$y_1^n + (y_o y_2)^m = (2^k y_o y_3)^l, \tag{12}$$

where $\frac{y_1}{y_o} \notin \mathbb{N}^*$,
or:

$$y_1^n + (2^k y_o y_2)^m = (y_o y_3)^l, \tag{13}$$

where $\frac{y_1}{y_o} \notin \mathbb{N}^*$,
or:

$$(y_o y_1)^n + (2^k y_2)^m = (y_o y_3)^l, \tag{14}$$

where $\frac{y_2}{y_o} \notin \mathbb{N}^*$.

Let's transform the (12), (13) and (14):

$$y_o(y_o^{l-1} (2^k y_3)^l - y_o^{m-1} y_2^m) = y_1^n, \tag{15}$$

where $\frac{y_1}{y_o} \notin \mathbb{N}^*$,
or:

$$y_o(y_o^{l-1} y_3^l - y_o^{m-1} (2^k y_2)^m) = y_1^n, \tag{16}$$

where $\frac{y_1}{y_o} \notin \mathbb{N}^*$,
or:

$$y_o(y_o^{l-1} y_3^l - y_o^{n-1} y_1^n) = (2^k y_2)^m, \tag{17}$$

where $\frac{y_2}{y_o} \notin \mathbb{N}^*$.

The left side of (15), (16) and (17) will be a multiple of y_o, while the right side will not. I.e.:

$$y_1^n + (y_o y_2)^m \neq (2^k y_o y_3)^l, \quad (18)$$

where $\frac{y_1}{y_o} \notin \mathbb{N}^*$,
or:

$$y_1^n + (2^k y_o y_2)^m \neq (y_o y_3)^l, \quad (19)$$

where $\frac{y_1}{y_o} \notin \mathbb{N}^*$,
or:

$$(y_o y_1)^n + (2^k y_2)^m \neq (y_o y_3)^l, \quad (20)$$

where $\frac{y_2}{y_o} \notin \mathbb{N}^*$.

The Beale's Conjecture for (II-D) is true.

E. Proof of the converse for a variant without a common divisor

Let the terms and the sum in (3) and (4) not have a common divisor.

II.*E*.1. Features of addition and subtraction of odd numbers

The proof for (3) and (4) without a common divisor should consider the following:
Let $y_{x1} > y_{x2}$. Then, if:

$$y_{x1} + y_{x2} = 2^z y_{x3}, \quad (21)$$

where $(z \geq 2) \in \mathbb{N}^*$;
then:

$$y_{x1} - y_{x2} = 2 y_{x4}. \quad (22)$$

And vice versa, if:

$$y_{x1} + y_{x2} = 2 y_{x3}, \quad (23)$$

then:

$$y_{x1} - y_{x2} = 2^z y_{x4}, \quad (24)$$

where $(z \geq 2) \in \mathbb{N}^*$.

II.*E*.2. The option of (3) without a divisor, when $(2^k y_3) < y_2 < y_1$.

Let's consider the (3). Let:

$$y_1 + y_2 = 2^d y_4, \quad (25)$$

where $d \in \mathbb{N}^*$.
Let $(2^k y_3) < y_2 < y_1$ in (3). Then:

$$(2^k y_3 + y_5)^n + (2^k y_3 + y_6)^m = (2^k y_3)^l, \quad (26)$$

where:

$$y_5 \text{ and } y_6 \text{ correspond to (2).} \quad (27)$$

Let's substitute the terms in brackets in (26) instead of y_1 and y_2 in (25):

$$2^k y_3 + y_5 + 2^k y_3 + y_6 = 2^d y_4. \quad (28)$$

Let's express $(2^k y_3)$ from (28):

$$2^k y_3 = \frac{2^d y_4 - y_5 - y_6}{2}. \quad (29)$$

Let's substitute (29) into (26):

$$\left(\frac{2^d y_4 + (y_5 - y_6)}{2}\right)^n + \left(\frac{2^d y_4 - (y_5 - y_6)}{2}\right)^m = \\ = \left(\frac{2^d y_4 - (y_5 + y_6)}{2}\right)^l. \quad (30)$$

II.*E*.2.1. *The option when* $d = 1$ *in* (30), *and the Addition Method*

Let $d = 1$ in (30). Then, according to (23) and (24):

$$y_5 + y_6 = 2 y_7, \quad (31)$$

and

$$y_5 - y_6 = 2^e y_8, \quad (32)$$

where $(e \geq 2) \in \mathbb{N}^*$.

Substituting (31) and (32) into (30) and dividing the expressions in brackets by 2, we obtain the following expression:

$$(y_4 + 2^{e-1} y_8)^n + (y_4 - 2^{e-1} y_8)^m = (y_4 - y_7)^l, \quad (33)$$

where:

$$2^{e-1} y_8 = \frac{y_5 - y_6}{2}, \\ y_7 = \frac{y_5 + y_6}{2}. \quad (34)$$

The Beale's Conjecture will now be proved using the Addition Method.

Addition Method (on the example of (II.E.2.1))

Let's consider the (33). Then:
$(y_4 + 2^{e-1} y_8)^n$ is the same as the expression $(y_4 + 2^{e-1} y_8)$, folded $(y_4 + 2^{e-1} y_8)^{n-1}$ times;
$(y_4 - 2^{e-1} y_8)^m$ is the same as the expression $(y_4 - 2^{e-1} y_8)$, folded $(y_4 - 2^{e-1} y_8)^{m-1}$ times;
$(y_4 - y_7)^l$ is the same as the expression $(y_4 - y_7)$, folded $(y_4 - y_7)^{l-1}$ times.

II.*E*.2.1.1

Let in (33):

$$(y_4 + 2^{e-1} y_8)^{n-1} > (y_4 - 2^{e-1} y_8)^{m-1}. \quad (35)$$

Then in (33) it is possible:

all $(y_4 - 2^{e-1} y_8)$ to add with the same number of $(y_4 + 2^{e-1} y_8)$ in pairs. (36)

Let's change the terms of (36) so that a term appears that is equal to the expression in brackets on the right side of (33).

To do this, let's decompose all the terms into parts and select the term we need using the expressions from (34):

$$\begin{aligned}&(y_4+2^{e-1}y_8)+(y_4-2^{e-1}y_8)=\\&=y_4+\frac{y_5}{2}-\frac{y_6}{2}+y_4+\frac{y_5}{2}+\frac{y_6}{2}=\\&=\Big(y_4-\frac{y_5}{2}-\frac{y_6}{2}\Big)+\Big(y_4+\frac{y_5}{2}+\frac{y_6}{2}\Big)=\\&=(y_4-y_7)+(y_4+y_7).\end{aligned} \tag{37}$$

Let's substitute (37) into the left side of (33) taking into account (36):

$$\begin{aligned}&(y_4-y_7)(y_4-2^{e-1}y_8)^{m-1}+(y_4+y_7)\cdot\\&\cdot(y_4-2^{e-1}y_8)^{m-1}+(y_4+2^{e-1}y_8)\cdot\\&\cdot\Big((y_4+2^{e-1}y_8)^{n-1}-(y_4-2^{e-1}y_8)^{m-1}\Big).\end{aligned} \tag{38}$$

Let's try to select (y_4-y_7) from the last two terms of (38), so that as a result we can only get expression (y_4-y_7), folded $(y_4-y_7)^{l-1}$ times.
For this, let us present the third term of (38) as expression $(y_4+2^{e-1}y_8)$, folded $((y_4+2^{e-1}y_8)^{n-1}-(y_4-2^{e-1}y_8)^{m-1})$ times, the second term of (38) - as expression $(y_4-2^{e-1}y_8)$, folded $(y_4+y_7)(y_4-2^{e-1}y_8)^{m-2}$ times.
Let's add each $(y_4-2^{e-1}y_8)$ with the same number of expressions $(y_4+2^{e-1}y_8)$ in pairs.
Then, according to (37), let's divide the sum into two terms and add expression $(y_4-y_7)(y_4+y_7)(y_4-2^{e-1}y_8)^{m-2}$ to the first term of (38).
We get the following expression:

$$\begin{aligned}&(y_4-y_7)(y_4-2^{e-1}y_8)^{m-2}(2y_4+y_7-2^{e-1}y_8)+\\&+(y_4+y_7)^2(y_4-2^{e-1}y_8)^{m-2}+(y_4+2^{e-1}y_8)\cdot\\&\cdot\Big((y_4+2^{e-1}y_8)^{n-1}-(y_4-2^{e-1}y_8)^{m-2}\cdot\\&\cdot(2y_4+y_7-2^{e-1}y_8)\Big).\end{aligned} \tag{39}$$

Let's try to select (y_4-y_7) from the last two terms of (39) in the same way, in order to achieve only expression (y_4-y_7), folded $(y_4-y_7)^{l-1}$ times.
We get the following expression:

$$\begin{aligned}&(y_4-y_7)(y_4-2^{e-1}y_8)^{m-3}\Big((y_4-2^{e-1}y_8)\cdot\\&\cdot(2y_4+y_7-2^{e-1}y_8)+(y_4+y_7)^2\Big)+(y_4+y_7)^3\cdot\\&\cdot(y_4-2^{e-1}y_8)^{m-3}+(y_4+2^{e-1}y_8)\cdot\\&\cdot\Bigg((y_4+2^{e-1}y_8)^{n-1}-(y_4-2^{e-1}y_8)^{m-3}\cdot\\&\cdot\Big((y_4-2^{e-1}y_8)(2y_4+y_7-2^{e-1}y_8)+\\&+(y_4+y_7)^2\Big)\Bigg).\end{aligned} \tag{40}$$

Expression (40) has only become more complicated.
If we continue to extract expression (y_4-y_7) from the sum of the second and third terms, then the expression will become more complicated and the moment will come when x becomes greater than m in the exponent $m-x$.
Then all terms will leave the set of natural numbers without zero.
Thus, in addition to the term multiple of (y_4-y_7), there will always be two other terms not multiple of (y_4-y_7).
That is, a situation is unattainable when (40) will be equal to only $(y_4-y_7)^l$.

