Kantowski-Sachs Cosmology, Weyl Geometry and Asymptotic Safety in Quantum Gravity *

Carlos Castro Perelman
Center for Theoretical Studies of Physical Systems
Clark Atlanta University, Atlanta, GA. 30314
Ronin Institute, 127 Haddon Pl., NJ. 07043
perelmanc@hotmail.com

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Abstract

A brief review of the essentials of Asymptotic Safety and the Renormalization Group (RG) improvement of the Schwarzschild Black Hole that removes the $r = 0$ singularity is presented. It is followed with a RG-improvement of the Kantowski-Sachs metric associated with a Schwarzschild black hole interior and such that there is *no* singularity at $t = 0$ due to the running Newtonian coupling $G(t)$ (vanishing at $t = 0$). Two temporal horizons at $t_- \simeq t_P$ and $t_+ \simeq t_H$ are found. For times below the Planck scale $t < t_P$, and above the Hubble time $t > t_H$, the components of the Kantowski-Sachs metric exhibit a key sign *change*, so the roles of the spatial z and temporal t coordinates are *exchanged*, and one recovers a *repulsive* inflationary de Sitter-like core around $z = 0$, and a Schwarzschild-like metric in the exterior region $z > R_H = 2G_oM$. The inclusion of a running cosmological constant $\Lambda(t)$ follows. We proceed with the study of a dilaton-gravity (scalar-tensor theory) system within the context of Weyl's geometry that permits to single out the expression for the *classical* potential $V(\phi) = \kappa\phi^4$, instead of being introduced by hand, and find a family of metric solutions which are conformally equivalent to the (Anti) de Sitter metric. To conclude, an ansatz for the truncated effective average action of ordinary dilaton-gravity in Riemannian geometry is introduced, and a RG-improved Cosmology based on the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric is explored where instead of recurring to the cutoff identification $k = k(t) = \xi H(t)$, based on the Hubble function $H(t)$, with ξ a positive constant, one has now $k = k(t) = \xi\phi(t)$, when ϕ is a positive-definite dilaton scalar field which is monotonically decreasing with time.

*Dedicated to the loving memory of Irina Novikova, a brilliant and heavenly creature who met a tragic death at a young age.

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1 Introduction

The problem of dark energy and the solution to the cosmological constant problem is one of the most challenging problems facing Cosmology today. There are a vast numerable proposals for its solution. Two proposed forms for dark energy are the cosmological constant, representing a constant energy density filling space homogeneously, and scalar fields such as quintessence or moduli, dynamic quantities whose energy density can vary in time and space. The nature of dark energy is more hypothetical than that of dark matter, and many things about the nature of dark energy remain matters of speculation. Dark energy is thought to be very homogeneous, not very dense and is not known to interact through any of the fundamental forces other than gravity. In the models based on the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric, it can be shown that a strong constant negative pressure in all the universe causes an acceleration in universe expansion if the universe is already expanding, or a deceleration in universe contraction if the universe is already contracting. This accelerating expansion effect is sometimes labeled "gravitational repulsion".

A major outstanding problem is that quantum field theories predict a huge cosmological constant, more than 100 orders of magnitude too large. This would need to be almost, but not exactly, cancelled by an equally large term of the opposite sign. Some supersymmetric theories require a cosmological constant that is exactly zero, which does not help because supersymmetry must be broken. Nonetheless, the cosmological constant is the most economical solution to the problem of cosmic acceleration. Thus, the current standard model of cosmology, the Lambda-CDM (cold dark matter) model, includes the cosmological constant as an essential feature. We refer to [23], [24], [30] and many references therein.

In quintessence models of dark energy, the observed acceleration of the scale factor is caused by the potential energy of a dynamical field, referred to as quintessence field. Quintessence differs from the cosmological constant in that it can vary in space and time. In order for it not to clump and form structure like matter, the field must be very light so that it has a large Compton wavelength. This class of theories attempts to come up with an all-encompassing theory of both dark matter and dark energy as a single phenomenon that modifies the laws of gravity at various scales.

The Asymptotic Safety program initiated by Weinberg [1] is based on a non-Gaussian (interacting) fixed point of the gravitational renormalization group flow. It provides a mechanism for completing the gravitational force at very high energies. The ultraviolet (UV) fixed point controls the scaling of couplings such that unphysical divergences are absent while the emergence of classical low-energy physics is linked to a crossover between two renormalization group fixed points. These features make Asymptotic Safety an attractive framework for cosmological model building leading to scenarios which may naturally give rise to a quantum gravity driven inflationary phase in the very early universe and an almost scale-free fluctuation spectrum [3], [13].

The main motivation for this work is three-fold. (i) To regularize the black-hole Schwarzschild and Kantowski-Sachs metrics at $r = 0, t = 0$, respectively. (ii) To explore further the Asymptotic Safety program in cosmology. (iii) To describe the role that Weyl gravity plays in determining the observed vacuum energy density and which is based in the quintessence models of dark energy. The main finding in the study of a dilaton-Weyl gravity system is that it allows to derive the functional form of the scalar potential and obtain the observed vacuum energy density after fixing the Weyl scale invariance. The other relevant finding is that the observed vacuum energy density result can also be obtained from the Renormalization Group (RG) improvement of the Kantowski-Sachs metric (associated with a Schwarzschild black hole interior) after relating the running Newtonian coupling $G(t)$ to the dilaton scalar field $\phi(t)$ as prescribed by Jordan-Brans-Dicke (JBD) gravity and given by $G(t) = \frac{1}{16\pi\phi^2(t)}$ [19].

This work is organized as follows. In **2.1** we briefly review the essentials of Asymptotic Safety [1], [3] and the Renormalization Group (RG) improvement of the Schwarzschild Black Hole [3], [5]. In **2.2** we model our Universe as a homogeneous *anisotropic* self-gravitating fluid consistent with the Kantowski-Sachs homogeneous anisotropic cosmology. A dynamical regularization of the Kantowski-Sachs metric associated with a black hole interior is performed such that there is *no* singularity at $t = 0$. Two temporal horizons at $t_- \simeq t_P$ and $t_+ \simeq t_H$ are found. For times below the Planck scale $t < t_P$, and above the Hubble time $t > t_H$, the components of the Kantowski-Sachs metric exhibit a key sign *change*, so the roles of the spatial z and temporal t coordinates are *exchanged*, and one recovers a *repulsive* inflationary de Sitter-like core around $z = 0$, and a Schwarzschild-like metric in the exterior region $z > R_H = 2G_oM$.

In **2.3** the inclusion of a running Cosmological Constant $\Lambda(t)$ is studied. A coupled system of two first-order non-linear differential equations is found. Their origin stems from the *running* gravitational coupling $G(t)$, and cosmological constant $\Lambda(t)$, combined with the RG improvement of the Einstein field equations with a cosmological constant, which is associated with another Kantowski-Sachs-like metric. The solutions to these first-order non-linear differential equations furnish the temporal dependence of $G(t), \Lambda(t)$. Consistency requires that one should recover the observed vacuum energy density in the asymptotic $t \to \infty$ limit : $\rho_{vac}(t \to \infty) \to \frac{\Lambda_o}{8\pi G_o} \simeq 10^{-122} M_P^4$.

In **3.1** a dilaton-gravity (scalar-tensor theory) system within the context of Weyl's geometry is studied, a quartic scalar potential is singled out and a family of metric solutions which are conformally equivalent to the (Anti) de Sitter metric are found. Finally, section **3.2** is devoted to the study of dilaton-gravity based entirely on Riemannian geometry. A typical ansatz for the *truncated* effective average action in dilaton-gravity (DG) of the Jordan frame is introduced, and a RG-improved Cosmology based on the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric is explored where instead of recurring to the cutoff identification $k = k(t) = \xi H(t)$, based on the Hubble function $H(t)$, with ξ a positive constant [13], we have now $k = k(t) = \xi\phi(t)$ in the particular case where ϕ is a positive-definite dilaton scalar field which is monotonically decreasing with time, and reaching zero at $t = \infty$. Solutions for $\phi(t)$ and the Hubble function $H(t) = \frac{\dot{a}}{a}$ are found.

2 Renormalization Group-Improved Gravity

2.1 Asymptotic Safety and RG improvement of the Schwarzschild Black Hole

Testing Asymptotic Safety [1] at the conceptual level requires the ability to construct approximations of the gravitational renormalization group (RG) flow beyond the realm of perturbation theory. A very powerful framework for carrying out such computations is the Wetterich-Morris functional renormalization group equation (FRGE) for the gravitational effective average action Γ_k [2]

$$k\frac{\partial \Gamma_k[g,\bar{g}]}{\partial k} = \frac{1}{2}\, Tr\ [\ \frac{k\partial_k \mathcal{R}_k}{\Gamma^{(2)} + \mathcal{R}_k}] \tag{2.1}$$

where k is the RG mass scale. The construction of the FRGE uses the background field formalism, splitting the metric $g_{\mu\nu}$ into a fixed background $\bar{g}_{\mu\nu}$ and fluctuations $h_{\mu\nu}$. The Hessian $\Gamma_k^{(2)}$ is the second functional derivative of Γ_k with respect to the fluctuation field at a fixed background. The infrared regulator $\mathcal{R}_k$ provides a scale-dependent mass term suppressing fluctuactions with momenta $p^2 < k^2$, while integrating out those with $p^2 > k^2$. The functional trace (matrix-valued operator trace) Tr stands for summation over internal indices, and integration over spacetime and momenta. It appears with positive sign for bosonic fields; a negative sign for fermionic ones, Grassmann odd fields (ghosts), and a factor of two for complex fields.

The arguably simplest approximation of the gravitational RG flow is obtained from projecting the FRGE onto the Einstein-Hilbert action approximating Γ_k by [3], [13]

$$\Gamma_k = \int d^4x\ \sqrt{|g|}\ \frac{1}{16\pi G(k)}\ [\ R(g_{\mu\nu})\ - 2\ \Lambda(k)\ ]\ +\ \cdots \tag{2.2}$$

where the ellipsis $\cdots$ denote the gauge fixing and ghost terms. This ansatz comprises two scale-dependent coupling constants, Newton's constant G_k and a cosmological constant Λ_k. The scale-dependence of these couplings is conveniently expressed in terms of their dimensionless counterparts $\lambda_k \equiv \Lambda_k k^{-2}$; $g_k \equiv G_k k^2$, and captured by the beta functions

$$\beta_g(g_k,\lambda_k) = k\partial_k g_k, \quad \beta_\lambda(g_k,\lambda_k) = k\partial_k\lambda_k \tag{2.3}$$

Eq-(1) yields a system of coupled differential equations determining the scale-dependence of $G(k), \Lambda(k)$. The interacting (non-Gaussian) ultra-violet (UV) fixed points are determined by the conditions $\beta_g(g_*, \lambda_*) = 0; \beta_\lambda(g_*, \lambda_*) = 0$, with $g_* \neq 0; \lambda_* \neq 0$ and are postulated to correspond to a scale invariant field theory.

