THE NEW MATRIX MULTIPLICATION

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Abstract. In this article, we are giving the meaning of a 'New Multiplication' for the matrices. I have studied the properties of this multiplication in two cases, in the case of 2−D matrices and in the case of 3−D matrices, with elements from over whatever field F.

1. Introduction, Historical overview of the Common Matrix-Multiplication

When matrix multiplication had first appeared in history? This is a difficult question! During the search for a response I encountered the following facts: 200 BC: Han dynasty, coefficients are written on a counting board [16]. In year 1545 Cardan: Cramer rule for 2x2 matrices [16]. 1683 Seki and Leibnitz independently first appearance of Determinants 1750 Cramer (1704-1752) rule for solving systems of linear equations using determinants [16]. 1764 Bezout rule to determine determinants. 1772 Laplace expansion of determinants. 1801 Gauss first introduces determinants [16]. 1812 Cauchy multiplication formula of determinant. Independent of Binet. 1812 Binet (1796-1856) discovered the rule $det(AB) = det(A) \cdot det(B)$ [18]. 1826 Cauchy Uses term "tableau" for a matrix [16]. 1844 Grassman, geometry in n dimensions [18], (50 years ahead of its epoch ([14] p. 204-205). In year 1850 Sylvester first use of term "matrix" (matrice=pregnant animal in old french or matrix=womb in latin as it generates determinants). 1858 Cayley matrix algebra [16], but still in 3 dimensions [18]. 1888 Giuseppe Peano (1858-1932) axioms of abstract vector space. I have also encountered these facts: In his 1867 treatise on determinants, C. L. Dodgson objected to the use of the term "matrix", stating, "I am aware that the word 'Matrix' is already in use to express the very meaning for which i use the word 'Block'; but surely the former word means rather the mould, or form, into which algebraical quantities may be introduced, than an actual assemblage of such quantities." However, Dodgson's objections have passed unheeded and the term "matrix" has stuck (see [10]).

A matrix is a concise and useful way of uniquely representing and working with linear transformations. In particular, every linear transformation can be represented by a matrix, and every matrix corresponds to a unique linear transformation. The matrix, and its close relative the determinant, are extremely important concepts in linear algebra, and were first formulated by Sylvester (1851) and Cayley. In his 1851 paper, Sylvester wrote, "For this purpose we must commence, not with a square, but with an oblong arrangement of terms consisting, suppose, of

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m lines and n columns (see [11]). This will not in itself represent a determinant, but is, as it were, a Matrix out of which we may form various systems of determinants by fixing upon a number p, and selecting at will p lines and p columns, the squares corresponding of p th order." Because Sylvester was interested in the determinant formed from the rectangular array of number and not the array itself (see [13] p. 804), Sylvester used the term "matrix" in its conventional usage to mean "the place from which something else originates" (see [12]). Sylvester (1851) subsequently used the term matrix informally (see [14]), stating "Form the rectangular matrix consisting of n rows and $(n+1)$ columns. Then all the $n+1$ determinants that can be formed by rejecting any one column at pleasure out of this matrix are identically zero." However, it remained up to Sylvester's collaborator Cayley to use the terminology in its modern form in papers of 1855 and 1858 (see [12]).

1.1. **General Definitions of Matrix-Multiplication.** Let's have two matrices $\mathbf{A}$ and $\mathbf{B}$, as follows

$$\mathbf{A}=\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nm} \end{pmatrix}, \quad \mathbf{B}=\begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1p} \\ B_{21} & B_{22} & \cdots & B_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ B_{m1} & B_{m2} & \cdots & B_{mp} \end{pmatrix}$$

the "matrix product" $\mathbf{AB}$ (denoted without multiplication signs or dots) is defined to be the "$n\times p$" matrix

$$\mathbf{AB}=\begin{pmatrix} (\mathbf{AB})_{11} & (\mathbf{AB})_{12} & \cdots & (\mathbf{AB})_{1p} \\ (\mathbf{AB})_{21} & (\mathbf{AB})_{22} & \cdots & (\mathbf{AB})_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ (\mathbf{AB})_{n1} & (\mathbf{AB})_{n2} & \cdots & (\mathbf{AB})_{np} \end{pmatrix}$$

where each "i,j" entry is given by multiplying the entries $\mathbf{A}_{ik}$ by the entries $\mathbf{B}_{kj}$, for k=$1,2,...,m$, and summing the results over k:

$$(\mathbf{AB})_{ij}=\sum_{k=1}^{m}\mathbf{A}_{ik}\mathbf{B}_{kj}$$

Thus the product $\mathbf{AB}$ is defined only if the number of columns in $\mathbf{A}$ is equal to the number of rows in $\mathbf{B}$, in this case m. Each entry may be computed one at a time. Sometimes, the summation convention is used as it is understood to sum over the repeated index "k". To prevent any ambiguity, this convention will not be used in the article.

Definition 1. *The product* $\mathbf{C}$ *of two matrices* $\mathbf{A}$ *and* $\mathbf{B}$ *is defined as*

$$\mathbf{C}_{i,k}=\mathbf{A}_{i,j}\mathbf{B}_{j,k}$$

where "j" is summed over for all possible values of "i" and "k" and the notation above uses the Einstein summation convention.

The implied summation over repeated indices without the presence of an explicit sum sign is called Einstein summation, and is commonly used in both matrix and tensor analysis. Therefore, in order for matrix multiplication to be defined, the dimensions of the matrices must satisfy $(m\times n)(n\times p)=(m\times p)$ where $(m\times n)$ denotes a matrix with m rows and n columns (see [2], [3], [4], [5]).

Matrix multiplication is one of the most fundamental tasks in mathematics and computer science.

1.2. **Other Forms of Matrix Multiplication.** The term "matrix multiplication" is most commonly reserved for the definition given in this article. It could be more loosely applied to other definitions (see [5]).

