Ramanujan's mathematics applied to several topics of Theoretical Physics and Cosmology

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Abstract

In this paper we have described several Ramanujan's formulas and obtained some mathematical connections with various equations concerning different sectors of Theoretical Physics and Cosmology

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Ramanujan indefinite integrals

(2) As a definite integral from which we can calculate for all values [see formula 1, p. xxvii, bottom].

$$(2)\quad \int_0^\infty \frac{1}{\left\{1+\left(\frac{x}{a}\right)^2\right\}\left\{1+\left(\frac{x}{a+1}\right)^2\right\}\cdots}\,\frac{dx}{\left\{1+\left(\frac{x}{b}\right)^2\right\}\left\{1+\left(\frac{x}{b+1}\right)^2\right\}\cdots} = \frac{1}{2}\sqrt{\pi}\,\frac{\Gamma\left(a+\frac{1}{2}\right)}{\Gamma(a)}\,\frac{\Gamma\left(b+\frac{1}{2}\right)}{\Gamma(b)}\,\frac{\Gamma(a+b)}{\Gamma\left(a+b+\frac{1}{2}\right)}$$

$$(8)\quad \frac{1}{\left(25+\frac{1^4}{100}\right)(e^{\pi}+1)}+\frac{3}{\left(25+\frac{3^4}{100}\right)(e^{3\pi}+1)}+\frac{5}{\left(25+\frac{5^4}{100}\right)(e^{5\pi}+1)}+\ldots=\frac{\pi}{8}\coth^2\frac{5\pi}{2}-\frac{4689}{11890}$$

The integral that we want analyze is, for a = 1 and b = 2:

$$\int_0^\infty \frac{1}{1+\left(\frac{x}{a}\right)^2\cdot 1+\left(\frac{x}{a+1}\right)^2\ldots}\,\frac{dx}{1+\left(\frac{x}{b}\right)^2\cdot\ 1+\left(\frac{x}{b+1}\right)^2\ldots}$$

$$\int \frac{1}{(1+x^2)\left(1+\frac{x^2}{4}\right)\left(1+\frac{x^2}{9}\right)\left(1+\frac{x^2}{4}\right)\left(1+\frac{x^2}{9}\right)\left(1+\frac{x^2}{4}\right)\left(1+\frac{x^2}{4}\right)}\,dx =$$

$$\frac{1}{25\,000}\left(-\frac{30\,x\,(3773\,x^6+65\,221\,x^4+360\,464\,x^2+687\,792)}{(x^2+4)^3\,(x^2+9)} - 33\,888\tan^{-1}\left(\frac{x}{3}\right) - 55\,763\tan^{-1}\left(\frac{x}{2}\right) + 100\,000\tan^{-1}(x)\right) + \text{constant}$$

Alternate forms of the integral:

$$\left(x^8\left(-\left(33\,888\tan^{-1}\left(\frac{x}{3}\right)+55\,763\tan^{-1}\left(\frac{x}{2}\right)\right)\right) - 113\,190\,x^7 - x^6\left(711\,648\tan^{-1}\left(\frac{x}{3}\right) + 1\,171\,023\tan^{-1}\left(\frac{x}{2}\right)\right) - 1\,956\,630\,x^5 - x^4\left(5\,286\,528\tan^{-1}\left(\frac{x}{3}\right) + 8\,699\,028\tan^{-1}\left(\frac{x}{2}\right)\right) - 10\,813\,920\,x^3 - x^2\left(16\,808\,448\tan^{-1}\left(\frac{x}{3}\right) + 27\,658\,448\tan^{-1}\left(\frac{x}{2}\right)\right) + 100\,000\,(x^2+4)^3\,(x^2+9)\tan^{-1}(x) - 20\,633\,760\,x - 19\,519\,488\tan^{-1}\left(\frac{x}{3}\right) - 32\,119\,488\tan^{-1}\left(\frac{x}{2}\right)\right) \Big/ \left(25\,000\,(x^2+4)^3\,(x^2+9)\right) + \text{constant}$$

$$-\frac{515\,844\,x}{625\,(x^2+4)^3\,(x^2+9)} - \frac{11\,319\,x^7}{2500\,(x^2+4)^3\,(x^2+9)} - \frac{195\,663\,x^5}{2500\,(x^2+4)^3\,(x^2+9)} - \frac{270\,348\,x^3}{625\,(x^2+4)^3\,(x^2+9)} - \frac{2118\,i\log\left(1-\frac{i\,x}{3}\right)}{3125} + \frac{2118\,i\log\left(1+\frac{i\,x}{3}\right)}{3125} - \frac{55\,763\,i\log\left(1-\frac{i\,x}{2}\right)}{50\,000} + \frac{55\,763\,i\log\left(1+\frac{i\,x}{2}\right)}{50\,000} + 2\,i\log(1-i\,x) - 2\,i\log(1+i\,x) + \text{constant}$$

$$-\left(\left(33\,888\,x^8\tan^{-1}\left(\frac{x}{3}\right) + 55\,763\,x^8\tan^{-1}\left(\frac{x}{2}\right) - 100\,000\,x^8\tan^{-1}(x) + 113\,190\,x^7 + 711\,648\,x^6\tan^{-1}\left(\frac{x}{3}\right) + 1\,171\,023\,x^6\tan^{-1}\left(\frac{x}{2}\right) - 2\,100\,000\,x^6\tan^{-1}(x) + 1\,956\,630\,x^5 + 5\,286\,528\,x^4\tan^{-1}\left(\frac{x}{3}\right) + 8\,699\,028\,x^4\tan^{-1}\left(\frac{x}{2}\right) - 15\,600\,000\,x^4\tan^{-1}(x) + 10\,813\,920\,x^3 + 16\,808\,448\,x^2\tan^{-1}\left(\frac{x}{3}\right) + 27\,658\,448\,x^2\tan^{-1}\left(\frac{x}{2}\right) - 49\,600\,000\,x^2\tan^{-1}(x) + 20\,633\,760\,x + 19\,519\,488\tan^{-1}\left(\frac{x}{3}\right) + 32\,119\,488\tan^{-1}\left(\frac{x}{2}\right) - 57\,600\,000\tan^{-1}(x)\right) \Big/ \left(25\,000\,(x^2+4)^3\,(x^2+9)\right)\right) + \text{constant}$$

$\log(x)$ is the natural logarithm

Expanded form of the integral:

$$-\frac{515\,844\,x}{625\,(x^2+4)^3\,(x^2+9)}-\frac{11\,319\,x^7}{2500\,(x^2+4)^3\,(x^2+9)}-\frac{195\,663\,x^5}{2500\,(x^2+4)^3\,(x^2+9)}-\frac{270\,348\,x^3}{625\,(x^2+4)^3\,(x^2+9)}-\frac{4236\tan^{-1}\left(\frac{x}{3}\right)}{3125}-\frac{55\,763\tan^{-1}\left(\frac{x}{2}\right)}{25\,000}+4\tan^{-1}(x)+\text{constant}$$

Series expansion of the integral at x=0:

$$x-\frac{20\,x^3}{27}+\frac{671\,x^5}{1080}+O\left(x^6\right)$$

(Taylor series)

Series expansion of the integral at x=-i:

$$\left(\left(2\,i\log(x+i)+\frac{i\left(161\,990+101\,664\tanh^{-1}\left(\frac{1}{3}\right)+167\,289\tanh^{-1}\left(\frac{1}{2}\right)-150\,000\log(2)\right)}{75\,000}+\pi\right)-\frac{16\,(x+i)}{3}+O\left((x+i)^2\right)\right)-4\,\pi\left\lfloor\frac{3}{4}-\frac{\arg(x+i)}{2\,\pi}\right\rfloor$$

arg(z) is the complex argument

⌊x⌋ is the floor function

$\tanh^{-1}(x)$ is the inverse hyperbolic tangent function

Series expansion of the integral at x=i:

$$\left(\left(-2\,i\log(x-i)-\frac{i\left(161\,990+101\,664\tanh^{-1}\left(\frac{1}{3}\right)+167\,289\tanh^{-1}\left(\frac{1}{2}\right)-150\,000\log(2)\right)}{75\,000}+\pi\right)-\frac{16\,(x-i)}{3}+O\left((x-i)^2\right)\right)+4\,\pi\left\lfloor\frac{\pi-2\arg(x-i)}{4\,\pi}\right\rfloor$$

Series expansion of the integral at x=-2 i:

$$\frac{1}{25\,000}\left(\left(\frac{9000}{(x+2\,i)^3}-\frac{9900\,i}{(x+2\,i)^2}-\frac{53\,715}{x+2\,i}+\frac{1}{8}\left(-223\,052\,i\log(x+2\,i)+i\left(-95\,073+271\,104\tanh^{-1}\left(\frac{2}{3}\right)-800\,000\tanh^{-1}(2)+446\,104\log(2)\right)+688\,474\,\pi\right)-\frac{6\,470\,759}{96}\,(x+2\,i)+O\left((x+2\,i)^2\right)\right)-44\,237\,\pi\left\lfloor\frac{3}{4}-\frac{\arg(x+2\,i)}{2\,\pi}\right\rfloor-100\,000\,\pi\left\lfloor\frac{\arg(x+2\,i)}{2\,\pi}+\frac{3}{4}\right\rfloor\right)$$

Series expansion of the integral at x=2 i:

$$\frac{1}{25000}\left(\left(\frac{9000}{(x-2i)^3}+\frac{9900\,i}{(x-2i)^2}-\frac{53715}{x-2i}-\frac{1}{8}i\left(-223052\log(x-2i)+446104\log(2)-800000\tanh^{-1}(2)+271104\tanh^{-1}\left(\frac{2}{3}\right)-111526\,i\pi-95073\right)-\frac{6470759}{96}(x-2i)+O\left((x-2i)^2\right)\right)+44237\pi\left\lfloor\frac{\pi-2\arg(x-2i)}{4\pi}\right\rfloor+100000\pi\left\lfloor\frac{2\arg(x-2i)+\pi}{4\pi}\right\rfloor\right)$$

Definite integral:

$$\int_0^\infty \frac{1}{\left(1+\frac{x^2}{9}\right)^2\left(1+\frac{x^2}{4}\right)^4\left(1+x^2\right)}\,dx=\frac{10349\,\pi}{50000}\approx 0.650247$$

We also have this other way of calculating the same integral for repeated values of a and b:

$$\int_0^\infty \frac{1}{1+\left(\frac{x}{a}\right)^2\cdot 1+\left(\frac{x}{a+1}\right)^2\ldots}\frac{dx}{1+\left(\frac{x}{b}\right)^2\cdot 1+\left(\frac{x}{b+1}\right)^2\ldots}$$

$$\int \frac{1}{1+x^2\left(1+\frac{x^2}{4}\right)\left(1+\frac{x^2}{4}\right)\left(1+\frac{x^2}{9}\right)\left(1+\frac{x^2}{4}\right)^8\left(1+\frac{x^2}{9}\right)}\,dx=42467328\sum_{\{\omega:\ \omega^{26}+58\omega^{24}+1521\omega^{22}+23880\omega^{20}+250320\omega^{18}+1847808\omega^{16}+9859584\omega^{14}+38350848\omega^{12}+108011520\omega^{10}+214958080\omega^{8}+287113216\omega^{6}+231211008\omega^{4}+84934656\omega^{2}+84934656=0\}}\log(x-\omega)/(13\omega^{25}+696\omega^{23}+16731\omega^{21}+238800\omega^{19}+2252880\omega^{17}+14782464\omega^{15}+69017088\omega^{13}+230105088\omega^{11}+540057600\omega^{9}+859832320\omega^{7}+861339648\omega^{5}+462422016\omega^{3}+84934656\omega)+\text{constant}$$

Definite integral:

$$\int_0^\infty \frac{1}{1+x^2\left(1+\frac{x^2}{9}\right)^2\left(1+\frac{x^2}{4}\right)^{10}}\,dx\approx 0.625697064767\ldots$$

where 0,625697... is a value even closer to 0.6283185 ...