Thus, returning to (3):

$$y_1^n+y_2^m\neq(2^ky_3)^l,$$

without a common divisor, when $(2^ky_3)<y_2<y_1$ in (3), $d=1$ (30) and $(y_4+2^{e-1}y_8)^{n-1}>(y_4-2^{e-1}y_8)^{m-1}$ in (33).

The Beale's Conjecture for (II.E.2.1.1) is true.

II.*E*.2.1.2

Let in (33):

$$(y_4+2^{e-1}y_8)^{n-1}<(y_4-2^{e-1}y_8)^{m-1}.$$

Calculating by the Addition Method (II.E.2.1.1), we obtain the following expression:

$$\begin{aligned}&(y_4-y_7)(y_4+2^{e-1}y_8)^{n-3}\Big((y_4+2^{e-1}y_8)\cdot\\&\cdot(2y_4+y_7+2^{e-1}y_8)+(y_4+y_7)^2\Big)+(y_4+y_7)^3\cdot\\&\cdot(y_4+2^{e-1}y_8)^{n-3}+(y_4-2^{e-1}y_8)\cdot\\&\cdot\Bigg((y_4-2^{e-1}y_8)^{m-1}-(y_4+2^{e-1}y_8)^{n-3}\cdot\\&\cdot\Big((y_4+2^{e-1}y_8)(2*y_4+y_7+2^{e-1}y_8)+\\&+(y_4+y_7)^2\Big)\Bigg)\neq(y_4-y_7)^l.\end{aligned} \tag{41}$$

Thus, returning to (3):

$$y_1^n+y_2^m\neq(2^ky_3)^l,$$

without a common divisor, when $(2^ky_3)<y_2<y_1$ in (3), $d=1$ in (30) and $(y_4+2^{e-1}y_8)^{n-1}<(y_4-2^{e-1}y_8)^{m-1}$ in (33).

The Beale's Conjecture for (II.E.2.1.2) is true.

II.*E*.2.1.3

Let in (33):

$$(y_4+2^{e-1}y_8)^{n-1}=(y_4-2^{e-1}y_8)^{m-1}.$$

Then (33) becomes a special case of the option of the multiplicity of both terms, for which Beale's Conjecture was proved in (II-C).
Therefore, in the future, expressions, where $A^{n-1}=B^{m-1}$, will not be considered.

The Beale's Conjecture for (II.E.2.1) is true.

II.*E*.2.2. *The option when $d > 1$ in* (30)

Let $d > 1$ in (30). Then, according to (23) and (24):

$$y_5 - y_6 = 2y_8, \tag{42}$$

and

$$y_5 + y_6 = 2^e y_7, \tag{43}$$

where $(e \geq 2) \in \mathbb{N}^*$.

Substituting (42) and (43) into (30) and dividing the expressions in brackets by 2, we obtain the following expression:

$$(2^{d-1}y_4+y_8)^n+(2^{d-1}y_4-y_8)^m = (2^{d-1}y_4-2^{e-1}y_7)^l, \tag{44}$$

where:

$$y_8 = \frac{y_5 - y_6}{2},$$
$$2^{e-1}y_7 = \frac{y_5 + y_6}{2}.$$

II.*E*.2.2.1

Let in (44):

$$(2^{d-1}y_4 + y_8)^{n-1} < (2^{d-1}y_4 - y_8)^{m-1}.$$

Calculating by the Addition Method (II.E.2.1.1), we obtain the following expression:

$$\begin{aligned}&(2^{d-1}y_4 - 2^{e-1}y_7)(2^{d-1}y_4 - y_8)^{m-3} \cdot \\ &\cdot \Big((2^{d-1}y_4 - y_8)(2^{d-1}y_4 + 2^{e-1}y_7 - y_8) + \\ &+ (2^{d-1}y_4 + 2^{e-1}y_7)^2\Big)+(2^{d-1}y_4 + 2^{e-1}y_7)^3 \cdot \\ &\cdot (2^{d-1}y_4 - y_8)^{m-3} + (2^{d-1}y_4 + y_8) \cdot \\ &\cdot \Big((2^{d-1}y_4 + y_8)^{n-1} - (2^{d-1}y_4 - y_8)^{m-3} \cdot \\ &\cdot \Big((2^{d-1}y_4 - y_8)(2^{d-1}y_4 + 2^{e-1}y_7 - y_8) + \\ &+ (2^{d-1}y_4 + 2^{e-1}y_7)^2\Big)\Big) \neq (2^{d-1}y_4 - 2^{e-1}y_7)^l.\end{aligned} \tag{45}$$

Thus, returning to (3):

$$y_1^n + y_2^m \neq (2^k y_3)^l,$$

without a common divisor, when $(2^k y_3) < y_2 < y_1$ in (3), $d > 1$ in (30) and $(2^{d-1}y_4 + y_8)^{n-1} > (2^{d-1}y_4 - y_8)^{m-1}$ in (44).

The Beale's Conjecture for (II.E.2.2.1) is true.

II.*E*.2.2.2

Let in (44):

$$(2^{d-1}y_4 + y_8)^{n-1} < (2^{d-1}y_4 - y_8)^{m-1}.$$

Calculating by the Addition Method (II.E.2.1.1), we obtain the following expression:

$$\begin{aligned}&(2^{d-1}y_4 - 2^{e-1}y_7)(2^{d-1}y_4 + y_8)^{n-3}\Big((2^{d-1}y_4 + y_8) \cdot \\ &\cdot (2^{d-1}y_4 + 2^{e-1}y_7 + y_8) + (2^{d-1}y_4 + 2^{e-1}y_7)^2\Big)+ \\ &+ (2^{d-1}y_4 + 2^{e-1}y_7)^3(2^{d-1}y_4 + y_8)^{n-3} + \\ &+ (2^{d-1}y_4 - y_8)\Big((2^{d-1}y_4 - y_8)^{m-1} - \\ &- (2^{d-1}y_4 + y_8)^{n-3}\Big((2^{d-1}y_4 + y_8) \cdot \\ &\cdot (2^{d-1}y_4 + 2^{e-1}y_7 + y_8) + \\ &+ (2^{d-1}y_4 + 2^{e-1}y_7)^2\Big)\Big) \neq (2^{d-1}y_4 - 2^{e-1}y_7)^l.\end{aligned} \tag{46}$$

Thus, returning to (3):

$$y_1^n + y_2^m \neq (2^k y_3)^l,$$

without a common divisor, when $(2^k y_3) < y_2 < y_1$ in (3).

The Beale's Conjecture for (II.E.2) is true.

II.*E*.3. The option of (3) without a divisor, when $(2^k y_3) < y_1 < y_2$.

Let $(2^k y_3) < y_2 < y_1$ in (3). Then:

$$(2^k y_3 + y_5)^n + (2^k y_3 + y_6)^m = (2^k y_3)^l, \tag{47}$$

where (27) is true.

Expression (47) coincides with (26).
Let's substitute the terms in brackets from (47) instead of y_1 and y_2 in (25). Let's express $(2^k y_3)$ from the resulting expression. Substituting it into (47), we obtain the following expression:

$$\begin{aligned}&\left(\frac{2^d y_4 - (y_5 - y_6)}{2}\right)^n + \left(\frac{2^d y_4 + (y_5 - y_6)}{2}\right)^m = \\ &= \left(\frac{2^d y_4 - (y_5 + y_6)}{2}\right)^l.\end{aligned} \tag{48}$$

Problem (II.E.3) is solved in the same way as (II.E.2); only n and m, y_5 and y_6 are swapped .
Thus, returning to (3):

$$y_1^n + y_2^m \neq (2^k y_3)^l,$$

without a common divisor, when $(2^k y_3) < y_1 < y_2$ in (3).

The Beale's Conjecture for (II.E.3) is true.

II.*E*.4. The option of (3) without a divisor, when $y_1 < y_2 < (2^k y_3)$.

Let $y_1 < y_2 < (2^k y_3)$ in (3). Then:

$$(2^k y_3 - y_5)^n + (2^k y_3 - y_6)^m = (2^k y_3)^l, \tag{49}$$

where (27) is true.

