A Generator for Sums of Powers of Recursive Integer Sequences*

Ryan Zielinski
ryan_zielinski@fastmail.com

1 March 2020†

Abstract

In this paper we will prove a relationship for sums of powers of recursive integer sequences. Also, we will give a possible path to discovery. As corollaries of the main result we will derive relationships for familiar integer sequences like the Fibonacci, Lucas, and Pell numbers. Last, we will discuss some applications and look at related work.

1 Introduction

Let $(W_n)_{n \geqq 1}$ be a recursive integer sequence with initial values of

$$a = W_1 \leqq W_2 = b,$$

where $a > 0$, and, for $n \geqq 2$, a general term of

$$W_{n-1} + p \cdot W_n = W_{n+1},$$

where p is a positive integer. Then we have the following result:

Proposition 1. *Given* $(W_n)_{n \geqq 1}$,

$$\sum_{k=1}^{n} W_k^{m+1} + \sum_{k=1}^{n} (W_{k+1} - W_k) \sum_{l=1}^{k} W_l^m = W_{n+1} \sum_{k=1}^{n} W_k^m,$$

where $n \geqq 1$ *and* m *is a positive integer.*

In this paper we will prove this result rigorously. But, before we do that, we will give a possible scenario for how someone might discover it.

†Updated on 12 March 2020. The main result was condensed into a single case, related work was discussed, and some minor changes were made.

2 Discovery

In order to illustrate how someone might discover such a result, and to offer a concrete case to be kept in mind for later material, we discuss an example from the Fibonacci numbers.

Let $(F_n)_{n \geqq 1}$ be the sequence of Fibonacci numbers with initial values of

$$1 = F_1 = F_2$$

and, for $n \geqq 2$, a general term of

$$F_{n-1} + F_n = F_{n+1}.$$

There is an argument from antiquity [5, section 2], using little more than a simple diagram, which leads to the fundamental result of

$$1 + 2 + \cdots + n = \frac{n\,(n+1)}{2}.$$

We modify it for our present purpose.

Let us start with

$$\begin{aligned}\sum_{k=1}^{4} F_k^5 &= F_1^5 + F_2^5 + F_3^5 + F_4^5 = F_1F_1^4 + F_2F_2^4 + F_3F_3^4 + F_4F_4^4 \\ &= F_1^4 + F_2^4 + 2F_3^4 + 3F_4^4.\end{aligned}$$

We place it in a table as follows:

F_1^4	F_2^4	F_3^4	F_4^4
		F_3^4	F_4^4
			F_4^4

In order to fill in the table we write

F_1^4	F_2^4		
F_1^4	F_2^4	F_3^4	
F_1^4	F_2^4	F_3^4	F_4^4
F_1^4	F_2^4	F_3^4	F_4^4

This sum is equal to

$$\begin{aligned}&(2-1)\left(F_1^4 + F_2^4\right) + (3-2)\left(F_1^4 + F_2^4 + F_3^4\right) + (5-3)\left(F_1^4 + F_2^4 + F_3^4 + F_4^4\right) \\ &= (F_3 - F_2)\left(F_1^4 + F_2^4\right) + (F_4 - F_3)\left(F_1^4 + F_2^4 + F_3^4\right) + (F_5 - F_4)\left(F_1^4 + F_2^4 + F_3^4 + F_4^4\right),\end{aligned}$$

which is

$$\sum_{k=1}^{3}(F_{k+2}-F_{k+1})\sum_{l=1}^{k+1}F_l^4.$$

For the entire table

F_1^4	F_2^4	F_3^4	F_4^4
F_1^4	F_2^4	F_3^4	F_4^4
F_1^4	F_2^4	F_3^4	F_4^4
F_1^4	F_2^4	F_3^4	F_4^4
F_1^4	F_2^4	F_3^4	F_4^4

we write the sum in a different way:

$$5\sum_{k=1}^{4}F_k^4=F_5\sum_{k=1}^{4}F_k^4.$$

Together we have

$$\sum_{k=1}^{4}F_k^5+\sum_{k=1}^{3}(F_{k+2}-F_{k+1})\sum_{l=1}^{k+1}F_l^4=F_5\sum_{k=1}^{4}F_k^4.$$