- **Hadamard Product**

  In mathematics, the Hadamard product is a binary oper ation that takes two matrices of the same dimensions, and produces another matrix where each element i, j is the product of elements ij of the original two matrices. It should not be confused with the more common matrix product. It is attributed to, and named after, either French mathematician Jacques Hadamard, or German mathematician Issai Schur.

  The Hadamard product is **associative** and **distributive**, and unlike the matrix product it is also **commutative** (see [6]).

  Definition 2. *For two matrices,* $\mathbf{A}, \mathbf{B}$ *of the same dimension,* $m \times n$*, the Hadamard product,* $\mathbf{A} \circ \mathbf{B}$*, is a matrix, of the same dimension as the operands, with elements given by*

  $$(\mathbf{A} \circ \mathbf{B})_{i,j} = (\mathbf{A})_{i,j} \circ (\mathbf{B})_{i,j} .$$

  For matrices of different dimensions ($m \times n$ and $p \times q$, where $m \neq p$ or $n \neq q$ or both) the Hadamard product is undefined.

- **Frobenius Inner Product**

  In mathematics, the Frobenius inner product is a binary operation that takes two matrices and returns a number. It is often denoted $\langle \mathbf{A}, \mathbf{B} \rangle_F$ The operation is a component-wise inner product of two matrices as though they are vectors. The two matrices must have the same dimension—same number of rows and columns—but are not restricted to be square matrices (see [7]).

  Definition 3. *For two matrices,* $\mathbf{A}, \mathbf{B}$ *of the same dimension,* $m \times n$*,the Frobenius inner product is defined by the following summation* Σ *of matrix elements*

  $$\langle \mathbf{A}, \mathbf{B} \rangle_F = Trace(\mathbf{A}^T \mathbf{B}) = \sum_i \sum_j A_{ij} B_{ij}.$$

  We have the properties

  $$\langle \mathbf{A}, \mathbf{B} \rangle_F = \langle \mathbf{B}, \mathbf{A} \rangle_F ; \langle \mathbf{A}, \mathbf{A} \rangle_F \geq 0,$$

  for all $\mathbf{A}$; $\langle \mathbf{A}, \mathbf{A} \rangle_F = 0 \Leftrightarrow \mathbf{A} = \mathbf{0}$.

- **Kronecker Product**

  In mathematics, the *Kronecker product*, denoted by $\otimes$, is an operation on two matrices of arbitrary size resulting in a block matrix (see [8]). It is a generalization of the outer product (which is denoted by the same symbol) from vectors to matrices, and gives the matrix of the tensor product with respect to a standard choice of basis. The Kronecker product should not be confused with the usual matrix multiplication, which is an entirely different operation.

  The Kronecker product is named after Leopold Kronecker, even though there is little evidence that he was the first to define and use it. Indeed, in the past the Kronecker product was sometimes called the Zehfuss matrix, after Johann Georg Zehfuss who in 1858 described the matrix operation we now know as the Kronecker product (see [15]).

Definition 4. *If* $\mathbf{A}$ *is an* $m \times n$ *matrix, and* $\mathbf{B}$ *is an* $p \times q$ *matrix, then the Kronecker product* $\mathbf{A} \otimes \mathbf{B}$ *is an* $mp \times nq$ *block matrix*

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix}$$

Property: The Kronecker product is a special case of the tensor product, so it is **bilinear** and **associative**:
(1) $\mathbf{A} \otimes (\mathbf{B} + \mathbf{C}) = \mathbf{A} \otimes \mathbf{B} + \mathbf{A} \otimes \mathbf{C}$;
(2) $(\mathbf{A} + \mathbf{B}) \otimes \mathbf{C} = \mathbf{A} \otimes \mathbf{C} + \mathbf{B} \otimes \mathbf{C}$;
(3) $(k\mathbf{A}) \otimes \mathbf{B} = k(\mathbf{A} \otimes \mathbf{B})$; $(\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C} = \mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C})$,
where $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are matrices and k is a scalar.

Non-commutative: In general, $\mathbf{A} \otimes \mathbf{B}$ and $\mathbf{B} \otimes \mathbf{A}$ are different matrices. However, $\mathbf{A} \otimes \mathbf{B}$ and $\mathbf{B} \otimes \mathbf{A}$ are permutation equivalent, meaning that there exist permutation matrices $\mathbf{P}$ and $\mathbf{Q}$ (so called commutation matrices) such that: $\mathbf{A} \otimes \mathbf{B} = \mathbf{P}(\mathbf{B} \otimes \mathbf{A})\mathbf{Q}$.If $\mathbf{A}$ and $\mathbf{B}$ are square matrices, then $\mathbf{A} \otimes \mathbf{B}$ and $\mathbf{B} \otimes \mathbf{A}$ are even permutation similar, meaning that we can take $\mathbf{P} = \mathbf{Q}^{\mathbf{T}}$.

The mixed-product property: If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$ are matrices of such size that one can form the matrix products $\mathbf{AC}$ and $\mathbf{BD}$, then $(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = (\mathbf{AC}) \otimes (\mathbf{BD})$. This is called the *mixed-product property*, because it mixes the ordinary matrix product and the Kronecker product.

The inverse of a Kronecker product: It follows that $\mathbf{A} \otimes \mathbf{B}$ is invertible if and only if both $\mathbf{A}$ and $\mathbf{B}$ are invertible, in which case the inverse is given by $(\mathbf{A} \otimes \mathbf{B})^{-1} = \mathbf{A}^{-1} \otimes \mathbf{B}^{-1}$. Transposition and conjugate transposition are distributive over the Kronecker product: $(\mathbf{A} \otimes \mathbf{B})^T = \mathbf{A}^T \otimes \mathbf{B}^T$.

- **Cracovian Product**

The Cracovian products of two matrices, say $\mathbf{A}$ and $\mathbf{B}$, is defined by

$$\mathbf{A} \wedge \mathbf{B} = \mathbf{B}^{\mathbf{T}}\mathbf{A},$$

where $\mathbf{B}^{\mathbf{T}}$ and $\mathbf{A}$ are assumed compatible for the common (Cayley) type of matrix multiplication (see [9]).