Let's substitute the terms in brackets from (49) instead of y_1 and y_2 in (25). Let's express $(2^k y_3)$ from the resulting expression. Substituting it into (49), we obtain the following expression:

$$\left(\frac{2^d y_4 + (y_6 - y_5)}{2}\right)^n + \left(\frac{2^d y_4 - (y_6 - y_5)}{2}\right)^m = \\ = \left(\frac{2^d y_4 - (y_6 + y_5)}{2}\right)^l \tag{50}$$

II.*E*.4.1. *The option when $d = 1$ in (50)*

Let $d = 1$ in (50). Then, according to (23) and (24):

$$y_5 - y_6 = 2^e y_8, \tag{51}$$

where $(e \geq 2) \in \mathbb{N}^*$,

and

$$y_5 + y_6 = 2y_7. \tag{52}$$

Substituting (51) and (52) into (50) and dividing the expressions in brackets by 2, we obtain the following expression:

$$(y_4 - 2^{e-1}y_8)^n + (y_4 + 2^{e-1}y_8)^m = (y_4 + y_7)^l, \tag{53}$$

where:

$$2^{e-1}y_8 = \frac{y_5 - y_6}{2}, \\ y_7 = \frac{y_5 + y_6}{2}.$$

II.*E*.4.1.1

Let in (53):

$$(y_4 - 2^{e-1}y_8)^{n-1} < (y_4 + 2^{e-1}y_8)^{m-1}.$$

Calculating by the Addition Method (II.E.2.1.1), we obtain the following expression:

$$(y_4 + y_7)(y_4 - 2^{e-1}y_8)^{n-3}\Big((y_4 - 2^{e-1}y_8) \cdot \\ \cdot (2 * y_4 - y_7 - 2^{e-1}y_8) + (y_4 - y_7)^2\Big) + (y_4 - y_7)^3 \cdot \\ \cdot (y_4 - 2^{e-1} * y_8)^{n-3} + (y_4 + 2^{e-1}y_8) \cdot \\ \cdot \Bigg((y_4 + 2^{e-1}y_8)^{m-1} - (y_4 - 2^{e-1}y_8)^{n-3} \cdot \\ \cdot \Big((y_4 - 2^{e-1}y_8)(2y_4 - y_7 - 2^{e-1}y_8) + \\ + (y_4 - y_7)^2\Big)\Bigg) \neq (y_4 + y_7)^l. \tag{54}$$

Thus, returning to (3):

$$y_1^n + y_2^m \neq (2^k y_3)^l,$$

without a common divisor, when $y_1 < y_2 < (2^k y_3)$ in (3), $d = 1$ in (50) and $(y_4 - 2^{e-1}y_8)^{n-1} < (y_4 + 2^{e-1}y_8)^{m-1}$ in (53).

The Beale's Conjecture for (II.E.4.1.1) is true.

II.*E*.4.1.2

Let in (53):

$$(y_4 - 2^{e-1}y_8)^{n-1} > (y_4 + 2^{e-1}y_8)^{m-1}.$$

Calculating by the Addition Method (II.E.2.1.1), we obtain the following expression:

$$(y_4 + y_7)(y_4 + 2^{e-1}y_8)^{m-3}\Big((y_4 + 2^{e-1}y_8) \cdot \\ \cdot (2y_4 - y_7 + 2^{e-1}y_8) + (y_4 - y_7)^2\Big) + (y_4 - y_7)^3 \cdot \\ \cdot (y_4 + 2^{e-1}y_8)^{m-3} + (y_4 + 2^{e-1}y_8) \cdot \\ \cdot \Bigg((y_4 - 2^{e-1}y_8)^{n-1} - (y_4 + 2^{e-1}y_8)^{m-3} \cdot \\ \cdot \Big((y_4 + 2^{e-1}y_8)(2y_4 - y_7 + 2^{e-1}y_8) + \\ + (y_4 - y_7)^2\Big)\Bigg) \neq (y_4 + y_7)^l. \tag{55}$$

Thus, returning to (3):

$$y_1^n + y_2^m \neq (2^k y_3)^l,$$

without a common divisor, when $y_1 < y_2 < (2^k y_3)$ in (3) and $d = 1$ in (50).

The Beale's Conjecture for (II.E.4.1) is true.

II.*E*.4.2. *The option when $d > 1$ in (50)*

Let $d > 1$ in (50). Then, according to (21) and (22):

$$y_5 - y_6 = 2y_8, \tag{56}$$

and

$$y_5 + y_6 = 2^e y_7, \tag{57}$$

where $(e \geq 2) \in \mathbb{N}^*$.

Substituting (56) and (57) into (50) and dividing the expressions in brackets by 2, we obtain the following expression:

$$(2^{d-1}y_4 - y_8)^n + ((2^{d-1}y_4 + y_8)^m = \\ = (2^{d-1}y_4 + (2^{e-1}y_7))^l, \tag{58}$$

where:

$$y_8 = \frac{y_5 - y_6}{2}, \\ 2^{e-1}y_7 = \frac{y_5 + y_6}{2}.$$

II.*E*.4.2.1

Let in (58):

$$(2^{d-1}y_4 - y_8)^{n-1} < (2^{d-1}y_4 + y_8)^{m-1}.$$

Calculating by the Addition Method (II.E.2.1.1), we obtain the following expression:

$$\begin{aligned}&(2^{d-1}y_4 + 2^{e-1}y_7)(2^{d-1}y_4 - y_8)^{n-3}\Big((2^{d-1}y_4 - y_8)\cdot\\ &\cdot(2^{d-1}y_4 - 2^{e-1}y_7 - y_8) + (2^{d-1}y_4 - 2^{e-1}y_7)^2\Big) +\\ &+(2^{d-1}y_4 - 2^{e-1}y_7)^3(2^{d-1}y_4 - y_8)^{n-3} +\\ &+(2^{d-1}y_4 + y_8)\Bigg((2^{d-1}y_4 + y_8)^{m-1} -\\ &-(2^{d-1}y_4 - y_8)^{n-3}\Big((2^{d-1}y_4 - y_8)\cdot\\ &\cdot(2^{d-1}y_4 - 2^{e-1}y_7 - y_8) +\\ &+(2^{d-1}y_4 - 2^{e-1}y_7)^2\Big)\Bigg) \neq (2^{d-1}y_4 + 2^{e-1}y_7)^l.\end{aligned} \tag{59}$$

Thus, returning to (3):

$$y_1^n + y_2^m \neq (2^k y_3)^l,$$

without a common divisor, when $y_1 < y_2 < (2^k y_3)$ in (3), $d > 1$ in (50) and $(2^{d-1}y_4 - y_8)^{n-1} < (2^{d-1}y_4 + y_8)^{m-1}$ in (58).

The Beale's Conjecture for (II.E.4.2.1) is true.

II.*E*.4.2.2

Let in (58):

$$(2^{d-1}y_4 - y_8)^{n-1} > (2^{d-1}y_4 + y_8)^{m-1}.$$

Calculating by the Addition Method (II.E.2.1.1), we obtain the following expression:

$$\begin{aligned}&(2^{d-1}y_4 + 2^{e-1}y_7)(2^{d-1}y_4 + y_8)^{m-3}\cdot\\ &\cdot\Big((2^{d-1}y_4 + y_8)(2^{d-1}y_4 - 2^{e-1}y_7 + y_8) +\\ &+(2^{d-1}y_4 - 2^{e-1}y_7)^2\Big) + (2^{d-1}y_4 - 2^{e-1}y_7)^3\cdot\\ &\cdot(2^{d-1}y_4 + y_8)^{m-3} + (2^{d-1}y_4 - y_8)\cdot\\ &\cdot\Bigg((2^{d-1}y_4 - y_8)^{n-1} - (2^{d-1}y_4 + y_8)^{m-3}\cdot\\ &\cdot\Big((2^{d-1}y_4 + y_8)(2^{d-1}y_4 - 2^{e-1}y_7 + y_8) +\\ &+(2^{d-1}y_4 - 2^{e-1}y_7)^2\Big)\Bigg) \neq (2^{d-1}y_4 + 2^{e-1}y_7)^l.\end{aligned} \tag{60}$$

Thus, returning to (3):

$$y_1^n + y_2^m \neq (2^k y_3)^l,$$

without a common divisor, when $y_1 < y_2 < (2^k y_3)$ in (3).

The Beale's Conjecture for (II.E.4) is true.

II.*E*.5. <u>The option of (3) without a divisor, when $y_2 < y_1 < (2^k y_3)$.</u>

Let $y_2 < y_1 < (2^k y_3)$ in (3). Then:

$$(2^k y_3 - y_5)^n + (2^k y_3 - y_6)^m = (2^k y_3)^l, \tag{61}$$

where (27) is true.

Expression (61) coincides with (49).
Let's substitute the terms in brackets from (61) instead of y_1 and y_2 in (25). Let's express $(2^k y_3)$ from the resulting expression. Substituting it into (61), we obtain the following expression:

$$\begin{aligned}&\left(\frac{2^d y_4 - (y_5 - y_6)}{2}\right)^n + \left(\frac{2^d y_4 + (y_5 - y_6)}{2}\right)^m =\\ &= \left(\frac{2^d y_4 + (y_5 + y_6)}{2}\right)^l.\end{aligned} \tag{62}$$

Problem (II.E.5) is solved in the same way as (II.E.4); only n and m, y_5 and y_6 are swapped.
Thus, returning to (3):

$$y_1^n + y_2^m \neq (2^k y_3)^l,$$

without a common divisor, when $y_2 < y_1 < (2^k y_3)$ in (3).

The Beale's Conjecture for (II.E.5) is true.

II.*E*.6. <u>The option of (3) without a divisor, when $y_1 < (2^k y_3) < y_2$.</u>

Let $y_1 < (2^k y_3) < y_2$ in (3). Then:

$$(2^k y_3 - y_5)^n + (2^k y_3 + y_6)^m = (2^k y_3)^l, \tag{63}$$

where (27) is true.

Let's substitute the terms in brackets from (63) instead of y_1 and y_2 in (25). Let's express $(2^k y_3)$ from the resulting expression. Substituting it into (63), we obtain the following expression:

$$\begin{aligned}&\left(\frac{2^d y_4 - (y_5 + y_6)}{2}\right)^n + \left(\frac{2^d y_4 + (y_5 + y_6)}{2}\right)^m =\\ &= \left(\frac{2^d y_4 + (y_5 - y_6)}{2}\right)^l.\end{aligned} \tag{64}$$

II.*E*.6.1. <u>*The option when* $y_5 > y_6$ *in* (63)</u>

Let $y_5 > y_6$ in (63).