The Renormalization group flow of the gravitational coupling and cosmological constant in Asymptotic Safety was studied by [3]. An approximate solution to the scale dependence of $G(k)$ and $\Lambda(k)$ was found to be

$$G(k^2) \ = \ \frac{G_o}{1 \ + \ g_*^{-1} \ G_o \ k^2}, \ \ \Lambda(k) \ = \ \Lambda_o \ + \ \frac{b \ G(k)}{4} \ k^4, \ \ \Lambda_o > 0, \ \ b > 0 \quad (2.4a)$$

In $D = 4$, the dimensionless gravitational coupling has a nontrivial fixed point $g = G(k)k^2 \to g_*$ in the $k \to \infty$ limit, and the dimensionless variable $\lambda = \Lambda(k)k^{-2}$ has also a nontrivial ultraviolet fixed point $\lambda_* \neq 0$ [3].

The interacting (non-Gaussian) fixed points $g_k = G(k) \ k^2$, and $\lambda_k = \Lambda(k) \ k^{-2}$ in the ultraviolet limit $k \to \infty$ turned out to be, respectively, [4]

$$g_* \ = \ 0.707, \ \ \lambda_* \ = \ 0.193, \ \ b \ = \ 4\frac{\lambda_*}{g_*}, \quad (2.4b)$$

G_o and Λ_o are the present day value of the Newtonian gravitational coupling and the cosmological constant. The infrared limits are $\Lambda(k \to 0) = \Lambda_0 > 0, \ G(k \to 0) = G_o = G_N$. Whereas the ultraviolet limits are $\Lambda(k = \infty) = \infty; G(k = \infty) = 0$. One should note that as the fixed-point values actually depend on the regularization scheme employed, and the gauge chosen, the fixed point values presented in (2.4b) should not be taken at face value. However, this fact shall not affect our results which follow.

The results in eq-(2.4b) have been used by several authors, see [3], [5] and references therein, to construct a renormalization group (RG) improvement of the Schwarzschild Black-Hole Spacetime by recurring to the correspondence $k^2 \to k^2(r)$, which is based in constructing a judicious monotonically decreasing function $k^2 = k^2(r)$, and which in turn allows to replace $G(k^2) \to G(r)$.

Let us start with the renormalization-group improved Schwarzschild black-hole metric [3]

$$(ds)^2 \ = \ - \ (1 - \frac{2G(r)M_o}{r})(dt)^2 \ + \ (1 - \frac{2G(r)M_o}{r})^{-1}(dr)^2 \ + \ r^2(d\Omega_2)^2 \quad (2.5)$$

based on the Renormalization group flow of $G(r)$ in the Asymptotic Safety program [1]. The metric (2.5) is *not* a solution of the vacuum field equations but instead is a solution to the modified Einstein equations $G^\mu_\nu = 8\pi G(r) T^\mu_\nu$ where the *running* Newtonian coupling $G(r)$ and an effective stress energy tensor

$$T^\mu_\nu \ \equiv \ diag \ (-\rho(r), \ p_r(r), \ p_\theta(r), \ p_\varphi(r)) \quad (2.6)$$

appears in the right hand side. The components of T^μ_ν associated to the modified Einstein equations $G^\mu_\nu = 8\pi G(r) T^\mu_\nu$ are respectively given by

$$\rho \ = \ - \ p_r \ = \ \frac{M}{4\pi r^2 G(r)} \ \frac{dG(r)}{dr}, \ \ p_\theta \ = \ p_\varphi \ = \ - \ \frac{M}{8\pi r G(r)} \ \frac{d^2 G(r)}{dr^2} \tag{2.7}$$

The energy-momentum tensor is in this case an $effective$ stress energy tensor resulting from vacuum polarizations effects of the quantum gravitational field [8] (like a quantum-gravitational self-energy). As explained by [5], the quantum system is self-sustaining: a small variation of the Newton's constant triggers a ripple effect, consisting of successive back-reactions of the semi-classical background spacetime which, in turn, provokes further variations of the Newton's coupling and so forth.

As a result, the sequence of RG improvements is completely determined by a series of recursive relations. In the limiting case, after choosing the following monotonically decreasing function $k^2 = k^2(r) = \xi G_o \rho(r) = \frac{\xi G_o M}{4\pi r^2 G(r)} \frac{dG(r)}{dr}$, where ξ is a positive constant, and upon substituting $k^2 = k^2[\rho(r)]$ into the right-hand side of the running gravitational coupling $G(k^2(r))$ in eq-(4), leads to the differential equation for the sought-after functional form of $G(r)$

$$G(r) \ = \ \frac{G_o}{1 \ + \ g_*^{-1} \ G_o \ k^2[\rho(r)]}, \ \ k^2[\rho(r)] \ = \ \xi \ G_o \ \rho(r) \ = \ \frac{\xi G_o M}{4\pi r^2 G(r)} \ \frac{dG(r)}{dr} \tag{2.8}$$

The solution to the differential equation is [5]

$$G(r) \ = \ G_o \ (1 - e^{-r^3/r_s l^2_{cr}}); \ \ r_s \ = \ 2 G_o M, \ \ l_{cr} \ = \ \sqrt{\frac{3\xi}{8\pi g_*}} \ L_{Planck} \tag{2.9}$$

leading to a Dymnikova-type of metric [16] in eq-(5). We shall choose $\xi = (8\pi g_*/3) \Rightarrow l_{cr} = L_P$, where L_P is the PLanck length scale. A simple inspection reveals that there is *no* singularity at $r = 0$. An expansion of the exponential gives for very small values of $r: \ 1 - (2G(r)M)/r) \simeq 1 - (2G_o M r^2/2G_o M L_P^2) = 1 - (r^2/L_P^2)$, and one recovers a *repulsive* de Sitter core around the origin $r = 0$. Hence, the key result of [5] is that if the gravitational renormalization group (RG) flow attains a non-trivial fixed point at high energies, the back-reaction effects produced by the running Newton's coupling leads to an iterated sequence of recurrence relations which converges to a "renormalized" black-hole spacetime of the Dymnikova-type, which is free of singularities.

A more detailed analysis of the quantum-improved Ricci and Kretschmann scalars for the Schwarzschild black hole can be found in [5] where due to a cancellation between the classical tidal forces, described by the classical Kretschmann scalar K_{class}, and their quantum counterpart, controlled by the tension generated by the self-energy density ρ, the resulting "renormalized" tidal forces are finite and the spacetime is devoid of singularities.

In particular, a repulsive de Sitter behavior has also been been found in [7], and also in the gravastar (gravitational vacuum star) picture, proposed by [6] where the gravastar has an effective phase transition at/near where the event horizon is expected to form, and the interior is replaced by a de Sitter condensate. Based on these ideas, and the RG-improved black-hole solutions resulting from Asymptotic Safety, we shall proceed with our proposal that our Universe could be seen as Gravitating Vacuum State inside a Black-Hole.

2.2 Kantowski-Sachs metric and Schwarzschild Black Hole Interior

Adopting the units $c = 1$, and after replacing the radial variable r for the spatial coordinate z (for reasons explained below), the Kantowski-Sachs metric associated with the *interior* region of Schwarzschild black-hole is given by

$$(ds)^2 \;=\; - \;\left(\frac{2G_oM}{t} \;-\; 1 \right)^{-1} (dt)^2 \;+\; \left(\frac{2G_oM}{t} \;-\; 1 \right) (dz)^2 \;\;+\; t^2 (d\Omega_2)^2 \tag{2.10}$$

Such metric is a solution of the Einstein vacuum field equations and was analyzed in full detail by the authors [9]. As it is well known to the experts inside the black-hole horizon region the roles of r and t are exchanged. The Kantowski-Sachs metrics [11] are associated with spatially homogeneous *anisotropic* relativistic cosmological models [11].

The black hole mass parameter M in (10) assumes the role now of a characteristic *time* (divided by G_o) for the existence of universes inside the interior Schwarzschild solution, as may be inferred from the cosmological interpretation of the interior metric [9]. For example, in Black-Hole Cosmology [17] one sets M to coincide with the mass of the Universe enclosed inside the Hubble horizon radius R_H, and which also coincides with the Schwarzschild radius $2G_oM$. Hence, the characteristic time will be set equal to the Hubble horizon time $t_H \equiv 2G_oM$. An interesting numerical cincidence is that the uniform density over a spherical ball of radius R_H given by $M/(4\pi/3)R_H^3 = \frac{3}{8\pi G_o R_H^2}$ coincides precisely with the observed critical density (also vacuum density) of our universe. For more details of Black Hole cosmology see [17].

Given the metric (10), the scalar Kretschmann invariant polynomial is

$$K = R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} = 48 \; \frac{(G_oM)^2}{t^6}$$

showing that a curvature singularity occurs at $t = 0$. This is our main motivation to recur to Asymptotic Safety in Quantum Gravity in oder to provide a Renormalization-Group improved version of the Kantowski-Sachs metric (10), and show that there is *no* longer a singularity at $t = 0$. Furthermore, we shall also include a running cosmological "constant" $\Lambda(t)$, besides a running gravitational coupling $G(t)$, when we evaluate the variable vacuum energy density, and which asymptotically should tend to the observed (extremely small) vacuum energy density $10^{-122} M_P^4$.

The most salient features of the metric (10) were rigorously examined by [9]. In particular, the study of null and timelike geodesics. In the null geodesic case, when $\theta = \frac{\pi}{2}$ and $dz = 0$, they noted that these are *not* circular orbits, as the z coordinate can *no* longer be considered as a radial coordinate. Another surprising result, considering the interior point of view, is that the trajectories of particles at rest are geodesics, contrary to the exterior region where particles at rest are necessarily accelerated. This fact is due to the non-static character of the interior geometry. For explicit details we refer to [9].

The Renormalization Group-improved Kantowski-Sachs metric associated with the interior of a black-hole is given by

$$(ds)^2 \ = \ - \ \left(\frac{2G(t)M}{t} \ - \ 1 \right)^{-1} (dt)^2 \ + \ \left(\frac{2G(t)M}{t} \ - \ 1 \right) (dz)^2 \ + \ t^2 (d\Omega_2)^2 \tag{2.11}$$

The modified Einstein equations are $G^\mu_\nu \ = \ 8\pi G(t) T^\mu_\nu$, where as before, the *running* Newtonian coupling $G(t)$, and the effective stress energy tensor due to vacuum polarizations effects of the quantum gravitational field [8] appear in the right hand side[1].