Since $(\mathbf{AB})^{\mathbf{T}} = \mathbf{B}^{\mathbf{T}}\mathbf{A}^{\mathbf{T}}$, the products $(\mathbf{A} \wedge \mathbf{B}) \wedge \mathbf{C}$ and $\mathbf{A} \wedge (\mathbf{B} \wedge \mathbf{C})$ will generally be different; thus, Cracovian multiplication is **non-associative**. Cracovians are an example of a **quasigroup**.

2. My Results, My New Matrix-Multiplication and their Properties

2.1. Zaka-Product of 2−D Matrix.

Definition 5. *Let* $\mathbf{U} = (U_{i,j})_{i=\overline{1.m};j=\overline{1.n}}$ *and* $\mathbf{A} = (A_{i,j})_{i=\overline{1.m};j=\overline{1.n}}$ *,two 2D matrices of the same size from the* $\mathcal{M}_{m\times n}(\mathrm{F})$*, the* ***ZAKA product*** *of 2D matrices* $\mathbf{U}$*,* $\mathbf{A}$*, we will call matrix* $\mathbf{C} = \mathbf{U} \odot \mathbf{A}$*, it is easy to check that the matrix has the same size* $\mathbf{C} = (C_{i,j})_{i=\overline{1.m};j=\overline{1.n}}$ *. Where, the coefficients of this matrix are* *calculated as follows*

$$C_{i,j} = \left\{ \begin{array}{c} A_{i,j}U_{i,j} + A_{i-1,j}U_{i-1,j} + A_{i+1,j}U_{i+1,j} \\ +A_{i,j-1}U_{i,j-1} + A_{i,j+1}U_{i,j+1} \end{array} \right\}$$

where

$$\begin{array}{l} A_{i-1,j} = 0 \text{ and } U_{i-1,j} = 0 \text{ for } i = 1; \\ A_{i,j-1} = 0 \text{ and } U_{i,j-1} = 0 \text{ for } j = 1; \\ A_{i+1,j} = 0 \text{ and } U_{i+1,j} = 0 \text{ for } i = m; \\ A_{i,j+1} = 0 \text{ and } U_{i,j+1} = 0 \text{ for } j = n; \end{array}$$

it is clear that,

$$\circledcirc : \mathcal{M}_{m\times n}(F) \times \mathcal{M}_{m\times n}(F) \longrightarrow \mathcal{M}_{m\times n}(F).$$

Example 1. *Let's have the* 3×3 *matrices*

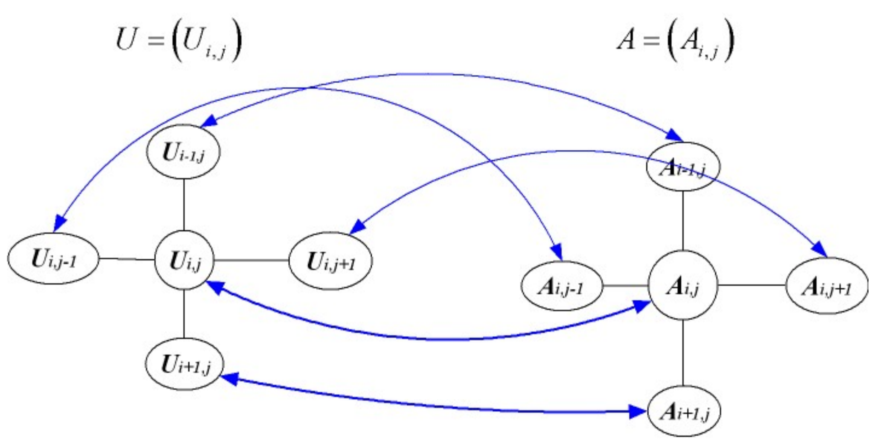
$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 9 & 6 & 3 \\ 8 & 5 & 2 \\ 7 & 4 & 1 \end{bmatrix}$$

The Zaka product matrix $\mathbf{C} = \mathbf{A} \circledcirc \mathbf{B}$ *is:*

$$\mathbf{C} = \mathbf{A} \circledcirc \mathbf{B} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \circledcirc \begin{bmatrix} 9 & 6 & 3 \\ 8 & 5 & 2 \\ 7 & 4 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 9\cdot 1 + 4\cdot 6 + 2\cdot 8 & 4\cdot 6 + 7\cdot 3 + 5\cdot 5 + 1\cdot 9 & 7\cdot 3 + 8\cdot 2 + 4\cdot 6 \\ 2\cdot 8 + 1\cdot 9 + 5\cdot 5 + 3\cdot 7 & 5\cdot 5 + 4\cdot 6 + 8\cdot 2 + 6\cdot 4 + 2\cdot 8 & 8\cdot 2 + 7\cdot 3 + 9\cdot 1 + 5\cdot 5 \\ 3\cdot 7 + 2\cdot 8 + 6\cdot 4 & 6\cdot 4 + 5\cdot 5 + 9\cdot 1 + 3\cdot 7 & 9\cdot 1 + 8\cdot 2 + 6\cdot 4 \end{bmatrix}$$

$$\Rightarrow \mathbf{C} = \mathbf{A} \circledcirc \mathbf{B} = \begin{bmatrix} 49 & 79 & 61 \\ 71 & 105 & 71 \\ 61 & 79 & 49 \end{bmatrix}.$$



The Zaka matrix-multiplication, for 2D matrices

FIGURE 1. The Zaka matrix-multiplication, for 2−D matrix

Remark 1. *To have the Zaka product site between the $2-D$ matrices, the matrices should have the same size.*

For example if we have the matrix $\mathbf{A}_{m\times n}$ and $\mathbf{B}_{k\times l}$ to have the Zaka product $\odot$ site between these two matrices, then $m=k$ and $n=l$.

