Riemann's Zeta Function is defined as,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad Re(s) > 1 \qquad \text{(Ref. 1)}$$

Riemann's Xi function is defined as

$$\xi(s) = \frac{s(s-1)}{2} \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) \qquad \text{(Ref. 1)}$$

$\xi(s)$ is an <u>Entire function</u> (Ref. 1)

& its zeroes are the non trivial zeros of the Riemann Zeta Function (Ref. 1)

Non trivial zeroes $\rightarrow 0 \le Re(s) \le 1$ (Ref. 1)

- Riemann Hypothesis – All the non trivial zeros of the Riemann Zeta function lie on the critical line with real part equal to $1/2$.

Proof :-

Riemann's Xi Function [2, p. 47] is defined as

$$\xi(s) = \xi(0) \prod_{\rho} \left(1 - \frac{s}{\rho}\right)$$

where ρ ranges over the zeros ρ of $\xi(\rho) = 0$ and

If we Combine the factors $\left(1 - \frac{s}{\rho}\right)$ and $\left(1 - \frac{s}{1-\rho}\right)$ then $\xi(s)$ is Absolutely Cgt. infinite product.

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From, [1, p. 42, section 2.5]

$$\prod_{\rho}\left(1-\frac{s}{\rho}\right) = \prod_{\operatorname{Im}\rho>0} 1-\frac{s(1-s)}{\rho(1-\rho)} = \prod_{\operatorname{Im}\rho>0}\left(1-\frac{s}{\rho}\right)\left(1-\frac{s}{1-\rho}\right)$$

Let, $\epsilon(\rho_0) = 0$

$$\left[\because\ \epsilon(s) = \epsilon(0)\prod_{\operatorname{Im}\rho>0}\left(1-\frac{s}{\rho}\right)\left(1-\frac{s}{1-\rho}\right)\right] \quad \text{—} \ (*)$$

$$|\epsilon(\overline{\rho_0})| = |\epsilon(\rho_0)| \quad \text{———} \ \text{Ref. [3]}$$

$$|\epsilon(\overline{\rho_0})| = 0$$

$$\epsilon(\overline{\rho_0}) = 0$$

$$(*) \Rightarrow \epsilon(\overline{\rho_0}) = \epsilon(0)\prod_{\operatorname{Im}\rho>0}\left(1-\frac{\overline{\rho_0}}{\rho}\right)\left(1-\frac{\overline{\rho_0}}{1-\rho}\right) = 0$$

$$\epsilon(0) = \frac{1}{2} \quad \text{———} \ [\text{Ref. } 4]$$

$$\frac{1}{2}\prod_{\operatorname{Im}\rho>0}\left(1-\frac{\overline{\rho_0}}{\rho}\right)\left(1-\frac{\overline{\rho_0}}{1-\rho}\right) = 0$$

$$\left(1-\frac{\overline{\rho_0}}{\rho_0}\right)\left(1-\frac{\overline{\rho_0}}{1-\rho_0}\right)\prod_{\substack{\operatorname{Im}\rho>0 \\ \rho\neq\rho_0}}\left(1-\frac{\overline{\rho_0}}{\rho}\right)\left(1-\frac{\overline{\rho_0}}{1-\rho}\right) = 0$$

$$\frac{(\rho_0 - \overline{\rho_0})(1 - \rho_0 - \overline{\rho_0})}{\rho_0(1-\rho_0)} \prod_{\substack{\operatorname{Im}\rho>0 \\ \rho \neq \rho_0}} \left(1 - \frac{\overline{\rho_0}}{\rho}\right)\left(1 - \frac{\overline{\rho_0}}{1-\rho}\right) = 0$$

$$\frac{2i \operatorname{Im}(\rho_0)\left[1 - 2\operatorname{Re}(\rho_0)\right]}{\rho_0(1-\rho_0)} \prod_{\substack{\operatorname{Im}\rho>0 \\ \rho \neq \rho_0}} \left(1 - \frac{\overline{\rho_0}}{\rho}\right)\left(1 - \frac{\overline{\rho_0}}{1-\rho}\right) = 0$$

$$2i \operatorname{Im}(\rho_0)\left[1 - 2\operatorname{Re}(\rho_0)\right] \prod_{\operatorname{Im}\rho>0} \left(1 - \frac{\overline{\rho_0}}{\rho}\right)\left(1 - \frac{\overline{\rho_0}}{1-\rho}\right) = 0 \quad \text{—}(\#)$$