The energy-momentum tensor corresponding to the modified Einstein equations is

$$T^\mu_\nu \ \equiv \ diag\ (-\rho(t), \ p_z(t), \ p_\theta(t), \ p_\varphi(t)) \tag{2.12}$$

and whose components are respectively given by

$$\rho \ = \ - \ p_z \ = \ \frac{M}{4\pi \ t^2 \ G(t)} \ \frac{dG(t)}{dt}, \quad p_\theta \ = \ p_\varphi \ = \ - \ \frac{M}{8\pi \ t \ G(t)} \ \frac{d^2G(t)}{dt^2} \tag{2.13}$$

After choosing the monotonically decreasing function of time

$$k^2 \ = \ k^2(t) \ = \ \xi \ G_o \ \rho(t) \ = \ \xi \ G_o M \ \frac{(dG(t)/dt)}{4\pi \ t^2 G(t)} \tag{2.14}$$

the running gravitational coupling $G(t)$ obtained in the dynamical renormalization of the Kantowski-Scachs-like metric (10) is given by the solution to the differential equation

$$G(t) \ = \ \frac{G_o}{1 \ + \ g_*^{-1} \ G_o \ k^2[G(t)]}, \quad k^2[G(t)] \ = \ \xi \ G_o \ \rho(t) \ = \ \frac{\xi G_o M}{4\pi t^2 G(t)} \ \frac{dG(t)}{dt} \tag{2.15}$$

The solution to the above differential equation has the same functional form as before

[1]Dirac proposed long ago the possibility of the temporal variation of the fundamental constants

$$G(t) \ = \ G_o \ (1 - e^{-t^3/t_s t_{cr}^2}); \ \ t_s \ = \ 2G_oM = t_H, \ \ t_{cr} \ = \ \sqrt{\frac{3\xi}{8\pi g_*}} \ t_{Planck} \quad (2.16)$$

We shall set again $\xi = (8\pi g_*/3) \Rightarrow t_{cr} = t_P$, Planck's time. Note also that the expression $G(t)$ (16) has the following correspondence (in natural units $\hbar = c = 1$)

$$r \ \leftrightarrow \ t, \ \ l_{cr} = L_P \ \leftrightarrow \ t_P, \ \ r_s = 2G_oM \ \leftrightarrow \ t_H \quad (2.17)$$

with the prior solution $G(r)$ of eq-(2.9). The Schwarzschild radius r_s (black hole horizon) corresponds now to the cosmological horizon R_H (Hubble radius), and the Planck scale L_P corresponds to the Planck time t_P.

Once again, we find that when $G(t)$ is given by eq-(2.16) there is *no* singularity at $t = 0$. Given that $t_{cr} = t_P$; $t_s = 2G_oM = t_H$, a simple expansion of the exponential for very small values of t gives $(2G(t)M/t) - 1 \simeq (t^2/t_P^2) - 1$, leading to no singularity of the metric at $t = 0$. There are two temporal horizons, $t_- \simeq t_P$; $t_+ \simeq t_H$ around the Planck and Hubble time, respectively. Taking the trace of $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G(t) T_{\mu\nu}$ yields the scalar curvature R in terms of the trace of the stress energy tensor $T = -\rho_r + p_z + p_\theta + p_\phi$, which in turn gives

$$R(t) \ = \ \frac{6G_oM}{t_H t_P^2} \ \left(4 \ - \ \frac{3t^3}{t_P^2 t_H} \right) \ e^{-(t^3/t_P^2 t_H)} \quad (2.18a)$$

At $t = 0 \Rightarrow R = \frac{24G_oM}{t_H t_P^2}$ the scalar curvature is finite.

Given $h(t) \equiv \frac{G(t)M}{t}$, after some lengthy but straightforward algebra the modified scalar Kretschmann invariant is given by

$$K \ = \ 16 \ \frac{h(t)^2}{t^4} \ + \ 16 \ \frac{(dh/dt)^2}{t^2} \ + \ 4 \ (\frac{d^2h}{dt^2})^2 \quad (2.18b)$$

Hence, the modified scalar Kretschmann invariant evaluated at $t = 0$ is indeed finite and given by $K(t = 0) = 96(G_oM/t_H t_P^2)^2$. Therefore, one has achieved in eq-(2.11) a dynamical regularization of the Kantowski-Sachs metric : there is *no* singularity at $t = 0$. The scalar curvature (2.48) vanishes in the $t = \infty$ limit.

It is important to emphasize that the cosmic time t in the Kantowski-Sachs metric ranges from 0 to ∞. If the cosmic time could be extended to $-\infty$, in this limit the Ricci scalar (2.18a) would be singular and the problem of the cosmological singularity is only shifted.

Inserting the solution (2.16) found for $G(t)$ into eq-(2.14), $k^2(t)$ becomes

$$k^2(t) \ = \ \xi \ G_oM \ \frac{3}{4\pi t_P^2 t_H} \ \frac{exp(-t^3/t_P^2 t_H)}{(\ 1 \ - \ exp(-t^3/t_P^2 t_H) \ )} \quad (2.19a)$$

with $G_o = t_P^2$, and $2G_oM = t_H$. And the variable energy density is

$$\rho(t) \ = \ M \ \frac{(dG(t)/dt)}{4\pi \ t^2 G(t)} \ = \ \frac{3M}{4\pi t_P^2 t_H} \ \frac{exp(-t^3/t_P^2 t_H)}{(\ 1 \ - \ exp(-t^3/t_P^2 t_H) \ )} \quad (2.19b)$$

The density *blows* up at $t = 0$, and is zero at $t = \infty$. However the Ricci tensor and the scalar curvature (Einstein tensor) are *finite* at $t = 0$. The reason being that when $G(t = 0) = 0; \rho(t = 0) = \infty$, their product $G(t = 0)\rho(t = 0) = \frac{3}{8\pi t_P^2}$ is *finite*.

To sum up : one has attained a dynamical regularization of the Kantowski-Sachs metric (2.11) associated with a black hole interior and that there is *no* singularity at $t = 0$. Two temporal horizons at $t_- \simeq t_P$ and $t_+ \simeq t_H$ are found. For times below the Planck scale $t < t_P$, and above the Hubble time $t > t_H$, the components of the Kantowski-Sachs metric (11) exhibit a key sign *change*, so the roles of the spatial z and temporal coordinates t are *exchanged*, and one recovers a *repulsive* inflationary de Sitter-like core around $z = 0$, and a Schwarzschild-like metric in the exterior region $z > R_H = 2G_oM$.

Concluding, we have modeled our Universe as a homogeneous *anisotropic* self-gravitating fluid consistent with the Kantowski-Sachs homogeneous anisotropic cosmology and Black-Hole cosmology [17]. If one wishes, one can repeat the whole calculations and *include* the running cosmological constant $\Lambda(k^2(t))$ if one desires to identify the running cosmological constant with the running vacuum energy density. Below we shall include the running cosmological constant in order to suitably *modify* the Kantowski-Sachs-like metric (2.10). This will *change* the expression for $k^2(t)$ in (2.19a), and in turn, lead to a very different expression for $\rho(t)$ than the one provided by eq-(2.19b).

2.3 Inclusion of the Running Cosmological Constant

It is known that the running of the cosmological constant re-introduces the black-hole singularity in Schwarzschild-de Sitter spacetimes, as it diverges when the renormalization-group scale approaches infinity, see for instance [14]. This result was based in setting the cutoff identification scale to be $k = \xi d^{-1}(r)$, where $d(r)$ is the classical geodesic proper radial distance which is coordinate invariant. In the proximity to $r = 0$, the classical proper distance scales as $d(r) \sim r^{3/2}$. We shall explore next another possibility based in setting $k^2 = \xi G_o \rho$. Let us introduce the running cosmological constant $\Lambda(t)$ into the following RG improved and modified Kantowski-Sachs metric

$$(ds)^2 = -\left(\frac{2G(t)M}{t} + \frac{\Lambda(t)}{3}t^2 - 1\right)^{-1}(dt)^2 + \left(\frac{2G(t)M}{t} + \frac{\Lambda(t)}{3}t^2 - 1\right)(dz)^2 + t^2\,(d\Omega_2)^2 \tag{2.20}$$

Given the effective stress energy tensor associated with a self-gravitating anisotropic fluid Universe

$$T_\mu^\nu = diag\,(-\,\rho(t),\ p_z(t),\ p_\theta(t),\ p_\phi(t)) \tag{2.21}$$

the modified Einstein equations with a running cosmological and gravitational constant

$$R^{\nu}_{\mu} \ - \frac{1}{2} \ \delta^{\nu}_{\mu} \ R \ + \ \Lambda(t) \ \delta^{\nu}_{\mu} \ = \ 8\pi G(t) \ T^{\nu}_{\mu} \tag{2.22}$$

and corresponding to the metric (20) become

$$\Lambda(t) \ - \ \frac{2M}{t^2} \ \frac{d}{dt} \ \left(\ G(t) \ + \ \frac{\Lambda(t)t^3}{6M} \ \right) \ = \ - \ 8\pi G(t) \ \rho(t) \tag{2.23a}$$

$$\Lambda(t) \ - \ \frac{2M}{t^2} \ \frac{d}{dt} \ \left(\ G(t) \ + \ \frac{\Lambda(t)t^3}{6M} \ \right) \ = \ 8\pi G(t) \ p_z(t) \tag{2.23b}$$

$$\Lambda(t) \ - \ \frac{M}{t} \ \frac{d^2}{dt^2} \ \left(\ G(t) \ + \ \frac{\Lambda(t)t^3}{6M} \ \right) \ = \ 8\pi G(t) \ p_{\theta}(t) \tag{2.23c}$$

$$\Lambda(t) \ - \ \frac{M}{t} \ \frac{d^2}{dt^2} \ \left(\ G(t) \ + \ \frac{\Lambda(t)t^3}{6M} \ \right) \ = \ 8\pi G(t) \ p_{\phi}(t) \tag{2.23d}$$

From eqs-(2.23) one can read-off the expressions for the density and pressure

$$\rho(t) \ = \ - \ p_z(t) \ = \ \frac{M}{4\pi \ t^2 \ G(t)} \ \frac{d}{dt} \ \left(\ G(t) \ + \ \frac{\Lambda(t)t^3}{6M} \ \right) \ - \ \frac{\Lambda(t)}{8\pi G(t)} \ =$$

$$\frac{M}{4\pi t^2 G(t)} \frac{dG(t)}{dt} \ + \ \frac{1}{24\pi} \ \frac{t}{G(t)} \ \frac{d\Lambda(t)}{dt} \tag{2.24}$$

$$p_{\theta\theta} \ = \ p_{\phi\phi} \ = \ \frac{\Lambda(t)}{8\pi G(t)} \ - \ \frac{M}{8\pi \ t \ G(t)} \ \frac{d^2}{dt^2} \ \left(\ G(t) \ + \ \frac{\Lambda(t)t^3}{6M} \ \right) \tag{2.25}$$

