AN INTRODUCTION TO MULTIVARIATE EXPANSION

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Abstract. We introduce the notion of an expansion in a specified and mixed directions. This is a piece of an extension program of **single variable expansivity theory** developed by the author.

1. **Introduction**

The notion of a **single variable expansion** had been developed earlier on by the author [2]. This notion surprisingly turns out to be useful in studying the sendov conjecture. For the paper detailing this study, see ([1]). This theory also has a wide range of applications in determining the insolubility of certain systems of single variable differential equations. In the current paper, We launch an extension program where the problem is studied in the ring $\mathbb{C}[x_1, x_2, \ldots, x_n]$ with the complex base field $\mathbb{C}$. It turns out that the basic notions under study in the single variable mostly carry over to this setting. As an application, we obtain one of the many potential results

Theorem 1.1. *Let $\mathcal{F} := \{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of the ring $\mathbb{C}[x_1, x_2, \ldots, x_n]$. Let $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(\mathcal{S})$ for all $1 \leq i \leq n$ be destabilized stage $k \geq 1$ and*

$$\Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(\mathcal{S})] = \Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S})] = w$$

for all $1 \leq i, j \leq n$. Consider the function

$$\overrightarrow{O\mathcal{S}} \cdot \overrightarrow{O\mathcal{S}_e} : \mathbb{R}^n \longrightarrow \mathcal{N} \subset \mathbb{C}.$$

Then any $h \in \mathcal{N}$ has the representation

$$h := a_1^w + a_2^w + \cdots + a_n^w.$$

Throughout this paper, we keep the usual standard notion $\mathcal{S}$ for all tuples whose entries belong to the ring $\mathbb{C}[x_1, x_2, \ldots, x_n]$. Occasionally we might choose to index these tuples by $\mathcal{S}_j$ over the natural numbers $\mathbb{N}$ if we have two or more and we want to keep them distinct from each other. The tuples $\mathcal{S}_0 = (0, 0 \ldots, 0)$ and $\mathcal{S}_e = (1, 1, \ldots, 1)$ are still reserved for the null and the unit tuple respectively.

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2. **Expansion in mixed and specified directions**

In this section we introduce the notion of an expansion in a mixed and a specified direction. We launch the following languages.

Definition 2.1. Let $\mathcal{F} := \{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of $\mathbb{C}[x_1, x_2, \ldots, x_n]$. Then by an expansion on $\mathcal{F} := \{\mathcal{S}_i\}_{i=1}^{\infty}$ in the direction x_i for $1 \leq i \leq n$, we mean the composite map

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]} : \mathcal{F} \longrightarrow \mathcal{F},$$

where

$$\gamma(\mathcal{S}) = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} \quad \text{and} \quad \beta(\gamma(\mathcal{S})) = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & \cdots & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}$$

with

$$\nabla_{[x_i]} = \left(\frac{\partial f_1}{\partial x_i}, \frac{\partial f_2}{\partial x_i}, \ldots, \frac{\partial f_n}{\partial x_i}\right).$$

The value of the lth expansion at a given value a of x_i is given by

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{l}_{[x_i](a)} \in \mathbb{C}[x_1, \ldots x_{i-1}, x_{i+1}, \ldots, x_n].$$

Similarly by an expansion in the mixed direction we mean

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=2}^{l}[x_{\sigma(i)}]} \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{\sigma(1)}]}$$

for some permutation $\sigma : \{1, 2, \ldots, l\} \longrightarrow \{1, 2, \ldots, l\}$. The value of this expansion on a given value a_i of $x_{\sigma(i)}$ for all $\sigma(1) \leq i \leq \sigma(l)$ is given by

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}](a_i)}(\mathcal{S}) \in \mathbb{C}.$$

Remark 2.2. Next we prove a fundamental result which shows that an expansion is commutative. This reinforces the very notion that there is no need to give precedence to the direction of an expansion. In essence, it gives some flexibility to the way and manner an expansion could be carried out.