We note that for a =1 and b = 2, we have:

$$\frac{1}{2}\pi^{1/2}\frac{\Gamma\left(a+\frac{1}{2}\right)\Gamma\left(b+\frac{1}{2}\right)\Gamma(a+b)}{\Gamma(a)\Gamma(b)\Gamma\left(a+b+\frac{1}{2}\right)}=\frac{1}{2}\pi^{\frac{1}{2}}\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{5}{2}\right)\Gamma(3)}{\Gamma(1)\Gamma\left(\frac{7}{2}\right)\Gamma(2)}=$$

$$\frac{1}{2}\pi^{\frac{1}{2}}\frac{\frac{1}{2}\sqrt{\pi}\cdot\frac{3}{4}\sqrt{\pi}\cdot 2}{\frac{15}{8}\sqrt{\pi}}=0{,}628318530638\ldots=\frac{2\pi}{10}=\frac{\pi}{5}$$

Then, in conclusion, we have the following expression (for a = 1 and b = 2):

$$\int_0^\infty \frac{1}{1+\left(\frac{x}{a}\right)^2\cdot 1+\left(\frac{x}{a+1}\right)^2\ldots}\frac{dx}{1+\left(\frac{x}{b}\right)^2\cdot 1+\left(\frac{x}{b+1}\right)^2\ldots}=$$

$$\frac{1}{2}\pi^{1/2}\frac{\Gamma\left(a+\frac{1}{2}\right)\Gamma\left(b+\frac{1}{2}\right)\Gamma(a+b)}{\Gamma(a)\Gamma(b)\Gamma\left(a+b+\frac{1}{2}\right)}=0{,}628318530638\ldots=\frac{\pi}{5}$$

Practically the obtained value is exactly 10 times 2π which represents the circumference of unit radius or π /5

Now, we have also:

$$(1.5)\quad \int_0^\infty \frac{1+\left(\frac{x}{b+1}\right)^2}{1+\left(\frac{x}{a}\right)^2}\cdot\frac{1+\left(\frac{x}{b+2}\right)^2}{1+\left(\frac{x}{a+1}\right)^2}\ldots dx=\tfrac{1}{2}\pi^{\frac{1}{2}}\frac{\Gamma(a+\frac{1}{2})\,\Gamma(b+1)\,\Gamma(b-a+\frac{1}{2})}{\Gamma(a)\,\Gamma(b+\frac{1}{2})\,\Gamma(b-a+1)}.$$

$$(1.6)\quad \int_0^\infty \frac{dx}{(1+x^2)(1+r^2x^2)(1+r^4x^2)\ldots}=\frac{\pi}{2(1+r+r^3+r^6+r^{10}+\ldots)}.$$

The first integral to the left of equation (1.5) is:

$$\int_0^\infty \frac{1+\left(\frac{x}{b+1}\right)^2}{1+\left(\frac{x}{a}\right)^2} \cdot \frac{1+\left(\frac{x}{b+2}\right)^2}{1+\left(\frac{x}{a+1}\right)^2} \dots dx$$

The value of the function to the right of (1.5):

$$\frac{1}{2}\pi^{1/2} \frac{\Gamma\left(a+\frac{1}{2}\right)\Gamma(b+1)\Gamma\left(b-a+\frac{1}{2}\right)}{\Gamma(a)\Gamma\left(b+\frac{1}{2}\right)\Gamma(b-a+1)}$$

for a = 1 e b = 2, we obtain:

$$\frac{1}{2}\pi^{\frac{1}{2}} \frac{\Gamma\left(\frac{3}{2}\right)\Gamma(3)\Gamma\left(\frac{3}{2}\right)}{\Gamma(1)\Gamma\left(\frac{5}{2}\right)\Gamma(2)} = \frac{1}{2}\sqrt{\pi}\frac{\frac{1}{2}\sqrt{\pi}\cdot 2\cdot\frac{1}{2}\sqrt{\pi}}{\frac{3}{4}\sqrt{\pi}}$$

$$= 0.886226925452758 \cdot \frac{1.570796326794897}{1.329340388179137} = 1{,}047197551196598 \dots$$

Thence:

$$\int_0^\infty \frac{1+\left(\frac{x}{b+1}\right)^2}{1+\left(\frac{x}{a}\right)^2} \cdot \frac{1+\left(\frac{x}{b+2}\right)^2}{1+\left(\frac{x}{a+1}\right)^2} \dots dx = \frac{1}{2}\pi^{1/2} \frac{\Gamma\left(a+\frac{1}{2}\right)\Gamma(b+1)\Gamma\left(b-a+\frac{1}{2}\right)}{\Gamma(a)\Gamma\left(b+\frac{1}{2}\right)\Gamma(b-a+1)} =$$

$$= 1{,}047197551196598 \dots$$

We note that: $\left(\frac{1}{2}\sqrt{\pi}\right)^2 = 0.8862269^2 = 0{,}7853981 = \frac{\pi}{4}$

Now:

$$\frac{1}{2}\pi^{1/2}\frac{\Gamma\left(a+\frac{1}{2}\right)\Gamma(b+1)\Gamma\left(b-a+\frac{1}{2}\right)}{\Gamma(a)\Gamma\left(b+\frac{1}{2}\right)\Gamma(b-a+1)} = \int_0^{\infty}\frac{1+\left(\frac{x}{b+1}\right)^2}{1+\left(\frac{x}{a}\right)^2}\cdot\frac{1+\left(\frac{x}{b+2}\right)^2}{1+\left(\frac{x}{a+1}\right)^2}\dots dx$$

from which:

$$\left(\frac{1}{2}\pi^{\frac{1}{2}}\right)^2 = \left[\int_0^{\infty}\frac{1+\left(\frac{x}{b+1}\right)^2}{1+\left(\frac{x}{a}\right)^2}\cdot\frac{1+\left(\frac{x}{b+2}\right)^2}{1+\left(\frac{x}{a+1}\right)^2}\dots dx\right]^2 \times$$

$$\times\left[\frac{1}{\frac{\Gamma\left(a+\frac{1}{2}\right)\Gamma(b+1)\Gamma\left(b-a+\frac{1}{2}\right)}{\Gamma(a)\Gamma\left(b+\frac{1}{2}\right)\Gamma(b-a+1)}}\right]^2 = 0{,}7853981 = \frac{\pi}{4}$$

Indeed for

$$\int_0^{2.898159}\frac{\left(1+\frac{x^2}{9}\right)\left(1+\frac{x^2}{16}\right)x}{\left(1+x^2\right)\left(1+\frac{x^2}{4}\right)}\,dx = 1.04719$$

we obtain :

$$(0.8862269)^2 = (1{,}04719)^2\cdot\left(\frac{1}{1{,}1816359}\right)^2 =$$

$$0{,}7853981182 = 1{,}0966068961\cdot 0{,}8462843757539^2$$

$$0{,}7853981182 = 1{,}0966068961\cdot 0{,}7161972446453$$

$$0{,}785398 \cong 0{,}7853868$$

The integral

$$\int_0^\infty \frac{1+\left(\frac{x}{b+1}\right)^2}{1+\left(\frac{x}{a}\right)^2}\cdot\frac{1+\left(\frac{x}{b+2}\right)^2}{1+\left(\frac{x}{a+1}\right)^2}\ldots dx$$

for a =1 and b = 2, become:

Indefinite integral:

$$\int \frac{\left(1+\frac{x^2}{9}\right)\left(1+\frac{x^2}{16}\right)\left(1+\frac{x^2}{25}\right)\left(1+\frac{x^2}{64}\right)}{(1+x^2)\left(1+\frac{x^2}{4}\right)\left(1+\frac{x^2}{9}\right)\left(1+\frac{x^2}{144}\right)}dx = \frac{9}{400}\left(x-\frac{2176}{429}\tan^{-1}\left(\frac{x}{12}\right)-18\tan^{-1}\left(\frac{x}{2}\right)+\frac{7560}{143}\tan^{-1}(x)\right)+\text{constant}$$

Alternate forms of the integral:

$$\frac{3\left(429x-2176\tan^{-1}\left(\frac{x}{12}\right)-7722\tan^{-1}\left(\frac{x}{2}\right)+22680\tan^{-1}(x)\right)}{57200}+\text{constant}$$

$$\frac{3\left(429x-2\left(1088\tan^{-1}\left(\frac{x}{12}\right)+3861\tan^{-1}\left(\frac{x}{2}\right)\right)+22680\tan^{-1}(x)\right)}{57200}+\text{constant}$$

$$\frac{9x}{400}-\frac{204\,i\log\left(1-\frac{ix}{12}\right)}{3575}+\frac{204\,i\log\left(1+\frac{ix}{12}\right)}{3575}-\frac{81}{400}\,i\log\left(1-\frac{ix}{2}\right)+\frac{81}{400}\,i\log\left(1+\frac{ix}{2}\right)+\frac{1701\,i\log(1-ix)}{2860}-\frac{1701\,i\log(1+ix)}{2860}+\text{constant}$$

Expanded form of the integral:

$$\frac{9x}{400}-\frac{408\tan^{-1}\left(\frac{x}{12}\right)}{3575}-\frac{81}{200}\tan^{-1}\left(\frac{x}{2}\right)+\frac{1701\tan^{-1}(x)}{1430}+\text{constant}$$

Series expansion of the integral at x=0:

$$x-\frac{16399x^3}{43200}+\frac{610081x^5}{2592000}+O(x^6)$$

(Taylor series)

Series expansion of the integral at x=-i:

$$\frac{1}{1430}\left(\left(\frac{3}{40}\left(11\,340\,i\log(x+i)+i\left(-429+2176\tanh^{-1}\left(\frac{1}{12}\right)+7722\tanh^{-1}\left(\frac{1}{2}\right)-11\,340\log(2)\right)+5670\,\pi\right)+\frac{329\,643\,(x+i)}{5720}-\frac{384\,618\,321\,i\,(x+i)^2}{1\,635\,920}+\frac{15\,134\,599\,739\,(x+i)^3}{233\,936\,560}+\frac{4\,538\,344\,203\,557\,i\,(x+i)^4}{53\,524\,684\,928}-\frac{15\,165\,844\,268\,932\,187\,(x+i)^5}{287\,026\,122\,926\,400}+O\left((x+i)^6\right)\right)-1701\,\pi\left\lfloor\frac{3}{4}-\frac{\arg(x+i)}{2\,\pi}\right\rfloor\right)$$

arg(z) is the complex argument

$\lfloor x \rfloor$ is the floor function

$\tanh^{-1}(x)$ is the inverse hyperbolic tangent function

Big-O notation

Series expansion of the integral at x=i:

$$\frac{1}{1430}\left(\left(\frac{3}{40}\left(-11\,340\,i\log(x-i)-i\left(-429+2176\tanh^{-1}\left(\frac{1}{12}\right)+7722\tanh^{-1}\left(\frac{1}{2}\right)-11\,340\log(2)\right)+5670\,\pi\right)+\frac{329\,643\,(x-i)}{5720}+\frac{384\,618\,321\,i\,(x-i)^2}{1\,635\,920}+\frac{15\,134\,599\,739\,(x-i)^3}{233\,936\,560}-\frac{4\,538\,344\,203\,557\,i\,(x-i)^4}{53\,524\,684\,928}-\frac{15\,165\,844\,268\,932\,187\,(x-i)^5}{287\,026\,122\,926\,400}+O\left((x-i)^6\right)\right)+1701\,\pi\left\lfloor\frac{\pi-2\arg(x-i)}{4\,\pi}\right\rfloor\right)$$

Big-O notation

Series expansion of the integral at x=-2 i:

$$\frac{1}{28\,600}\left(\left(\frac{3}{4}\left(-7722\,i\log(x+2\,i)+4\,i\left(-429+1088\tanh^{-1}\left(\frac{1}{6}\right)-11\,340\tanh^{-1}(2)+3861\log(2)\right)+41\,499\,\pi\right)-\frac{3\,478\,761}{280}(x+2\,i)+\frac{606\,579\,831\,i\,(x+2\,i)^2}{78\,400}+\frac{30\,134\,093\,847\,(x+2\,i)^3}{5\,488\,000}-\frac{12\,925\,064\,989\,551\,i\,(x+2\,i)^4}{3\,073\,280\,000}-\frac{1\,822\,758\,154\,735\,661\,(x+2\,i)^5}{537\,824\,000\,000}+O\left((x+2\,i)^6\right)\right)-22\,437\,\pi\left\lfloor\frac{3}{4}-\frac{\arg(x+2\,i)}{2\,\pi}\right\rfloor-34\,020\,\pi\left\lfloor\frac{\arg(x+2\,i)}{2\,\pi}+\frac{3}{4}\right\rfloor\right)$$

Big-O notation

Series expansion of the integral at x=2 i:

$$\frac{1}{28\,600}\left(\left(-\frac{3}{4}\,i\left(-7722\log(x-2\,i)+15\,444\log(2)-45\,360\tanh^{-1}(2)+4352\tanh^{-1}\left(\frac{1}{6}\right)-3861\,i\,\pi-1716\right)-\frac{3\,478\,761}{280}(x-2\,i)-\frac{606\,579\,831\,i\,(x-2\,i)^2}{78\,400}+\frac{30\,134\,093\,847\,(x-2\,i)^3}{5\,488\,000}+\frac{12\,925\,064\,989\,551\,i\,(x-2\,i)^4}{3\,073\,280\,000}-\frac{1\,822\,758\,154\,735\,661\,(x-2\,i)^5}{537\,824\,000\,000}+O\left((x-2\,i)^6\right)\right)+22\,437\,\pi\left\lfloor\frac{\pi-2\arg(x-2\,i)}{4\,\pi}\right\rfloor+34\,020\,\pi\left\lfloor\frac{2\arg(x-2\,i)+\pi}{4\,\pi}\right\rfloor\right)$$

Big-O notation

Series expansion of the integral at x=-12 i:

$$\frac{1}{28600}\Bigg(\Bigg(3\,(-544\,i\log(x+12\,i)+544\,i\log(3)+1632\,i\log(2)-11340\,i\tanh^{-1}(12)+3861\,i\tanh^{-1}(6)+7207\,\pi-2574\,i)+\frac{2517862\,(x+12\,i)}{5005}+\frac{2163484619\,i\,(x+12\,i)^2}{300600300}+\frac{26623398263003\,(x+12\,i)^3}{54162162054000}-\frac{309607706021256161\,i\,(x+12\,i)^4}{8674611874568640000}-\frac{3351103859049825276557\,(x+12\,i)^5}{1302492972966481296000000}+O\left((x+12\,i)^6\right)\Bigg)+11583\,\pi\left\lfloor\frac{3}{4}-\frac{\arg\left(\frac{x}{2}+6\,i\right)}{2\,\pi}\right\rfloor+11583\,\pi\left\lfloor\frac{\arg\left(\frac{x}{2}+6\,i\right)}{2\,\pi}+\frac{3}{4}\right\rfloor+3264\,\pi\left\lfloor\frac{\frac{3\pi}{2}-\arg(x+12\,i)}{2\,\pi}\right\rfloor-34020\,\pi\left\lfloor\frac{3}{4}-\frac{\arg(x+12\,i)}{2\,\pi}\right\rfloor-34020\,\pi\left\lfloor\frac{\arg(x+12\,i)}{2\,\pi}+\frac{3}{4}\right\rfloor\Bigg)$$

Definite integral after subtraction of diverging parts:

$$\int_0^\infty\left(\frac{\left(1+\frac{x^2}{64}\right)\left(1+\frac{x^2}{25}\right)\left(1+\frac{x^2}{16}\right)}{\left(1+\frac{x^2}{144}\right)\left(1+\frac{x^2}{4}\right)(1+x^2)}-\frac{9}{400}\right)dx=\frac{1743\,\pi}{5200}\approx 1.05304$$