The relation $\rho(t) = -p_z(t)$ bears the same form as the dark energy equation of state $\rho = -p$. The conservation of energy $\nabla_{\nu}(8\pi G(t) T^{\nu}_{\mu} - \Lambda(t)\delta^{\nu}_{\mu}) = 0$ follows directly from the Bianchi identities and leads to the relation between $\rho = -p_z$ and the tangential pressure components of the anisotropic fluid (Universe) $p_{\theta} = p_{\phi}$

$$\frac{d}{dt} \left(\ G(t)\rho(t) + \frac{\Lambda(t)}{8\pi} \ \right) \ + \ \frac{2 \ G(t) \ (\ \rho(t) + p_{\theta}(t) \ )}{t} \ = \ 0 \tag{2.26}$$

The $k^2 \leftrightarrow \rho(t)$ relation is postulated to be of the same form as before $k^2 = \xi G_o \rho$ (ξ is a positive numerical constant) but where now $\rho(t)$ is given by eq-(2.24)

$$k^2(t) \ = \ k^2[\rho(t)] \ = \ k^2[G(t); \Lambda(t)] \ = \ \xi \ G_o \ \rho(t) \ =$$

$$\xi \ G_o \ \left(\ \frac{M}{4\pi \ t^2 \ G(t)} \ \frac{d}{dt} \ [\ G(t) \ + \ \frac{\Lambda(t)t^3}{6M} \ ] \ - \ \frac{\Lambda(t)}{8\pi G(t)} \ \right) \ =$$

$$\xi \ G_o \ \left(\frac{M}{4\pi t^2 G(t)} \frac{dG(t)}{dt} \ + \ \frac{1}{24\pi} \ \frac{t}{G(t)} \ \frac{d\Lambda(t)}{dt} \right) \tag{2.27}$$

Upon substituting the above expression (2.27) for $k^2[\rho(t)] = k^2[G(t);\Lambda(t)]$ (given in terms of $G(t), \Lambda(t)$ and their first order derivatives) into the right hand side of the running gravitational coupling

$$G(t) \ = \ \frac{G_o}{1 \ + \ g_*^{-1} \ G_o \ k^2[\rho(t)]} \ = \ \frac{G_o}{1 \ + \ g_*^{-1} \ G_o \ k^2[G(t);\Lambda(t)]} \tag{2.28a}$$

it furnishes one differential equation involving $G(t)$ and $\Lambda(t)$

$$G(t) \ + \ \xi \ \frac{G_o^2}{g_*} \ \left(\frac{M}{4\pi t^2} \frac{dG(t)}{dt} \ + \ \frac{t}{24\pi} \ \frac{d\Lambda(t)}{dt} \right) \ - \ G_o \ = \ 0 \tag{2.28b}$$

The second differential equation is obtained from the running cosmological constant

$$\Lambda(t) \ = \ \Lambda_o \ + \ \frac{b}{4} \ G(t) \ k^4[\rho(t)] \ = \ \Lambda_o \ + \ \frac{b}{4} \ G(t) \ k^4[G(t);\Lambda(t)] \tag{2.29a}$$

where k^4 is the square of the expression $k^2[G(t);\Lambda(t)]$ displayed in eq-(2.27)

$$\Lambda(t) \ - \ \Lambda_o \ - \ \frac{b}{4} \ \xi^2 \ \frac{G_o^2}{G(t)} \ \left(\frac{M}{4\pi t^2} \frac{dG(t)}{dt} \ + \ \frac{t}{24\pi} \ \frac{d\Lambda(t)}{dt} \right)^2 \ = \ 0 \tag{2.29b}$$

The differential equations (2.28b,2.29b) comprise a very complicated coupled system of two first-order non-linear differential equations whose origin stems from the *running* gravitational coupling, and cosmological constant, combined with the RG improvement of the Einstein field equations with a cosmological constant, and associated with the Kantowski-Sachs-like metric of eq-(2.20).

By eliminating one of the functions, the two $first$-order nonlinear differential equations (NLDE) can be reduced to a single *second*-order NLDE. Eliminating $\Lambda(t)$ from eqs-(2.28b, 2.29b) yields the following second order NLDE for $G(t)$:

$$\frac{24\pi g_*}{\xi G_o^2 t} \ (G_o - G(t)) \ - \ \frac{6M}{t^3} \ \frac{dG(t)}{dt} \ +$$

$$\frac{b\xi^2 G_o^2}{4} \ \frac{(dG(t)/dt)}{G(t)^2} \ \left(\frac{M}{4\pi t^2} \frac{dG(t)}{dt} \ + \ \frac{t}{24\pi} [\ \frac{24\pi g_*}{\xi G_o^2 t} \ (\ G_o - G(t)) \ - \ \frac{6M}{t^3} \ \frac{dG(t)}{dt} \ ] \right)^2$$

$$- \ \frac{b\xi^2 G_o^2}{4G(t)} \ \frac{d}{dt} \ \left(\frac{M}{4\pi t^2} \frac{dG(t)}{dt} \ + \frac{t}{24\pi} \ [\ \frac{24\pi g_*}{\xi G_o^2 t} \ (\ G_o - G(t)) \ - \ \frac{6M}{t^3} \ \frac{dG(t)}{dt} \ ] \right)^2 \ = \ 0 \tag{2.30}$$

Since the NLDE eq-(2.30) is of second order one requires to impose boundary conditions on $G(t)$ and $(dG(t)/dt)$. The choice of boundary conditions must be consistent with the FRGE flow solutions (2.4) which require that for very late times $t \rightarrow \infty$: $G(t) \rightarrow G_o$; $\Lambda(t) \rightarrow \Lambda_o$. Therefore, the solutions to the coupled system of differential equations (2.28b,2.29b) must obey the boundary conditions :

In the asymptotic $t \rightarrow \infty$ limit one should have : $G(t) \rightarrow G_o$; $\Lambda(t) \rightarrow \Lambda_o$; $\frac{dG(t)}{dt} \rightarrow 0$; $\frac{d\Lambda(t)}{dt} \rightarrow 0$. And, in this way $\rho_{vac}(t) \rightarrow \frac{\Lambda_o}{8\pi G_o} \simeq 10^{-122} M_P^4$ one recovers the observed vacuum energy density.

And, when $t \rightarrow 0$: $\Lambda(t) \rightarrow \infty$, $(d\Lambda/dt) \rightarrow -\infty$; while $G(t)$ and $(dG/dt) \rightarrow 0$; and $\rho_{vac} \rightarrow \infty$. Having solved the complicated system of differential equations for $G(t), \Lambda(t)$, the temporal behavior of the *running* vacuum energy density is given by $\rho_{vac}(t) = \frac{\Lambda(t)}{8\pi G(t)}$. It will be extremely small $\frac{\Lambda_o}{8\pi G_o} \sim 10^{-122} M_P^4$ at $t \rightarrow \infty$, and it blows up at $t = 0$. Due to the regularization effects the curvature scalar R obtained from evaluating the trace of eqs-(2.22)

$$R \ = \ 4\Lambda(t) \ - \ 16\pi G(t) \ (p_z + p_\theta) \ =$$

$$2\Lambda(t) \ + \ \frac{4M}{t^2} \ \frac{dG}{dt} \ + \ \frac{2}{3} \ t \ \frac{d\Lambda(t)}{dt} \ + \ \frac{2M}{t} \ \frac{d^2}{dt^2} \ \left(\ G(t) \ + \ \frac{\Lambda(t) t^3}{6M} \ \right) \quad (2.31)$$

should be $finite$ at $t = 0$. When one evaluates eq-(2.31) in the region close to $t = 0$ there should be terms with positive and negative sign in order to render R finite at $t = 0$. Note that as $t \rightarrow 0$, the derivative $(d\Lambda/dt) \rightarrow -\infty$. If the metric (2.20) is regularized at $t = 0$ one expects that $\Lambda(t)t^2$, and $(G(t)/t)$ should be $finite$ at $t = 0$.

Whereas, the $t \rightarrow \infty$ limit of eq-(2.31) gives $R \rightarrow 4\Lambda_o$. The scalar curvature attains a maximum (but finite regularized) value at $t = 0$, and decreases with time reaching the minimum nonzero value $4\Lambda_o$ at $t = \infty$. A de Sitter space in $D = 4$ has a constant $R = 4\Lambda_o$ (at all times). This result should be contrasted with the value of R found in eq-(2.18), in the absence of a running cosmological constant, where it vanishes in the $t = \infty$ limit, while being finite at $t = 0$, $R \sim \frac{1}{t_P^2}$. It is beyond the scope of this work to analytically and numerically solve eq-(2.30). For these reasons we can only provide the simplest solutions to eqs-(2.28b, 2.29b) as follows. After defining

$$2h(t) \ = \ \frac{2G(t)M}{t} \ + \ \frac{\Lambda(t)}{3} t^2 \quad (2.32)$$

one can show that the Ricci curvature scalar and Kretschmann invariant corresponding to the metric (2.20) are

$$R \ = \ 4 \ \frac{h(t)}{t^2} \ + \ 8 \ \frac{(dh(t)/dt)}{t} \ + \ 2 \ \frac{d^2 h(t)}{dt^2} \quad (2.33)$$

$$K \ = \ 16 \ \frac{h(t)^2}{t^4} \ + \ 16 \ \frac{(dh(t)/dt)^2}{t^2} \ + \ 4 \ (\frac{d^2 h(t)}{dt^2})^2 \quad (2.34)$$

In order to make R and K finite at $t = 0$, a careful inspection reveals that as $t \rightarrow 0$ it requires that $G(t)$ scales as $t^{3+\alpha}$, and $\Lambda(t)$ scales as t^β with

$\alpha, \beta \geq 0$. And having $\Lambda(t)$ scale as t^β, for $\beta \geq 0$, implies that the UV fixed point $\lambda_* = \Lambda(k)k^{-2}$ *vanishes* as $k \to \infty$ ($t \to 0$). This vanishing behavior of λ_* agrees also with the results in [14].