2.2. **Properties of the Zaka-product of $2-$D Matrices.**

Proposition 1. *(Zaka product is **comutativ**)*

$$\forall \mathbf{U}, \mathbf{A} \in \mathcal{M}_{m\times n}(\mathrm{F}), \mathbf{U} \circledcirc \mathbf{A} = \mathbf{A} \circledcirc \mathbf{U}.$$

Proof. By Definition 5 we have:

$$\begin{aligned}
&C_{i,j} = (\mathbf{U} \circledcirc \mathbf{A})_{i,j} \\
&= A_{i,j}\cdot U_{i,j} + (A_{i-1,j}\cdot U_{i-1,j} + A_{i+1,j}\cdot U_{i+1,j} + A_{i,j-1}\cdot U_{i,j-1} + A_{i,j+1}\cdot U_{i,j+1}) \\
&= U_{i,j}\cdot A_{i,j} + (U_{i-1,j}\cdot A_{i-1,j} + U_{i+1,j}\cdot A_{i+1,j} + U_{i,j-1}\cdot A_{i,j-1} + U_{i,j+1}\cdot A_{i,j+1}) \\
&= (\mathbf{A} \circledcirc \mathbf{U})_{i,j}
\end{aligned}$$
□

Proposition 2. *(Zaka product is **distributive**)*

$$\forall \mathbf{U}, \mathbf{A}, \mathbf{B} \in \mathcal{M}_{m\times n}(\mathrm{F}), \quad \begin{array}{l} \mathbf{1}. (\mathbf{U}+\mathbf{B}) \circledcirc \mathbf{A} = \mathbf{U} \circledcirc \mathbf{A} + \mathbf{B} \circledcirc \mathbf{A}; \\ \mathbf{2}. \mathbf{U} \circledcirc (\mathbf{B}+\mathbf{A}) = \mathbf{U} \circledcirc \mathbf{B} + \mathbf{U} \circledcirc \mathbf{A}; \end{array}$$

Proof. By Definition 5 we have:

$$\begin{aligned}
&((\mathbf{U}+\mathbf{B}) \circledcirc \mathbf{A})_{i,j} = A_{i,j}\cdot[U_{i,j}+B_{i,j}] + A_{i-1,j}\cdot[U_{i-1,j}+B_{i-1,j}] + A_{i+1,j}\cdot \\
&[U_{i+1,j}+B_{i+1,j}] + A_{i,j-1}\cdot[U_{i,j-1}+B_{i,j-1}] + A_{i,j+1}\cdot[U_{i,j+1}+B_{i,j+1}] \\
&= \{A_{i,j}\cdot U_{i,j} + (A_{i-1,j}\cdot U_{i-1,j} + A_{i+1,j}\cdot U_{i+1,j} + A_{i,j-1}\cdot U_{i,j-1} + A_{i,j+1}\cdot U_{i,j+1})\} + \\
&\{A_{i,j}\cdot B_{i,j} + (A_{i-1,j}\cdot B_{i-1,j} + A_{i+1,j}\cdot B_{i+1,j} + A_{i,j-1}\cdot B_{i,j-1} + A_{i,j+1}\cdot B_{i,j+1})\} \\
&= (\mathbf{U} \circledcirc \mathbf{A})_{i,j} + (\mathbf{B} \circledcirc \mathbf{A})_{i,j} \\
&= (\mathbf{U} \circledcirc \mathbf{A} + \mathbf{B} \circledcirc \mathbf{A})_{i,j}.
\end{aligned}$$
□

Proposition 3. *(Zaka product is **non-associative**)*

*For three matrices $\forall \mathbf{U}, \mathbf{A}, \mathbf{B} \in \mathcal{M}_{m\times n}(\mathrm{F})$, different from the **zero** matrix, have the inequality:*

$$(\mathbf{A} \circledcirc \mathbf{U}) \circledcirc \mathbf{B} \neq \mathbf{A} \circledcirc (\mathbf{U} \circledcirc \mathbf{B})$$

Proof. By following the definition 5, have:

$$\begin{aligned}
&[(\mathbf{A} \circledcirc \mathbf{U}) \circledcirc \mathbf{B}]_{i,j} = [\mathbf{C} \circledcirc \mathbf{B}]_{i,j} \\
&= B_{i,j}\cdot C_{i,j} + B_{i-1,j}\cdot C_{i-1,j} + B_{i+1,j}\cdot C_{i+1,j} + +B_{i,j-1}\cdot C_{i,j-1} + B_{i,j+1}\cdot C_{i,j+1} \\
&= B_{i,j}\cdot\{A_{i,j}\cdot U_{i,j} + A_{i-1,j}\cdot U_{i-1,j} + A_{i+1,j}\cdot U_{i+1,j} + A_{i,j-1}\cdot U_{i,j-1} + A_{i,j+1}\cdot U_{i,j+1}\} + \\
&B_{i-1,j}\cdot\{A_{i-1,j}\cdot U_{i-1,j} + A_{i-2,j}\cdot U_{i-2,j} + A_{i,j}\cdot U_{i,j} + A_{i-1,j-1}\cdot U_{i-1,j-1} + A_{i-1,j+1}\cdot U_{i-1,j+1}\} + \\
&B_{i+1,j}\cdot\{A_{i+1,j}\cdot U_{i+1,j} + A_{i,j}\cdot U_{i,j} + A_{i+2,j}\cdot U_{i+2,j} + A_{i+1,j-1}\cdot U_{i+1,j-1} + A_{i+1,j+1}\cdot U_{i+1,j+1}\} + \\
&B_{i,j-1}\cdot\{A_{i,j-1}\cdot U_{i,j-1} + A_{i-1,j-1}\cdot U_{i-1,j-1} + +A_{i+1,j-1}\cdot U_{i+1,j-1} + A_{i,j-2}\cdot U_{i,j-2} + A_{i,j}\cdot U_{i,j}\} + \\
&B_{i,j+1}\cdot\{A_{i,j+1}\cdot U_{i,j+1} + A_{i-1,j+1}\cdot U_{i-1,j+1} + +A_{i+1,j+1}\cdot U_{i+1,j+1} + A_{i,j}\cdot U_{i,j} + A_{i,j+2}\cdot U_{i,j+2}\}
\end{aligned}$$