II.*E*.6.1.1 <u>*The option when* $d = 1$ *in* (64)</u>

Let $d = 1$ in (64). Then, according to (23) and (24):

$$y_5 + y_6 = 2^e y_8, \tag{65}$$

where $(e \geq 2) \in \mathbb{N}^*$,
and

$$y_5 - y_6 = 2y_7. \tag{66}$$

Substituting (65) and (66) into (64) and dividing the expressions in brackets by 2, we obtain the following expression:

$$(y_4 - 2^{e-1}y_8)^n + (y_4 + 2^{e-1}y_8)^m = (y_4 + y_7)^l, \tag{67}$$

where:

$$2^{e-1}y_8 = \frac{y_5 + y_6}{2},$$
$$y_7 = \frac{y_5 - y_6}{2}.$$

Expression (67) coincides with (53), therefore, problem (II.E.6.1.1) is solved in the same way as (II.E.4.1).
Thus, returning to (3):

$$y_1^n + y_2^m \neq (2^k y_3)^l,$$

without a common divisor, when $y_1 < (2^k y_3) < y_2$ in (3), and $d = 1$ in (64).

The Beale's Conjecture for (II.E.6.1.1) is true.

II.*E*.6.1.2 *The option when* $d > 1$ *in* (64)

Let $d > 1$ in (64). Then, according to (23) and (24):

$$y_5 + y_6 = 2y_8, \tag{68}$$

and

$$y_5 - y_6 = 2^e y_7, \tag{69}$$

where $(e \geq 2) \in \mathbb{N}^*$. Substituting (68) and (69) into (64) and dividing the expressions in brackets by 2, we obtain the following expression:

$$(2^{d-1}y_4 - y_8)^n + (2^{d-1}y_4 + y_8)^m = (2^{d-1}y_4 + 2^{e-1}y_7)^l, \tag{70}$$

where:

$$y_8 = \frac{y_5 + y_6}{2},$$
$$2^{e-1}y_7 = \frac{y_5 - y_6}{2}.$$

Expression (70) coincides with (58), therefore, problem (II.E.6.1.2) is solved in the same way as (II.E.4.2).
Thus, returning to (3):

$$y_1^n + y_2^m \neq (2^k y_3)^l,$$

without a common divisor, when $y_1 < (2^k y_3) < y_2$ in (3), and $y_5 > y_6$ in (63).

The Beale's Conjecture for (II.E.6.1) is true.

II.*E*.6.2. *The option when* $y_5 < y_6$ *in* (63)

Let $y_5 < y_6$ in (63).
Then (64) can be represented as follows:

$$\left(\frac{2^d y_4 - (y_6 + y_5)}{2}\right)^n + \left(\frac{2^d y_4 + (y_6 + y_5)}{2}\right)^m = \\ = \left(\frac{2^d y_4 - (y_6 - y_5)}{2}\right)^l. \tag{71}$$

II.*E*.6.2.1 *The option when* $d = 1$ *in* (71)

Let $d = 1$ in (71). Then, according to (21) and (22):

$$y_6 + y_5 = 2^e y_8, \tag{72}$$

where $(e \geq 2) \in \mathbb{N}^*$,
and

$$y_6 - y_5 = 2y_7. \tag{73}$$

Substituting (72) and (73) into (71) and dividing the expressions in brackets by 2, we obtain the following expression:

$$(y_4 - 2^{e-1}y_8)^n + (y_4 + 2^{e-1}y_8)^m = (y_4 - y_7)^l, \tag{74}$$

where:

$$2^{e-1}y_8 = \frac{y_6 + y_5}{2},$$
$$y_7 = \frac{y_6 - y_5}{2}.$$

II.*E*.6.2.1.1

Let in (74):

$$(y_4 - 2^{e-1}y_8)^{n-1} < (y_4 + 2^{e-1}y_8)^{m-1}.$$

Calculating by the Addition Method (II.E.2.1.1), we obtain the following expression:

$$(y_4 - y_7)(y_4 - 2^{e-1}y_8)^{n-3}\Big((y_4 - 2^{e-1}y_8) \cdot \\ \cdot (2y_4 + y_7 - 2^{e-1}y_8) + (y_4 + y_7)^2\Big) + (y_4 + y_7)^3 \cdot \\ \cdot (y_4 - 2^{e-1}y_8)^{n-3} + (y_4 + 2^{e-1}y_8) \cdot \\ \cdot \Big((y_4 + 2^{e-1}y_8)^{m-1} - (y_4 - 2^{e-1}y_8)^{n-3} \cdot \\ \cdot \Big((y_4 - 2^{e-1}y_8)(2y_4 + y_7 - 2^{e-1}y_8) + \\ + (y_4 + y_7)^2\Big)\Big) \neq (y_4 - y_7)^l. \tag{75}$$

Thus, returning to (3):

$$y_1^n + y_2^m \neq (2^k y_3)^l,$$

without a common divisor, when $y_1 < (2^k y_3) < y_2$ in (3), $y_5 < y_6$ in (63), $d = 1$ in (71) and $(y_4 - 2^{e-1}y_8)^{n-1} < (y_4 + 2^{e-1}y_8)^{m-1}$ in (74).

The Beale's Conjecture for (II.E.6.2.1.1) is true.

II.*E*.6.2.1.2

Let in (74):

$$(y_4 - 2^{e-1}y_8)^{n-1} > (y_4 + 2^{e-1}y_8)^{m-1}.$$

Calculating by the Addition Method (II.E.2.1.1), we obtain the following expression:

$$\begin{aligned}&(y_4 - y_7)(y_4 + 2^{e-1}y_8)^{m-3}\Big((y_4 + 2^{e-1}y_8)\cdot\\&\cdot(2y_4 + y_7 + 2^{e-1}y_8) + (y_4 + y_7)^2\Big) + (y_4 + y_7)^3\cdot\\&\cdot(y_4 + 2^{e-1}y_8)^{m-3} + (y_4 - 2^{e-1}y_8)\cdot\\&\cdot\Bigg((y_4 - 2^{e-1}y_8)^{n-1} - (y_4 + 2^{e-1}y_8)^{m-3}\cdot\\&\cdot\Big((y_4 + 2^{e-1}y_8)(2y_4 + y_7 + 2^{e-1}y_8) +\\&+ (y_4 + y_7)^2\Big)\Bigg) \neq (y_4 - y_7)^l.\end{aligned} \tag{76}$$

Thus, returning to (3):

$$y_1^n + y_2^m \neq (2^k y_3)^l,$$

without a common divisor, when $y_1 < (2^k y_3) < y_2$ in (3), $y_5 < y_6$ in (63) and $d = 1$ in (71).

The Beale's Conjecture for (II.E.6.2.1) is true.

II.*E*.6.2.2 <u>*The option when* $d > 1$ *in* (71)</u>

Let $d > 1$ in (71). Then, according to (23) and (24):

$$y_6 + y_5 = 2y_8, \tag{77}$$

and

$$y_6 - y_5 = 2^e y_7, \tag{78}$$

where $(e \geq 2) \in \mathbb{N}^*$.

Substituting (77) and (78) into (71) and dividing the expressions in brackets by 2, we obtain the following expression:

$$(y_4 - 2^{e-1}y_8)^n + (y_4 + 2^{e-1}y_8)^m = (2^{d-1}y_4 - 2^{e-1}y_7)^l, \tag{79}$$

where:

$$y_8 = \frac{y_6 + y_5}{2},$$
$$2^{e-1}y_7 = \frac{y_6 - y_5}{2}.$$

II.*E*.6.2.2.1

Let in (79):

$$(2^{d-1}y_4 - y_8)^{n-1} < (2^{d-1}y_4 + y_8)^{m-1}.$$

Calculating by the Addition Method (II.E.2.1.1), we obtain the following expression:

$$\begin{aligned}&(2^{d-1}y_4 - 2^{e-1}y_7)(2^{d-1}y_4 - y_8)^{n-3}\Big((2^{d-1}y_4 - y_8)\cdot\\&\cdot(2^d y_4 + 2^{e-1}y_7 - y_8) + (2^{d-1}y_4 + 2^{e-1}y_7)^2\Big) +\\&+ (2^{d-1}y_4 + 2^{e-1}y_7)^3(2^{d-1}y_4 - y_8)^{n-3} +\\&+ (2^{d-1}y_4 + y_8)\Bigg((2^{d-1}y_4 + y_8)^{m-1} -\\&- (2^{d-1}y_4 - y_8)^{n-3}\Big((2^{d-1}y_4 - y_8)\cdot\\&\cdot\Big((2^{d-1}y_4 - y_8)(2^d y_4 + 2^{e-1}y_7 - y_8) +\\&+ (2^{d-1}y_4 + 2^{e-1}y_7)^2\Big)\Bigg) \neq (2^{d-1}y_4 - 2^{e-1}y_7)^l.\end{aligned} \tag{80}$$

Thus, returning to (3):

$$y_1^n + y_2^m \neq (2^k y_3)^l,$$

without a common divisor, when $y_1 < (2^k y_3) < y_2$ in (3), $y_5 < y_6$ in (63), $d > 1$ in (71) and $(2^{d-1}y_4 - y_8)^{n-1} < (2^{d-1}y_4 + y_8)^{m-1}$ in (79).

The Beale's Conjecture for (II.E.6.2.2.1) is true.

