On the distribution of the nontrivial zeros of Riemann Zeta on the critical line Re(z)=1/2

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Abstract:

The Riemann Zeta function or Euler–Riemann Zeta function, ζ(s), is a function of a complex variable z that analytically continues the sum of the Dirichlet series:

[1] $$\zeta(z) = \sum_{k=1}^{\infty} k^{-z}$$

The Riemann zeta function is a meromorphic function on the whole complex z-plane, which is holomorphic everywhere except for a simple pole at z = 1 with residue 1. One of the most important advance in the study of Prime numbers was the paper by Bernhard Riemann in November 1859 called "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse" (On the number of primes less than a given quantity). In this paper, Riemann gave a formula for the number of primes less than x in terms the integral of 1/log(x), and also provided insights into the roots (zeros) of the zeta function, formulating a conjecture about the location of the zeros of $\zeta(z)$ in the critical line Re(z)=1/2.

The Riemann Zeta function is one of the most studied and well-known mathematical functions in history. In this paper, we will prove that the distribution of the nontrivial zeros of the Riemann Zeta function in the critical line (Re(z)=1/2) is not random. There is a relationship between the values of those zeros and the Harmonic function that leads to an algebraic relationship between any two zeros. We will also show a simple code to obtain zeros based on the Harmonic function.

1. Nomenclature and conventions

   a. $\zeta(z) = \sum_{k=1}^{n} k^{-z}$ is the Zeta function of Riemann

   b. z*: any nontrivial solution of the Zeta function in the critical line Re(z)=1/2. By default, a reference to zero of ζ(z) will refer to a nontrivial zero of ζ(z).

   c. ß(n) is the n^{th} zero of the Riemann function in the critical line Re(z)=1/2 . e.g. b1=14.134725…

   d. α=Re(z) is the real part of z

   e. ß=Im(z) is the imaginary part of z

   f. If z=α+iß, we define Modulus(z)= |z|= $\sqrt{(\alpha^2+ß^2)}$

2. An analytic continuation for $\zeta(z)$ for Re(z)>0

From (2, Caceres) one can express ζ(z) as the difference of two functions using the following decomposition in Re(z)≥0, z≠1:

[1] $\zeta(z) = X(z) - Y(z)$, where:

[2] $X(z,n) = (\sum_{k=1}^{n} k^{-\alpha} (\cos(ß * \ln(k)) + \frac{1}{2} n^{-\alpha} \cos(ß \ln(n)) + $
$+ i * (\sum_{k=1}^{n} k^{-\alpha} (\sin(ß * \ln(k)) + \frac{1}{2} n^{-\alpha} \sin(ß \ln(n)))$

and: $X(z) = \lim_{n\to\infty} X(z,n)$

[3] $Y(z,n) = n^{(1-\alpha)} \frac{1}{[(1-\alpha)^2 + ß^2]} [((1-\alpha) * \cos(ß \ln(n)) + ß * \sin(ß \ln(n))) +$
$+ i \ (ß * \cos(ß \ln(n)) - (1-\alpha) * \sin(ß \ln(n)))]$

and: $Y(z) = \lim_{n\to\infty} Y(z,n)$

From [2] one can calculate the module of X(z) and the following limit:

[4] $\lim_{n\to\infty} (|X(z,n)|^2 / n) = \lim_{n\to\infty} \frac{1}{n} (\sum_{k=1}^{n} k^{-2\alpha} + \sum_{k=1}^{n} \sum_{j\neq k}^{n} k^{-\alpha} * j^{-\alpha} * \cos\left(ß \left(\ln\left(\frac{k}{j}\right)\right)\right))$

For α=1/2, one can express ($|X(z,n)|^2/n$) as: (Konrad)