Concerning the uneven indices, if we add

$$(1-1)\left(F_1^4\right)=(F_2-F_1)\left(F_1^4\right)=0$$

to both sides then we get

$$\sum_{k=1}^{4}F_k^5+\sum_{k=1}^{4}(F_{k+1}-F_k)\sum_{l=1}^{k}F_l^4=F_5\sum_{k=1}^{4}F_k^4.$$

This suggests the general case will be

$$\sum_{k=1}^{n}F_k^{m+1}+\sum_{k=1}^{n}(F_{k+1}-F_k)\sum_{l=1}^{k}F_l^m=F_{n+1}\sum_{k=1}^{n}F_k^m, \tag{1}$$

where $n \geqq 1$ and m is a positive integer.

3 Main Result and Corollaries

Now we prove the main result of the paper.

Proof of Proposition 1

Proof. we proceed by mathematical induction. Again, the relationship we want to establish is

$$\sum_{k=1}^{n} W_k^{m+1} + \sum_{k=1}^{n} (W_{k+1} - W_k) \sum_{l=1}^{k} W_l^m = W_{n+1} \sum_{k=1}^{n} W_k^m. \tag{2}$$

For the base case of $n = 1$,

$$\sum_{k=1}^{1} W_k^{m+1} + \sum_{k=1}^{1} (W_{k+1} - W_k) \sum_{l=1}^{k} W_l^m = W_1^{m+1} + (W_2 - W_1) W_1^m.$$

$W_1 = W_2$ or $W_1 < W_2$. If $W_1 = W_2$ then

$$W_1^{m+1} + (W_2 - W_1) W_1^m = W_1^{m+1} = W_1 W_1^m = W_2 W_1^m = W_2 \sum_{k=1}^{1} W_k^m.$$

If $W_1 < W_2$ then

$$W_1^{m+1} + (W_2 - W_1) W_1^m = W_1^{m+1} + W_2 W_1^m - W_1^{m+1} = W_2 \sum_{k=1}^{1} W_k^m.$$

For the inductive step, assume that (2) is true for some $n \geqq 1$. Then

$$\begin{aligned}
\sum_{k=1}^{n+1} W_k^{m+1} + \sum_{k=1}^{n+1} (W_{k+1} - W_k) \sum_{l=1}^{k} W_l^m \quad = \quad & \sum_{k=1}^{n} W_k^{m+1} + W_{n+1}^{m+1} \\
& + \sum_{k=1}^{n} (W_{k+1} - W_k) \sum_{l=1}^{k} W_l^m + (W_{n+2} - W_{n+1}) \sum_{l=1}^{n+1} W_l^m \\
& = \sum_{k=1}^{n} W_k^{m+1} + \sum_{k=1}^{n} (W_{k+1} - W_k) \sum_{l=1}^{k} W_l^m \\
& + W_{n+1} W_{n+1}^m + (W_{n+2} - W_{n+1}) \sum_{l=1}^{n+1} W_l^m \\
& = W_{n+1} \sum_{k=1}^{n} W_k^m + W_{n+1} W_{n+1}^m + (W_{n+2} - W_{n+1}) \sum_{l=1}^{n+1} W_l^m \\
& = W_{n+1} \sum_{k=1}^{n+1} W_k^m + (W_{n+2} - W_{n+1}) \sum_{l=1}^{n+1} W_l^m.
\end{aligned}$$

Notice that $\sum_{k=1}^{n+1} W_k^m = \sum_{l=1}^{n+1} W_l^m$. The same sum is expressed in two different notations. Therefore

$$(W_{n+1} + W_{n+2} - W_{n+1}) \sum_{k=1}^{n+1} W_k^m = W_{n+2} \sum_{k=1}^{n+1} W_k^m.$$

□

Corollaries

Now we state the main result in terms of more familiar integer sequences like the Fibonacci, Lucas, and Pell numbers. ([1, 2] contain background information on these sequences.) For the Fibonacci numbers this will establish the conjecture of the previous section.