II.*E*.6.2.2.2

Let in (79):

$$(2^{d-1}y_4 - y_8)^{n-1} > (2^{d-1}y_4 + y_8)^{m-1}.$$

Calculating by the Addition Method (II.E.2.1.1), we obtain the following expression:

$$\begin{aligned}&(2^{d-1}y_4 - 2^{e-1}y_7)(2^{d-1}y_4 + y_8)^{m-3}\cdot\\&\cdot\Big((2^{d-1}y_4 + y_8)(2^d y_4 + 2^{e-1}y_7 + y_8) +\\&+ (2^{d-1}y_4 + 2^{e-1}y_7)^2\Big) + (2^{d-1}y_4 + 2^{e-1}y_7)^3\cdot\\&\cdot(2^{d-1}y_4 + y_8)^{m-3} + (2^{d-1}y_4 - y_8)\cdot\\&\cdot\Bigg((2^{d-1}y_4 - y_8)^{n-1} - (2^{d-1}y_4 + y_8)^{m-3}\cdot\\&\cdot\Big((2^{d-1}y_4 + y_8)(2^d y_4 + 2^{e-1}y_7 + y_8) +\\&+ (2^{d-1}y_4 + 2^{e-1}y_7)^2\Big)\Bigg) \neq (2^{d-1}y_4 - 2^{e-1}y_7)^l.\end{aligned} \tag{81}$$

Thus, returning to (3):

$$y_1^n + y_2^m \neq (2^k y_3)^l,$$

without a common divisor, when $y_1 < (2^k y_3) < y_2$ in (3).

The Beale's Conjecture for (II.E.6) is true.

II.*E*.7. <u>The option of (3) without a divisor, when $y_2 < (2^k y_3) < y_1$</u>

Let $y_2 < (2^k y_3) < y_1$ in (3). Then:

$$(2^k y_3 + y_5)^n + (2^k y_3 - y_6)^m = (2^k y_3)^l, \tag{82}$$

where (27) is true.

Problem (II.E.7) is solved in the same way as (II.E.6); only n and m, y_5 and y_6 are swapped.
Thus, returning to (3):

$$y_1^n + y_2^m \neq (2^k y_3)^l,$$

without a common divisor, when $y_2 < (2^k y_3) < y_1$ in (3).

The Beale's Conjecture for (II.E.7) is true.

II.*E*.8. The option of (4) without a divisor, when $y_1 < (2^k y_2) < y_3$

Let's consider the (4). Let:

$$y_1 + (2^k y_2) = y_4. \tag{83}$$

Let $y_1 < (2^k y_2) < y_3$ in (4). Then:

$$(y_3 - 2^f y_5)^n + (y_3 - y_6)^m = y_3^l, \tag{84}$$

where (27) is true and $f \in \mathbb{N}^*$.

Let's substitute the terms in brackets from (84) instead of y_1 and $2^k y_2$ in (83):

$$y_3 - 2^f y_5 + y_3 - y_6 = y_4. \tag{85}$$

Let's express y_3 from (85):

$$y_3 = \frac{y_4 + 2^f y_5 + y_6}{2}. \tag{86}$$

Let's substitute (86) into (84):

$$\left(\frac{(y_4 + y_6) - 2^f y_5}{2}\right)^n + \left(\frac{(y_4 - y_6) + 2^f y_5}{2}\right)^m = \\ = \left(\frac{(y_4 + y_6) + 2^f y_5}{2}\right)^l. \tag{87}$$

II.*E*.8.1 *The option when $f = 1$ in (87)*

Let $f = 1$ in (87). Then, according to (21) and (22):

$$y_4 - y_6 = 2y_8, \tag{88}$$

and

$$y_4 + y_6 = 2^e y_7, \tag{89}$$

where $(e \geq 2) \in \mathbb{N}^*$.

Substituting (88) and (89) into (87) and dividing the expressions in brackets by 2, we obtain the following expression:

$$(2^{e-1} y_7 - y_5)^n + (y_8 + y_5)^m = (2^{e-1} y_7 + y_5)^l, \tag{90}$$

where:

$$y_8 = \frac{y_4 - y_6}{2},$$
$$2^{e-1} y_7 = \frac{y_4 + y_6}{2}.$$

II.*E*.8.1.1

Let in (90):

$$(2^{e-1} y_7 - y_5)^{n-1} < (y_8 + y_5)^{m-1}.$$

Calculating by the Addition Method (II.E.2.1.1), we obtain the following expression:

$$\begin{aligned}&(2^{e-1} y_7 + y_5)(2^{e-1} y_7 - y_5)^{n-3}\Big((2^{e-1} y_7 - y_5) \cdot \\ &\cdot (2^{e-1} y_7 - 2y_5 + y_8) + (y_8 + y_5)^2\Big) + \\ &+ (y_8 + y_5)^3 (2^{e-1} y_7 - y_5)^{n-3} + (y_8 + y_5) \cdot \\ &\cdot \Bigg((y_8 + y_5)^{m-1} - (2^{e-1} y_7 - y_5)^{n-3} \cdot \\ &\cdot \Big((2^{e-1} y_7 - y_5)(2^{e-1} y_7 - 2y_5 + y_8) + \\ &+ (y_8 + y_5)^2\Big)\Bigg) \neq (2^{e-1} y_7 + y_5)^l.\end{aligned} \tag{91}$$

Thus, returning to (4):

$$y_1^n + (2^k y_2)^m \neq y_3^l,$$

without a common divisor, when $y_1 < (2^k y_2) < y_3$ in (4), $f = 1$ in (87) and $(2^{e-1} y_7 - y_5)^{n-1} < (y_8 + y_5)^{m-1}$ in (90).

The Beale's Conjecture for (II.E.8.1.1) is true.

II.*E*.8.1.2

Let in (90):

$$(2^{e-1} y_7 - y_5)^{n-1} > (y_8 + y_5)^{m-1}.$$

Calculating by the Addition Method (II.E.2.1.1), we obtain the following expression:

$$\begin{aligned}&(2^{e-1} y_7 + y_5)(y_8 + y_5)^{m-3}\Big((y_8 + y_5)2y_8 + \\ &+ (y_8 - y_5)^2\Big) + (y_8 - y_5)^3 (y_8 + y_5)^{m-3} + \\ &+ (2^{e-1} y_7 - y_5)\Bigg((2^{e-1} y_7 - y_5)^{n-1} - \\ &- (y_8 + y_5)^{m-3}\Big((y_8 + y_5)2y_8 + (y_8 - y_5)^2\Big)\Bigg) \neq \\ &\neq (2^{e-1} y_7 + y_5)^l.\end{aligned} \tag{92}$$

Thus, returning to (4):

$$y_1^n + (2^k y_2)^m \neq y_3^l,$$

without a common divisor, when $y_1 < (2^k y_2) < y_3$ in (4) and $f = 1$ in (87).

The Beale's Conjecture for (II.E.8.1) is true.

II.*E*.8.2 *The option when* $f > 1$ *in* (87)

Let $f > 1$ in (87). Then, according to (23) and (24):

$$y_4 - y_6 = 2^e y_8, \quad (93)$$

where $(e \geq 2) \in \mathbb{N}^*$,
and

$$y_4 + y_6 = 2y_7. \quad (94)$$

Substituting (93) and (94) into (87) and dividing the expressions in brackets by 2, we obtain the following expression:

$$(y_7 - 2^{f-1}y_5)^n + (2^{e-1}y_8 + 2^{f-1}y_5)^m = (y_7 + 2^{f-1}y_5)^l, \quad (95)$$

where:

$$2^{e-1}y_8 = \frac{y_4 + y_6}{2},$$
$$y_7 = \frac{y_4 - y_6}{2}.$$

II.*E*.8.2.1

Let in (95):

$$(y_7 - 2^{f-1}y_5)^{n-1} < (2^{e-1}y_8 + 2^{f-1}y_5)^{m-1}.$$

Calculating by the Addition Method (II.E.2.1.1), we obtain the following expression:

$$\begin{aligned}
&(y_7 + 2^{f-1}y_5)(y_7 - 2^{f-1}y_5)^{n-3}\Big((y_7 - 2^{f-1}y_5)\cdot\\
&\cdot(y_7 - 2^f y_5 + 2^{e-1}y_8) + (2^{e-1}y_8 - 2^{f-1}y_5)^2\Big)+\\
&+(2^{e-1}y_8 - 2^{f-1}y_5)^2\Big) + (2^{e-1}y_8 - 2^{f-1}y_5)^3\cdot\\
&\cdot(y_7 - 2^{f-1}y_5)^{n-3} + (2^{e-1}y_7 + 2^{f-1}y_5)\cdot\\
&\cdot\Bigg((2^{e-1}y_8 + 2^{f-1}y_5)^{m-1} - (y_7 - 2^{f-1}y_5)^{n-3}\cdot\\
&\cdot\Big((y_7 - 2^{f-1}y_5)(y_7 - 2^f y_5 + 2^{e-1}y_8) +\\
&+(2^{e-1}y_8 - 2^{f-1}y_5)^2\Big)\Bigg) \neq (y_7 + 2^{f-1}y_5)^l.
\end{aligned} \quad (96)$$

Thus, returning to (4):

$$y_1^n + (2^k y_2)^m \neq y_3^l,$$

without a common divisor, when $y_1 < (2^k y_2) < y_3$ in (4), $f > 1$ in (87) and $(y_7 - 2^{f-1}y_5)^{n-1} < (2^{e-1}y_8 + 2^{f-1}y_5)^{m-1}$ in (95).