One also finds that by setting $\Lambda(t) = \Lambda_o$ (constant), and $\lambda_* = 0 \Rightarrow b = 0$, into eqs-(2.28b, 2.29b) yields the same solution for $G(t)$ as the one found in eq-(2.16). Therefore, $\Lambda(t) = \Lambda_o$ (leading to $\lambda_* = b = 0$) and $G(t) = G_o(1 - e^{-t^3/t_H t_P^2})$ are the simplest solutions to eqs-(2.28b, 2.29b) which furnish $finite$ values for R and K at $t = 0$. These solutions correspond to setting $\alpha = \beta = 0$.

One still has to verify if other solutions to eqs-(2.28a, 2.29b) are plausible leading to finite R and K at $t = 0$, such that $G(t) \sim t^{3+\alpha}$ and $\Lambda(t) \sim t^\beta$, with $\alpha, \beta > 0$, in the region surrounding $t = 0$. What one has learned from this analysis is that one cannot impose the boundary condition $\Lambda(t = 0) = \infty$ giving $\lambda_* \neq 0$, but instead $\Lambda(t = 0) = 0$ leading to $\lambda_* = 0$, if one desires to have R and K finite at $t = 0$.

A closing remark, if one were to use the correspondence $k^2 = \xi R$, instead of $k^2 = \xi G_o \rho$, directly into the action (2.2), it will lead to $R + R^2$ Starobinksi-like inflationary actions. Mapping the latter to the Einstein frame yields a scalar-tensor theory involving the graviton and a scalar field ϕ with the inclusion of a self-interacting scalar potential $V(\phi)$ [13].

3 Weyl Geometry and Dilaton-Gravity

3.1 Classical Dilaton-Weyl Cosmology

Before discussing dilaton-gravity within the context of Weyl geometry, given the Lorentzian signature $(-, +, +, +)$, let us begin with an action in a curved *Riemannian* background

$$S = \int d^4x \sqrt{|g|} \left(\frac{R}{16\pi G_o} - \frac{g^{\mu\nu}}{2} (\partial_\mu \Phi) (\partial_\nu \Phi) - V(\Phi) \right) \tag{3.1}$$

and associated with a canonical real scalar field Φ with a known prescribed potential $V(\Phi)$. Varying the action with respect to the two fields $g_{\mu\nu}, \Phi$ yields

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R =$$

$$8\pi G_o \left((\partial_\mu \Phi)(\partial_\nu \Phi) - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} (\partial_\alpha \Phi)(\partial_\beta \Phi) - g_{\mu\nu} V(\Phi) \right) \tag{3.2}$$

$$\frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|} g^{\mu\nu} \partial_\nu \Phi \right) - \frac{\partial V(\Phi)}{\partial \Phi} = 0 \tag{3.3}$$

The two equations (3.2, 3.3) (Einstein-Klein-Gordon system) are now *coupled* and induce a *nonlinear* Klein-Gordon-like equation for Φ. One should note that even in the trivial case when $V = 0$, or $\frac{\partial V(\Phi)}{\partial \Phi}$ is a *linear* function of Φ, eq-(3.3) still will remain *nonlinear* after solving eqs-(3.2) for the metric $g_{\mu\nu}[\Phi]$ in terms of Φ. This intrinsic nonlinearity is also reflected when the authors [28] found that the *nonrelativistic* limit of the two coupled equations (3.2,3.3) furnish the *nonlinear* Newton-Schrödinger equation

$$i\hbar \frac{\partial \Psi(\vec{r},t)}{\partial t} \ =$$

$$- \ \frac{\hbar^2}{2m} \nabla^2 \Psi(\vec{r},t) \ + \ V \ \Psi(\vec{r},t) \ - \ \left(\ Gm^2 \ \int \ \frac{|\Psi(\vec{r}',t)|^2}{|\vec{r}-\vec{r}'|} \ d^3r' \right) \ \Psi(\vec{r},t) \quad (3.4)$$

which is obtained after solving the Poisson equation

$$\nabla^2 U \ = \ 4\pi G_o m \rho \ = \ 4\pi G_o m \ \Psi^* \Psi \quad (3.5)$$

for the Newtonian potential $U = U(\Psi, \Psi^*)$ and substituting its value into the Schrödinger equation.

The immediate advantage of recurring to Weyl geometry is that it will allow us to find exact solutions to the very complicated coupled system of equations (3.2, 3.3). References on Weyl's geometry can be found in [18], [**?**], [20], [21], [29], [31], among many others. A review of the the connection between JBD gravity and Weyl gravity can be found in [19]. Weyl's geometry main feature is that the norm of vectors under parallel infinitesimal displacement going from x^μ to $x^\mu + dx^\mu$ change as follows $\delta ||V|| \ \sim \ ||V|| A_\mu dx^\mu$ where A_μ is the Weyl gauge field of scale calibrations that behaves as a connection under Weyl transformations :

$$A'_\mu = A_\mu \ - \ \partial_\mu \ \Omega(x). \quad g_{\mu\nu} \ \rightarrow \ e^{2\Omega} \ g_{\mu\nu}. \quad (3.6)$$

involving the Weyl scaling parameter $\Omega(x^\mu)$. The Weyl covariant derivative operator acting on a tensor $\mathbf{T}$ is defined by $D_\mu \mathbf{T} = (\ \nabla_\mu \ + \ \omega(\mathbf{T} \ ) \ A_\mu \ ) \ \mathbf{T}$; where $\omega(\mathbf{T})$ is the Weyl weight of the tensor $\mathbf{T}$ and the derivative operator $\nabla_\mu \ = \partial_\mu + \Gamma_\mu$ involves a connection Γ_μ which is comprised of the ordinary Christoffel symbols $\{^{\rho}_{\mu\nu}\}$ *plus* the A_μ terms

$$\Gamma^\rho_{\mu\nu} \ = \ \{^{\rho}_{\mu\nu}\} \ + \ \delta^\rho_\mu \ A_\nu \ + \ \delta^\rho_\nu \ A_\mu \ - \ g_{\mu\nu} \ g^{\rho\sigma} \ A_\sigma \quad (3.7)$$

The Weyl gauge covariant operator $\partial_\mu + \Gamma_\mu + w(\mathbf{T}) A_\mu$ obeys the condition

$$D_\mu \ (g_{\nu\rho}) \ = \ \nabla_\mu \ (g_{\nu\rho}) \ + \ 2 \ A_\mu \ g_{\nu\rho} \ = \ 0. \quad (3.8)$$

where $\nabla_\mu (g_{\nu\rho}) \ = \ - 2 \ A_\mu \ g_{\nu\rho} \ = \ Q_{\mu\nu\rho}$ is the non-metricity tensor. Torsion can be added [20] if one wishes but for the time being we refrain from doing so. The connection $\Gamma^\rho_{\mu\nu}$ is Weyl *invariant* so that the geodesic equation in Weyl spacetimes is Weyl-covariant under Weyl gauge transformations (scalings)

$$ds \to e^{\Omega}\, ds; \; \frac{dx^\mu}{ds} \to e^{-\Omega}\, \frac{dx^\mu}{ds}; \; \frac{d^2x^\mu}{ds^2} \; \to \; e^{-2\Omega}\, [\, \frac{d^2x^\mu}{ds^2} \; - \; \frac{dx^\mu}{ds}\, \frac{dx^\nu}{ds}\, \partial_\nu \Omega \,] \quad (3.9)$$

$$g_{\mu\nu} \to e^{2\Omega}\, g_{\mu\nu}; \;\; A_\mu \to A_\mu - \partial_\mu \Omega; \;\; A^\mu \to e^{-2\Omega}\, (A^\mu - \partial^\mu \Omega); \;\; \Gamma^\rho_{\mu\nu} \to \Gamma^\rho_{\mu\nu}. \quad (3.10)$$

The Weyl connection and curvatures scale as

$$\Gamma^\rho_{\mu\nu} \to \Gamma^\rho_{\mu\nu}, \;\; \mathcal{R}^\rho_{\mu\nu\sigma} \to \mathcal{R}^\rho_{\mu\nu\sigma}, \;\; \mathcal{R}_{\mu\nu} \to \mathcal{R}_{\mu\nu}, \;\; \mathcal{R} \to e^{-2\Omega}\, \mathcal{R} \quad (3.11)$$

Thus, the Weyl covariant geodesic equation transforms under Weyl scalings as

$$\frac{d^2x^\rho}{ds^2} \; + \; \Gamma^\rho_{\mu\nu}\, \frac{dx^\mu}{ds}\, \frac{dx^\nu}{ds} \; - \; A_\mu\, \frac{dx^\mu}{ds}\, \frac{dx^\rho}{ds} \; = \; 0 \; \to$$

$$e^{-2\Omega}\, [\, \frac{d^2x^\rho}{ds^2} \; + \; \Gamma^\rho_{\mu\nu}\, \frac{dx^\mu}{ds}\, \frac{dx^\nu}{ds} \; - \; A_\mu\, \frac{dx^\mu}{ds}\, \frac{dx^\rho}{ds} \,] \; = \; 0. \quad (3.12)$$

The Weyl weight of the metric $g_{\nu\rho}$ is 2. The meaning of $D_\mu(g_{\nu\rho}) = 0$ is that the angle formed by two vectors remains the same under parallel transport despite that their lengths may change. This also occurs in conformal mappings of the complex plane. The Weyl covariant derivative acting on a scalar ϕ of Weyl weight $\omega(\phi) = -1$ is defined by

$$D_\mu \phi = \partial_\mu\, \phi \; + \; \omega(\phi) A_\mu\, \phi = \partial_\mu\, \phi \; - \; A_\mu\, \phi. \quad (3.13)$$

The Weyl scalar curvature in D dimensions and signature $(-,+,+,+....)$ is [2]

$$\mathcal{R}_{Weyl} \; = \; R_{Riemann} \; - \; (d-1)(d-2) A_\mu A^\mu \; - \; 2(d-1)\nabla_\mu A^\mu. \quad (3.14)$$

Having introduced the basics of Weyl geometry our starting action is the Weyl-invariant Jordan-Brans-Dicke-like action involving the scalar ϕ field and the scalar Weyl curvature $\mathcal{R}_{Weyl}$

$$S[g_{\mu\nu}, A_\mu, \phi] \; = \; S[g'_{\mu\nu}, A'_\mu, \phi'] \; \Rightarrow$$

$$\int d^4x\, \sqrt{|g|}\, [\, \phi^2\, \mathcal{R}_{Weyl}(g_{\mu\nu}, A_\mu) \; - \; \frac{1}{2} g^{\mu\nu}\, (D_\mu\phi)(D_\nu\phi) \; - \; V(\phi) \,] \; =$$

$$\int d^4x\, \sqrt{|g'|}\, [\, (\phi')^2\, \mathcal{R}_{Weyl}(g'_{\mu\nu}, A'_\mu) \; - \; \frac{1}{2} g'^{\mu\nu}\, (D'_\mu\phi')(D'_\nu\phi') \; - \; V(\phi') \,] \quad (3.15)$$

where under Wey scalings one has

$$\phi' \; = \; e^{-\Omega}\, \phi; \;\; g'_{\mu\nu} \; = \; e^{2\Omega}\, g_{\mu\nu}; \;\; \mathcal{R}_{Weyl}(g'_{\mu\nu}, A'_\mu) \; = \; e^{-2\Omega}\, \mathcal{R}_{Weyl}(g_{\mu\nu}, A_\mu)$$