Hence,

$$\begin{aligned}
&[(\mathbf{A} \circledcirc \mathbf{U}) \circledcirc \mathbf{B}]_{i,j} = B_{i,j}\cdot A_{i,j}\cdot U_{i,j} + B_{i,j}\cdot A_{i-1,j}\cdot U_{i-1,j} + B_{i,j}\cdot A_{i+1,j}\cdot U_{i+1,j} + \\
&B_{i,j}\cdot A_{i,j-1}\cdot U_{i,j-1} + B_{i,j}\cdot A_{i,j+1}\cdot U_{i,j+1} + B_{i-1,j}\cdot A_{i-1,j}\cdot U_{i-1,j} + B_{i-1,j}\cdot A_{i-2,j}\cdot \\
&U_{i-2,j} + B_{i-1,j}\cdot A_{i,j}\cdot U_{i,j} + B_{i-1,j}\cdot A_{i-1,j-1}\cdot U_{i-1,j-1} + B_{i-1,j}\cdot A_{i-1,j+1}\cdot U_{i-1,j+1} + \\
&B_{i+1,j}\cdot A_{i+1,j}\cdot U_{i+1,j} + B_{i+1,j}\cdot A_{i,j}\cdot U_{i,j} + B_{i+1,j}\cdot A_{i+2,j}\cdot U_{i+2,j} + B_{i+1,j}\cdot A_{i+1,j-1}\cdot \\
&U_{i+1,j-1} + B_{i+1,j}\cdot A_{i+1,j+1}\cdot U_{i+1,j+1} + B_{i,j-1}\cdot A_{i,j-1}\cdot U_{i,j-1} + B_{i,j-1}\cdot A_{i-1,j-1}\cdot \\
&U_{i-1,j-1} + B_{i,j-1}\cdot A_{i+1,j-1}\cdot U_{i+1,j-1} + B_{i,j-1}\cdot A_{i,j-2}\cdot U_{i,j-2} + B_{i,j-1}\cdot A_{i,j}\cdot U_{i,j} +
\end{aligned}$$

$B_{i,j+1} \cdot A_{i,j+1} \cdot U_{i,j+1} + B_{i,j+1} \cdot A_{i-1,j+1} \cdot U_{i-1,j+1} + B_{i,j+1} \cdot A_{i+1,j+1} \cdot U_{i+1,j+1} + B_{i,j+1} \cdot A_{i,j} \cdot U_{i,j} + B_{i,j+1} \cdot A_{i,j+2} \cdot U_{i,j+2}$

and

$[\mathbf{A} \circledcirc (\mathbf{U} \circledcirc \mathbf{B})]_{i,j} = [\mathbf{A} \circledcirc D]_{i,j} = [\mathbf{D} \circledcirc A]_{i,j}$
$= A_{i,j} \cdot D_{i,j} + (A_{i-1,j} \cdot D_{i-1,j} + A_{i+1,j} \cdot D_{i+1,j} + A_{i,j-1} \cdot D_{i,j-1} + A_{i,j+1} \cdot D_{i,j+1})$

where
$D_{i,j} = B_{i,j} \cdot U_{i,j} + (B_{i-1,j} \cdot U_{i-1,j} + B_{i+1,j} \cdot U_{i+1,j} + B_{i,j-1} \cdot U_{i,j-1} + B_{i,j+1} \cdot U_{i,j+1})$

$= A_{i,j} \cdot [B_{i,j} \cdot U_{i,j} + (B_{i-1,j} \cdot U_{i-1,j} + B_{i+1,j} \cdot U_{i+1,j} + B_{i,j-1} \cdot U_{i,j-1} + B_{i,j+1} \cdot U_{i,j+1}) + A_{i-1,j} \cdot [B_{i-1,j} \cdot U_{i-1,j} + (B_{i-2,j} \cdot U_{i-2,j} + B_{i,j} \cdot U_{i,j} + B_{i-1,j-1} \cdot U_{i-1,j-1} + B_{i-1,j+1} \cdot U_{i-1,j+1})] + A_{i+1,j} \cdot [B_{i+1,j} \cdot U_{i+1,j} + (B_{i,j} \cdot U_{i,j} + B_{i+2,j} \cdot U_{i+2,j} + B_{i+1,j-1} \cdot U_{i+1,j-1} + B_{i+1,j+1} \cdot U_{i+1,j+1})] + A_{i,j-1} \cdot [B_{i,j-1} \cdot U_{i,j-1} + (B_{i-1,j-1} \cdot U_{i-1,j-1} + B_{i+1,j-1} \cdot U_{i+1,j-1} + B_{i,j-2} \cdot U_{i,j-2} + B_{i,j} \cdot U_{i,j})] + A_{i,j+1} \cdot D_{i,j+1}[B_{i,j+1} \cdot U_{i,j+1} + (B_{i-1,j+1} \cdot U_{i-1,j+1} + B_{i+1,j+1} \cdot U_{i+1,j+1} + B_{i,j} \cdot U_{i,j} + B_{i,j+2} \cdot U_{i,j+2})]$
$= A_{i,j} \cdot B_{i,j} \cdot U_{i,j} + A_{i,j} \cdot B_{i-1,j} \cdot U_{i-1,j} + A_{i,j} \cdot B_{i+1,j} \cdot U_{i+1,j} + A_{i,j} \cdot B_{i,j-1} \cdot U_{i,j-1} + A_{i,j} \cdot B_{i,j+1} \cdot U_{i,j+1} + A_{i-1,j} \cdot B_{i-1,j} \cdot U_{i-1,j} + A_{i-1,j} \cdot B_{i-2,j} \cdot U_{i-2,j} + A_{i-1,j} \cdot B_{i,j} \cdot U_{i,j} + A_{i-1,j} \cdot B_{i-1,j-1} \cdot U_{i-1,j-1} + A_{i-1,j} \cdot B_{i-1,j+1} \cdot U_{i-1,j+1} + A_{i+1,j} \cdot B_{i+1,j} \cdot U_{i+1,j} + A_{i+1,j} \cdot B_{i,j} \cdot U_{i,j} + A_{i+1,j} \cdot B_{i+2,j} \cdot U_{i+2,j} + A_{i+1,j} \cdot B_{i+1,j-1} \cdot U_{i+1,j-1} + A_{i+1,j} \cdot B_{i+1,j+1} \cdot U_{i+1,j+1} + A_{i,j-1} \cdot B_{i,j-1} \cdot U_{i,j-1} + A_{i,j-1} \cdot B_{i-1,j-1} \cdot U_{i-1,j-1} + A_{i,j-1} \cdot B_{i+1,j-1} \cdot U_{i+1,j-1} + A_{i,j-1} \cdot B_{i,j-2} \cdot U_{i,j-2} + A_{i,j-1} \cdot B_{i,j} \cdot U_{i,j} + A_{i,j+1} \cdot B_{i,j+1} \cdot U_{i,j+1} + A_{i,j+1} \cdot B_{i-1,j+1} \cdot U_{i-1,j+1} + A_{i,j+1} \cdot B_{i+1,j+1} \cdot U_{i+1,j+1} + A_{i,j+1} \cdot B_{i,j} \cdot U_{i,j} + A_{i,j+1} \cdot B_{i,j+2} \cdot U_{i,j+2}.$