Or:

$$\int\frac{\left(1+\frac{x^2}{9}\right)\left(1+\frac{x^2}{16}\right)\left(1+\frac{x^2}{25}\right)\left(1+\frac{x^2}{256}\right)}{(1+x^2)\left(1+\frac{x^2}{4}\right)\left(1+\frac{x^2}{9}\right)\left(1+\frac{x^2}{400}\right)}\,dx=$$

$$\frac{1463\,x-9600\tan^{-1}\left(\frac{x}{20}\right)-39102\tan^{-1}\left(\frac{x}{2}\right)+112200\tan^{-1}(x)}{93632}+\text{constant}$$

$$\frac{1463\,x-6\left(1600\tan^{-1}\left(\frac{x}{20}\right)+6517\tan^{-1}\left(\frac{x}{2}\right)\right)+112200\tan^{-1}(x)}{93632}+\text{constant}$$

$$\frac{x}{64}-\frac{75\,i\log\left(1-\frac{i\,x}{20}\right)}{1463}+\frac{75\,i\log\left(1+\frac{i\,x}{20}\right)}{1463}-\frac{147}{704}\,i\log\left(1-\frac{i\,x}{2}\right)+\frac{147}{704}\,i\log\left(1+\frac{i\,x}{2}\right)+\frac{1275\,i\log(1-i\,x)}{2128}-\frac{1275\,i\log(1+i\,x)}{2128}+\text{constant}$$

Expanded form of the integral:

$$\frac{x}{64} - \frac{150\tan^{-1}\left(\frac{x}{20}\right)}{1463} - \frac{147}{352}\tan^{-1}\left(\frac{x}{2}\right) + \frac{1275\tan^{-1}(x)}{1064} + \text{constant}$$

Series expansion of the integral at x=0:

$$x - \frac{489 x^3}{1280} + \frac{151\,713 x^5}{640\,000} + O\left(x^6\right)$$

(Taylor series)

Series expansion of the integral at x=-i:

$$\frac{1}{1064}\left(\left(\frac{1}{88}\left(56\,100\,i\log(x+i) + i\left(-1463 + 9600\tanh^{-1}\left(\frac{1}{20}\right) + 39\,102\tanh^{-1}\left(\frac{1}{2}\right) - 56\,100\log(2)\right) + 28\,050\,\pi\right) + \frac{35\,835\,(x+i)}{1064} - \frac{151\,511\,489\,i\,(x+i)^2}{849\,072} + \frac{51\,062\,687\,009\,(x+i)^3}{1\,016\,339\,184} + \frac{70\,093\,070\,068\,265\,i\,(x+i)^4}{1\,081\,384\,891\,776} - \frac{43\,829\,608\,754\,773\,817\,(x+i)^5}{1\,078\,681\,429\,546\,560} + O\left((x+i)^6\right)\right) - 1275\,\pi\left\lfloor\frac{3}{4} - \frac{\arg(x+i)}{2\pi}\right\rfloor\right)$$

arg(z) is the complex argument

⌊x⌋ is the floor function

$\tanh^{-1}(x)$ is the inverse hyperbolic tangent function

Series expansion of the integral at x=i:

$$\frac{1}{1064}\left(\left(\frac{1}{88}\left(-56\,100\,i\log(x-i) - i\left(-1463 + 9600\tanh^{-1}\left(\frac{1}{20}\right) + 39\,102\tanh^{-1}\left(\frac{1}{2}\right) - 56\,100\log(2)\right) + 28\,050\,\pi\right) + \frac{35\,835\,(x-i)}{1064} + \frac{151\,511\,489\,i\,(x-i)^2}{849\,072} + \frac{51\,062\,687\,009\,(x-i)^3}{1\,016\,339\,184} - \frac{70\,093\,070\,068\,265\,i\,(x-i)^4}{1\,081\,384\,891\,776} - \frac{43\,829\,608\,754\,773\,817\,(x-i)^5}{1\,078\,681\,429\,546\,560} + O\left((x-i)^6\right)\right) + 1275\,\pi\left\lfloor\frac{\pi - 2\arg(x-i)}{4\pi}\right\rfloor\right)$$

Series expansion of the integral at x=-2 i:

$$\frac{1}{46816}\left(\left(\left(-\frac{19551}{2}\, i \log(x+2 i)+i\left(-1463+4800 \tanh^{-1}\left(\frac{1}{10}\right)-56100 \tanh^{-1}(2)+19551 \log(2)\right)+\frac{204849 \pi}{4}\right)-\frac{5452867}{264}(x+2 i)+\frac{2670247517\, i (x+2 i)^2}{209088}+\frac{1124594248183 (x+2 i)^3}{124198272}-\frac{454805094467717\, i (x+2 i)^4}{65576687616}-\frac{181415774152444783 (x+2 i)^5}{32460460369920}+O\left((x+2 i)^6\right)\right)-36549 \pi\left\lfloor\frac{3}{4}-\frac{\arg(x+2 i)}{2 \pi}\right\rfloor-56100 \pi\left\lfloor\frac{\arg(x+2 i)}{2 \pi}+\frac{3}{4}\right\rfloor\right)$$

Series expansion of the integral at x=2 i:

$$\frac{1}{46816}\left(\left(-\frac{1}{4}\, i\left(-39102 \log(x-2 i)+78204 \log(2)-224400 \tanh^{-1}(2)+19200 \tanh^{-1}\left(\frac{1}{10}\right)-19551\, i \pi-5852\right)-\frac{5452867}{264}(x-2 i)-\frac{2670247517\, i (x-2 i)^2}{209088}+\frac{1124594248183 (x-2 i)^3}{124198272}+\frac{454805094467717\, i (x-2 i)^4}{65576687616}-\frac{181415774152444783 (x-2 i)^5}{32460460369920}+O\left((x-2 i)^6\right)\right)+36549 \pi\left\lfloor\frac{\pi-2 \arg(x-2 i)}{4 \pi}\right\rfloor+56100 \pi\left\lfloor\frac{2 \arg(x-2 i)+\pi}{4 \pi}\right\rfloor\right)$$

Series expansion of the integral at x=-20 i:

$$\frac{1}{46816}\left(\left(\left(-2400\, i \log(x+20 i)+10\, i\left(-1463-5610 \tanh^{-1}(20)+720 \log(2)+240 \log(5)\right)+19551\, i \tanh^{-1}(10)+35349 \pi\right)+\frac{2763494 (x+20 i)}{4389}+\frac{649721759\, i (x+20 i)^2}{231159852}+\frac{20700898140173 (x+20 i)^3}{182620906277040}-\frac{658978172907416321\, i (x+20 i)^4}{128243705223988569600}-\frac{20382181232342744926607 (x+20 i)^5}{84429243334212874796160000}+O\left((x+20 i)^6\right)\right)-31749 \pi\left\lfloor\frac{3}{4}-\frac{\arg(x+20 i)}{2 \pi}\right\rfloor-36549 \pi\left\lfloor\frac{\arg(x+20 i)}{2 \pi}+\frac{3}{4}\right\rfloor\right)$$

Series expansion of the integral at x=20 i:

$$\frac{1}{46816}\Bigg(\Bigg(-i\,(-2400\log(x-20\,i)+2400\log(5)+7200\log(2)-56100\tanh^{-1}(20)+19551\tanh^{-1}(10)-1200\,i\,\pi-14630)+\frac{2763494\,(x-20\,i)}{4389}-\frac{649721759\,i\,(x-20\,i)^2}{231159852}+\frac{20700898140173\,(x-20\,i)^3}{182620906277040}+\frac{658978172907416321\,i\,(x-20\,i)^4}{128243705223988569600}-\frac{20382181232342744926607\,(x-20\,i)^5}{84429243334212874796160000}+O\big((x-20\,i)^6\big)\Bigg)+31749\,\pi\left\lfloor\frac{\pi-2\arg(x-20\,i)}{4\pi}\right\rfloor+36549\,\pi\left\lfloor\frac{2\arg(x-20\,i)+\pi}{4\pi}\right\rfloor\Bigg)$$

Series expansion of the integral at x=∞:

$$\frac{x}{64}+\frac{1671\,\pi}{4928}+\frac{27}{16\,x}+O\left(\left(\frac{1}{x}\right)^3\right)$$

(Laurent series)

Definite integral after subtraction of diverging parts:

$$\int_0^\infty\left(\frac{\left(1+\frac{x^2}{256}\right)\left(1+\frac{x^2}{25}\right)\left(1+\frac{x^2}{16}\right)}{\left(1+\frac{x^2}{400}\right)\left(1+\frac{x^2}{4}\right)(1+x^2)}-\frac{1}{64}\right)dx=\frac{1671\,\pi}{4928}\approx 1.06526$$

We note that 1,05304 and 1,06526 are nearly to the value:

$$\frac{1}{2}\pi^{\frac{1}{2}}\frac{\Gamma\left(\frac{3}{2}\right)\Gamma(3)\Gamma\left(\frac{3}{2}\right)}{\Gamma(1)\Gamma\left(\frac{5}{2}\right)\Gamma(2)}=\frac{1}{2}\sqrt{\pi}\,\frac{\frac{1}{2}\sqrt{\pi}\cdot 2\cdot\frac{1}{2}\sqrt{\pi}}{\frac{3}{4}\sqrt{\pi}}=1{,}047197551196598\ldots=$$

$$= \frac{1}{2}\pi^{\frac{1}{2}}\frac{\Gamma\left(\frac{3}{2}\right)\Gamma(3)\Gamma\left(\frac{3}{2}\right)}{\Gamma(1)\Gamma\left(\frac{5}{2}\right)\Gamma(2)} = \frac{\pi}{3}$$

And since the integral is of the form:

$$\int_0^\infty \frac{1+\left(\frac{x}{b+1}\right)^2}{1+\left(\frac{x}{a}\right)^2}\cdot\frac{1+\left(\frac{x}{b+2}\right)^2}{1+\left(\frac{x}{a+1}\right)^2}\ldots dx$$

we note that the further one goes on in the series of fractions the more the value of the integral tends to the number 1.04719755119 ... which, we note, multiplied 3 gives the following value: 3 × 1.04719755119 = 3.14159265357 ... that if we go to check it is exactly the value of π. Ramanujan, with the value of that integral multiplied 3 times, by a = 1 and b = 2, wanted to give us another mathematical expression that provides the value of π. Finally, we note that the value 1.05304 * 3 is equal to 3.15912 value close to π.

So in conclusion:

$$3\int_0^\infty \frac{1+\left(\frac{x}{b+1}\right)^2}{1+\left(\frac{x}{a}\right)^2}\cdot\frac{1+\left(\frac{x}{b+2}\right)^2}{1+\left(\frac{x}{a+1}\right)^2}\ldots dx = 3\left(\frac{1}{2}\pi^{\frac{1}{2}}\frac{\Gamma\left(a+\frac{1}{2}\right)\Gamma(b+1)\Gamma\left(b-a+\frac{1}{2}\right)}{\Gamma(a)\Gamma\left(b+\frac{1}{2}\right)\Gamma(b-a+1)}\right)$$

$$= 3 \times 1{,}047197551196598\ldots = 3{,}14159265357\ldots$$

for the integral from 0 to 2,898159 we have:

integrate [(1+(x^2)/9))/(1+(x^2))]*[(1+(x^2)/16))/(1+(x^2)/4)] x, [0,2.898159]

Definite integral

$$\int_0^{2.898159} \frac{\left(1+\frac{x^2}{9}\right)\left(1+\frac{x^2}{16}\right)x}{\left(1+x^2\right)\left(1+\frac{x^2}{4}\right)}\,dx = 1.04719$$

Indefinite integral

$$\int \frac{\left(1+\frac{x^2}{9}\right)\left(1+\frac{x^2}{16}\right)x}{\left(1+x^2\right)\left(1+\frac{x^2}{4}\right)}\,dx = \frac{1}{36}\left(\frac{x^2}{2}+20\log\left(x^2+1\right)-10\log\left(x^2+4\right)\right) + \text{constant}$$

Also for the definite integral

integrate [(1+(x^2)/225))/(1+(x^2)/169))]*[(1+(x^2)/256))/(1+(x^2)/196))] x, [0,1.449222]

$$\int_0^{1.449222} \frac{\left(1+\frac{x^2}{225}\right)\left(1+\frac{x^2}{256}\right)x}{\left(1+\frac{x^2}{169}\right)\left(1+\frac{x^2}{196}\right)}\,dx = 1.0472$$

and indefinite

$$\int \frac{\left(1+\frac{x^2}{225}\right)\left(1+\frac{x^2}{256}\right)x}{\left(1+\frac{x^2}{169}\right)\left(1+\frac{x^2}{196}\right)}\,dx = \frac{8281\left(\frac{x^2}{2}+\frac{812}{9}\log(x^2+169)-\frac{290}{9}\log(x^2+196)\right)}{14400} + \text{constant}$$

And further for even more varied:

integrate [(1+(x^2)/9))/(1+(x^2))]*[(1+(x^2)/16))/(1+(x^2)/4)] * [(1+(x^2)/225))/(1+(x^2)/169))]*[(1+(x^2)/256))/(1+(x^2)/196))] x, [0,2.92386]

$$\int_0^{2.92386} \frac{\left(1+\frac{x^2}{9}\right)\left(1+\frac{x^2}{16}\right)\left(1+\frac{x^2}{225}\right)\left(1+\frac{x^2}{256}\right)x}{\left(1+x^2\right)\left(1+\frac{x^2}{4}\right)\left(1+\frac{x^2}{169}\right)\left(1+\frac{x^2}{196}\right)}\,dx = 1.0472$$

$$\int \frac{\left(1+\frac{x^2}{9}\right)\left(1+\frac{x^2}{16}\right)\left(1+\frac{x^2}{225}\right)\left(1+\frac{x^2}{256}\right)x}{\left(1+x^2\right)\left(1+\frac{x^2}{4}\right)\left(1+\frac{x^2}{169}\right)\left(1+\frac{x^2}{196}\right)}\,dx = \frac{1}{518400}8281\left(\frac{x^2}{2}+\frac{1360}{39}\log(x^2+1)-\frac{1547}{88}\log(x^2+4)+\frac{7888}{99}\log(x^2+169)-\frac{27115}{936}\log(x^2+196)\right) + \text{constant}$$

And again:

integrate [(1+(x^2)/9))/(1+(x^2))]*[(1+(x^2)/16))/(1+(x^2)/4)] * [(1+(x^2)/225))/(1+(x^2)/169))]*[(1+(x^2)/256))/(1+(x^2)/196))] x, [0,0.93069)π]

$$\int_0^{0.93069\pi} \frac{\left(1+\frac{x^2}{9}\right)\left(1+\frac{x^2}{16}\right)\left(1+\frac{x^2}{225}\right)\left(1+\frac{x^2}{256}\right)x}{\left(1+x^2\right)\left(1+\frac{x^2}{4}\right)\left(1+\frac{x^2}{169}\right)\left(1+\frac{x^2}{196}\right)}\,dx = 1.0472$$

Thence:

$$\int_0^\infty \frac{1+\left(\frac{x}{b+1}\right)^2}{1+\left(\frac{x}{a}\right)^2}\cdot\frac{1+\left(\frac{x}{b+2}\right)^2}{1+\left(\frac{x}{a+1}\right)^2}\dots dx = \frac{1}{2}\pi^{1/2}\frac{\Gamma\left(a+\frac{1}{2}\right)\Gamma(b+1)\Gamma\left(b-a+\frac{1}{2}\right)}{\Gamma(a)\Gamma\left(b+\frac{1}{2}\right)\Gamma(b-a+1)} =$$

= 1,04719

For various definition intervals.