Proposition 2.1. *An expansion is commutative.*

Proof. Consider $\mathcal{F} := \{\mathcal{S}_i\}_{i=1}^{\infty}$ the collection of tuples of the ring $\mathbb{C}[x_1, x_2, \ldots, x_n]$. It suffices to show that for any $\mathcal{S} \in \mathcal{F}$ then the relation

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i] \otimes [x_j]}(\mathcal{S}) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j] \otimes [x_i]}(\mathcal{S})$$

is valid. We observe that

$$\begin{aligned}(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]\otimes[x_i]}(\mathcal{S}) &= \left(\frac{\partial}{\partial x_j}\left(\sum_{\substack{t\in[1,n]\\t\neq 1}}\sum_{k\neq t}\frac{\partial f_k}{\partial x_i}\right),\ldots,\frac{\partial}{\partial x_j}\left(\sum_{\substack{t\in[1,n]\\t\neq n}}\sum_{k\neq t}\frac{\partial f_k}{\partial x_i}\right)\right)\\
&= \left(\sum_{\substack{t\in[1,n]\\t\neq 1}}\sum_{k\neq t}\frac{\partial^2 f_k}{\partial x_j\partial x_i},\ldots,\sum_{\substack{t\in[1,n]\\t\neq n}}\sum_{k\neq t}\frac{\partial^2 f_k}{\partial x_j\partial x_i}\right)\\
&= \left(\sum_{\substack{t\in[1,n]\\t\neq 1}}\sum_{k\neq t}\frac{\partial^2 f_k}{\partial x_i\partial x_j},\ldots,\sum_{\substack{t\in[1,n]\\t\neq n}}\sum_{k\neq t}\frac{\partial^2 f_k}{\partial x_i\partial x_j}\right)\\
&= \left(\frac{\partial}{\partial x_i}\left(\sum_{\substack{t\in[1,n]\\t\neq 1}}\sum_{k\neq t}\frac{\partial f_k}{\partial x_j}\right),\ldots,\frac{\partial}{\partial x_i}\left(\sum_{\substack{t\in[1,n]\\t\neq n}}\sum_{k\neq t}\frac{\partial f_k}{\partial x_j}\right)\right)\\
&= (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]\otimes[x_j]}(\mathcal{S})\end{aligned}$$

since each entry of the tuple is contained in the polynomial ring $\mathbb{C}[x_1,x_2,\ldots,x_n]$. □

3. The totient and the residue of an expansion

In this section we introduce the notion of the residue and the totient of an expansion. These two notions are analogous to the notion of the rank and the degree of an expansion under the single variable theory. We launch more formally the following languages.

Definition 3.1. Let $\mathcal{F}=\{\mathcal{S}_i\}_{i=1}^{\infty}$ be the collection of tuples of the ring $\mathbb{C}[x_1,x_2,\ldots,x_n]$. Then we say the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S})$ is free with totient k, denoted $\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S})]$, if

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k}_{[x_i]}(\mathcal{S})=\mathcal{S}_0.$$

We call the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k-1}_{[x_i]}(\mathcal{S})$ the residue of the expansion, denoted $\Theta[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S})]$. Similarly we say a mixed expansion is free with totient k, denoted $\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})]$ if

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}\otimes_{l=1}^{k}[x_{\sigma(i)}]}(\mathcal{S})=\mathcal{S}_0$$

for $k\geq l$.

3.1. The dropler effect induced by an expansion.

In this section we introduce the notion of the dropler effect induced by an expansion. This phenomena is mostly induced by expansion on several other expansions in a specific direction.

Definition 3.2. Let $\mathcal{F}:=\{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of the ring $\mathbb{C}[x_1,x_2,\ldots,x_n]$. Then the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})$ is said to induce a dropler effect with intensity k, denoted $\mathcal{I}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S})]$ on the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S})$ if

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k}_{[x_j]}\circ(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})$$

is free with $k<\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S})]$ and k is the smallest such number. In other words, we say the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S})$ admits a dropler effect from the source $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})$ with intensity k. The energy $\mathbb{E}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S})]$ saved by the expansion under the dropler effect is given by

$$\mathbb{E}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S})]=\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S})]-\mathcal{I}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S})].$$

This is the energy-dropler effect intensity equation.