This seems clear that is different from the first result. □

2.3. Zaka Multiplication of 3−D Matrix. We recall the definition of 3−D matrix addition, (see [1])

Definition 6. [1] *Let* $\mathbf{U} = (U_{i,j,k})_{i=\overline{1.m};j=\overline{1.n};k=\overline{1.p}}$ *and* $\mathbf{A} = (A_{i,j,k})_{i=\overline{1.m};j=\overline{1.n};k=\overline{1.p}}$, *two 3D matrices of the same size, the* ***addition*** *of 3D matrix* [1], *to matrices* $\mathbf{U}$, $\mathbf{A}$, *we will call matrix* $\mathbf{C} = (C_{i,j,k})_{i=\overline{1.m};j=\overline{1.n};k=\overline{1.p}}$, *where* $C_{i,j,k} = U_{i,j,k} + A_{i,j,k}$, $\forall i = \overline{1.m}; j = \overline{1.n}; k = \overline{1.p}$.

Definition 7. *Let* $\mathbf{U} = (U_{i,j,k})_{i=\overline{1.m};j=\overline{1.n};k=\overline{1.p}}$ *and* $\mathbf{A} = (A_{i,j,k})_{i=\overline{1.m};j=\overline{1.n};k=\overline{1.p}}$, *two 3D matrices of the same size [1], the* ***Zaka product matrix****, to 3D-matrices* $\mathbf{U}, \mathbf{A}$, *we will call 3D-matrix* $\mathbf{C} = (C_{i,j,k})_{i=\overline{1.m};j=\overline{1.n};k=\overline{1.p}}$, *where the coefficients of this matrix are calculated as follows:*

$$C_{i,j,k} = \left\{ \begin{array}{c} A_{i,j,k} \cdot U_{i,j,k} + A_{i-1,j,k} \cdot U_{i-1,j,k} + A_{i+1,j,k} \cdot U_{i+1,j,k} \\ +A_{i,j-1,k} \cdot U_{i,j-1,k} + A_{i,j+1,k} \cdot U_{i,j+1,k} + \\ A_{i,j,k-1} \cdot U_{i,j,k-1} + A_{i,j,k+1} \cdot U_{i,j,k+1} \end{array} \right\}$$

Where

$A_{i-1,j,k} = 0$ and $U_{i-1,j,k} = 0$, for $i = 1$;
$A_{i+1,j,k} = 0$ and $U_{i+1,j,k} = 0$, for $i = m$.
$A_{i,j-1,k} = 0$ and $U_{i,j-1,k} = 0$, for $j = 1$;
$A_{i,j+1,k} = 0$ and $U_{i,j+1,k} = 0$, for $j = n$.
$A_{i,j,k-1} = 0$ and $U_{i,j,k-1} = 0$, for $k = 1$;
$A_{i,j,k+1} = 0$ and $U_{i,j,k+1} = 0$, for $k = p$.

it is clear that,

$$\circledcirc : \mathcal{M}_{m\times n\times p}(F) \times \mathcal{M}_{m\times n\times p}(F) \longrightarrow \mathcal{M}_{m\times n\times p}(F).$$

Remark 2. *To have the ZAKA product site between the 3D matrices, the matrices should have the same size.*

For example if we have the matrix $\mathbf{A}_{m\times n\times p}$ and $\mathbf{B}_{k\times l\times q}$ to have the ZAKA product "$\circledcirc$" site between these two matrices, then $m = k$, $n = l$ and $p = q$.

Example 2. *Let's have the $3 \times 3 \times 3$ matrices*

$$\mathbf{A} = \begin{bmatrix} \begin{pmatrix} 2 & 3 & 7 \\ 1 & 1 & 5 \\ 2 & 4 & 0 \end{pmatrix} \\ \\ \begin{pmatrix} 0 & 1 & 1 \\ -1 & 5 & 2 \\ 3 & 2 & 1 \end{pmatrix} \\ \\ \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \end{bmatrix} ; \mathbf{B} = \begin{bmatrix} \begin{pmatrix} 6 & 5 & 2 \\ 3 & 4 & 1 \\ 1 & 7 & 9 \end{pmatrix} \\ \\ \begin{pmatrix} 0 & 4 & 7 \\ 3 & 1 & 8 \\ 1 & 2 & 5 \end{pmatrix} \\ \\ \begin{pmatrix} 9 & 6 & 3 \\ 8 & 5 & 2 \\ 7 & 4 & 1 \end{pmatrix} \end{bmatrix}$$