We have then:

integrate [(1+(x^2)/9))/(1+(x^2))]*[(1+(x^2)/16))/(1+(x^2)/4)] * [(1+(x^2)/225))/(1+(x^2)/169))]*[(1+(x^2)/256))/(1+(x^2)/196))] x, [0,1.639851]

$$\int_0^{1.639851} \frac{\left(1+\frac{x^2}{9}\right)\left(1+\frac{x^2}{16}\right)\left(1+\frac{x^2}{225}\right)\left(1+\frac{x^2}{256}\right)x}{\left(1+x^2\right)\left(1+\frac{x^2}{4}\right)\left(1+\frac{x^2}{169}\right)\left(1+\frac{x^2}{196}\right)}\,dx = 0.618034$$

Which is the same as before, but with a different definition range: that is from 0 to 1.639851. The end result is the conjugate of the golden ratio 0.61803398 = 0.618034. This placing b = 2 + 1, 2 + 2, 2 + 13, 2 + 14 = 3, 4, 15 and 16 and a = 1, 1 + 1, 1 + 12, 1 + 13 = 1, 2, 13, 14.

We take now the integral (1.6):

$$\int_0^\infty \frac{dx}{(1+x^2)(1+r^2x^2)(1+r^4x^2)\dots} = \frac{\pi}{2(1+r+r^3+r^6+r^{10}+\cdots)}$$

and place r = 2. We will have for the function to the right of the equation:

$$\frac{\pi}{2(1+r+r^3+r^6+r^{10}+\cdots)} = \frac{\pi}{2(1+2+8+64+1024\dots)} =$$

$$= \frac{\pi}{2+4+16+128+2048\dots} = \frac{\pi}{2198} = 0,001429296\dots$$

For the integral, we have:

integrate 1/[(1+(x^2))(1+(4x^2))(1+(16x^2)]

$$\int \frac{1}{(1+x^2)(1+4x^2)(1+16x^2)}\,dx = \frac{1}{45}\left(\tan^{-1}(x) - 10\tan^{-1}(2x) + 16\tan^{-1}(4x)\right) + \text{constant}$$

$$\frac{1}{45}\left(\tan^{-1}(x) - 2\left(5\tan^{-1}(2x) - 8\tan^{-1}(4x)\right)\right) + \text{constant}$$

$$\frac{1}{90}\,i\log(1-i\,x) - \frac{1}{90}\,i\log(1+i\,x) - \frac{1}{9}\,i\log(1-2\,i\,x) + \frac{1}{9}\,i\log(1+2\,i\,x) + \frac{8}{45}\,i\log(1-4\,i\,x) - \frac{8}{45}\,i\log(1+4\,i\,x) + \text{constant}$$

$$\frac{1}{45}\tan^{-1}(x) - \frac{2}{9}\tan^{-1}(2x) + \frac{16}{45}\tan^{-1}(4x) + \text{constant}$$

$$x - 7x^3 + \frac{357x^5}{5} + O\left(x^6\right)$$

(Taylor series)

$$\frac{1}{45}\left(\left(\left(8\,i\log\left(x+\frac{i}{4}\right) - i\left(\tanh^{-1}\left(\frac{1}{4}\right) - 2\left(5\tanh^{-1}\left(\frac{1}{2}\right) + \log(16)\right)\right) + 4\pi\right) - \frac{48}{5}\left(x+\frac{i}{4}\right) - \frac{11536}{225}\,i\left(x+\frac{i}{4}\right)^2 + \frac{99904\left(x+\frac{i}{4}\right)^3}{1125} + \frac{17602592\,i\left(x+\frac{i}{4}\right)^4}{50625} - \frac{411986176\left(x+\frac{i}{4}\right)^5}{421875} + O\left(\left(x+\frac{i}{4}\right)^6\right)\right) - 16\pi\left\lfloor\frac{3}{4} - \frac{\arg(4x+i)}{2\pi}\right\rfloor\right)$$

$$\frac{1}{45}\left(\left(\left(-8\,i\log\left(x-\frac{i}{4}\right) + i\left(\tanh^{-1}\left(\frac{1}{4}\right) - 2\left(5\tanh^{-1}\left(\frac{1}{2}\right) + \log(16)\right)\right) + 4\pi\right) - \frac{48}{5}\left(x-\frac{i}{4}\right) + \frac{11536}{225}\,i\left(x-\frac{i}{4}\right)^2 + \frac{99904\left(x-\frac{i}{4}\right)^3}{1125} - \frac{17602592\,i\left(x-\frac{i}{4}\right)^4}{50625} - \frac{411986176\left(x-\frac{i}{4}\right)^5}{421875} + O\left(\left(x-\frac{i}{4}\right)^6\right)\right) + 16\pi\left\lfloor\frac{\pi - 2\arg(4x-i)}{4\pi}\right\rfloor\right)$$

$$\frac{1}{45}\left(\left(\left(-5\,i\log\left(x+\frac{i}{2}\right) - i\left(\tanh^{-1}\left(\frac{1}{2}\right) + 16\tanh^{-1}(2)\right) + \frac{27\pi}{2}\right) - 25\left(x+\frac{i}{2}\right) + \frac{1085}{18}\,i\left(x+\frac{i}{2}\right)^2 + \frac{4445}{27}\left(x+\frac{i}{2}\right)^3 - \frac{164885}{324}\,i\left(x+\frac{i}{2}\right)^4 - \frac{14665}{9}\left(x+\frac{i}{2}\right)^5 + O\left(\left(x+\frac{i}{2}\right)^6\right)\right) + 10\pi\left\lfloor\frac{3}{4} - \frac{\arg(2x+i)}{2\pi}\right\rfloor - 16\pi\left\lfloor\frac{3}{4} - \frac{\arg(4x+2i)}{2\pi}\right\rfloor - 16\pi\left\lfloor\frac{\arg(4x+2i)}{2\pi} + \frac{3}{4}\right\rfloor\right)$$

$$\frac{1}{45}\left(\left(\frac{1}{2}\,i\left(10\log\left(x-\frac{i}{2}\right)+32\tanh^{-1}(2)+2\tanh^{-1}\left(\frac{1}{2}\right)+5\,i\,\pi\right)-25\left(x-\frac{i}{2}\right)-\frac{1085}{18}\,i\left(x-\frac{i}{2}\right)^2+\frac{4445}{27}\left(x-\frac{i}{2}\right)^3+\frac{164\,885}{324}\,i\left(x-\frac{i}{2}\right)^4-\frac{14\,665}{9}\left(x-\frac{i}{2}\right)^5+O\left(\left(x-\frac{i}{2}\right)^6\right)\right)-10\,\pi\left\lfloor\frac{\pi-2\arg(2\,x-i)}{4\,\pi}\right\rfloor+16\,\pi\left\lfloor\frac{\pi-2\arg(4\,x-2\,i)}{4\,\pi}\right\rfloor+16\,\pi\left\lfloor\frac{2\arg(4\,x-2\,i)+\pi}{4\,\pi}\right\rfloor\right)$$

$$\frac{1}{45}\left(\left(\frac{1}{4}\left(2\,i\log(x+i)+2\,i\left(20\tanh^{-1}(2)-32\tanh^{-1}(4)-\log(2)\right)+25\,\pi\right)+\frac{53\,(x+i)}{20}-\frac{15\,841\,i\,(x+i)^2}{3600}-\frac{426\,853\,(x+i)^3}{54\,000}+\frac{92\,399\,041\,i\,(x+i)^4}{6\,480\,000}+\frac{692\,640\,151\,(x+i)^5}{27\,000\,000}+O\left((x+i)^6\right)\right)-7\,\pi\left\lfloor\frac{3}{4}-\frac{\arg(x+i)}{2\,\pi}\right\rfloor-6\,\pi\left\lfloor\frac{\arg(x+i)}{2\,\pi}+\frac{3}{4}\right\rfloor\right)$$

$$\frac{1}{45}\left(\left(\frac{1}{4}\left(-2\,i\log(x-i)+2\,i\log(2)+64\,i\tanh^{-1}(4)-40\,i\tanh^{-1}(2)+\pi\right)+\frac{53\,(x-i)}{20}+\frac{15\,841\,i\,(x-i)^2}{3600}-\frac{426\,853\,(x-i)^3}{54\,000}-\frac{92\,399\,041\,i\,(x-i)^4}{6\,480\,000}+\frac{692\,640\,151\,(x-i)^5}{27\,000\,000}+O\left((x-i)^6\right)\right)+7\,\pi\left\lfloor\frac{\pi-2\arg(x-i)}{4\,\pi}\right\rfloor+6\,\pi\left\lfloor\frac{2\arg(x-i)+\pi}{4\,\pi}\right\rfloor\right)$$

$$\frac{7\pi}{90}-\frac{1}{320\,x^5}+O\left(\left(\frac{1}{x}\right)^6\right)$$

(Laurent series)

$$\int_0^\infty \frac{1}{(1+x^2)(1+4\,x^2)(1+16\,x^2)}\,dx=\frac{7\,\pi}{90}\approx 0.244346$$

For the same integral, from 0 to 700,00098

integrate 1/((1+(x^2))(1+(4x^2))(1+(16x^2)) x, [0, 700.00098]

we have:

$$\int \frac{1}{\{(1+x^2)(1+4\,x^2)(1+16\,x^2)\,x,\,\{0,\,700.00098\}\}}\,dx=$$
$$\{-0.0111111\log(x^2+1)+0.222222\log(4\,x^2+1)-0.711111\log(16\,x^2+1)+\log(x),\ \{\tilde{\infty},\,0.00142857\,x\}\}+\text{constant}$$

$$\{-0.0111111\log(x^2+1)+0.222222\log(4\,x^2+1)-0.711111\log(16\,x^2+1)+\log(x),\ \{\tilde{\infty},\,0.00142857\,x\}\}+\text{constant}$$

The value 0.0014285 is very close to 0.0014292. It should be noted that 0.00142921 / 16 = 0.66403858 which is very close to the Planck constant value 6.626 * 10 ^-34 = 0.6626 * (10 ^-33) and close to 0.66274 which is the Laplace limit provided by the solution of the equation:

$$\frac{x\ e^{\sqrt{x^2+1}}}{\sqrt{x^2+1}+1} = 1$$

For r = 3, the integral that we calculate, is:

$$\int \frac{1}{(1+x^2)(1+9x^2)(1+81x^2)}\,dx = \frac{1}{640}\left(\tan^{-1}(x) - 30\tan^{-1}(3x) + 81\tan^{-1}(9x)\right) + \text{constant}$$

$$\frac{1}{640}\left(\tan^{-1}(x) - 3\left(10\tan^{-1}(3x) - 27\tan^{-1}(9x)\right)\right) + \text{constant}$$

$$\frac{i\log(1-ix)}{1280} - \frac{i\log(1+ix)}{1280} - \frac{3}{128}\,i\log(1-3ix) + \frac{3}{128}\,i\log(1+3ix) + \frac{81\,i\log(1-9ix)}{1280} - \frac{81\,i\log(1+9ix)}{1280} + \text{constant}$$

$$\frac{1}{640}\tan^{-1}(x) - \frac{3}{64}\tan^{-1}(3x) + \frac{81}{640}\tan^{-1}(9x) + \text{constant}$$

$$x - \frac{91x^3}{3} + \frac{7462x^5}{5} + O\left(x^6\right)$$

(Taylor series)

$$\frac{1}{640}\left(\left(\left(\frac{81}{4}\left(2i\log\left(x+\frac{i}{9}\right) + 2i\log\left(\frac{9}{2}\right) + \pi\right) + 30i\tanh^{-1}\left(\frac{1}{3}\right) - i\tanh^{-1}\left(\frac{1}{9}\right)\right) + \frac{6561}{80}\left(x+\frac{i}{9}\right) - \frac{3\,352\,671\,i\left(x+\frac{i}{9}\right)^2}{6400} - \frac{91\,899\,927\left(x+\frac{i}{9}\right)^3}{128\,000} + \frac{114\,552\,638\,991\,i\left(x+\frac{i}{9}\right)^4}{20\,480\,000} + \frac{9\,459\,894\,794\,301\left(x+\frac{i}{9}\right)^5}{1\,024\,000\,000} + O\left(\left(x+\frac{i}{9}\right)^6\right)\right) - 81\pi\left\lfloor\frac{3}{4} - \frac{\arg(9x+i)}{2\pi}\right\rfloor\right)$$