[2] Some authors define their A_μ field with the opposite sign as $-A_\mu$ which changes the sign in the last term of the Weyl scalar curvature (3.14)

$$V(\phi') \ = \ e^{-4\Omega} \, V(\phi), \ \ \sqrt{|g'|} \ = \ e^{4\Omega} \, \sqrt{|g|}; \ \ D'_\mu \phi' \ = \ e^{-\Omega} \, D_\mu \phi; \ \ A'_\mu \ = \ A_\mu \ - \partial_\mu \Omega. \tag{3.16}$$

Despite that one has *not* introduced any explicit dynamics to the A_μ field (there are no $F_{\mu\nu} F^{\mu\nu}$ terms in the action (3.15) with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$) one still has to take into account the equation obtained from the variation of the action in $d = 4$ w.r.t to the A^μ field and which leads to the pure-gauge configurations provided $\phi \neq 0$

$$\frac{\delta S}{\delta A_\mu} \ = \ 0 \ \Rightarrow \ \phi^2 \ \frac{\delta \mathcal{R}_{Weyl}}{\delta A_\mu} \ + \ \frac{\delta (D_\mu \phi)}{\delta A_\mu} \ \frac{\delta S_{matter}}{\delta (D_\mu \phi)} \ = \ 0 \ \Rightarrow$$

$$g^{\mu\nu} D_\nu \phi^2 \ = \ 0 \ \Rightarrow \ D_\mu \phi \ = \ 0 \ \Rightarrow \ A_\mu \ = \ \partial_\mu \ ln \ (\phi). \tag{3.17}$$

Hence, a variation of the action w.r.t the A_μ field leads to the pure gauge solutions (3.17) which is tantamount to saying that the scalar ϕ is Weyl-covariantly *constant* $D_\mu \phi = 0$ in any gauge $D_\mu \phi = 0 \rightarrow e^{-\Omega} D_\mu \phi = D'_\mu \phi' = 0$ (for non-singular gauge functions $\Omega \neq \pm \infty$).

Therefore, the scalar ϕ does not have true local dynamical degrees of freedom from the Weyl spacetime perspective. Since the gauge field is a total derivative, under a local gauge transformation with a gauge function $\Omega = ln(\phi/\phi_o)$, one can gauge away (locally) the gauge field A_μ and have $A'_\mu = 0$ in the new gauge. Globally, however, this may not be the case because there may be *topological* obstructions. Therefore, the gauge $A'_\mu = 0$ implies that $\phi' = \phi_o =$ constant, and which can be chosen such that $16 \pi G_N = \phi_o^{-2}$, where G_N is the observed Newtonian gravitational coupling, and one recovers the Einstein-Hilbert action with a cosmological constant $\Lambda_o = 8 \pi G_N V(\phi_o)$.

The pure-gauge configurations leads to the Weyl integrability condition $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = 0$ when $A_\mu = \partial_\mu \Omega$, and means physically that if we parallel transport a vector under a closed loop in a $flat$ spacetime, as we come back to the starting point, the *norm* of the vector has not changed; i.e, the rate at which a clock ticks does not change after being transported along a closed loop back to the initial point; and if we transport a clock from A to B along different paths, the clocks will tick at the same rate upon arrival at the same point B. This will ensure, for example, that the observed spectral lines of identical atoms will not change when the atoms arrive at the laboratory after taking different paths (histories) from their coincident starting point. In this way on can bypass Einstein's objections to Weyl. If $F_{\mu\nu} \neq 0$ the Weyl geometry is no longer integrable. This can occur if one adds explicit $F_{\mu\nu} F^{\mu\nu}$ terms to the action which may lead to true dynamical degrees of freedom for the gauge field A_μ.

This result $D_\mu \phi = 0$ also follows in other dimensions. Substituting

$$A_\mu \ = \ \frac{2}{d-2} \ \partial_\mu ln \phi \tag{3.18}$$

into

$$\mathcal{R}_{Weyl} \ = \ R_{Riemann} \ - \ (d-1)(d-2) A_\mu A^\mu \ - \ 2(d-1) \nabla_\mu A^\mu \tag{3.19}$$

gives

$$\mathcal{R}_{Weyl} \ = \ R_{Riemann} \ - \ 4 \ \frac{d-1}{d-2} \ \frac{\nabla_\mu \nabla^\mu \phi}{\phi} \tag{3.20}$$

The last term in (3.20) has a similar functional form as Bohm's quantum potential [25], [26]. From now we shall denote R for the Riemannian scalar curvature $R_{Riemann}$. The covariant derivative ∇_μ appearing in (3.19,3.20) is the one defined in terms of the Christoffel conection $\{\}$, and *not* based on the Weyl connection Γ.

Given the action (3.15) in $d = 4$ the field equations are obtained after the variations of the action with respect to the 3 fields $g_{\mu\nu}, A_\mu, \phi$, respectively

$$\phi^2 \ \left(\mathcal{R}^{Weyl}_{\mu\nu} \ - \ \frac{1}{2} \ g_{\mu\nu} \ \mathcal{R}_{Weyl} \right) \ - \ D_\mu D_\nu \phi^2 \ + \ g_{\mu\nu} \ g^{\alpha\beta} D_\alpha D_\beta \phi^2 \ =$$

$$\frac{1}{2} \ (D_\mu\phi)(D_\nu\phi) \ - \ \frac{1}{4} \ g_{\mu\nu} \ g^{\alpha\beta}(D_\alpha\phi)(D_\beta\phi) \ - \frac{1}{2} \ g_{\mu\nu} V(\phi) \tag{3.21}$$

$$D_\mu\phi^2 \ = \ 2 \ D_\mu\phi \ = \ 0 \ \Rightarrow \ A_\mu \ = \ \partial_\mu ln(\phi) \tag{3.22}$$

$$2\phi \ \mathcal{R}_{Weyl} \ - \ \frac{\partial V(\phi)}{\partial\phi} \ + \ D_\mu D^\mu \phi \ = \ 0 \tag{3.23}$$

As stated earlier, the field equation $D_\mu\phi = 0$ just states the ϕ is Weyl-covariantly constant. This result when followed by taking the trace of (3.21) gives $\phi^2 \mathcal{R}_{Weyl} = 2V(\phi)$ which allows to eliminate $\mathcal{R}_{Weyl} = 2\phi^{-2}V(\phi)$, and inserting it in eq-(3.23) yields $4\phi^{-1}V(\phi) - V'(\phi) = 0$, *singling* out the quartic potential $V(\phi) = \kappa\phi^4$ in $d = 4$, out of an infinity of possible choices for the *classical* potential. For example, one could have potentials of the form $V = \sum_n c_n M^{4-n} \phi^n$ where M is mass-like parameter (a scalar moduli parameter) which scales as $M \rightarrow e^{-\Omega} M$ in order to render the action Weyl invariant.

To sum up, in this Weyl geometric approach the choice for the *classical* potential $V = \kappa\phi^4$ is *not* ad hoc but can be inferred from the field equations themselves. One must emphasize that we are focusing solely on the *classical* potential. It is known that quantum fluctuations lead to an $effective$ scalar potential that will introduce *corrections* to the classical quartic potential, see for example [22] for technical details.

Eq-(3.23) in $d = 4$ can be rewritten in terms of the Riemannian scalar curvature, after using $D_\mu\phi = 0$, as

$$2 \ R \ \phi \ - \ \frac{12}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|} \ g^{\mu\nu} \ \partial_\nu\phi \right) \ - \ \frac{\partial V(\phi)}{\partial\phi} \ = \ 0 \tag{3.24}$$

Upon inserting the derived expression for $V(\phi) = \kappa\phi^4$ above, it gives

$$R \ \phi \ - \ \frac{6}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|} \ g^{\mu\nu} \ \partial_\nu\phi \right) \ - 2 \ \kappa \ \phi^3 \ = \ 0 \tag{3.25}$$

It remains now to solve eq-(3.21) given $D_\mu \phi = 0$ and $V(\phi) = \kappa \phi^4$. After factoring out ϕ^2 and substituting $\mathcal{R}_{Weyl} = 2\kappa\phi^2$ leads to

$$\mathcal{R}^{Weyl}_{\mu\nu} \;=\; \frac{1}{2}\; g_{\mu\nu}\; \kappa\phi^2 \tag{3.26}$$

with

$$\mathcal{R}^{Weyl}_{\mu\nu} \;=\; R_{\mu\nu} \;-2\;\nabla_\mu A_\nu - g_{\mu\nu}\; g^{\alpha\beta}\;\nabla_\alpha A_\beta \;+2\;A_\mu\;A_\nu \;-2\;g_{\mu\nu}\;g^{\alpha\beta}\;A_\alpha A_\beta \tag{3.27}$$

Before proceeding, it is relevant to mention that since the Weyl weight of $\mathcal{R}^{Weyl}_{\mu\nu}$ is 0, from eq-(3.27) after some straightforward lengthy algebra, one can infer the transformation law of the Riemannian Ricci tensor $R_{\mu\nu}$ in $d = 4$ under scalings $g_{\mu\nu} \rightarrow e^{2\Omega} g_{\mu\nu}$

$$R'_{\mu\nu} \;=$$

$$R_{\mu\nu} \;-2\;\nabla_\mu\nabla_\nu\Omega - g_{\mu\nu}\;g^{\alpha\beta}\;(\nabla_\alpha\nabla_\beta\Omega) \;+2\;(\nabla_\mu\Omega)(\nabla_\nu\Omega) \;-2\;g_{\mu\nu}\;g^{\alpha\beta}(\nabla_\alpha\Omega)(\nabla_\beta\Omega) \tag{3.28}$$

From eq-(3.26) one then arrives at the Weyl invariant field equation

$$\mathcal{R}^{Weyl}_{\mu\nu}(\mathbf{g},\mathbf{A}) \;=\; \mathcal{R}^{Weyl}_{\mu\nu}(\mathbf{g}',\mathbf{A}') \;=\; \frac{1}{2}\;g_{\mu\nu}\;\kappa\;\phi^2 \;=\; \frac{1}{2}\;g'_{\mu\nu}\;\kappa\;\phi'^2 \tag{3.29}$$

with $\mathbf{g} = g_{\mu\nu}, \mathbf{A} = A_\mu, \cdots$. The dimensionless parameter κ is inert under scalings.