[5] $\lim_{n\to\infty} (|X(z,n)|^2 / n) =$

$$= \lim_{n\to\infty} \frac{1}{n} (\sum_{k=1}^{n} k^{-1} + \sum_{k=1}^{n} \sum_{j\neq k}^{n} k^{-1/2} * j^{-1/2} * \cos\left(ß \left(\ln\left(\frac{k}{j}\right)\right)\right))$$

$$= \lim_{n\to\infty} \frac{1}{n} \left(\sum_{k=1}^{n} k^{-1}\right) + \lim_{n\to\infty} \frac{1}{n} (\sum_{k=1}^{n} \sum_{j\neq k}^{n} k^{-1/2} * j^{-1/2} * \cos\left(ß \left(\ln\left(\frac{k}{j}\right)\right)\right)) =$$

$$= 0 + \lim_{n\to\infty} \frac{1}{n} (\sum_{k=1}^{n} \sum_{j\neq k}^{n} k^{-1/2} * j^{-1/2} * \cos\left(ß \left(\ln\left(\frac{k}{j}\right)\right)\right)) =$$

$$= \lim_{n\to\infty} \frac{2n}{n} (\sum_{j=1}^{n-1} n^{-1/2} * j^{-1/2} * \cos\left(ß \left(\ln\left(\frac{n}{j}\right)\right)\right)) =$$

$$= \lim_{n\to\infty} 2(n^{-\frac{1}{2}} \sum_{j=1}^{n-1} * j^{-\frac{1}{2}} * \cos\left(ß \left(\ln\left(\frac{n}{j}\right)\right)\right)) =$$

Using the integral approximation of the infinite series

$$= 2 * \lim_{n\to\infty} \frac{2*\sqrt{n}*\cos\left(ß*ln\left(\frac{n}{n}\right)\right) - 2*ß*\sin\left(ß*\ln\left(\frac{n}{n}\right)\right)}{4*ß^2+1} * n^{-\frac{1}{2}}$$

$$= 2 * \frac{2*\sqrt{n}}{4*ß^2+1} n^{-\frac{1}{2}} = 2 * \frac{2}{4*ß^2+1} = \frac{1}{ß^2+1/4}$$

So, if $\lim_{n\to\infty}(|X(z,n)|^2/n)$ exists will be equal to:

[6] $\lim_{n\to\infty}(|X(z,n)|^2/n) = \frac{1}{ß^2+1/4}$ if z=1/2+iß

3. Calculating the nontrivial zeros of ζ(z) using the Harmonic function

From [4] and [6], and for any z*=α+ßi , a nontrivial zero of Zeta in the critical line α=1/2, one can write:

[7] $\sum_{k=1}^{n} k^{-1} \rightarrow \frac{n}{(ß^2+\frac{1}{4})} - \sum_{k=1}^{n}\sum_{j\neq k}^{n} k^{-1/2} * j^{-1/2} * cos\ (ß\,(ln\left(\frac{k}{j}\right))$ when $n \rightarrow \infty$

And simplifying the expression creating functions $O(n)$ and $X(n)$:

[8] $O(n) = -\sum_{k=1}^{n}\sum_{j\neq k}^{n} k^{-1/2} * j^{-1/2} * cos\ (ß\,(ln\left(\frac{k}{j}\right)$

And

[9] $P(n) = \frac{n}{(ß^2+\frac{1}{4})}$

To write:

[10] $\sum_{k=1}^{n} k^{-1} \rightarrow O(n) + P(n)$ when $n \rightarrow \infty$

Where $\sum_{k=1}^{n} k^{-1}$ is the Harmonic function *H(n)*.

[11] $H(n) = \sum_{k=1}^{n} k^{-1}$

From the definition of limit, one can say that for any ε arbitrarily small, there exists and *N* such that for any *n>N*:

[12] $H(n) - \left(O(n) + P(n)\right) < \varepsilon$

Representation of H(n), O(n), and P(n)
P(n) is a straight line with slope $\frac{1}{(ß^2+1/4)}$