$$\frac{1}{640}\left(\left(\left(\frac{1}{2}\,i\left(81\left(-\log(9\,x-i)+\log(2)-\frac{i\,\pi}{2}\right)-60\tanh^{-1}\left(\frac{1}{3}\right)+2\tanh^{-1}\left(\frac{1}{9}\right)\right)+\frac{6561}{80}\left(x-\frac{i}{9}\right)+\frac{3\,352\,671\,i\left(x-\frac{i}{9}\right)^2}{6400}-\frac{91\,899\,927\left(x-\frac{i}{9}\right)^3}{128\,000}-\frac{114\,552\,638\,991\,i\left(x-\frac{i}{9}\right)^4}{20\,480\,000}+\frac{9\,459\,894\,794\,301\left(x-\frac{i}{9}\right)^5}{1\,024\,000\,000}+O\left(\left(x-\frac{i}{9}\right)^6\right)\right)+81\,\pi\left\lfloor\frac{\pi-2\arg(9\,x-i)}{4\,\pi}\right\rfloor\right)\right)$$

$$\frac{1}{640}\left(\left(\left(\left(-15\,i\log(3\,x+i)-i\left(\tanh^{-1}\left(\frac{1}{3}\right)+81\tanh^{-1}(3)-15\log(2)\right)+\frac{147\,\pi}{2}\right)-\frac{225}{2}\left(x+\frac{i}{3}\right)+\frac{10\,395}{32}\,i\left(x+\frac{i}{3}\right)^2+\frac{34\,965}{32}\left(x+\frac{i}{3}\right)^3-\frac{4\,005\,855\,i\left(x+\frac{i}{3}\right)^4}{1024}-\frac{7\,424\,865}{512}\left(x+\frac{i}{3}\right)^5+O\left(\left(x+\frac{i}{3}\right)^6\right)\right)+30\,\pi\left\lfloor\frac{3}{4}-\frac{\arg(3\,x+i)}{2\,\pi}\right\rfloor-81\,\pi\left\lfloor\frac{3}{4}-\frac{\arg(9\,x+3\,i)}{2\,\pi}\right\rfloor-81\,\pi\left\lfloor\frac{\arg(9\,x+3\,i)}{2\,\pi}+\frac{3}{4}\right\rfloor\right)\right)$$

$$\frac{1}{640}\left(\left(\left(\left(15\,i\log(3\,x-i)+i\left(\tanh^{-1}\left(\frac{1}{3}\right)+81\tanh^{-1}(3)-15\log(2)\right)-\frac{15\,\pi}{2}\right)-\frac{225}{2}\left(x-\frac{i}{3}\right)-\frac{10\,395}{32}\,i\left(x-\frac{i}{3}\right)^2+\frac{34\,965}{32}\left(x-\frac{i}{3}\right)^3+\frac{4\,005\,855\,i\left(x-\frac{i}{3}\right)^4}{1024}-\frac{7\,424\,865}{512}\left(x-\frac{i}{3}\right)^5+O\left(\left(x-\frac{i}{3}\right)^6\right)\right)-30\,\pi\left\lfloor\frac{\pi-2\arg(3\,x-i)}{4\,\pi}\right\rfloor+81\,\pi\left\lfloor\frac{\pi-2\arg(9\,x-3\,i)}{4\,\pi}\right\rfloor+81\,\pi\left\lfloor\frac{2\arg(9\,x-3\,i)+\pi}{4\,\pi}\right\rfloor\right)\right)$$

$$\frac{1}{640}\left(\left(\frac{1}{4}\left(2\,i\log(x+i)+2\,i\left(60\tanh^{-1}(3)-162\tanh^{-1}(9)-\log(2)\right)+205\,\pi\right)+\frac{191\,(x+i)}{80}-\frac{22\,351\,i\,(x+i)^2}{6400}-\frac{2\,076\,011\,(x+i)^3}{384\,000}+\frac{168\,558\,271\,i\,(x+i)^4}{20\,480\,000}+\frac{12\,553\,182\,131\,(x+i)^5}{1\,024\,000\,000}+O\left((x+i)^6\right)\right)-52\,\pi\left\lfloor\frac{3}{4}-\frac{\arg(x+i)}{2\,\pi}\right\rfloor-51\,\pi\left\lfloor\frac{\arg(x+i)}{2\,\pi}+\frac{3}{4}\right\rfloor\right)$$

$$\frac{1}{640}\left(\left(\frac{1}{4}\left(-2\,i\log(x-i)+2\,i\log(2)+324\,i\tanh^{-1}(9)-120\,i\tanh^{-1}(3)+\pi\right)+\frac{191\,(x-i)}{80}+\frac{22\,351\,i\,(x-i)^2}{6400}-\frac{2\,076\,011\,(x-i)^3}{384\,000}-\frac{168\,558\,271\,i\,(x-i)^4}{20\,480\,000}+\frac{12\,553\,182\,131\,(x-i)^5}{1\,024\,000\,000}+O\left((x-i)^6\right)\right)+52\,\pi\left\lfloor\frac{\pi-2\arg(x-i)}{4\,\pi}\right\rfloor+51\,\pi\left\lfloor\frac{2\arg(x-i)+\pi}{4\,\pi}\right\rfloor\right)$$

$$\frac{13\,\pi}{320}-\frac{1}{3645\,x^5}+O\left(\left(\frac{1}{x}\right)^6\right)$$

(Laurent series)

$$\int_0^\infty \frac{1}{(1+x^2)(1+9\,x^2)(1+81\,x^2)}\,dx=\frac{13\,\pi}{320}\approx 0.127627$$

The value of this integral, for r = 3 is equal to 0.127627. The function to the right of the equation is:

$$\frac{\pi}{2(1+r+r^3+r^6+r^{10}+\cdots)}=\frac{\pi}{2(1+3+27+729+59049\ldots)}=$$

$$=\frac{\pi}{2+6+54+1458+118098\ldots}=\frac{\pi}{119618}=2{,}6263544396\times 10^{-5}\ldots=$$

= 0,00002626354…

Let's see with a definition interval how we can equalize the integral.

Let

integrated 1 / [(1 (x^2)) (1 (9x^2)) (1 (81x^2))] x, [0, 0.0072568]

We will have the integral that goes from 0 to 0.0072568:

$$\int_0^{0.0072568} \frac{x}{(1+x^2)(1+9\,x^2)(1+81\,x^2)}\,dx=0.0000262677$$

$$\int \frac{x}{(1+x^2)(1+9\,x^2)(1+81\,x^2)}\,dx=\frac{\log(x^2+1)}{1280}-\frac{1}{128}\log(9\,x^2+1)+\frac{9\log(81\,x^2+1)}{1280}+\text{constant}$$

And the value 0.0000262677 is very close to 0.0000262635. This written value 2.6263544396 ×10^(-5) is very close to the square of the golden ratio which is equal to 2.61803398. The value is also equal to 2.62205755 which is the Lemniscate constant ϖ:

$$\pi G = 4\sqrt{\tfrac{2}{\pi}}\,\Gamma\left(\tfrac{5}{4}\right)^2 = \tfrac{1}{4}\sqrt{\tfrac{2}{\pi}}\,\Gamma\left(\tfrac{1}{4}\right)^2 = 4\sqrt{\tfrac{2}{\pi}}\left(\tfrac{1}{4}!\right)^2$$

We resume the integral:

$$\frac{1}{2}\pi^{1/2}\frac{\Gamma\left(a+\frac{1}{2}\right)\Gamma(b+1)\Gamma\left(b-a+\frac{1}{2}\right)}{\Gamma(a)\Gamma\left(b+\frac{1}{2}\right)\Gamma(b-a+1)} = \int_0^\infty \frac{1+\left(\frac{x}{b+1}\right)^2}{1+\left(\frac{x}{a}\right)^2}\cdot\frac{1+\left(\frac{x}{b+2}\right)^2}{1+\left(\frac{x}{a+1}\right)^2}\ldots dx$$

from which we obtained and verified:

$$\left(\frac{1}{2}\sqrt{\pi}\right)^2 = \left[\int_0^\infty \frac{1+\left(\frac{x}{b+1}\right)^2}{1+\left(\frac{x}{a}\right)^2}\cdot\frac{1+\left(\frac{x}{b+2}\right)^2}{1+\left(\frac{x}{a+1}\right)^2}\ldots dx\right]^2 \times \left[\frac{1}{\frac{\Gamma\left(a+\frac{1}{2}\right)\Gamma(b+1)\Gamma\left(b-a+\frac{1}{2}\right)}{\Gamma(a)\Gamma\left(b+\frac{1}{2}\right)\Gamma(b-a+1)}}\right]^2$$

$$= 0{,}785398$$

Now let's analyze the wave function of the universe of the paper

Wave function of the Universe

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$$\psi_0(a_0) = 2\cos\left[\frac{(H^2a_0^2-1)^{3/2}}{3H^2} - \frac{\pi}{4}\right], \quad Ha_0 > 1 \qquad (6.18)$$

By introducing "imaginary time", a mathematical abstraction that allows to relate two events that are casually unconnected or not temporally connected - that is, a time without the presence of space and spacetime - singularity can be studied. Hawking proposes an infinite universe or more precisely without outline or boundaries, expanding and with an beginning in imaginary time

In boundless state theory, the wave function of the universe - that is, an image to explain how the universe was born - is calculated through the paths integral, a tensor metric function defined at (D-1) of compact surface, where D is the space-time dimension. [1] This wave function of the universe can satisfy the Wheeler-DeWitt equation

The state without borders can be positioned even before a multiverse or a world-brane or wherever it is necessary to speak of infinity or singularity; in the ecpirotic model of branes the boundless state is hyperspace where d-branes fluctuate. Ahmed Farag Ali and Saurya Das instead replaced Hawking's boundless dome model, with an eternal fluid of gravitons, in their cosmology of quantum potential, while for Roger Penrose the universe consists of infinite Big Bang-expansion cycles (conformal cyclic cosmology) and does not require an initial state.

In our model, the informal state has several phases: in the first phase there is the uncreated vacuum of motionless light and time, which contains the perturbative vacum of string in idea-form, i.e. the infinite-dimensional Hilbert space that contains infinite potential bulk in 26 dimensions with infinite pairs of strings + and - (we note that the perturbative vacuum of string, therefore the quantum foam, is not chaotic but only an infinite set that contains infinite possibilities arranged in an orderly albeit in continuous vibration), in the second phase these infinite potential bulk, reduce their size from 26 to 10 (actually 24 and 8) actually compacting 16 into a 16d torus (wave function collapse) and in the third phase, intermediate between the informal and the formal state, between the infinite pairs of strings, only one pair is formalized for the collapse of the wave function in two D3-branes.

According to the "no-boundary proposal", the three phases that we describe take place, in imaginary time and in the ecpirotic model of branes the borderless state is identified with the hyperspace where the d-branes fluctuate, which is our Hilbert space / infinite dimensional torus. (A. Nardelli)

The equation that we are going to develop is:

with

$Ha_0 > 1$

and

be compact and that the fields should be regular on these geometries. In the case of a positive cosmological constant Λ any regular Euclidean solution of the field equations is necessarily compact.[8] In particular, the solution of greatest symmetry is the four-sphere of radius $3/\Lambda$, whose metric we write as

$$ds^2 = (\sigma/H)^2(d\theta^2 + \sin^2\theta\, d\Omega_3^2) , \qquad (3.2)$$

where $d\Omega_3^2$ is the metric on the three-sphere. $H^2 = \sigma^2\Lambda/3$ and we have introduced the normalization factor $\sigma^2 = l^2/24\pi^2$ for later convenience. Thus, it is clear that

them when $\Lambda < 0$. We shall therefore consider only the case $\Lambda > 0$ in this paper and shall regard $\Lambda = 0$ as a limiting case of $\Lambda > 0$.

We have:

$Ha_0 > 1$

$H^2 = \sigma^2\Lambda/3$

$\sigma^2 = l^2/24\pi^2$

for:

$Ha_0 > 1 \quad \Gamma\left(\frac{5}{2}\right) = \frac{3}{4}\sqrt{\pi} \approx 1.329\,340\,388\,179\,137\,0205 ,$

from which: $H^2 a_0^2 = 9\pi/16$

we have $\Lambda > 0$, thence we put: $\Lambda = 1$

from which $H^2 = l^2/72\pi^2$

after the calculations, we place $l^2 = \frac{1}{2} = 0{,}5$ that is equal to:

$$\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = \frac{\sqrt{\pi}}{2} \cdot \frac{1}{\sqrt{\pi}} = \frac{1}{2}$$

We remember also that:

arccos(0.707106700000000) = 0.785398278212558, dove 0,7071067 è 1/√2 e 0,78539827… è $\pi/4$.

thence:

$$\psi_0(a_0) = 2\cos\left[\frac{(H^2 {a_0}^2 - 1)^{3/2}}{3H^2} - \frac{\pi}{4}\right]$$

$$\psi_0 a_0 = 2\cos\left[\frac{\left(\frac{9\pi}{16} - 1\right)^{3/2}}{\frac{3l^2}{72\pi^2}} - \frac{\pi}{4}\right] = 2\cos\left(\frac{0{,}671919}{\frac{l^2}{24\pi^2}} - \frac{\pi}{4}\right) =$$

$$= 2\cos\left(\frac{0{,}671919 \cdot 24\pi^2}{l^2} - \frac{\pi}{4}\right) = 2\cos 159{,}1577932698\frac{1}{l^2} - 2\cos\frac{\pi}{4} =$$

$$\psi_0 a_0 = -0{,}971750458594\frac{1}{l^2} - 2 \cdot 0{,}7071067811$$

$$\psi_0 a_0 = -0{,}971750458594\frac{1}{l^2} - 1{,}4142135622$$

$$-0{,}971750458594\frac{1}{l^2} - \sqrt{2} = \psi_0 a_0$$

For $l^2 = 0{,}5$ we have:

$$-\sqrt{2} = \psi_0 a_0 + \frac{0{,}971750458594}{0{,}5}$$

$$1{,}943500917 + \psi_0 a_0 = -\sqrt{2}$$

$$\psi_0 a_0 = -\sqrt{2} - 1{,}943500917 = -3{,}357714479$$

thence

$$\psi_0 a_0 = -3{,}357714479.$$

Now, we note that:

$$\frac{\psi_0 a_0}{2} = 1{,}6788572396 \quad \text{and} \quad \frac{\psi_0 a_0}{5} = 0{,}6715428958$$

We observe that 0.6715428958 + 1 = 1.6715428958 is a value practically equalt to the proton mass.