Proposition 3.1. *Let $\mathcal{F}:=\{\mathcal{S}_i\}_{i=1}^{\infty}$ be the collection of tuples of the ring $\mathbb{C}[x_1,x_2,\ldots,x_n]$. If the expansions $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_s]}(\mathcal{S})$ and $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_t]}(\mathcal{S})$ each admits a dropler effect from the same source with intensity k_1 and k_2, respectively, then the expansion*

$$[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_s]}+(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_t]}](\mathcal{S})$$

admits a dropler effect from the same source with intensity $\max\{k_1,k_2\}$.

Proof. Suppose the expansions $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_s]}(\mathcal{S})$ and $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_t]}(\mathcal{S})$ each admits a dropler effect from the same source with intensity k_1 and k_2, respectively, and let $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})$ be the source. Then it follows from Definition 3.2

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k_1}_{[x_s]}\circ(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})=\mathcal{S}_0$$

and

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k_2}_{[x_t]}\circ(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})=\mathcal{S}_0$$

with $k_1<\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_s]}(\mathcal{S})]$ and $k_2<\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_t]}(\mathcal{S})]$. By replacing k_1 and k_2 with $\max\{k_1,k_2\}<\Phi[\{(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_s]}+(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_t]}\}(\mathcal{S})]$, we see that

$$\left[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{\max\{k_1,k_2\}}_{[x_s]}+(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{\max\{k_1,k_2\}}_{[x_t]}\right]\circ(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})=\mathcal{S}_0$$

and it is easy to see that

$$\left[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{\max\{k_1,k_2\}-r}_{[x_s]}+(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{\max\{k_1,k_2\}-z}_{[x_t]}\right]\circ(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})\neq\mathcal{S}_0$$

for any $r,z\geq 1$. This completes the proof. □

3.2. **Destabilization of an expansion.** In this section we introduce the notion of destabilization induced by an expansion. This notion will form an essential toolbox in proving some result in this sequel. We launch more formally the following languages.

Definition 3.3. Let $\mathcal{F}:=\{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of $\mathbb{C}[x_1,x_2,\ldots,x_n]$. Then we say the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S})$ is said to undergo natural destabilization if $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{0}_{[x_i](0)}(\mathcal{S})\neq\mathcal{S}_0$. We say the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S})$ is said to undergo a destabilization at stage $k\geq 1$ if $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{j}_{[x_i](0)}(\mathcal{S})=\mathcal{S}_0$ for all $1\leq j\leq k-1$ and $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k}_{[x_i](0)}(\mathcal{S})\neq\mathcal{S}_0$. In other words, we say the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S})$ admits a destabilization at stage $k\geq 0$. We say it is strongly destabilized if $\mathrm{Id}_m[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k}_{[x_i](0)}(\mathcal{S})]\neq 0$ for all $1\leq m\leq n$.

Remark 3.4. Next we prove a result that tells us that destabilization should by necessity happen in an expansion. The following result confines this stage to a certain range.

Proposition 3.2. *Let $\mathcal{F} := \{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of the ring $\mathbb{C}[x_1, x_2, \ldots, x_n]$. Then the stage of destabilization $k \geq 0$ of the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(\mathcal{S})$ must satisfy the inequality*

$$0 \leq k < \Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(\mathcal{S})].$$

Proof. If the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(\mathcal{S})$ admits a natural destabilization, then the stage $k = 0$. Thus we may assume that the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(\mathcal{S})$ do not admit a natural destabilization. Suppose on the contrary that the stage of destabilization satisfies $k \geq \Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(\mathcal{S})]$. Then it follows that

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i](0)}^{\Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(\mathcal{S})]-1}(\mathcal{S}) = \mathcal{S}_0.$$

This is absurd since the expansion

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}^{\Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(\mathcal{S})]-1}(\mathcal{S})$$

is the residue of the expansion in the direction x_i. □

Remark 3.5. Next we relate the dropler effect of an expansion with a given intensity to the stage of destabilization in the following proposition.