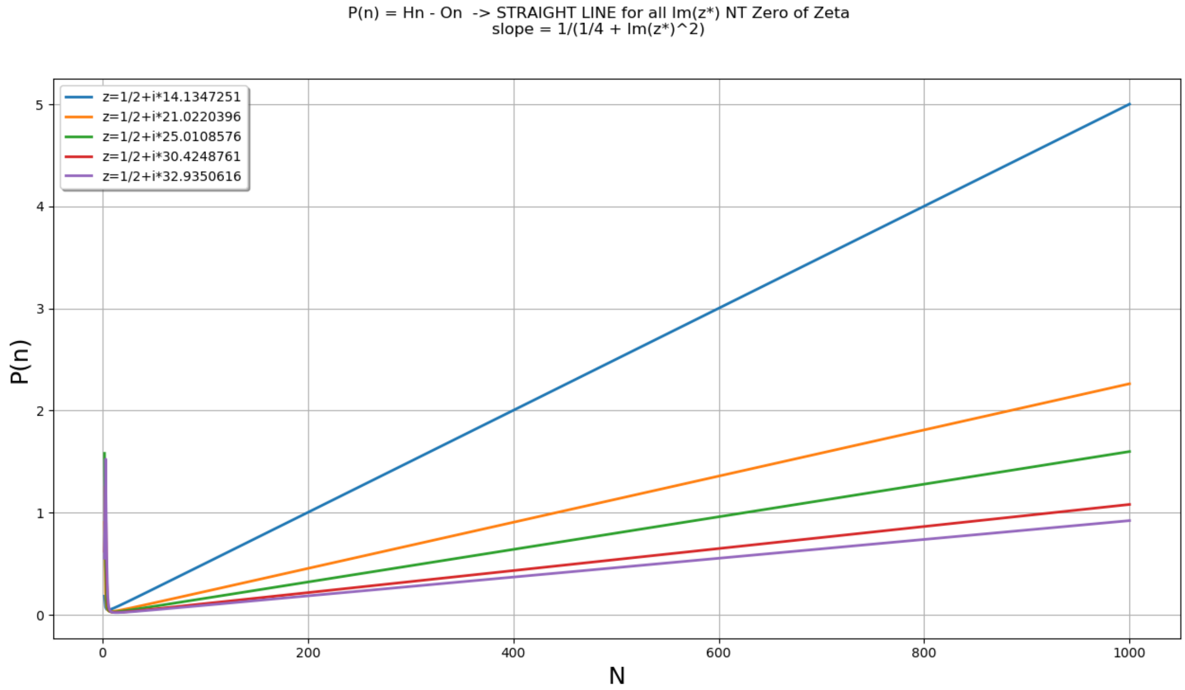

Figure 1. Straight Lines P(n)

[12] can be used to create an algorithm to find the nontrivial zeros of zeta in the critical line without knowing any of them based on their connection to the Harmonic function.

An example of a Python code to calculate the zeros of zeta in the critical line with 1 decimal places accuracy based on [12]:

```
# __Python 3.7
# __Pedro Caceres__ 2020 Feb 17
#Rough code to find zeros of Riemann Zeta using the Harmonic function
harmo   = 0
epsilon = 0.01
nn      = 50

for j in range(1,nn):
     harmo += 1 / j
print('Harmonic(',nn,')=', harmo)

for b in range (1,500):
     b = b / 10
     a1   = nn/((1-alfa)**2 + b**2)
     b1 = 0
     for k in range(1,nn):
         for j in range(1,nn):
             if j!=k:
                 b1 +=  (k*j)**(-alfa)  * m.cos(b * m.log(k/j))
     h1=a1-b1
```

```
        if abs(h1-harmo) < epsilon:
            print('------> Solution beta=',b, '  ... and->', h1-harmo)
#end_of_code
```

This code tends the following results:

```
Harmonic( 50 )= 4.4792053383294235

--------------> Solution beta= 14.1   ... and error -> 0.0067952158225219605
--------------> Solution beta= 25.0   ... and error -> -0.008460202279115592
--------------> Solution beta= 30.4   ... and error -> 0.002423758745334403 4
--------------> Solution beta= 37.6   ... and error -> 0.0012958863904977136
--------------> Solution beta= 40.9   ... and error -> -0.009083573623293262
--------------> Solution beta= 48.0   ... and error -> -0.0027214317425938717
--------------> Solution beta= 49.6   ... and error -> 0.0024275253143217768
```

These values compared to:

ß(1) = 14.134725142
ß(3) = 25.010857580
ß(4) = 30.424876126
ß(6) = 37.586178159
ß(7) = 40.918719012
ß(9) = 48.005150881
ß(10) = 49.773832478

Changing the values of "n" and epsilon, we can increase the accuracy of the results.