The zero gauge choice $A'_\mu = 0$ leads to

$$A'_\mu \;=\; \partial_\mu[ln(\frac{\phi'}{\phi_o})] \;=\; A_\mu \;-\; \partial_\mu\Omega \;=\; \partial_\mu[ln(\frac{\phi}{\phi_o})] - \partial_\mu\Omega \;= 0 \;\Rightarrow$$

$$\phi' = \phi_o;\;\; e^{\Omega} \;=\; \frac{\phi}{\phi_o} \tag{3.30}$$

which resulted from $D'_\mu\phi' = D_\mu\phi = 0$.

Consequently, one arrives finally at

$$\mathcal{R}^{Weyl}_{\mu\nu}(\mathbf{g}';\mathbf{A}'=\mathbf{0}) = \;R'_{\mu\nu} \;=\; \frac{1}{2}\;g'_{\mu\nu}\;\kappa\;\phi_o^2 \;\Rightarrow\; R' \;=\; 2\kappa\;\phi_o^2 \tag{3.31}$$

leading to a family of spacetime backgrounds which are all conformally equivalent to backgrounds of *constant* Riemannian scalar curvature : (Anti) de Sitter spaces. The solutions to the scalar ϕ field equation (3.25) defined in spacetime backgrounds which are conformally equivalent to a (Anti) de Sitter background

$$g_{\mu\nu} \;=\; e^{2\;\Omega'(x)}\;g'^{(A)dS}_{\mu\nu} \;=\; e^{-2\;\Omega(x)}\;g'^{(A)dS}_{\mu\nu},\;\;\Omega \;= -\;\Omega' \tag{3.32}$$

are of the form $\phi = e^{-\Omega'(x)}\phi_o = e^{\Omega(x)}\phi_o$; $\phi_o = (16\pi G_N)^{-1/2}$ is the constant directly related to the observed Newtonian coupling G_N. Given

$$g_{\mu\nu} \;=\; e^{2\;\Omega'(x)}\;g'^{(A)dS}_{\mu\nu} \;=\; e^{-2\;\Omega(x)}\;g'^{(A)dS}_{\mu\nu} \tag{3.33a}$$

under Weyl scalings the constant Riemannian scalar curvature of (Anti) de Sitter space in $d = 4$ transforms as

$$R \ = \ e^{2\Omega(x)} \ \Big(\ R'_{(A)dS} \ + 6 \ (\nabla_\mu \nabla^\mu \Omega) \ - 6 \ (\nabla_\mu \Omega) \ (\nabla^\mu \Omega) \ \Big) \qquad (3.33b)$$

such that

$$R \ \phi \ - \ \frac{6}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|} \ g^{\mu\nu} \ \partial_\nu \phi \right) \ - 2 \ \kappa \ \phi^3 \ = $$

$$e^{3\Omega(x)} \ \Big(\ R'_{(A)dS} \ \phi_o \ - 2 \ \kappa \ \phi_o^3 \ \Big) \ = \ 0 \qquad (3.34)$$

as expected.

To sum up, starting with a dilaton-gravity (scalar-tensor theory) system within the context of Weyl's geometry, permits to single out the expression for the *classical* potential $V(\phi) = \kappa\phi^4$, instead of being introduced by hand, and find the following family of metric solutions to the field equations (3.21-3.23) which are conformally equivalent to the (Anti) de Sitter metric

$$g_{\mu\nu}[\phi] \ = \ e^{-2\Omega} \ g'^{(A)dS}_{\mu\nu}[\phi_o] \ = \ (\frac{\phi_o}{\phi})^2 \ g'^{(A)dS}_{\mu\nu}[\phi_o] \qquad (3.35a)$$

and where the Weyl field is

$$A_\mu[\phi] \ = \ \partial_\mu[ln(\frac{\phi}{\phi_o})] \qquad (3.35b)$$

The (Anti) de Sitter metric $g'^{(A)dS}_{\mu\nu}[\phi_o]$ has an explicit dependence on ϕ_o via the cosmological constant Λ : $R' = 4\Lambda = 2\kappa\phi_o^2$. $\kappa < 0$ for Anti de Sitter space; $\kappa > 0$ for de Sitter space. The solutions with $\kappa = 0$ lead, for example, to the Schwarzschild ($R'_{\mu\nu} = R' = 0$) and Reisnner-Nordstrom ($R' = 0$) metrics corresponding to static spherically symmetric backgrounds.

The prime example of a de Sitter background of constant Riemannian scalar curvature is the observed accelerated-expanding universe $R' = 12H_o^2 \ = \ \frac{12}{R_H^2}$ where R_H is the present day Hubble radius. Substituting $R' = 12H_o^2$ into eq-(3.34) fixes the numerical coefficient κ of the potential $V(\phi') = \kappa\phi'^4$,

$$12H_o^2 \ = \ 2\kappa \ \phi_o^2 \ \Rightarrow \ \kappa \ = \ \frac{6}{\phi_o^2 R_H^2} \qquad (3.36)$$

Therefore, by evaluating the potential at $\phi' = \phi_o$, after fixing the Weyl scale invariance by setting $\phi'^2 = \phi_o^2 = (16\pi G_N)^{-1}$ ($G_o = G_N =$ Newton's constant) it gives

$$V(\phi_o) \ = \ \kappa \ \phi_o^4 \ = \ \frac{6}{\phi_o^2 R_H^2} \ \phi_o^4 \ = \ 6 \ \frac{\phi_o^2}{R_H^2} \ = \ \frac{6}{16\pi G_N R_H^2} \ = \ \frac{3}{8\pi G_N R_H^2} \ = \ \rho_{cr} \qquad (3.37)$$

and one recovers, in a straightforward fashion, the Universe's observed critical mass density with the precise numerical factor, which agrees also with the observed vacuum energy density ρ_{vac}. The reason one has found exact solutions to the field equations in a straightforward fashion is due to the fact that one did not introduce dynamical degrees of freedom for the Weyl field A_μ. Including $F_{\mu\nu}F^{\mu\nu}$ into the action (3.15) would have considerably affected matters. In general the RG flow will induce dynamics for the vector field A_μ.

3.2 Dilaton-Gravity and RG-improved Cosmology

This last section is devoted to the study of dilaton-gravity [27] based entirely on Riemannian geometry and its application to Cosmology. A typical ansatz for the *truncated* effective average action in dilaton-gravity (DG) in the Jordan frame is

$$\Gamma_k^{DG} \;=\; \int d^4x\;\sqrt{|g|}\;[\;F_k(\phi^2)\;R\;-\frac{1}{2}\;K_k(\phi^2)\;(\partial_\mu\phi)\;(\partial^\mu\phi)\;\;-\;V_k(\phi^2)\;]\;+\;\cdots \tag{3.38}$$

where the ellipsis $\cdots$ denote the gauge fixing and ghost terms. The functions F_k, K_k, V_k depend on the scalar field ϕ and the RG scale k. One can expand these functions in a power series with k-dependent coefficients

$$F_k \;=\; \sum_{n=-\infty}^{\infty} a_n(k)\;\phi^n,\;\; V_k \;=\; \sum_{n=-\infty}^{\infty}\;\lambda_n(k)\;\phi^n,\;\; K_k \;=\; \sum_{n=-\infty}^{\infty}\;b_n(k)\;\phi^n \tag{3.39}$$

By substituting the ansatz Γ_k^{DG} into the FRGE equation (2.1) [3], and scaling the coefficients $a_n(k), b_n(k), \lambda_n(k)$ by suitable powers of k in order to generate dimensionless couplings, yields a system of coupled partial differential equations determining the scale-dependence of F_k, K_k, V_k. Converting Γ_k^{DG} to the Einstein frame the predictions for the cosmological observables may then be constructed [13], [15].

Let us simplify the problem considerably by choosing $a_0(k) = \frac{1}{16\pi G(k)}$, $K_k = 1$, and $\lambda_4(k) = \frac{\lambda(k)}{4}$, $V_k(\phi) = \frac{\lambda(k)}{4}\phi^4$, while setting *all* the other coefficients in eq-(3.39) to zero. Concentrating on a Friedmann-Lemaitre-Robertson-Walker (FLRW) metric when the spatial curvature is flat, the time evolution of ϕ and the scaling factor $a(t)$ are obtained from solving the RG-improved Friedmann and Klein-Gordon-like equations

$$(\frac{\dot a}{a})^2 \;=\; \frac{8\pi G(k)}{3}\;\left(\;\frac{\dot\phi^2}{2}\;+\;V_k(\phi)\;\right) \tag{3.40}$$

$$\ddot\phi \;+\; 3\frac{\dot a}{a}\;\dot\phi\;+\;\frac{dV_k(\phi)}{d\phi}\;\;=\;0,\;\;V_k(\phi) = \frac{\lambda(k)}{4}\phi^4 \tag{3.41}$$

The RG-improvement version of the Friedmann and Klein-Gordon-like equations that we shall propose in this work *differs* from the one studied in [13]. It is obtained by assuming that $G(k)$ and $\lambda(k)$ are converted to functions of the cosmological time $G(t), \lambda(t)$ by an appropriate cutoff identification $k = k(t)$ given now by $k = \xi\phi(t)$, instead of the usual identification $k = \xi H(t)$ in [13]. The Hubble function is defined by $H(t) \equiv \frac{\dot a}{a}$. $k = \xi\phi(t)$, with $\xi > 0$. The choice $k = \xi\phi(t)$ is valid [3] in the particular case when the scalar field $\phi(t)$ is positive-definite and monotonically *decreases* with time reaching zero at $t = \infty$.