Proposition 3.3. *Let $\mathcal{F} := \{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of $\mathbb{C}[x_1, x_2, \ldots, x_n]$. Suppose the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S})$ admits a dropler effect from the source $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})$. Then the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S})$ is destabilized at stage $\mathbb{E}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S})]$ if and only if*

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}](0)}(\mathcal{S}) \neq \mathcal{S}_0$$

and

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}^{-t} \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}](0)}(\mathcal{S}) = \mathcal{S}_0$$

for all $1 \leq t \leq \mathbb{E}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S})] - 1$.

Proof. Pick $\mathcal{S} \in \mathcal{F}$ and suppose the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S})$ admits a dropler effect from the source $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})$. Then by definition 3.2, It follows that

$$\begin{aligned}(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}^{\mathcal{I}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S})]} \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}) &= \mathcal{S}_0 \\ &= (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}^{\Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S})]}.\end{aligned}$$

It follows that

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}^{\Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S})] - \mathcal{I}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S})]}$$

since an expansion is a bijective map. Since

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}](0)}(\mathcal{S}) \neq \mathcal{S}_0$$

and

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{-t}_{[x_j]}\circ(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}](0)}(\mathcal{S})=\mathcal{S}_0$$

for all $1\leq t\leq \mathbb{E}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S})]-1$, the result follows by appealing to definition 3.3. □

Next we expose an important relationship that exists between the totient of the mixed expansion and the underlying expansion in specific directions. One could view this result as a sub-additivity property of the totient of an expansion.

Proposition 3.4. *Let $\mathcal{F}=:\{\mathcal{S}_i\}_{i=1}^{\infty}$ be the collection of tuples of the ring $\mathbb{C}[x_1,x_2,\ldots,x_n]$. Then we have*

$$\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})]\leq\sum_{i=1}^{\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})]}\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(i)}]}(\mathcal{S})].$$

4. **Diagonalization and sub-expansion of an expansion**

In this section we introduce the notion of diagonalization of an expansion and sub-expansion of an expansion. This notion is mostly applied to expansions in mixed directions. We launch the following languages to ease our work.

Definition 4.1. Let $\mathcal{F}:=\{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of the ring $\mathbb{C}[x_1,x_2,\ldots,x_n]$. Then for any $\mathcal{S}\in\mathcal{F}$, we say the mixed expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})$ is diagonalizable in the direction x_j $(1\leq j\leq n)$ of order k at the spot $\mathcal{S}_j\in\mathcal{F}$ with $\mathcal{S}\neq\mathcal{S}_j$ if

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k}_{[x_j]}(\mathcal{S}_j).$$

Remark 4.2. Next we launch the notion of the sub-expansion of an expansion. The same notion under the **single variable theory** still carries over to this setting.

Definition 4.3. Let $\mathcal{F}:=\{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of the ring $\mathbb{C}[x_1,x_2,\ldots,x_n]$. Then we say the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k}_{[x_j]}(\mathcal{S}_z)$ is a sub-expansion of the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{l}_{[x_j]}(\mathcal{S}_t)$ denoted $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k}_{[x_j]}(\mathcal{S}_z)\leq(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{l}_{[x_j]}(\mathcal{S}_t)$ if there exist some $0\leq m<l$ such that

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k}_{[x_j]}(\mathcal{S}_z)=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k+m}_{[x_j]}(\mathcal{S}_t).$$

We say the sub-expansion is proper if $m\geq 1$. We denote the proper sub-expansion by $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k}_{[x_j]}(\mathcal{S}_z)<(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{l}_{[x_j]}(\mathcal{S}_t)$

Remark 4.4. Next we relate the notion of the sub-expansion of an expansion to the notion of **Diagonalization of a mixed expansion**. We expose this profound relationship in the following proposition.