The fact that the Harmonic function, H_n , can be expressed in an infinite number of ways as a function of any ß=Im(z) imaginary part of a nontrivial solution of ζ(z), provides also an algorithm to calculate all nontrivial zeros from any known zero. If we define the function:

[12] $$H(\alpha, ß, n) -> \frac{n}{(ß^2+\frac{1}{4})} - \sum_{k=1}^{n}\sum_{j\neq k}^{n} k^{-\frac{1}{2}} * j^{-\frac{1}{2}} * \cos\left(ß * \left(ln\left(\frac{k}{j}\right)\right)\right) \text{ when } n \to \infty$$

For α=1/2, and ε arbitrarily small, for any two nontrivial zeros of zeta $(\alpha, ß_1)$ and $(\alpha, ß_2)$, there exists and N such that for any $n>N$:

[13] $$H\left(\alpha = \frac{1}{2}, ß_1, n\right) - H\left(\alpha = \frac{1}{2}, ß_2, n\right) < \varepsilon$$

This proposition means that the nontrivial zeros of the Riemann Zeta are not distributed randomly, and they follow a defined structure.

Sample code to show how [13] can be used to find zeros based on a known zero:

```
# Code to find zeros from any known zero
# __Pedro Caceres__ 2020 Feb 17
nn = 60 #Not really high. Used for a rough calculation

epsilon = 0.00002
# Known Zero ß(1)
zero= 14.134725142

#Calculating H(1/2,zero,n) = a - b
a2  = nn/((1-alfa)**2 + zero**2)
b2=0
```

```
for k in range(1, nn):
    for j in range(1, nn):
        if j != k:
            b2 += (k * j) ** (-alfa) * m.cos(zero * m.log(k / j))

h2 = a2-b2 #H2 to compare against

# range to find additional zeros of zeta
for b in range (245000,310000): #adding digits increases accuracy
    b = b / 10000

    #Calculating a, b
    a1   = nn/((1-alfa)**2 + b**2)
    b1 = 0
    for k in range(1,nn):
        for j in range(1,nn):
            if j!=k:
                b1 += (k*j)**(-alfa) * m.cos(b * m.log(k/j))

    #Calculating H1
    h1=a1-b1

    #If error < eosilon, then print potential zero
    if abs(h1-h2) < epsilon:
        print('-----------> Solution beta=',b, '  ... and error ->', h1-h2)

#end_of_code
```

Results:

```
-----> Solution beta= 25.0155   ... and error -> +1.442262027140373 e-05
-----> Solution beta= 30.4385   ... and error -> -1.140533215249206 e-05
```

These values compared to:

ß(3) = 25.010857580
ß(4) = 30.424876126

Changing the values of “nn” and epsilon in the code, the accuracy can be increased to more decimal places.

4. Conclusion

The distribution of the nontrivial zeros of the Riemann Zeta function in the critical line is not random. They are located in values of Im(z)=ß that verify that for any ß, and ε arbitrarily small, there exists and *N* such that for any *n>N*:

[14]

$$\sum_{k=1}^{n} k^{-1} - (\frac{n}{(ß^2+\frac{1}{4})} - \sum_{k=1}^{n}\sum_{j\neq k}^{n} k^{-\frac{1}{2}} * j^{-\frac{1}{2}} * cos\,(ß\,(ln\left(\frac{k}{j}\right))) < \varepsilon$$

REFERENCES

(1) B. Riemann, “Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse”, Monat. der Königl. Preuss. Akad. der Wissen. zu Berlin aus der Jahre 1859 (1860), 671–680; also, Gesammelte math. Werke und wissensch. Nachlass, 2. Aufl. 1892, 145–155.

(2) Caceres, Pedro. “$\zeta(z) = X(z) - Y(z)$ A decomposition of the Riemann Zeta function for $Re(z) > 0, z \neq 1$”, 2020. VIXRA: https://vixra.org/abs/2003.0189

(3) Kopp, Konrad. “Theory and Application of Infinite Series.” Blackie & Son Limited, London, 1928

(4) Sondow, Jonathan and Weisstein, Eric W. "Riemann Zeta Function." From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/RiemannZetaFunction.html