The ***Zaka-Product****,* $\mathbf{C} = \mathbf{A} \circledcirc \mathbf{B}$ *is:*

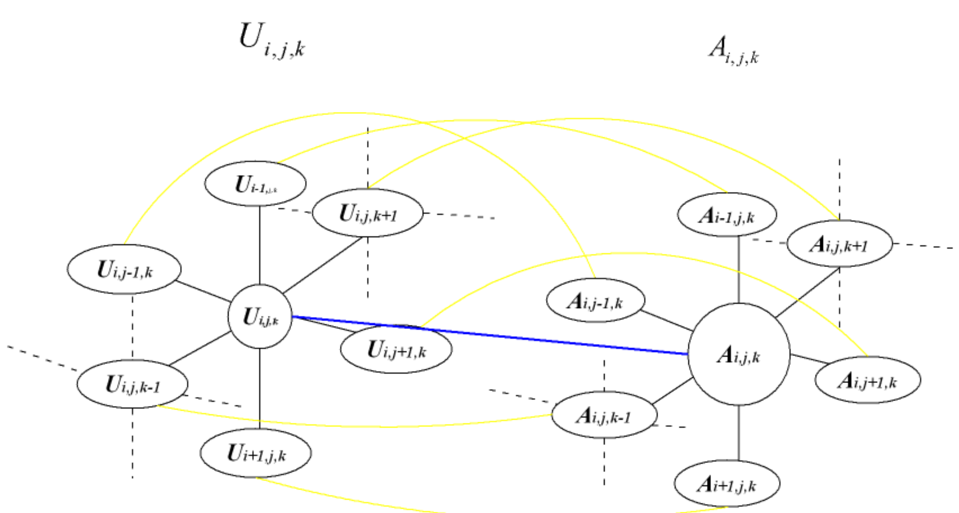


Figure 2. The Zaka matrix-multiplication, for 3−D matrices.

$$\mathbf{C} = \mathbf{A} \circledcirc \mathbf{B} = \begin{bmatrix} \begin{pmatrix} 2 & 3 & 7 \\ 1 & 1 & 5 \\ 2 & 4 & 0 \end{pmatrix} \\ \\ \begin{pmatrix} 0 & 1 & 1 \\ -1 & 5 & 2 \\ 3 & 2 & 1 \end{pmatrix} \\ \\ \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \end{bmatrix} \circledcirc \begin{bmatrix} \begin{pmatrix} 6 & 5 & 2 \\ 3 & 4 & 1 \\ 1 & 7 & 9 \end{pmatrix} \\ \\ \begin{pmatrix} 0 & 4 & 7 \\ 3 & 1 & 8 \\ 1 & 2 & 5 \end{pmatrix} \\ \\ \begin{pmatrix} 9 & 6 & 3 \\ 8 & 5 & 2 \\ 7 & 4 & 1 \end{pmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{pmatrix} 2\cdot 6+3\cdot 5+1\cdot 3 & 3\cdot 5+7\cdot 2+1\cdot 4+2\cdot 6 & 7\cdot 2+5\cdot 1+3\cdot 5 \\ 1\cdot 3+2\cdot 6+1\cdot 4+2\cdot 1 & 1\cdot 4+3\cdot 5+5\cdot 1+4\cdot 7+1\cdot 3 & 5\cdot 1+7\cdot 2+0\cdot 9+1\cdot 4 \\ 2\cdot 1+1\cdot 3+4\cdot 7 & 4\cdot 7+1\cdot 4+0\cdot 9+2\cdot 1 & 0\cdot 9+5\cdot 1+4\cdot 7 \end{pmatrix} \\ \\ \begin{pmatrix} 0\cdot 0+1\cdot 4+(-1)\cdot 3 & 1\cdot 4+1\cdot 7+5\cdot 1+0\cdot 0 & 1\cdot 7+2\cdot 8+1\cdot 4 \\ (-1)\cdot 3+0\cdot 0+5\cdot 1+3\cdot 1 & 5\cdot 1+1\cdot 4+2\cdot 8+2\cdot 2+(-1)\cdot 3 & 2\cdot 8+1\cdot 7+1\cdot 5+5\cdot 1 \\ 3\cdot 1+(-1)\cdot 3+2\cdot 2 & 2\cdot 2+5\cdot 1+1\cdot 5+3\cdot 1 & 1\cdot 5+2\cdot 8+2\cdot 2 \end{pmatrix} \\ \\ \begin{pmatrix} 1\cdot 9+4\cdot 6+2\cdot 8 & 4\cdot 6+7\cdot 3+5\cdot 5+1\cdot 9 & 7\cdot 3+8\cdot 2+4\cdot 6 \\ 2\cdot 8+1\cdot 9+5\cdot 5+3\cdot 7 & 5\cdot 5+4\cdot 6+8\cdot 2+6\cdot 4+2\cdot 8 & 8\cdot 2+7\cdot 3+9\cdot 1+5\cdot 5 \\ 3\cdot 7+2\cdot 8+6\cdot 4 & 6\cdot 4+5\cdot 5+9\cdot 1+3\cdot 7 & 9\cdot 1+8\cdot 2+6\cdot 4 \end{pmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{pmatrix} 12+15+3 & 15+14+4+12 & 14+5+15 \\ 3+12+4+2 & 4+15+5+28+3 & 5+14+9+4 \\ 2+1\cdot 3+4 & 28+4+0+2 & 0+5+28 \end{pmatrix} \\ \\ \begin{pmatrix} 0+4-3 & 4+7+5+0 & 7+16+4 \\ -3+0+5+3 & 5+4+16+4-3 & 16+7+5+5 \\ 3-3+4 & 4+5+5+3 & 5+16+4 \end{pmatrix} \\ \\ \begin{pmatrix} 9+24+16 & 24+21+25+9 & 21+16+24 \\ 16+9+25+21 & 25+24+16+24+16 & 16+21+9+25 \\ 21+16+24 & 24+25+9+21 & 9+16+24 \end{pmatrix} \end{bmatrix}$$

$$\Longrightarrow \mathbf{C} = \mathbf{A} \circledcirc \mathbf{B} = \begin{bmatrix} \begin{pmatrix} 30 & 45 & 34 \\ 21 & 55 & 32 \\ 33 & 34 & 33 \end{pmatrix} \\ \\ \begin{pmatrix} 1 & 16 & 27 \\ 5 & 26 & 33 \\ 4 & 17 & 25 \end{pmatrix} \\ \\ \begin{pmatrix} 49 & 79 & 61 \\ 71 & 105 & 71 \\ 61 & 79 & 49 \end{pmatrix} \end{bmatrix}$$