$\because \operatorname{Im}\rho > 0 \Rightarrow \operatorname{Im}\rho_0 > 0$

$\therefore \operatorname{Im}\rho_0 \neq 0$

$$(\#) \Rightarrow \quad 2i \operatorname{Im}(\rho_0)\left[1 - 2\operatorname{Re}(\rho_0)\right] \cdot I_{\rho, \overline{\rho_0}} = 0 \quad \text{—}(1)$$

where $I_{\rho_0, \overline{\rho_0}} = \prod_{\substack{\operatorname{Im}\rho>0 \\ \rho \neq \rho_0}} \left(1 - \frac{\overline{\rho_0}}{\rho}\right)\left(1 - \frac{\overline{\rho_0}}{1-\rho}\right)$

$$I_{\rho, \overline{\rho_0}} = 0 \Rightarrow \prod_{\substack{\operatorname{Im}\rho>0 \\ \rho \neq \rho_0}} \left(1 - \frac{\overline{\rho_0}}{\rho}\right)\left(1 - \frac{\overline{\rho_0}}{1-\rho}\right) = 0$$

$\because \xi(s)$ is Absolutely convergent infinite product so it is a convergent infinite product. [Ref. 5_2 Pg 290]

Value of a cgt. infinite product is zero iff atleast one of the factors is 0

$\Rightarrow$ $1 - \frac{\overline{\rho_0}}{\rho} = 0$ or $1 - \frac{\overline{\rho_0}}{1-\rho} = 0$; $\rho \neq \rho_0$

$\Rightarrow$ $\rho = \overline{\rho_0}$ or $1-\rho = \overline{\rho_0}$

$\rho = \overline{\rho_0}$ or $\rho = 1 - \overline{\rho_0}$

which is a contradiction [Ref. 6, Pg. 9]

The appearance of the ~~factor~~ zero $\overline{\rho_0}$ will imply inclusion of factor $1 - \frac{\overline{\rho}}{\overline{\rho_0}}$

~~which i~~ Now $g(\rho) = \overline{\rho}$ is not analytic w.r.t. ρ

$\therefore$ $\left(1 - \frac{\overline{\rho}}{\overline{\rho_0}}\right)$ is not differentiable w.r.t ρ.

$\therefore$ $\prod \left(1 - \frac{\overline{\rho}}{\overline{\rho_0}}\right)\left(1 - \frac{\overline{\rho}}{1-\overline{\rho_0}}\right)$ will not be Analytic.

But $E(s)$ was Analytic.

$\longrightarrow \longleftarrow$

|||ly, Appearance of the zero $1-\overline{\rho_0}$ will imply inclusion of factor $1 - \frac{\overline{\rho}}{1-\overline{\rho_0}}$ which makes $E(s)$ non analytic

$\therefore \quad I_{\rho, \overline{\rho_0}} = 0 \quad \Rightarrow$ A contradiction

$\therefore \; I_{\rho, \overline{\rho_0}} \neq 0$

(1) 2)

~~2i Im(ρ0)~~

$$2 i \, \mathrm{Im}(\rho_0) \cdot \left[1 - 2\,\mathrm{Re}(\rho_0)\right] I_{\rho, \overline{\rho_0}} = 0$$

$\because \; \mathrm{Im}(\rho) > 0 \quad \Rightarrow \mathrm{Im}\,\rho_0 > 0$

$I_{\rho, \rho_0} \neq 0$

$$\Rightarrow \quad 1 - 2\,\mathrm{Re}(\rho_0) = 0$$

$$\Rightarrow \boxed{\mathrm{Re}(\rho_0) = \frac{1}{2}}$$

$\therefore \; \epsilon(\rho_0) = 0 \quad \Rightarrow \quad \mathrm{Re}(\rho_0) = \frac{1}{2}$

This proves R.H.