Proposition 4.1. *Let $\mathcal{F}:=\{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of the ring $\mathbb{C}[x_1,x_2,\ldots,x_n]$. If the mixed expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})$ is diagonalizable in the direction x_j $(1\leq j\leq n)$ at the spots $\mathcal{S}_t,\mathcal{S}_r\in\mathcal{F}$ with orders k_r and k_t, respectively, with $\mathcal{S}_t\neq\mathcal{S}_r$ and $k_r>k_t$, then*

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k_t}_{[x_j]}(\mathcal{S}_t)\leq(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k_r}_{[x_j]}(\mathcal{S}_r).$$

Proof. Let $\mathcal{F} := \{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of the ring $\mathbb{C}[x_1, x_2, \ldots, x_n]$ and suppose the mixed expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})$ is diagonalizable in the direction x_j $(1 \leq j \leq n)$ at the spots $\mathcal{S}_t, \mathcal{S}_r \in \mathcal{F}$ with orders k_r and k_t, respectively. Then it follows that

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}^{k_r}(\mathcal{S}_r)$$

and

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}^{k_t}(\mathcal{S}_t).$$

It follows that

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}^{k_t}(\mathcal{S}_t) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}^{k_r}(\mathcal{S}_r).$$

Since $k_r > k_t$, it follows that there exist some $m \geq 1$ such that

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}^{k_t}(\mathcal{S}_t) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}^{k_t+m}(\mathcal{S}_r)$$

and It follows that

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}^{k_t}(\mathcal{S}_t) \leq (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}^{k_r}(\mathcal{S}_r).$$

This completes the proof. □

5. **The kernel of an expansion**

In this section we introduce the notion of the kernel of an expansion. One could draw some parallels with this notion and the notion of the boundary points of an expansion under the single variable theory. This choice of terminology is appropriate for this context, since we are no longer considering points as being solutions to our tuple equation but instead tuples consisting of solutions to certain partial differential equation. We launch formally the following languages.

Definition 5.1. Let $\mathcal{F} = \{\mathcal{S}_i\}_{i=1}^{\infty}$ be the collection of tuples of $\mathbb{C}[x_1, x_2, \ldots, x_n]$. Then by the kernel of the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}^{l}(\mathcal{S})$ denoted $\mathrm{Ker}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}^{l}(\mathcal{S})]$, we mean solutions to the equation

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}^{l}(\mathcal{S}) = \mathcal{S}_0.$$

Let us denote the associated quotient ring of $\mathbb{C}[x_1, x_2, \ldots, x_n]$ by $\overline{\mathbb{C}[x_1, x_2, \ldots, x_n]}$. Then it is easy to notice that

$$\mathrm{Ker}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}^{l}(\mathcal{S})] \subset \overline{\mathbb{C}[x_1, x_2, \ldots, x_j, x_{j+1} \ldots, x_n]}.$$

The kernel is a useful tool to study and could as well has a broad range of application. We do not study this into detail in the current paper, since this is the first of the series of papers carefully charted to study these things.

6. **Applications**

In this section we give one striking application of the theory to the area of number theory. We state the application in the following result.

Theorem 6.1. *Let* $\mathcal{F} := \{\mathcal{S}_i\}_{i=1}^{\infty}$ *be a collection of tuples of the ring* $\mathbb{C}[x_1, x_2, \ldots, x_n]$. *Let* $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(\mathcal{S})$ *for all* $1 \leq i \leq n$ *be destabilized stage* $k \geq 1$ *and*

$$\Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(\mathcal{S})] = \Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S})] = w$$

for all $1 \leq i, j \leq n$. *Consider the function*

$$\overrightarrow{O\mathcal{S}} \cdot \overrightarrow{O\mathcal{S}_e} : \mathbb{R}^n \longrightarrow \mathcal{N} \subset \mathbb{C}.$$

Then any $h \in \mathcal{N}$ *has the representation*

$$h := a_1^w + a_2^w + \cdots + a_n^w.$$

[1].

References

1. Theophilus, Agama. *A proof of Sendov's conjecture*, arXiv preprint arXiv:1907.12825, 2019.
2. Theophilus, Agama *Theory of expansivity*, researchgate, 2018.

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