Furthermore:

$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6} = 1.6449\ldots$$

and that:

"the **elementary charge**, usually denoted by ***e*** or sometimes q_e, is the electric charge carried by a single proton, or equivalently, the magnitude of the electric charge carried by a single electron. This elementary charge is a fundamental physical constant. The charge *e* is sometimes called the **elementary positive charge**. This charge has a measured value exactly of $1.602176634\times10^{-19}$ C (coulombs)".

Now:

$$\frac{\pi^2}{6} \cdot \boldsymbol{e} = 1{,}6449 \ldots \times -1{,}602176634 = -2{,}63547492637482242 \ldots$$

taking

$$\frac{1}{\psi_0 a_0} \cdot \frac{\pi^2 \boldsymbol{e}}{6} = -0{,}297821630235165686 3 \cdot -2{,}63547492637482242 =$$

$$\frac{1}{\psi_0 a_0} \cdot \frac{\pi^2 \boldsymbol{e}}{6} = 0{,}7849014390168 \ldots$$

Value practically equal to $\frac{\pi}{4} = 0{,}785398163397 \ldots$ Indeed: $0{,}78490 \cong 0{,}78539$.

In conclusion, we have the following fundamental connections:

From

$$\psi_0 a_0 = 2\cos\left[\frac{\left(\frac{9\pi}{16} - 1\right)^{3/2}}{\frac{3l^2}{72\pi^2}} - \frac{\pi}{4}\right]$$

For the inverse function muliplyed by $\frac{\pi^2 \boldsymbol{e}}{6}$ we obtain:

$$\frac{1}{\psi_0 a_0} \cdot \frac{\pi^2 \boldsymbol{e}}{6} = \frac{1}{2\cos\left[\frac{\left(\frac{9\pi}{16}-1\right)^{3/2}}{\frac{3l^2}{72\pi^2}} - \frac{\pi}{4}\right]} \cdot \frac{\pi^2 \boldsymbol{e}}{6} = \frac{\pi}{4} = 0{,}785398163397.$$

From the Ramanujan's integral equation (inverse formula),

$$\left(\frac{1}{2}\sqrt{\pi}\right)^2 = \left[\int_0^\infty \frac{1+\left(\frac{x}{b+1}\right)^2}{1+\left(\frac{x}{a}\right)^2} \cdot \frac{1+\left(\frac{x}{b+2}\right)^2}{1+\left(\frac{x}{a+1}\right)^2} \dots dx\right]^2 \times \left[\frac{1}{\frac{\Gamma\left(a+\frac{1}{2}\right)\Gamma(b+1)\Gamma\left(b-a+\frac{1}{2}\right)}{\Gamma(a)\Gamma\left(b+\frac{1}{2}\right)\Gamma(b-a+1)}}\right]^2$$

$$= 0{,}785398$$

We have:

$$\frac{\pi}{4} = \left[\int_0^\infty \frac{1+\left(\frac{x}{b+1}\right)^2}{1+\left(\frac{x}{a}\right)^2} \cdot \frac{1+\left(\frac{x}{b+2}\right)^2}{1+\left(\frac{x}{a+1}\right)^2} \dots dx\right]^2 \times \left[\frac{1}{\frac{\Gamma\left(a+\frac{1}{2}\right)\Gamma(b+1)\Gamma\left(b-a+\frac{1}{2}\right)}{\Gamma(a)\Gamma\left(b+\frac{1}{2}\right)\Gamma(b-a+1)}}\right]^2$$

$$= \frac{1}{\psi_0 a_0} \cdot \frac{\pi^2 \boldsymbol{e}}{6} = \frac{1}{2\cos\left[\frac{\left(\frac{9\pi}{16}-1\right)^{3/2}}{\frac{l^2}{24\pi^2}} - \frac{\pi}{4}\right]} \cdot \frac{\pi^2 \boldsymbol{e}}{6} = 0{,}785398$$

From:

$\frac{\psi_0 a_0}{2} = 1{,}6788572396$ and $\frac{\psi_0 a_0}{5} = 0{,}6715428958$

we have:

$$\frac{\psi_0 a_0}{2} = \frac{1}{2}\left\{2\cos\left[\frac{\left(\frac{9\pi}{16} - 1\right)^{3/2}}{\frac{3l^2}{72\pi^2}} - \frac{\pi}{4}\right]\right\} = 1{,}6788572396\ldots$$

$$\frac{\psi_0 a_0}{5} = \frac{1}{5}\left\{2\cos\left[\frac{\left(\frac{9\pi}{16} - 1\right)^{3/2}}{\frac{3l^2}{72\pi^2}} - \frac{\pi}{4}\right]\right\} = 0{,}6715428958\ldots$$

From:

Brane New World
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....we calculate the graviton correlator using the Hartle-Hawking "No Boundary" proposal and analytically continue to Lorentzian signature.
We have the Euclidean correlator defined as an infinite sum. However, the eigenspace of the Laplacian on de Sitter space suggests that the Lorentzian propagator is most naturally expressed as an integral over real p'....

The extra factor $ie^{p\pi}/\sinh p'\pi$ combines with the factor $-i\tanh p'\pi$ in the integrand to $e^{p'\pi}/\cosh p'\pi$. Furthermore, since $G(-p', y_0) = \bar{G}(p', y_0)$, we can rewrite the correlator as an integral from 0 to ∞. We finally obtain the Lorentzian tensor Feynman (time-ordered) correlator,

$$\begin{aligned}\langle h_{ij}(x)h_{i'j'}(x')\rangle &= \frac{128\pi^2R^4}{N^2}\left[\int_0^{+\infty} dp' \tanh p'\pi W^{L(p')}_{iji'j'}(\mu)\Re(G(p', y_0)^{-1})\right.\\ &\qquad \left.+\pi\sum_{k=1}^{2}\tan\Lambda_k\pi W^{L(i\Lambda_k)}_{iji'j'}(\mu)\mathbf{Res}(G(p', y_0)^{-1}; i\Lambda_k)\right]\\ &\quad +i\frac{128\pi^2R^4}{N^2}\left[\int_0^{+\infty} dp' W^{L(p')}_{iji'j'}(\mu)\Re(G(p', y_0)^{-1})\right.\\ &\qquad \left.-\pi\sum_{k=1}^{2} W^{L(i\Lambda_k)}_{iji'j'}(\mu)\mathbf{Res}(G(p', y_0)^{-1}; i\Lambda_k)\right]. \end{aligned} \tag{3.66}$$

In this integral the bitensor $W^{L(p')}_{iji'j'}(\mu(x,x'))$ may be written as the sum of the degenerate rank-two tensor harmonics on closed de Sitter space with eigenvalue $\lambda_{p'} = (p'^2 + 17/4)$ of the Laplacian. Note that the normalization factor $\tilde{Q}_{p'} = p'(4p'^2 + 25)/48\pi^2$ of the bitensor is imaginary at $p' = i\Lambda_k$ and the residues of G^{-1} are also imaginary, so the quantities in square brackets are all real. Both integrands in equation 3.66 vanish as $p' \to 0$, so the correlator is well-behaved in the infrared.

For cosmological applications, one is usually interested in the expectation of some quantity squared, like the microwave background multipole moments. For this purpose, all that matters is the symmetrized correlator, which is just the real part of the Feynman correlator.

Gravitational waves provide an extra source of time-dependence in the background in which the cosmic microwave background photons propagate. In particular, the contribution of gravitational waves to the CMB anisotropy is given by the integral in the Sachs-Wolfe formula, which is basically the integral along the photon trajectory of the time derivative of the tensor perturbation. Hence the resulting microwave multipole moments C_l can be directly determined from the graviton correlator.

We can therefore understand the effect of the strongly coupled CFT on the microwave fluctuation spectrum by comparing our result 3.66 with the transverse traceless part of the graviton propagator in four-dimensional de Sitter spacetime [41]. On the four-sphere, this is easily obtained by varying the Einstein-Hilbert action with a cosmological constant. In terms of the bitensor, this yields

$$\langle h_{ij}(\Omega)h_{i'j'}(\Omega')\rangle = 32\pi G_4 R^2 \sum_{p'=7i/2}^{i\infty} \frac{W^{(p')}_{iji'j'}(\mu(\Omega,\Omega'))}{\lambda_{p'} - 2}, \tag{3.67}$$

which continues to

$$\langle h_{ij}(x)h_{i'j'}(x')\rangle = 32\pi G_4 R^2 \int_0^{+\infty} \frac{dp'}{\lambda_{p'} - 2} W^{L(p')}_{iji'j'}(\mu(x,x')). \tag{3.68}$$

This can be compared with equation 3.66. Note that (apart from the pole at $p' = 3i/2$ corresponding to the gauge mode mentioned before) there are no supercurvature modes. We defer a detailed discussion of the effect of the CFT on the tensor perturbation spectrum in de Sitter space to the next section.

and:

The set of tensor eigenmodes on S^4 (or on de Sitter space) forms a representation of the symmetry group of the manifold. It follows in particular that their sum over the parity states $\mathcal{P} = \{e, o\}$ and the quantum numbers k, l and m on the three sphere defines a maximally symmetric bitensor on S^4 (or dS space) [43]:

$$W_{(p')}{}^{ij}{}_{i'j'}(\mu) = \sum_{\mathcal{P}klm} q^{(p')ij}_{\mathcal{P}klm}(\Omega) q^{(p')\mathcal{P}klm}_{i'j'}(\Omega')^*. \tag{B.2}$$

On S^4 the label p' takes the value $7i/2, 9i/2, ...$ It is related to a real label p by $p' = i(p+3/2)$. The ranges of the other labels are then $0 \leq k \leq p$, $0 \leq l \leq k$ and $-l \leq m \leq l$. On de Sitter space there is a continuum of eigenvalues $p' \in [0, \infty)$. We will assume from now on that the eigenmodes are normalized by the condition

$$\int \sqrt{\gamma} d^4\Omega q^{(p')ij}_{\mathcal{P}klm} q^{(p'')*}_{\mathcal{P}'k'l'm'ij} = \delta^{p'p''} \delta_{\mathcal{P}\mathcal{P}'} \delta_{ll'} \delta_{mm'} \tag{B.3}$$

We will determine below which solution corresponds to the bitensor defined by B.2.

Our discussion so far applies to either S^4 or de Sitter space. We now specialize to the case of S^4 and will later obtain results for de Sitter space by analytic continuation. The hypergeometric functions on S^4 may be expressed in terms of Legendre polynomials in $\cos\mu$ (eq. [15.4.19] in [45]),

$$\begin{cases} \alpha(\mu) & = Q_{p'} \Gamma(4) 2^3 (\sin\mu)^{-3} P^{-3}_{-1/2+ip'}(-\cos\mu), \\ \beta(\mu) & = Q_{p'} \Gamma(5) 2^4 (\sin\mu)^{-4} P^{-4}_{-1/2+ip'}(-\cos\mu). \end{cases} \tag{B.11}$$

The solutions for $\alpha(z)$ and $\beta(z)$ are singular at $z = 1$ (i.e. for coincident points on S^4) for generic values of p'. However, for the values of p' corresponding to the eigenvalues of the Laplacian on S^4, they are regular everywhere on S^4. Similarly, $\alpha(1-z)$ and $\beta(1-z)$ are generically singular for antipodal points on S^4 and regular for these special values of p'. For these special values, $\alpha(z)$ and $\alpha(1-z)$ are no longer linearly independent but related by a factor of $(-1)^{(n+1)/2}$ where $n = -2ip' = 7, 9, 11, \ldots$. This follows from the relation (eq.[8.2.3] in [45])

$$P_\nu^\mu(-z) = e^{i\nu\pi} P_\nu^\mu(z) - \frac{2}{\pi} e^{-i\mu\pi} \sin[\pi(\nu+\mu)] Q_\nu^\mu(z), \tag{B.12}$$

where the second term vanishes for $p' = 7i/2, 9i/2, \ldots$. In fact, the hypergeometric series terminates for these values of p' and the hypergeometric functions reduce to Gegenbauer polynomials $C_{n-7/2}^{(7/2)}(1-2z)$. We have a choice between using $\alpha(z)$ and $\alpha(1-z)$ in the bitensor for these values of p'. However, to obtain the Lorentzian correlator, we had to express the discrete sum 3.62 as a contour integral. Since the Euclidean correlator obeys a differential equation with a delta function source at $\mu = 0$, we must maintain regularity of the integrand at $\mu = \pi$ when extending the bitensor in the complex p'-plane. In other words, for generic p', we need to work with the solution $\alpha(z)$, rather then $\alpha(1-z)$. We shall therefore choose $\alpha(z)$, since this is the solution that we will analytically continue.