The Beale's Conjecture for (II.E.8.2.1) is true.

II.*E*.8.2.2

Let in (95):

$$(y_7 - 2^{f-1}y_5)^{n-1} > (2^{e-1}y_8 + 2^{f-1}y_5)^{m-1}.$$

Calculating by the Addition Method (II.E.2.1.1), we obtain the following expression:

$$\begin{aligned}
&(y_7 + 2^{f-1}y_5)(2^{e-1}y_8 + 2^{f-1}y_5)^{m-3}\cdot\\
&\cdot\Big((2^{e-1}y_8 + 2^{f-1}y_5)2^e y_8 + (2^{e-1}y_8 - 2^{f-1}y_5)^2\Big)+\\
&+(2^{e-1}y_8 - 2^{f-1}y_5)^3(2^{e-1}y_8 + 2^{f-1}y_5)^{m-3} +\\
&+(y_7 - 2^{f-1}y_5)\Bigg((y_7 - 2^{f-1}y_5)^{n-1} -\\
&-(2^{e-1}y_8 + 2^{f-1}y_5)^{m-3}\Big((2^{e-1}y_8 + 2^{f-1}y_5)2^e y_8 +\\
&+(2^{e-1}y_8 - 2^{f-1}y_5)^2\Big)\Bigg) \neq (y_7 + 2^{f-1}y_5)^l.
\end{aligned} \quad (97)$$

Thus, returning to (4):

$$y_1^n + (2^k y_2)^m \neq y_3^l,$$

without a common divisor, when $y_1 < (2^k y_2) < y_3$ in (4).

The Beale's Conjecture for (II.E.8) is true.

II.*E*.9. The option of (4) without a divisor, when $(2^k y_2) < y_1 < y_3$

Let $(2^k y_2) < y_1 < y_3$ in (4). Then:

$$(y_3 - 2^f y_5)^n + (y_3 - y_6)^m = y_3^l, \quad (98)$$

where (27) is true and $f \in \mathbb{N}^*$.

Expression (98) coincides with (84), therefore, problem (II.E.9) is solved in the same way as (II.E.8).
Thus, returning to (4):

$$y_1^n + (2^k y_2)^m \neq y_3^l,$$

without a common divisor, when $(2^k y_2) < y_1 < y_3$ in (4).

The Beale's Conjecture for (II.E.9) is true.

II.*E*.10. The option of (4) without a divisor, when $y_3 < y_1 < (2^k y_2)$

Let $y_3 < y_1 < (2^k y_2)$ in (4). Then:

$$(y_3 + 2^f y_5)^n + (y_3 + y_6)^m = y_3^l, \quad (99)$$

where (27) is true and $f \in \mathbb{N}^*$.

Let's substitute the terms in brackets from (99) instead of y_1 and $2^k y_2$ in (83):

$$y_3 + 2^f y_5 + y_3 + y_6 = y_4. \quad (100)$$

Let's express y_3 from (100):

$$y_3 = \frac{y_4 - 2^f y_5 - y_6}{2}. \quad (101)$$

Let's substitute (101) into (99):

$$\left(\frac{(y_4-y_6)+2^f y_5}{2}\right)^n+\left(\frac{(y_4+y_6)-2^f y_5}{2}\right)^m= \\ =\left(\frac{(y_4-y_6)-2^f y_5}{2}\right)^l. \qquad (102)$$

II.*E*.10.1 *The option when* $f=1$ *in* (102)

Let $f=1$ in (102). Then, according to (23) and (24):

$$y_5+y_6=2y_8, \qquad (103)$$

and

$$y_5-y_6=2^e y_7, \qquad (104)$$

where $(e\geq 2)\in\mathbb{N}^*$.

Substituting (103) and (104) into (102) and dividing the expressions in brackets by 2, we obtain the following expression:

$$(2^{e-1}y_7+y_5)^n+(y_8-y_5)^m=(2^{e-1}y_7-y_5)^l, \qquad (105)$$

where:

$$y_8=\frac{y_4+y_6}{2}, \\ 2^{e-1}y_7=\frac{y_4-y_6}{2}.$$

II.*E*.10.1.1

Let in (105):

$$(2^{e-1}y_7+y_5)^{n-1}>(y_8-y_5)^{m-1}.$$

Calculating by the Addition Method (II.E.2.1.1), we obtain the following expression:

$$(2^{e-1}y_7-y_5)(y_8-y_5)^{m-3}\Big((y_8-y_5)\cdot \\ \cdot(2^{e-1}y_7+y_8)+(2^{e-1}y_7+y_5)^2\Big)+ \\ +(2^{e-1}y_7+y_5)^3(y_8-y_5)^{m-3}+(2^{e-1}y_7+y_5)\cdot \\ \cdot\Big((2^{e-1}y_7+y_5)^{n-1}-(y_8-y_5)^{m-3}\cdot \\ \cdot\Big((y_8-y_5)(2^{e-1}y_7+y_8)+(2^{e-1}y_7+y_5)^2\Big)\Big)\neq \\ \neq(2^{e-1}y_7-y_5)^l. \qquad (106)$$

Thus, returning to (4):

$$y_1^n+(2^k y_2)^m\neq y_3^l,$$

without a common divisor, when $y_3<y_1<(2^k y_2)$ in (4), $f=1$ in (102) and $(2^{e-1}y_7+y_5)^{n-1}>(y_8-y_5)^{m-1}$ in (105).

The Beale's Conjecture for (II.E.10.1.1) is true.

II.*E*.10.1.2

Let in (105):

$$(2^{e-1}y_7+y_5)^{n-1}<(y_8-y_5)^{m-1}.$$

Calculating by the Addition Method (II.E.2.1.1), we obtain the following expression:

$$(2^{e-1}y_7-y_5)(2^{e-1}y_7+y_5)^{n-3}\Big((2^{e-1}y_7+y_5)\cdot \\ \cdot(2^{e-1}y_7+2y_5+y_8)+(y_8+y_5)^2\Big)+(y_8+y_5)^3\cdot \\ \cdot(2^{e-1}y_7+y_5)^{n-3}+(y_8-y_5)\Big((y_8-y_5)^{m-1}- \\ -(2^{e-1}y_7+y_5)^{n-3}\Big((2^{e-1}y_7+y_5)\cdot \\ \cdot(2^{e-1}y_7+2y_5+y_8)+(y_8+y_5)^2\Big)\Big)\neq \\ \neq(2^{e-1}y_7-y_5)^l. \qquad (107)$$

Thus, returning to (4):

$$y_1^n+(2^k y_2)^m\neq y_3^l,$$

without a common divisor, when $y_3<y_1<(2^k y_2)$ in (4) and $f=1$ in (102).

The Beale's Conjecture for (II.E.10.1) is true.

II.*E*.10.2 *The option when* $f>1$ *in* (102)

Let $f>1$ in (102). Then, according to (21) and (22):

$$y_4+y_6=2^e y_8, \qquad (108)$$

where $(e\geq 2)\in\mathbb{N}^*$,
and

$$y_4-y_6=2y_7. \qquad (109)$$

Substituting (108) and (109) into (102) and dividing the expressions in brackets by 2, we obtain the following expression:

$$(y_7+2^{f-1}y_5)^n+(2^{e-1}y_8-2^{f-1}y_5)^m= \\ =(y_7-2^{f-1}y_5)^l, \qquad (110)$$

where:

$$2^{e-1}y_8=\frac{y_4+y_6}{2}, \\ y_7=\frac{y_4-y_6}{2}.$$

II.*E*.10.2.1

Let in (110):

$$(y_7+2^{f-1}y_5)^{n-1}>(2^{e-1}y_8-2^{f-1}y_5)^{m-1}.$$

Calculating by the Addition Method (II.E.2.1.1), we obtain the following expression:

$$\begin{aligned}&(y_7-2^{f-1}y_5)(2^{e-1}y_8-2^{f-1}y_5)^{m-3}\cdot\\&\cdot\Big((2^{e-1}y_8-2^{f-1}y_5)2^e y_8+\\&+(2^{e-1}y_8+2^{f-1}y_5)^2\Big)+(2^{e-1}y_8+2^{f-1}y_5)^3\cdot\\&\cdot(2^{e-1}y_8-2^{f-1}y_5)^{m-3}+(y_7+2^{f-1}y_5)\cdot\\&\cdot\Big((y_7+2^{f-1}y_5)^{n-1}-(2^{e-1}y_8-2^{f-1}y_5)^{m-3}\cdot\\&\cdot\Big((2^{e-1}y_8-2^{f-1}y_5)2^e y_8+\\&+(2^{e-1}y_8+2^{f-1}y_5)^2\Big)\Big)\neq(y_7-2^{f-1}y_5)^l.\end{aligned}\tag{111}$$

Thus, returning to (4):

$$y_1^n+(2^k y_2)^m\neq y_3^l,$$

without a common divisor, when $y_3<y_1<(2^k y_2)$ in (4), $f>1$ in (102) and $(y_7+2^{f-1}y_5)^{n-1}>(2^{e-1}*y_8-2^{f-1}y_5)^{m-1}$ in (110).

The Beale's Conjecture for (II.E.10.2.1) is true.