For the Fibonacci numbers, $F_{k+1} - F_k = F_{k-1}$. If we set $F_0 = 0$ then we get

Corollary 1. *Given* $(F_n)_{n \geqq 1}$,

$$\sum_{k=1}^{n} F_k^{m+1} + \sum_{k=1}^{n} F_{k-1} \sum_{l=1}^{k} F_l^m = F_{n+1} \sum_{k=1}^{n} F_k^m,$$

where $n \geqq 1$ *and* m *is a positive integer.*

The Lucas numbers $(L_n)_{n \geqq 1}$ are defined identically as the Fibonacci numbers,

$$L_{n-1} + L_n = L_{n+1},$$

where $n \geqq 2$, but with the different initial values of

$$L_1 = 1 \text{ and } L_2 = 3.$$

Also, it is common to set $L_0 = 2$. Since $L_{k+1} - L_k = L_{k-1}$, we have

Corollary 2. *Given* $(L_n)_{n \geqq 1}$,

$$\sum_{k=1}^{n} L_k^{m+1} + \sum_{k=1}^{n} L_{k-1} \sum_{l=1}^{k} L_l^m = L_{n+1} \sum_{k=1}^{n} L_k^m,$$

where $n \geqq 1$ *and* m *is a positive integer.*

For the Pell numbers $(P_n)_{n \geqq 1}$ we remind ourselves that

$$P_1 = 1 \text{ and } P_2 = 2$$

and, for $n \geqq 2$,

$$P_{n-1} + 2P_n = P_{n+1}.$$

Also, we set $P_0 = 0$. Since $P_{k+1} - P_k = P_{k-1} + P_k$, we have

Corollary 3. *Given* $(P_n)_{n \geqq 1}$,

$$\sum_{k=1}^{n} P_k^{m+1} + \sum_{k=1}^{n} (P_{k-1} + P_k) \sum_{l=1}^{k} P_l^m = P_{n+1} \sum_{k=1}^{n} P_k^m,$$

where $n \geqq 1$ *and* m *is a positive integer.*

4 Applications and Related Work

Now we discuss some applications of our result and look at related work.

Subsequences

It is quite natural to apply these ideas to subsequences of recursive integer sequences. For example, suppose we look at Fibonacci numbers of even and odd indices, F_{2k} and F_{2k-1}. For even indices we have

$$\sum_{k=1}^{n} F_{2k}^{m+1} + \sum_{k=1}^{n} \left(F_{2(k+1)} - F_{2k}\right) \sum_{l=1}^{k} F_{2l}^{m} = F_{2(n+1)} \sum_{k=1}^{n} F_{2k}^{m}.$$

Since $F_{2k+2} - F_{2k} = F_{2k+1}$, we can write it also as

$$\sum_{k=1}^{n} F_{2k}^{m+1} + \sum_{k=1}^{n} F_{2k+1} \sum_{l=1}^{k} F_{2l}^{m} = F_{2(n+1)} \sum_{k=1}^{n} F_{2k}^{m}. \tag{3}$$

For odd indices we have

$$\sum_{k=1}^{n} F_{2k-1}^{m+1} + \sum_{k=1}^{n} \left(F_{2(k+1)-1} - F_{2k-1}\right) \sum_{l=1}^{k} F_{2l-1}^{m} = F_{2(n+1)-1} \sum_{k=1}^{n} F_{2k-1}^{m}.$$

Since $F_{2k+1} - F_{2k-1} = F_{2k}$, we can write it also as

$$\sum_{k=1}^{n} F_{2k-1}^{m+1} + \sum_{k=1}^{n} F_{2k} \sum_{l=1}^{k} F_{2l-1}^{m} = F_{2(n+1)-1} \sum_{k=1}^{n} F_{2k-1}^{m}. \tag{4}$$

For the Lucas and Pell numbers and other recursive integer sequences we can do the same thing.

Partial Summation Formula

Proposition 1 is not the only result of its kind. [3] contains what is called the "partial summation formula." We state it as follows:

Proposition 2. *Given two sets of positive integers, $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$, which are monotone increasing,*

$$\sum_{k=1}^{n} a_k b_k = A_n b_n - \sum_{k=1}^{n-1} A_k \left(b_{k+1} - b_k\right), \tag{5}$$

where $A_k = \sum_{l=1}^{k} a_l$ and $n \geqq 2$.