The above conditions leave the overall normalisation of the bitensor undetermined. To fix the normalisation constant $Q_{p'}$, consider the biscalar quantity

$$g^{ii'} g^{jj'} W_{iji'j'}^{(p')}(\mu) = 12 w_1^{(p')} - 6 w_2^{(p')} + 24 w_3^{(p')} \tag{B.13}$$

In the coincident limit $\Omega \to \Omega'$ and $z \to 1$ this yields

$$W_{ij}^{(p')\,ij}(\Omega,\Omega) = \sum_{Pklm} q_{ij}^{(p')Pklm}(\Omega) q^{(p')Plm\,ij}(\Omega)^* = -72\alpha(1). \tag{B.14}$$

Since $F(0) = 1$ we have $\alpha(1) = Q_{p'}(-1)^{(1+n)/2}$. By integrating over the four-sphere and using the normalization condition B.3 on the tensor harmonics one obtains, for $n = -2ip' = 7, 9, 11, \ldots$

$$Q_{p'} = \frac{ip'(4p'^2+25)}{48\pi^2(-1)^{(1+n)/2}} = \frac{p'(4p'^2+25)}{48\pi^2 \sinh p'\pi}. \tag{B.15}$$

We analyze the following equation:

$$\langle h_{ij}(x) h_{i'j'}(x') \rangle = 32\pi G_4 R^2 \int_0^{+\infty} \frac{dp'}{\lambda_{p'} - 2} W_{iji'j'}^{L(p')}(\mu(x,x')).$$

$$\lambda_{p'} = (p'^2 + 17/4)$$

where $p' = 7i/2$.

Regard the connection with $\frac{\pi}{4}$, from the integral:

$$\langle h_{ij}(x) h_{i'j'}(x') \rangle = 32\pi G_4 R^2 \int_0^{+\infty} \frac{dp'}{\lambda_{p'} - 2} W_{iji'j'}^{L(p')}(\mu(x,x')).$$

Dividing by 128 both the sides, with regard 32π we obtain: $\frac{32\pi}{128} = \frac{\pi}{4}$. thence:

$$\frac{\langle h_{ij}(x) h_{i'j'}(x') \rangle}{128} = \frac{\pi}{4} G_4 R^2 \int_0^{+\infty} \frac{dp'}{\lambda_{p'} - 2} W_{iji'j'}^{L(p')}\big(\mu(x,x')\big)$$

$$\frac{\pi}{4} G_4 R^2 \int_0^{+\infty} \frac{dp'}{\lambda_{p'} - 2} W_{iji'j'}^{L(p')}\big(\mu(x,x')\big) = \frac{\langle h_{ij}(x) h_{i'j'}(x') \rangle}{128}$$

$$\frac{\pi}{4} = \frac{\langle h_{ij}(x) h_{i'j'}(x') \rangle}{128} \cdot \frac{1}{G_4 R^2 \int_0^{+\infty} \frac{dp'}{\lambda_{p'} - 2} W_{iji'j}^{L(p')}\big(\mu(x,x')\big)}$$

Now, we have :

$$\langle h_{ij}(x) h_{i'j'}(x') \rangle = 32\pi G_4 R^2 \int_0^{+\infty} \frac{dp'}{\lambda_{p'} - 2} W_{iji'j'}^{L(p')}\big(\mu(x,x')\big)$$

where

$\lambda_{p'} = (p'^2 + 17/4)$

$p' = 7i/2.$

thence: $\lambda_{p'} - 2 = \left(\frac{7i}{2}\right)^2 + \frac{17}{4} - 2 = -\frac{49}{4} + \frac{17}{4} - 2 = -8 - 2 = -10;$

$$\int \frac{7i/2}{-10} W_{iji'j'}^{L(p')}\big(\mu(x,x')\big) = \int -\frac{3{,}5i}{10} W_{iji'j'}^{L(p')}\big(\mu(x,x')\big) = -\frac{7i}{20} W_{iji'j'}^{L(p')}\big(\mu(x,x')\big) =$$

$$= -0.35i W_{iji'j'}^{L(p')}\big(\mu(x,x')\big)$$

Now

$W_{iji'j'}^{L(p')}\big(\mu(x,x')\big)$; $W = -72\alpha(1);$

$\alpha(1) = Q_{p'}(-1)^{(1+n)/2}.$

$$Q_{p'} = \frac{ip'(4p'^2 + 25)}{48\pi^2(-1)^{(1+n)/2}} = \frac{p'(4p'^2 + 25)}{48\pi^2 \sinh p'\pi}.$$

$Q_{p'} = \frac{3.5i\left(4\cdot\left(-\frac{49}{4}\right)+25\right)}{48\ {}^2 \sin h\, 3{,}5i\pi} = \frac{(-24)}{48\ {}^2 \cdot \sinh \pi} = -\frac{24}{48\pi^2 \cdot 11{,}548739} = \frac{24}{5471{,}11} = 0{,}004386678 \ldots$

Thence: $-0.35i \times 0{,}004386678 = -0.0015353373i$

from which:

$$-0.0015353373i \times (-72) = -0{,}1105442856i$$

Now:

$$-0{,}1105442856i \times 32\pi G = -0{,}1105442856i \times (6{,}70541535982 \cdot 10^{-9})$$
$$= 0{,}00000000074124535$$

that multiplied by $\frac{\pi}{36}$ is equal to:

$$0{,}00000000074124535 \cdot \frac{\pi}{36} = 0{,}000000000064686 = 0{,}64686 \times 10^{-1}$$

value near to the golden ratio conjugate 0,61803398…

Now we calculate the integral for a certain interval:

$$32\pi G_4 R^2 \int_0^{+\infty} \frac{dp'}{\lambda_{p'} - 2} W_{iji'j}^{L(p')}\left(\mu(x, x')\right)$$

$$32\pi \cdot 6{,}67 \times 10^{-11} \int -0.0015353373i \times (-72)\, dx$$

integrate [(-0.0015353373)*(-72)] [0.021829] [0, 0.000000006705415]

$$\int (-0.0015353373\,(-72))\, 0.021829\, \{0,\, 6.705415 \times 10^{-9}\}\, dx =$$
$$\{0,\, 1.61806 \times 10^{-11}\, x\} + \text{constant}$$

integrate [(-0.0015353373)*(-72)] [0.01465686] [0, 0.000000006705415]

$$\int (-0.0015353373\,(-72))\, 0.01465686\, \{0,\, 6.705415 \times 10^{-9}\}\, dx =$$
$$\{0,\, 1.08643 \times 10^{-11}\, x\} + \text{constant}$$

Thence:

$$\langle h_{ij}(x) h_{i'j'}(x') \rangle = 32\pi G_4 R^2 \int_0^{+\infty} \frac{dp'}{\lambda_{p'} - 2} W_{iji'j}^{L(p')}\left(\mu(x, x')\right) = 1{,}61806 \times 10^{-11} x$$

for [0, 0.021829]

We note that 1.61806 (excluding zeros before the value) is practically equal to the golden ratio 1.61803398 ... and that the other value 1.08643 is the result of a Ramanujan integral that we have already analyzed and which allows us to obtain another very interesting mathematical connection.

We know that:

$$\int_0^\infty \frac{1+\left(\frac{x}{b+1}\right)^2}{1+\left(\frac{x}{a}\right)^2} \cdot \frac{1+\left(\frac{x}{b+2}\right)^2}{1+\left(\frac{x}{a+1}\right)^2} \dots dx = \frac{1}{2}\sqrt{\pi}\frac{\Gamma\left(a+\frac{1}{2}\right)\Gamma(b+1)\Gamma\left(b-a+\frac{1}{2}\right)}{\Gamma(a)\Gamma\left(b+\frac{1}{2}\right)\Gamma(b-a+1)} =$$

$$= 1{,}047197551196598 \dots$$

$$\frac{1}{2}\pi^{1/2}\frac{\Gamma\left(a+\frac{1}{2}\right)\Gamma(b+1)\Gamma\left(b-a+\frac{1}{2}\right)}{\Gamma(a)\Gamma\left(b+\frac{1}{2}\right)\Gamma(b-a+1)} = \int_0^\infty \frac{1+\left(\frac{x}{b+1}\right)^2}{1+\left(\frac{x}{a}\right)^2} \cdot \frac{1+\left(\frac{x}{b+2}\right)^2}{1+\left(\frac{x}{a+1}\right)^2} \dots dx$$

from which:

$$\left(\frac{1}{2}\pi^{\frac{1}{2}}\right)^2 = \left[\int_0^\infty \frac{1+\left(\frac{x}{b+1}\right)^2}{1+\left(\frac{x}{a}\right)^2} \cdot \frac{1+\left(\frac{x}{b+2}\right)^2}{1+\left(\frac{x}{a+1}\right)^2} \dots dx\right]^2 \times$$

$$\times \left[\frac{1}{\frac{\Gamma\left(a+\frac{1}{2}\right)\Gamma(b+1)\Gamma\left(b-a+\frac{1}{2}\right)}{\Gamma(a)\Gamma\left(b+\frac{1}{2}\right)\Gamma(b-a+1)}}\right]^2 = 0{,}7853981 = \frac{\pi}{4}$$

Indeed for

$$\int_0^{2.898159} \frac{\left(1+\frac{x^2}{9}\right)\left(1+\frac{x^2}{16}\right)x}{\left(1+x^2\right)\left(1+\frac{x^2}{4}\right)}\, dx = 1.04719$$

we obtain:

$$(0.8862269)^2 = (1{,}04719)^2 \cdot \left(\frac{1}{1{,}1816359}\right)^2 =$$

$$0{,}7853981182 = 1{,}0966068961 \cdot 0{,}8462843757539^2$$

$$0{,}7853981182 = 1{,}0966068961 \cdot 0{,}7161972446453$$

$$0{,}785398 \cong 0{,}7853868$$

And the following connection:

$$\langle h_{ij}(x)h_{i'j'}(x')\rangle = \left[32\pi G_4 R^2 \int_0^{+\infty} \frac{dp'}{\lambda_{p'} - 2} W_{iji'j'}^{L(p')}\left(\mu(x, x')\right)\right] \cdot \frac{\pi}{36}$$
$$= 0{,}64686 \times 10^{-10}$$

$$\frac{1}{9}\left(\frac{1}{2}\sqrt{\pi}\right)^2 = \frac{1}{9}\left[\int_0^{\infty} \frac{1+\left(\frac{x}{b+1}\right)^2}{1+\left(\frac{x}{a}\right)^2} \cdot \frac{1+\left(\frac{x}{b+2}\right)^2}{1+\left(\frac{x}{a+1}\right)^2} \dots dx\right]^2 \times \left[\frac{1}{\frac{\Gamma\left(a+\frac{1}{2}\right)\Gamma(b+1)\Gamma\left(b-a+\frac{1}{2}\right)}{\Gamma(a)\Gamma\left(b+\frac{1}{2}\right)\Gamma(b-a+1)}}\right]^2$$

$$= \frac{0{,}785398}{9} = 0{,}0872664$$

$$\frac{\pi}{36} = 0{,}64686 \times 10^{-10} \cdot \frac{1}{\left[32\pi G_4 R^2 \int_0^{+\infty} \frac{dp'}{\lambda_{p'} - 2} W_{iji'j'}^{L(p')}\left(\mu(x, x')\right)\right]}$$

Indeed:

$$\frac{0{,}000000000064686}{0{,}00000000074124535} = 0{,}087266\dots$$

$$\frac{1}{9}\left(\frac{1}{2}\sqrt{\pi}\right)^2 = \frac{1}{9}\left[\int_0^\infty \frac{1+\left(\frac{x}{b+1}\right)^2}{1+\left(\frac{x}{a}\right)^2}\cdot\frac{1+\left(\frac{x}{b+2}\right)^2}{1+\left(\frac{x}{a+1}\right)^2}\dots dx\right]^2 \times \left[\frac{1}{\frac{\Gamma\left(a+\frac{1}{2}\right)\Gamma(b+1)\Gamma\left(b-a+\frac{1}{2}\right)}{\Gamma(a)\Gamma\left(b+\frac{1}{2}\right)\Gamma(b-a+1)}}\right]^2$$

$$\Rightarrow 0{,}64686\times 10^{-10}\cdot\frac{1}{\left[32\pi G_4 R^2\int_0^{+\infty}\frac{dp'}{\lambda_{p'}-2}W_{iji'j'}^{L(p')}\left(\mu(x,x')\right)\right]} = \frac{\pi}{36}$$

while:

$$\langle h_{ij}(x)h_{i'j'}(x')\rangle = 32\pi G_4 R^2\int_0^{+\infty}\frac{dp'}{\lambda_{p'}-2}W_{iji'j'}^{L(p')}\left(\mu(x,x')\right)$$

integrate [(-0.0015353373)*(-72)] [0.01465686] [0, 0.000000006705415]

$$\int (-0.0015353373\,(-72))\,0.01465686\,\{0,\,6.705415\times 10^{-9}\}\,dx =$$
$$\{0,\,1.08643\times 10^{-11}\,x\} + \text{constant}$$

Indeed

$$0{,}00000000074124535\times 0{,}01465686 = 1{,}08643\times 10^{-11}$$

thence the following fundamental mathematical connection:

$$\frac{\pi^{1/4}}{\Gamma\left(\frac{3}{4}\right)} = 1+\int_0^\infty\frac{e^{-t^2/2}}{\sqrt{2\pi}}\left[\frac{4e^{\pi}\left(e^{2\pi}-\cos\left(\sqrt{2\pi}t\right)\right)}{e^{4\pi}-2e^{2\pi}\cos\left(\sqrt{2\pi}t\right)+1}\right] = 1{,}08643481 \Rightarrow$$

$$\Rightarrow 32\pi G_4 R^2\int_0^{+\infty}\frac{dp'}{\lambda_{p'}-2}W_{iji'j'}^{L(p')}\left(\mu(x,x')\right)\quad \text{x, [0, 0.01465686]}$$

We note, with great interest, as well as the result of the Ramanujan's integral written above, multiplied by the "Module of Infinite Tetration of *i*" that is $|{}^{\infty}i|$ or

$$\lim_{n\to\infty} |{}^n i| = \left| \lim_{n\to\infty} \underbrace{i^{i^{\cdot^{\cdot^{i}}}}}_{n} \right|$$

that is equal to 0.56755516330695782538, give us as result a number that is practically the reciprocal of the aurea ratio. Indeed, we have:

1,086434*0,567555 = 0,616611...