II.*E*.10.2.2

Let in (110):

$$(y_7+2^{f-1}y_5)^{n-1}<(2^{e-1}y_8-2^{f-1}y_5)^{m-1}.$$

Calculating by the Addition Method (II.E.2.1.1), we obtain the following expression:

$$\begin{aligned}&(y_7-2^{f-1}y_5)(y_7+2^{f-1}y_5)^{n-3}\Big((y_7+2^{f-1}y_5)\cdot\\&\cdot(2^{e-1}y_8+2^f y_5+y_7)+(2^{e-1}y_8+2^{f-1}y_5)^2\Big)+\\&+(2^{e-1}y_8+2^{f-1}y_5)^3(y_7+2^{f-1}y_5)^{n-3}+\\&+(2^{e-1}y_8-2^{f-1}y_5)\Big((2^{e-1}y_8-2^{f-1}y_5)^{m-1}-\\&-(y_7+2^{f-1}y_5)^{n-3}\Big((y_7+2^{f-1}y_5)\cdot\\&\cdot(2^{e-1}y_8+2^f y_5+y_7)+(2^{e-1}y_8+2^{f-1}y_5)^2\Big)\Big)\neq\\&\neq(y_7-2^{f-1}y_5)^l.\end{aligned}\tag{112}$$

Thus, returning to (4):

$$y_1^n+(2^k y_2)^m\neq y_3^l,$$

without a common divisor, when $y_3<y_1<(2^k y_2)$ in (4).

The Beale's Conjecture for (II.E.10) is true.

II.*E*.11. <u>The option of (4) without a divisor, when $y_3<(2^k y_2)<y_1$</u>

Let $y_3<(2^k y_2)<y_1$ in (4). Then:

$$(y_3+2^f y_5)^n+(y_3+y_6)^m=y_3^l,\tag{113}$$

where (27) is true and $f\in\mathbb{N}^*$.

Expression (113) coincides with (99), therefore, problem (II.E.11) is solved in the same way as (II.E.10).
Thus, returning to (4):

$$y_1^n+(2^k y_2)^m\neq y_3^l,$$

without a common divisor, when $y_3<(2^k y_2)<y_1$ in (4).

The Beale's Conjecture for (II.E.11) is true.

II.*E*.12. <u>The option of (4) without a divisor, when $y_1<y_3<(2^k y_2)$</u>

Let $y_1<y_3<(2^k y_2)$ in (4). Then:

$$(y_3-2^f y_5)^n+(y_3+y_6)^m=y_3^l,\tag{114}$$

where (27) is true and $f\in\mathbb{N}^*$.

Let's substitute the terms in brackets from (114) instead of y_1 and $(2^k y_2)$ in (83):

$$y_3-2^f y_5+y_3+y_6=y_4.\tag{115}$$

Let's express y_3 from (115):

$$y_3=\frac{y_4+2^f y_5-y_6}{2}.\tag{116}$$

Let's substitute (116) into (114):

$$\begin{aligned}&\left(\frac{(y_4-y_6)-2^f y_5}{2}\right)^n+\left(\frac{(y_4+y_6)+2^f y_5}{2}\right)^m=\\&=\left(\frac{(y_4-y_6)+2^f y_5}{2}\right)^l.\end{aligned}\tag{117}$$

II.*E*.12.1 <u>*The option when* $f=1$ *in* (117)</u>

Let $f=1$ in (117). Then, according to (23) and (24):

$$y_4+y_6=2y_8,\tag{118}$$

and

$$y_4-y_6=2^e y_7,\tag{119}$$

where $(e\geq 2)\in\mathbb{N}^*$.

Substituting (118) and (119) into (117) and dividing the expressions in brackets by 2, we obtain the following expression:

$$(2^{e-1}y_7-y_5)^n+(y_8+y_5)^m=(2^{e-1}y_7-y_5)^l,\tag{120}$$

where:

$$y_8=\frac{y_4+y_6}{2},$$
$$2^{e-1}y_7=\frac{y_4-y_6}{2}.$$

II.*E*.12.1.1

Let in (120):

$$(2^{e-1}y_7 - y_5)^{n-1} < (y_8 + y_5)^{m-1}.$$

Calculating by the Addition Method (II.E.2.1.1), we obtain the following expression:

$$\begin{aligned}&(2^{e-1}y_7 + y_5)(2^{e-1}y_7 - y_5)^{n-3}\Big((2^{e-1}y_7 - y_5)\cdot\\&\cdot(2^{e-1}y_7 + y_8 - 2y_5) + (y_8 - y_5)^2\Big) + (y_8 - y_5)^3\cdot\\&\cdot(2^{e-1}y_7 - y_5)^{n-3} + (y_8 + y_5)\Big((y_8 + y_5)^{m-1} -\\&-(2^{e-1}y_7 - y_5)^{n-3}\Big((2^{e-1}y_7 - y_5)\cdot\\&\cdot(2^{e-1}y_7 + y_8 - 2y_5) + (y_8 - y_5)^2\Big)\Big) \neq\\&\neq (2^{e-1}y_7 + y_5)^l.\end{aligned} \tag{121}$$

Thus, returning to (4):

$$y_1^n + (2^k y_2)^m \neq y_3^l,$$

without a common divisor, when $y_1 < y_3 < (2^k y_2)$ in (4), $f = 1$ in (117) and $(2^{e-1}y_7 - y_5)^{n-1} < (y_8 + y_5)^{m-1}$ in (120).

The Beale's Conjecture for (II.E.12.1.1) is true.

II.*E*.12.1.2

Let in (120):

$$(2^{e-1}y_7 - y_5)^{n-1} > (y_8 + y_5)^{m-1}.$$

Calculating by the Addition Method (II.E.2.1.1), we obtain the following expression:

$$\begin{aligned}&(2^{e-1}y_7 + y_5)(y_8 + y_5)^{m-3}\Big((y_8 + y_5)2y_8 +\\&+(y_8 - y_5)^2\Big) + (y_8 - y_5)^3(y_8 + y_5)^{m-3} +\\&+(2^{e-1}y_7 - y_5)\Big((2^{e-1}y_7 - y_5)^{n-1} -\\&-(y_8 + y_5)^{m-3}\Big((y_8 + y_5)2y_8 + (y_8 - y_5)^2\Big)\Big) \neq\\&\neq (2^{e-1}y_7 + y_5)^l.\end{aligned} \tag{122}$$

Thus, returning to (4):

$$y_1^n + (2^k y_2)^m \neq y_3^l,$$

without a common divisor, when $y_1 < y_3 < (2^k y_2)$ in (4) and $f = 1$ in (117).

The Beale's Conjecture for (II.E.12.1) is true.

II.*E*.12.2 <u>*The option when* $f > 1$ *in* (117)</u>

Let $f > 1$ in (117). Then, according to (21) and (22):

$$y_4 + y_6 = 2^e y_8, \tag{123}$$

where $(e \geq 2) \in \mathbb{N}^*$,
and

$$y_4 - y_6 = 2y_7. \tag{124}$$

Substituting (123) and (124) into (117) and dividing the expressions in brackets by 2, we obtain the following expression:

$$\begin{aligned}&(y_7 - 2^{f-1}y_5)^n + (2^{e-1}y_8 + 2^{f-1}y_5)^m =\\&= (y_7 + 2^{f-1}y_5)^l,\end{aligned} \tag{125}$$

where:

$$2^{e-1}y_8 = \frac{y_4 + y_6}{2},$$
$$y_7 = \frac{y_4 - y_6}{2}.$$

II.*E*.12.2.1

Let in (125):

$$(y_7 - 2^{f-1}y_5)^{n-1} < (2^{e-1}y_8 + 2^{f-1}y_5)^{m-1}.$$

Calculating by the Addition Method (II.E.2.1.1), we obtain the following expression:

$$\begin{aligned}&(y_7 + 2^{f-1}y_5)(y_7 - 2^{f-1}y_5)^{n-3}\Big((y_7 - 2^{f-1}y_5)\cdot\\&\cdot(2^{e-1}y_8 + y_7 - 2^f y_5) + (2^{e-1}y_8 - 2^{f-1}y_5)^2\Big) +\\&+(2^{e-1}y_8 - 2^{f-1}y_5)^3(y_7 - 2^{f-1}y_5)^{n-3} +\\&+(2^{e-1}y_8 + 2^{f-1}y_5)\Big((2^{e-1}y_8 + 2^{f-1}y_5)^{m-1} -\\&-(2^{e-1}y_7 - y_5)^{n-3}\Big((2^{e-1}y_7 - y_5)\cdot\\&\cdot(2^{e-1}y_8 + y_7 - 2^f y_5) + (2^{e-1}y_8 - 2^{f-1}y_5)^2\Big)\Big) \neq\\&\neq (y_7 + 2^{f-1}y_5)^l.\end{aligned} \tag{126}$$

Thus, returning to (4):

$$y_1^n + (2^k y_2)^m \neq y_3^l,$$

without a common divisor, when $y_1 < y_3 < (2^k y_2)$ in (4), $f > 1$ in (117) and $(y_7 - 2^{f-1}y_5)^{n-1} < (2^{e-1}y_8 + 2^{f-1}y_5)^{m-1}$ in (125).

The Beale's Conjecture for (II.E.12.2.1) is true.