The explicit expressions $G = G(k)$ and $\lambda = \lambda(k)$ are dictated by the RG equations of the dilaton-gravity system [15]. And, which in turn, fix the functional form $G = G(k = \xi\phi)$ and $\lambda = \lambda(k = \xi\phi)$ in terms of ϕ. Denoting the temporal derivatives by dots, the RG-improvement version of the Friedmann and Klein-Gordon-like equations become

$$(\frac{\dot a}{a})^2 \ = \ \frac{8\pi G(\xi\phi)}{3} \ \left(\ \frac{\dot\phi^2}{2} \ + \ \frac{\lambda(\xi\phi)}{4} \ \phi^4 \ \right) \tag{3.42}$$

$$\ddot\phi \ + \ 3\frac{\dot a}{a} \ \dot\phi \ + \lambda(\xi\phi) \ \phi^3 \ + \ \frac{\phi^4}{4} \ \frac{d\lambda(\xi\phi)}{d\phi} \ = \ 0 \tag{3.43}$$

Given the prescribed functional form of $G = G(\xi\phi)$ and $\lambda = \lambda(\xi\phi)$, eqs-(3.42,3.43) determine the temporal dependence of $a(t), \phi(t)$. Because the above equations are very difficult to solve analytically let us simplify matters by focusing in the deep ultraviolet region where $g = G(k)k^2$, and $\lambda(k)$ assume their non-Gaussian interacting fixed-point values $g_* \neq 0; \lambda_* \neq 0$, respectively. It then leads to $G \simeq \frac{g_*}{\xi^2\phi^2}$, and $V \simeq \frac{\lambda_*}{4}\phi^4$. Inserting these values into eqs-(3.42,3.43) , and eliminating $H = \frac{\dot a}{a}$, yields the nonlinear differential equation

$$\left(\ \frac{\ddot\phi \ + \ \lambda_*\phi^3}{3\dot\phi} \ \right)^2 \ \simeq \ \frac{8\pi}{3} \ \frac{g_*}{\xi^2\phi^2} \ \left(\ \frac{\dot\phi^2}{2} \ + \ \frac{\lambda_*}{4} \ \phi^4 \ \right) \tag{3.44}$$

which has a very simple solution that is monotonically decreasing with time when $A > 0$ and given by

$$\phi(t) \ = \ \frac{A(g_*, \lambda_*)}{t}, \quad A^2 \ = \ \frac{6\pi\xi^{-2}g_* \ - \ 2 \ + \ \sqrt{12 \ (3\pi^2\xi^{-4}g_*^2 + 3\pi\xi^{-2}g_* - 1)}}{2\lambda_*} \tag{3.45}$$

where the numerical coefficient A is an algebraic function of g_*, λ_* and the numerical parameter ξ. The latter parameter ξ can be fine-tuned such that A is real and positive for the numerical values obtained for g_*, λ_*. In particular, by simply choosing $3\pi\xi^{-2}g_* - 1 \geq 0$ in eq-(3.45), when $\lambda_* > 0$, one will have a real-valued solution for A. For example, by choosing $3\pi g_* = \xi^2$ one finds $A = \frac{1}{\sqrt{\lambda_*}}$.

[3] One could choose also $k^2 = \xi^2\phi^2(t)$ when the magnitude $|\phi|$ of $\phi(t)$ monotonically *decreases* with time reaching zero at $t = \infty$. In this case $\phi(t)$ may oscillate in sign as well

Inserting the solution $\phi(t) = \frac{1}{\sqrt{\lambda_* t}}$, and $\lambda = \lambda_*$ into eq-(3.43), one finds in the vicinity of $t = 0$ that the scaling factor is $a(t) \simeq a_o(\frac{t}{t_o})$. Choosing different values for ξ yields different values for A and leads to different scaling exponents $a(t) \simeq a_o(\frac{t}{t_o})^{\gamma/3}$, with $\gamma = 2+\lambda_* A^2$. The quartic potential $V \simeq A^4 t^{-4}$ evaluated at the Planck time $t = t_P$ is of the order of M_P^4, which agrees with the expected zero-point energy density of a scalar field when a Planck-mass cutoff is chosen.

The dilaton-gravity beta functions [15] $differ$ from the pure Einstein gravity plus cosmological-constant system. In the former case, the value of the UV fixed point g_* associated to the dimensionless coupling $g(k) = G(k)k^2$ is $shifted$ from the value in the latter case. However, one still has a similar functional expression for $G(k^2)$ of the form $G(k^2) = G_o(1 + g_*^{-1} G_o k^2)^{-1}$. Introducing the cut-off identification scale $k = \xi\phi$ yields

$$G = G[k^2(\phi)] = \frac{G_o}{1 + g_*^{-1} G_o \xi^2 \phi^2} \Rightarrow \phi^2 = \frac{g_*}{\xi^2} \left(G^{-1} - G_o^{-1} \right) \quad (3.46)$$

And one learns that ϕ^2 is infinite at $G = 0$ (when $k = \infty$, $t = 0$), and ϕ^2 is zero at $G = G_o$ (when $k = 0$, $t = \infty$). Therefore, the results in eqs-(3.45, 3.46) validate introducing the cut-off identification scale $k(t) = \xi\phi(t)$. One should add that one must *not* confuse fixing the Weyl scale invariance by setting $\phi_o^2 = \frac{1}{16\pi G_o}$ in the previous section with the RG flow of the gravitational coupling expressed in terms of $\phi(t)$ and described by eq-(3.46).

Fo instance, instead of studying the RG flow of the dilaton-gravity system and the RG-improved Cosmology based on the FLRW metric, we return to the Kantowski-Sachs metric (2.11) of the previous section, and for the sake of the argument in order to obtain a numerical *estimate* of the classical Higgs-like quartic potential $\kappa\phi^4$, where κ is given by eq-(3.36), let us set $\phi^2(t) = \frac{1}{16\pi G(t)}$, instead of using the relation in eq-(3.46) [4], and insert the expression for $G(t)$ found in eq-(2.16) that was derived within the context of the Kantowski-Sachs cosmology. The quartic potential becomes in this case

$$V(\phi(t)) = \kappa\, \phi^4(t) = \frac{6}{\phi_o^2 R_H^2} \frac{\phi_o^4}{(1 - e^{-t^3/R_H t_P^2})^2} \quad (3.47)$$

When $t = t_H = R_H$, $V(\phi(t_H)) \simeq V(\phi_o) = (3/8\pi G_N R_H^2) = \rho_{vac}$, one recovers the observed vacuum energy density [5]. Whereas at Planck's time $t = t_P$, one finds after a Taylor expansion of the exponential, in $c = 1$ units, the expected very large result

$$V(\phi(t_P)) = \frac{6}{\phi_o^2 R_H^2} \frac{\phi_o^4}{(1 - e^{-t_P/R_H})^2} \simeq \frac{6}{\phi_o^2 R_H^2} \frac{\phi_o^4}{(t_P/R_H)^2} = \frac{3}{8\pi} M_P^4 \quad (3.48)$$

[4] $\phi^2(t) = \frac{1}{16\pi G(t)}$ is the choice prescribed by JBD gravity [19]

[5] We use the $\simeq$ symbol because strictly speaking $(1 - e^{-R_H^2/L_P^2})^2 = (1-\epsilon)^2 \neq 1$

simply by substituting $16\pi\phi_o^2 = G_N^{-1} = L_P^{-2} = M_P^2$ ($\hbar = c = 1$). As the bubble expands it borrows energy from the vacuum, thus depleting its energy density to the extremely low value currently observed. Note that the potential (3.48) at $t = t_P$ has the same order of magnitude as the potential $V \simeq A^4 t^{-4}$ evaluated at the Planck time $t = t_P$ and derived from eq-(3.45).

To finalize, if one were to use the usual identification $k(t) = \xi H(t)$ in [13], instead of $k = \xi\phi(t)$, eqs-(3.42,3.43) would have turned out to be

$$(\frac{\dot{a}}{a})^2 \;=\; H^2 \;=\; \frac{8\pi G(\xi H)}{3} \left(\frac{\dot{\phi}^2}{2} \;+\; \frac{\lambda(\xi H)}{4}\; \phi^4 \right) \tag{3.49}$$

$$\ddot{\phi} \;+\; 3\; H\; \dot{\phi} \;+ \lambda(\xi H)\; \phi^3 \;=\; 0 \tag{3.50}$$

One way to solve eqs-(3.49,3.50) is by rewriting the temporal derivatives in terms of derivatives with respect to H as follows

$$\dot{\phi} \;=\; \frac{(d\phi/dH)}{(dt/dH)} \;=\; \frac{\phi'(H)}{t'(H)}, \;\; \phi' = \frac{d\phi}{dH}, \;\; t' = \frac{dt}{dH} \tag{3.51}$$

$$\ddot{\phi} \;=\; \frac{\phi''(H)\; t'(H) \;-\; t''(H)\; \phi'(H)}{(t'(H))^3} \tag{3.52}$$

such that eqs-(3.49,3.50) turn into two nonlinear differential equations for $\phi = \phi(H)$, and $t = t(H)$. After eliminating H (if possible) yields finally $\phi = \phi(t)$. Consequently, it is much easier to solve the system of eqs-(3.42,3.43) than (3.49-3.52). For this reason we opted to use the cut-off identification $k(t) = \xi\phi(t)$ rather than $k(t) = \xi H(t)$.

A thorough discussion of the possible implications of Asymptotic Safety in FLRW-cosmological models based on the cutoff identification $k = \xi H(t)$ can be found [13]. For instance, in the study of cosmology in the fixed point regime, quantum gravity-driven inflation and an almost scale-free fluctuation spectrum. In order to study the effective average action of the dilaton-gravity system within the context of Weyl geometry one could follow a similar and very technical procedure as the one taken in [15] involving the inclusion of N_S scalars, N_F fermions and N_V Abelian vectors. For simplicity we decided to discuss the dilaton-gravity system within the context of Riemannian geometry.

To conclude, the exploration of Kantowski-Sachs and FLRW-Cosmology, Weyl Geometry and Asymptotic Safety in Quantum Gravity has generated some interesting results in this work that deserve to be investigated further. The fact that one recovers the observed vacuum energy density in eqs-(3.37, 3.47) is not a numerical coincidence. Most notably, it remains to find nontrivial solutions to the first-order non-linear differential equations (2.28a, 2.29b) which furnish the temporal dependence of $G(t), \Lambda(t)$ in the Kantowski-Sachs metric (2.20). It was verified that $G(t) = G_o(1 - e^{-t^3/t_H t_P^2})$ and $\Lambda(t) = \Lambda_o$ was a trivial solution. Solutions with a running $\Lambda(t)$ are harder to find. In this case we found that one cannot impose the boundary condition $\Lambda(t = 0) = \infty$ giving $\lambda_* \neq 0$, but instead $\Lambda(t = 0) = 0$ leading to $\lambda_* = 0$, if one desires to have a finite Ricci scalar

R and Kretschmann invariant K at $t = 0$. Consistency of these sought-after solutions requires that one should recover the observed vacuum energy density in the asymptotic $t \to \infty$ limit : $\rho_{vac}(t \to \infty) \to \frac{\Lambda_o}{8\pi G_o} \simeq 10^{-122} M_P^4$. Namely that $\Lambda(t)$ is zero at $t = 0$ and asymptotically approaches $\Lambda_o = \frac{3}{R_H^2}$. This will be the subject of future investigation.

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