This result implies our main result. We rewrite it as follows:

$$\sum_{k=1}^{n} a_k b_k = b_n \sum_{l=1}^{n} a_l - \sum_{k=1}^{n-1} (b_{k+1} - b_k) \sum_{l=1}^{k} a_l,$$

or

$$\sum_{k=1}^{n} a_k b_k + \sum_{k=1}^{n-1} (b_{k+1} - b_k) \sum_{l=1}^{k} a_l = b_n \sum_{l=1}^{n} a_l.$$

If we add $(b_{n+1} - b_n) \sum_{l=1}^{n} a_l$ to both sides then we get

$$\sum_{k=1}^{n} a_k b_k + \sum_{k=1}^{n} (b_{k+1} - b_k) \sum_{l=1}^{k} a_l = b_{n+1} \sum_{l=1}^{n} a_l.$$

Now we choose $a_k = W_k^m$ and $b_k = W_k$.

(It makes little sense to mention proving the converse. Our result concerns a more specialized problem.)

An important distinction is that our approach is method-based and this approach is result-based. For example, suppose we want to evaluate

$$\sum_{k=1}^{n} (-1)^{k+1} F_k^5,$$

a sum we have not considered so far. By our approach, we apply the method of Section 2 and arrive at

$$\sum_{k=1}^{n} (-1)^{k+1} F_k^5 + \sum_{k=1}^{n} (F_{k+1} - F_k) \sum_{l=1}^{k} (-1)^{l+1} F_l^4 = F_{n+1} \sum_{k=1}^{n} (-1)^{k+1} F_k^4.$$

By this approach, we set $a_k = (-1)^{k+1} F_k^4$ and $b_k = F_k$, substitute them into (5), and arrive at an analogous expression.

Of course, we can make other substitutions. $a_k = (-1)^{k+1} F_k$ and $b_k = F_k^4$ lead to

$$\sum_{k=1}^{n} (-1)^{k+1} F_k^5 = F_n^4 \sum_{l=1}^{n} (-1)^{l+1} F_l - \sum_{k=1}^{n-1} \left(F_{k+1}^4 - F_k^4\right) \sum_{l=1}^{k} (-1)^{l+1} F_l.$$

Perhaps this makes the calculation easier, perhaps it makes it harder. For our approach, however, to proceed this way would be awkward.

Generating Sums of Powers

The title of the paper contains the word "generator." Up until now we have not said anything about that. It should go without saying that the positive integers,

$$1, 2, 3, 4, \ldots,$$

are the prototypical recursive integer sequence. The initial term is 1 and we derive all subsequent terms by adding 1 to the preceding term. Looking at our main result, we have the following:

Corollary 4. *Given $\mathbb{Z}$,*

$$\sum_{k=1}^{n} k^{m+1} + \sum_{k=1}^{n} \sum_{l=1}^{k} l^{m} = (n+1) \sum_{k=1}^{n} k^{m}, \tag{6}$$

where $n \geqq 1$ and m is a positive integer.

In [5] the author discovered and proved this relationship, and without any consideration of a more general setting. (At a later time he learned al-Haytham might have gotten there 1,000 years earlier ([4, A000537]).) His purpose was the following. Suppose we start with

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n\,(n+1)}{2} = \frac{n^2 + n}{2}.$$

We can "feed" this result into (6) to "generate" an explicit expression for $\sum_{k=1}^{n} k^2$. Then we can use the new result for $\sum_{k=1}^{n} k^2$ to generate an expression for the next case of $\sum_{k=1}^{n} k^3$. We can do this for as many powers as we please.

It is tempting to try to derive sums of powers using the more general results of Propositions 1 and 2. Unfortunately, simplifying the intermediate sums requires knowing intricate relationships for the underlying sequences. Therefore we will leave such a matter for another time.

References

[1] M. Bicknell, A primer on the Pell sequence and related sequences, *The Fibonacci Quarterly* 13 (1975) 345–349.

[2] V. E. Hoggatt, Jr., *Fibonacci and Lucas Numbers.* The Fibonacci Association, Santa Clara, CA. Available at https://www.fq.math.ca/fibonacci-lucas.html.

[3] W. Lang, A215037: Applications of the partial summation formula to some sums over cubes of Fibonacci numbers, https://www.itp.kit.edu/~wl/EISpub/A215037.pdf.

[4] OEIS Foundation Inc., The On-Line Encyclopedia of Integer Sequences, https://oeis.org.

[5] R. Zielinski, Induction and analogy in a problem of finite sums, https://arxiv.org/abs/1608.04006.