II.*E*.12.2.2

Let in (125):

$$(y_7 - 2^{f-1}y_5)^{n-1} > (2^{e-1}y_8 + 2^{f-1}y_5)^{m-1}.$$

Calculating by the Addition Method (II.E.2.1.1), we obtain the following expression:

$$
\begin{aligned}
&(y_7+2^{f-1}y_5)(2^{e-1}y_8+2^{f-1}y_5)^{m-3}\cdot\\
&\cdot\Big((2^{e-1}y_8+2^{f-1}y_5)2^ey_8+\\
&+(2^{e-1}y_8-2^{f-1}y_5)^2\Big)+(2^{e-1}y_8-2^{f-1}y_5)^2\Big)+\\
&+(2^{e-1}y_8-2^{f-1}y_5)^3(2^{e-1}y_8+2^{f-1}y_5)^{m-3}+\\
&+(y_7-2^{f-1}y_5)\Big((y_7-2^{f-1}y_5)^{n-1}-\\
&-(2^{e-1}y_8+2^{f-1}y_5)^{m-3}\Big((2^{e-1}y_8+2^{f-1}y_5)\cdot\\
&\cdot 2^ey_8++(2^{e-1}y_8-2^{f-1}y_5)^2\Big)\Big)\neq(y_7+2^{f-1}y_5)^l.
\end{aligned}
\tag{127}
$$

Thus, returning to (4):

$$y_1^n+(2^ky_2)^m\neq y_3^l,$$

without a common divisor, when $y_1<y_3<(2^ky_2)$ in (4).

The Beale's Conjecture for (II.E.12) is true.

II.*E*.13. <u>The option of (4) without a divisor, when $(2^ky_2)<y_3<y_1$</u>

Let $(2^ky_2)<y_3<y_1$ in (4). Then:

$$(y_3+2^fy_5)^n+(y_3-y_6)^m=y_3^l, \tag{128}$$

where (27) is true and $f\in\mathbb{N}^*$.

Let's substitute the terms in brackets from (128) instead of y_1 and (2^ky_2) in (83):

$$y_3+2^fy_5+y_3-y_6=y_4. \tag{129}$$

Let's express y_3 from (129):

$$y_3=\frac{y_4-2^fy_5+y_6}{2}. \tag{130}$$

Let's substitute (130) into (128):

$$
\begin{aligned}
&\left(\frac{(y_4+y_6)+2^fy_5}{2}\right)^n+\left(\frac{(y_4-y_6)-2^fy_5}{2}\right)^m=\\
&=\left(\frac{(y_4+y_6)-2^fy_5}{2}\right)^l.
\end{aligned}
\tag{131}
$$

II.*E*.13.1 <u>*The option when $f=1$ in* (131)</u>

Let $f=1$ in (131). Then, according to (23) and (24):

$$y_4-y_6=2y_8, \tag{132}$$

and

$$y_4-y_6=2^ey_7, \tag{133}$$

where $(e\geq 2)\in\mathbb{N}^*$.

Substituting (132) and (133) into (131) and dividing the expressions in brackets by 2, we obtain the following expression:

$$(2^{e-1}y_7+y_5)^n+(y_8-y_5)^m=(2^{e-1}y_7-y_5)^l, \tag{134}$$

where:

$$
\begin{aligned}
y_8&=\frac{y_4-y_6}{2},\\
2^{e-1}y_7&=\frac{y_4+y_6}{2}.
\end{aligned}
$$

II.*E*.13.1.1

Let in (134):

$$(2^{e-1}y_7+y_5)^{n-1}>(y_8-y_5)^{m-1}.$$

Calculating by the Addition Method (II.E.2.1.1), we obtain the following expression:

$$
\begin{aligned}
&(2^{e-1}y_7-y_5)(y_8-y_5)^{m-3}\Big((y_8-y_5)2y_8+\\
&+(y_8+y_5)^2\Big)+(y_8+y_5)^3(y_8-y_5)^{m-3}+\\
&+(2^{e-1}y_7+y_5)\Big((2^{e-1}y_7+y_5)^{n-1}-\\
&-(y_8-y_5)^{m-3}\Big((y_8-y_5)2y_8+(y_8+y_5)^2\Big)\Big)\neq\\
&\neq(2^{e-1}y_7-y_5)^l.
\end{aligned}
\tag{135}
$$

Thus, returning to (4):

$$y_1^n+(2^ky_2)^m\neq y_3^l,$$

without a common divisor, when $(2^ky_2)<y_3<y_1$ in (4), $f=1$ in (131) and $(2^{e-1}y_7+y_5)^{n-1}>(y_8-y_5)^{m-1}$ in (134).

The Beale's Conjecture for (II.E.13.1.1) is true.

II.*E*.13.1.2

Let in (134):

$$(2^{e-1}y_7+y_5)^{n-1}<(y_8-y_5)^{m-1}.$$

Calculating by the Addition Method (II.E.2.1.1), we obtain the following expression:

$$
\begin{aligned}
&(2^{e-1}y_7-y_5)(2^{e-1}y_7+y_5)^{n-3}\Big((2^{e-1}y_7+y_5)\cdot\\
&\cdot(2^{e-1}y_7+y_8+2y_5)+(y_8+y_5)^2\Big)+(y_8+y_5)^3\cdot\\
&\cdot(2^{e-1}y_7+y_5)^{n-3}+(2^{e-1}y_7+y_5)\cdot\\
&\cdot\Big((2^{e-1}y_7+y_5)^{m-1}-(2^{e-1}y_7+y_5)^{n-3}\cdot\\
&\cdot\Big((2^{e-1}y_7+y_5)(2^{e-1}y_7+y_8+2y_5)+\\
&+(y_8+y_5)^2\Big)\Big)\neq(2^{e-1}y_7-y_5)^l.
\end{aligned}
\tag{136}
$$

Thus, returning to (4):

$$y_1^n + (2^k y_2)^m \neq y_3^l,$$

without a common divisor, when $(2^k y_2) < y_3 < y_1$ in (4) and $f = 1$ in (131).

The Beale's Conjecture for (II.E.13.1) is true.

II.*E*.13.2 *The option when $f > 1$ in (131)*

Let $f > 1$ in (131). Then, according to (21) and (22):

$$y_4 - y_6 = 2^e y_8, \tag{137}$$

where $(e \geq 2) \in \mathbb{N}^*$,
and

$$y_4 + y_6 = 2y_7. \tag{138}$$

Substituting (137) and (138) into (131) and dividing the expressions in brackets by 2, we obtain the following expression:

$$(y_7 + 2^{f-1}y_5)^n + (2^{e-1}y_8 - 2^{f-1}y_5)^m = (y_7 - 2^{f-1}y_5)^l, \tag{139}$$

where:

$$2^{e-1}y_8 = \frac{y_4 - y_6}{2},$$
$$y_7 = \frac{y_4 + y_6}{2}.$$

II.*E*.13.2.1

Let in (139):

$$(y_7 + 2^{f-1}y_5)^{n-1} < (2^{e-1}y_8 - 2^{f-1}y_5)^{m-1}.$$

Calculating by the Addition Method (II.E.2.1.1), we obtain the following expression:

$$\begin{aligned}
&(y_7 - 2^{f-1}y_5)(2^{e-1}y_8 - 2^{f-1}y_5)^{m-3} \cdot \\
&\cdot \Big((2^{e-1}y_8 - 2^{f-1}y_5)2^e y_8 + (2^{e-1}y_8 + 2^{f-1}y_5)^2\Big) + \\
&+ (2^{e-1}y_8 + 2^{f-1}y_5)^3 (2^{e-1}y_8 - 2^{f-1}y_5)^{m-3} + \\
&+ (y_7 + 2^{f-1}y_5)\Bigg((y_7 + 2^{f-1}y_5)^{n-1} - \\
&- (2^{e-1}y_8 - y_5)^{m-3}\Big((2^{e-1} * y_8 - y_5)2^e y_8 + \\
&+ (2^{e-1}y_8 + 2^{f-1}y_5)^2\Big)\Bigg) \neq (y_7 - 2^{f-1}y_5)^l.
\end{aligned} \tag{140}$$

Thus, returning to (4):

$$y_1^n + (2^k y_2)^m \neq y_3^l,$$

without a common divisor, when $(2^k y_2) < y_3 < y_1$ in (4), $f > 1$ in (131) and $(y_7 + 2^{f-1}y_5)^{n-1} < (2^{e-1}y_8 - 2^{f-1}y_5)^{m-1}$ in (139).

The Beale's Conjecture for (II.E.13.2.1) is true.

II.*E*.13.2.2

Let in (139):

$$(y_7 + 2^{f-1}y_5)^{n-1} < (2^{e-1}y_8 - 2^{f-1}y_5)^{m-1}.$$

Calculating by the Addition Method (II.E.2.1.1), we obtain the following expression:

$$\begin{aligned}
&(y_7 - 2^{f-1}y_5)(y_7 + 2^{f-1}y_5)^{n-3}\Big((y_7 + 2^{f-1}y_5) \cdot \\
&\cdot (2^{e-1}y_8 + y_7 + 2^f * y_5) + (2^{e-1}y_8 + 2^{f-1}y_5)^2\Big) + \\
&+ (2^{e-1}y_8 + 2^{f-1}y_5)^3 (y_7 + 2^{f-1}y_5)^{n-3} + \\
&+ (2^{e-1}y_8 - 2^{f-1}y_5)\Bigg((2^{e-1}y_8 - 2^{f-1}y_5)^{m-1} - \\
&- (y_7 + 2^{f-1}y_5)^{n-3}\Big((y_7 + 2^{f-1}y_5) \cdot \\
&\cdot (2^{e-1}y_8 + y_7 + 2^f y_5) + (2^{e-1}y_8 + 2^{f-1}y_5)^2\Big)\Bigg) \neq \\
&\neq (y_7 - 2^{f-1}y_5)^l.
\end{aligned} \tag{141}$$

Thus, returning to (4):

$$y_1^n + (2^k y_2)^m \neq y_3^l,$$

without a common divisor.

The Beale's Conjecture for (II-E) is true.

Given the conclusions of (II-A), (II-C), (II-D) and (II-E), the Beale's Conjecture is proved.

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REFERENCES