2.4. **Properties of the Zaka-product of 3−D matrices.**

Proposition 4. *(Zaka product* ***is comutativ****)*

$$\forall \mathbf{U}, \mathbf{A} \in \mathcal{M}_{m\times n\times p}(\mathrm{F}) \Longrightarrow \mathbf{U} \circledcirc \mathbf{A} = \mathbf{A} \circledcirc \mathbf{U}$$

Proof. By Definition 7 we have:
$(\mathbf{U} \circledcirc \mathbf{A})_{i,j,k} = A_{i,j,k} \cdot U_{i,j,k} + A_{i-1,j,k} \cdot U_{i-1,j,k} + A_{i+1,j,k} \cdot U_{i+1,j,k} + A_{i,j-1,k} \cdot U_{i,j-1,k} + A_{i,j+1,k} \cdot U_{i,j+1,k} + A_{i,j,k-1} \cdot U_{i,j,k-1} + A_{i,j,k+1} \cdot U_{i,j,k+1}$
$= U_{i,j,k} \cdot A_{i,j,k} + U_{i-1,j,k} \cdot A_{i-1,j,k} + U_{i+1,j,k} \cdot A_{i+1,j,k} + U_{i,j-1,k} \cdot A_{i,j-1,k} + U_{i,j+1,k} \cdot A_{i,j+1,k} + U_{i,j,k-1} \cdot A_{i,j,k-1} + U_{i,j,k+1} \cdot A_{i,j,k+1}$
$= (\mathbf{A} \circledcirc \mathbf{U})_{i,j,k}$ □

Proposition 5. *(Zaka product* ***is distributive****)*

$$\forall \mathbf{U}, \mathbf{A}, \mathbf{B} \in \mathcal{M}_{m\times n\times p}(\mathrm{F}) \quad \begin{array}{l} \mathbf{1}.\ (\mathbf{U}+\mathbf{B}) \circledcirc \mathbf{A} = \mathbf{U} \circledcirc \mathbf{A} + \mathbf{B} \circledcirc \mathbf{A}. \\ \mathbf{2}.\ \mathbf{U} \circledcirc (\mathbf{B}+\mathbf{A}) = \mathbf{U} \circledcirc \mathbf{B} + \mathbf{U} \circledcirc \mathbf{A} \end{array}$$

Proof. By following the definition 5 and definition 7, have
$((\mathbf{U}+\mathbf{B}) \circledcirc \mathbf{A})_{i,j,k} = A_{i,j,k} \cdot [U_{i,j,k} + B_{i,j,k}] + A_{i-1,j,k} \cdot [U_{i-1,j,k} + B_{i-1,j,k}] + A_{i+1,j,k} \cdot [U_{i+1,j,k} + B_{i+1,j,k}] + A_{i,j-1,k} \cdot [U_{i,j-1,k} + B_{i,j-1,k}] + A_{i,j+1,k} \cdot [U_{i,j+1,k} + B_{i,j+1,k}] + A_{i,j,k-1} \cdot [U_{i,j,k-1} + B_{i,j,k-1}] + A_{i,j,k+1} \cdot [U_{i,j,k+1} + B_{i,j,k+1}]$

$= A_{i,j,k} \cdot U_{i,j,k} + A_{i-1,j,k} \cdot U_{i-1,j,k} + A_{i+1,j,k} \cdot U_{i+1,j,k} + A_{i,j-1,k} \cdot U_{i,j-1,k} + A_{i,j+1,k} \cdot U_{i,j+1,k} + A_{i,j,k-1} \cdot U_{i,j,k-1} + A_{i,j,k+1} \cdot U_{i,j,k+1} + A_{i,j,k} \cdot B_{i,j,k} + A_{i-1,j,k} \cdot B_{i-1,j,k} + A_{i+1,j,k} \cdot B_{i+1,j,k} + A_{i,j-1,k} \cdot B_{i,j-1,k} + A_{i,j+1,k} \cdot B_{i,j+1,k} + A_{i,j,k-1} \cdot B_{i,j,k-1} + A_{i,j,k+1} \cdot B_{i,j,k+1}$

$= (\mathbf{U} \circledcirc \mathbf{A} + \mathbf{B} \circledcirc \mathbf{A})_{i,j,k}$

$= (\mathbf{U} \circledcirc \mathbf{A})_{i,j,k} + (\mathbf{B} \circledcirc \mathbf{A})_{i,j,k}$. □

Proposition 6. *(Zaka product is* ***non-associative****)*

For three matrices $\forall \mathbf{U}, \mathbf{A}, \mathbf{B} \in \mathcal{M}_{m\times n\times p}(\mathrm{F})$,*different from the* ***zero*** *matrix, have the inequality:*

$$(\mathbf{A} \circledcirc \mathbf{U}) \circledcirc \mathbf{B} \neq \mathbf{A} \circledcirc (\mathbf{U} \circledcirc \mathbf{B})$$

Proof. The proof is obvious, and the same as in Proposition 3, of 2D matrix. □

Notation 1. *Hope and I think that the "ZAKA" multiplication, there will be good applications in differential equations with partial derivatives, perhaps even in different simulations.*

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