Thence, we obtain:

$$\frac{\pi^{1/4}}{\Gamma\left(\frac{3}{4}\right)} \cdot 0{,}56755 = 1 + \int_0^\infty \frac{e^{-t^2/2}}{\sqrt{2\pi}} \left[\frac{4e^{\pi}\left(e^{2\pi} - \cos\left(\sqrt{2\pi}t\right)\right)}{e^{4\pi} - 2e^{2\pi}\cos\left(\sqrt{2\pi}t\right) + 1} \right] \cdot 0{,}56755 = 0{,}616611537\ldots$$

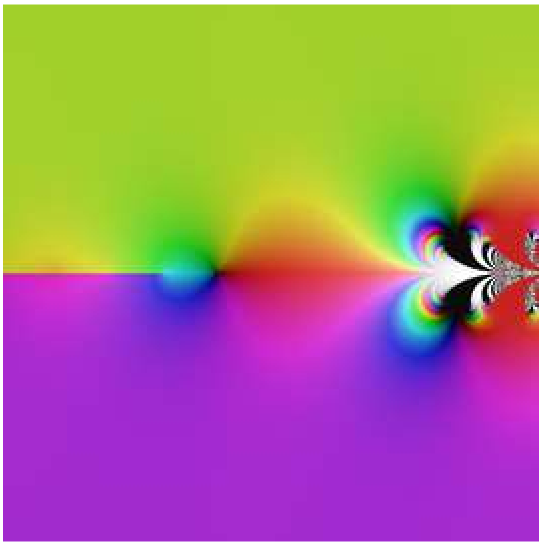
$$\pi^{1/4} = 1{,}3313353638\ldots$$

$$\Gamma\left(\frac{3}{4}\right)$$

$$\frac{\sqrt{2}\,\pi}{\Gamma\left(\frac{1}{4}\right)}$$

4,442882938158 / 3,625609 = 1,225417; 1,331335 / 1,225417 = 1,086434...

1.2254167024651776451290983033628905268512392481080706112З0...



From Wikipedia

Super-logarithm as inverse of tetration

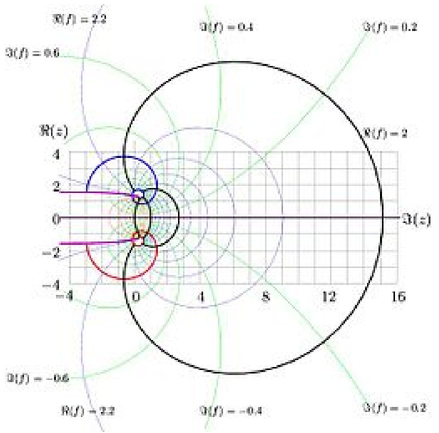
As [tetration](https://en.wikipedia.org/wiki/Tetration) (or super-exponential) $\mathrm{sexp}_b(z)$ is suspected to be an analytic function, at least for some values of b, the inverse function $\mathrm{slog}_b = \mathrm{sexp}_b^{-1}$ may also be analytic. Behavior of $\mathrm{slog}_b(z)$, defined in such a way, the complex z plane is sketched in Figure 1 for the case $b = e$. Levels of integer values of real and integer values of imaginary parts of the slog functions are shown with thick lines. If the existence and uniqueness of the analytic extension of tetration is provided by the condition of its asymptotic approach to the fixed points $L \approx 0.318 + 1.337\mathrm{i}$ and $L^* \approx 0.318 - 1.337\mathrm{i}$ of $L = \ln(L)$ in the upper and lower parts of the complex plane, then the inverse function should also be unique. Such a function is real at the real axis. It has two branch points at $z = L$ and $z = L^*$. It approaches its limiting value -2 in vicinity of the negative part of the real axis (all the strip between the cuts shown with pink lines in the figure), and slowly grows up along the positive direction of the real axis. As the derivative at the real axis is positive, the imaginary part of slog remains positive just above the real axis and negative just below the real axis. The existence, uniqueness and generalizations are under discussion.

We note that

$$(0{,}318^2 + 1{,}337i^2) = 0{,}101124 - 1{,}787569 = -1{,}686445$$

And that

– 1,686445 + 1,086434547 = – 0,600010453



From: **Gravitational waves from spin-3/2 fields Hunting SUSY in the sky** - Karim Benakli, Yifan Chen, Peng Cheng, Gaetan Lafforgue-Marmet:

3 Gravitational wave production

Here, we will compute the spectrum of energy density of gravitational waves produced by a gas of spin-3/2 states. We will be considering wave-lengths in the sub-horizon limit, the effects of curvature and torsion can be neglected.

The gravitational waves can be described as linear tensor perturbations, here in the transverse-traceless (TT) gauge, of the Friedman-Robertson-Walker (FRW) metric:

$$ds^2 = a^2(\tau)[-d\tau^2 + (\delta_{ij} + h_{ij})dx^i dx^j], \tag{3.1}$$

where τ is the conformal time. The linear perturbation part of Einstein equations lead to the gravitational wave equations of motion:

$$\ddot{h}_{ij} + 2\mathcal{H}\dot{h}_{ij} - \nabla h_{ij} = 16\pi G \Pi^{TT}_{ij}, \tag{3.2}$$

where the dot (.) stands for derivative with respect to the conformal time τ, $\mathcal{H} = \frac{\dot{a}}{a}$ is then the comoving Hubble rate, and Π^{TT}_{ij} is the **TT** part of the anisotropic stress tensor. In order to avoid manipulating non-local projection operator in configuration space, we perform a Fourier transform of the stress tensor $T_{\mu\nu}$ in terms of comoving wave-number **k**. Then $-\nabla$ gives $k^2 = |\mathbf{k}|^2$ and we can write

$$\Pi^{TT}_{ij}(\mathbf{k}, t) = \Lambda_{ij,lm}(\hat{\mathbf{k}})(T^{lm}(\mathbf{k}, t) - \mathcal{P}g^{lm}), \tag{3.3}$$

where $\mathcal{P}$ is the background pressure and $\Lambda_{ij,lm}$ is the TT projection tensor:

$$\Lambda_{ij,lm}(\hat{\mathbf{k}}) \equiv P_{il}(\hat{\mathbf{k}})P_{jm}(\hat{\mathbf{k}}) - \frac{1}{2}P_{ij}(\hat{\mathbf{k}})P_{lm}(\hat{\mathbf{k}}), \qquad P_{ij}(\hat{\mathbf{k}}) = \delta_{ij} - \hat{\mathbf{k}}_i\hat{\mathbf{k}}_j. \tag{3.4}$$

We assume the stochastic gravitational background to be isotropic, stationary and Gaussian, therefore completely specified by its power spectrum. For the sub-horizon modes $k \gg \mathcal{H}$, the spectrum of energy density per logarithmic frequency interval can be written as [10]:

$$\frac{d\rho_{GW}}{d\log k}(k, t) = \frac{2Gk^3}{\pi a^4(t)} \int_{t_I}^{t} dt' \int_{t_I}^{t} dt'' a(t')a(t'')\cos[k(t' - t'')]\Pi^2(k, t', t''), \tag{3.5}$$

where $\Pi^2(k, t', t'')$ is the unequal-time correlator of Π^{TT}_{ij} defined as:

$$\langle \Pi^{TT}_{ij}(\mathbf{k}, t)\Pi^{TTij}(\mathbf{k}', t')\rangle \equiv (2\pi)^3 \Pi^2(k, t, t')\delta^{(3)}(\mathbf{k} - \mathbf{k}'), \tag{3.6}$$

and $\langle ... \rangle$ denotes ensemble average.

To make Eq. (3.5) from massive particles non-zero, we expect the time-dependence of the wave-function to vary non-adiabatically with frequencies which we will discuss in the next section. We restrict to situation where $m_{3/2} \gg \mathcal{H}$ in which case we can use flat limit quantization. We choose to parametrize the time dependence by writing the spinor wave-functions as functions of time while keeping the vector polarizations $\epsilon^{\mu}_{\mathbf{p},l}$ constant:

$$\tilde{\psi}^{\mu}_{\mathbf{p},\lambda}(t) = \sum_{s=\pm 1, l=\pm 1,0} \langle 1, \frac{1}{2}, l, \frac{s}{2} | \frac{3}{2}, \lambda \rangle \epsilon^{\mu}_{\mathbf{p},l} \mathbf{u}^{(|\lambda|)}_{\mathbf{p},\frac{s}{2}}(t), \tag{3.7}$$

where we defined:

$$\mathbf{u}^{(|\lambda|)T}_{\mathbf{p},\frac{s}{2}}(t) = (u^{(|\lambda|)}_{\mathbf{p},+}(t)\chi^T_s(\mathbf{p}),\ s\, u^{(|\lambda|)}_{\mathbf{p},-}(t)\chi^T_s(\mathbf{p})), \tag{3.8}$$

expressed in terms of the (scalar) wave function $u^{(|\lambda|)}_{\mathbf{p},\pm}(t)$ and the two-component normalized eigenvectors $\chi_s(\mathbf{p})$ of the helicity operator.

We first consider the Hamiltonian of the fields, which is the space integral of the T^{00} component of the stress tensor (2.9):

$$\begin{aligned} H(t) &= \int d\mathbf{x}\, T^{00}(\mathbf{x},t) \\ &= \int d\mathbf{x}\, \frac{i}{4}\bar{\psi}_\mu(\mathbf{x},t)\gamma^0\partial_t\psi^\mu(\mathbf{x},t) + h.c. \\ &= \int d\mathbf{x}\, \frac{i}{4}\bar{\psi}^{(\frac{1}{2})}(\mathbf{x},t)\gamma^0\partial_t\psi^{(\frac{1}{2})}(\mathbf{x},t) + \frac{i}{4}\bar{\psi}^{(\frac{3}{2})}(\mathbf{x},t)\gamma^0\partial_t\psi^{(\frac{3}{2})}(\mathbf{x},t) + h.c., \end{aligned} \tag{3.9}$$

where in the second line the second term of Eq. (2.9) vanishes since we can do the integral by part leading to the constraint Eq. (2.4). In the last line, we used the property $\epsilon^\mu_{\mathbf{p},l}\epsilon^*_{\mu\mathbf{p},l'} = \delta_{l,l'}$ and $\chi^\dagger_s(\mathbf{p})\chi_{s'}(\mathbf{p}) = \delta_{s,s'}$. The two spinors are defined as:

$$\psi^{(|\lambda|)}(\mathbf{x},t) = \sum_{s=\pm1}\int \frac{d\mathbf{p}}{(2\pi)^3}e^{-i\mathbf{p}\cdot\mathbf{x}}\{\hat{a}_{\mathbf{p},\lambda}\mathbf{u}^{(|\lambda|)}_{\mathbf{p},\frac{s}{2}}(t) + \hat{a}^\dagger_{-\mathbf{p},\lambda}\mathbf{v}^{(|\lambda|)}_{\mathbf{p},\frac{s}{2}}(t)\}. \tag{3.10}$$

Substituting Eq. (3.10) into Eq. (3.9) doesn't give a diagonal form in terms of annihilation and creation operators. Thus we need to do the Bogoliubov transformation

$$\hat{\hat{a}}_{\mathbf{p},\lambda}(t) = \alpha^{(|\lambda|)}_{\mathbf{p}}(t)\,\hat{a}_{\mathbf{p},\lambda} + \beta^{(|\lambda|)}_{\mathbf{p}}(t)\,\hat{a}^\dagger_{-\mathbf{p},\lambda},$$

to make the Hamiltonian (3.9) diagonal:

$$H(t) = \int \frac{d\mathbf{p}}{(2\pi)^3}\sqrt{m^2_{3/2}+p^2}\sum_{\lambda=\pm\frac{1}{2},\pm\frac{3}{2}}\hat{\hat{a}}^\dagger_{\mathbf{p},\lambda}(t)\,\hat{\hat{a}}_{\mathbf{p},\lambda}(t), \tag{3.11}$$

where $p = |\mathbf{p}|$ and $\alpha^{(|\lambda|)}_{\mathbf{p}}(t)$, $\beta^{(|\lambda|)}_{\mathbf{p}}(t)$ are complex number satisfying $|\alpha^{(|\lambda|)}_{\mathbf{p}}(t)|^2 + |\beta^{(|\lambda|)}_{\mathbf{p}}(t)|^2 = 1$. In Heisenberg picture, the expectation value is defined by projecting the time-dependent operator on the initial vacuum $|0\rangle$ that corresponds to vanishing number density. Using $n^{(\lambda)}_{\mathbf{p}}(t) = \hat{\hat{a}}^\dagger_{\mathbf{p},\lambda}(t)\,\hat{\hat{a}}_{\mathbf{p},\lambda}(t)$ leads to the occupation number:

$$\begin{aligned} \langle 0|n^{(\lambda)}_{\mathbf{p}}(t)|0\rangle &= |\beta^{(|\lambda|)}_{\mathbf{p}}(t)|^2 \\ &= \frac{\sqrt{m^2_{3/2}+p^2} - p\,\mathrm{Re}(u^{(|\lambda|)*}_{\mathbf{p},+}(t)\,u^{(|\lambda|)}_{\mathbf{p},-}(t)) - m_{3/2}(1-|u^{(|\lambda|)}_{\mathbf{p},+}(t)|^2)}{2\sqrt{m^2_{3/2}+p^2}}. \end{aligned} \tag{3.12}$$

We also get a time-dependent physical vacuum satisfying

$$\hat{\hat{a}}_{\mathbf{p},\lambda}(t)|0_t\rangle = 0. \tag{3.13}$$

We next consider the sources of the gravitational waves. Plugging the mode decomposition (2.10) into Eq. (3.3) leads to

$$\Pi^{TT}_{ij}(\mathbf{k},t) = \frac{1}{4}\Lambda_{ij,lm}\int\frac{d\mathbf{p}}{(2\pi)^3}\{\hat{\Pi}^{lm}(\mathbf{p},t) + h.c.\}, \tag{3.14}$$

where $\mathbf{k}$ is the momentum mode of the gravitational wave and

$$\begin{aligned} \hat{\Pi}^{lm}(\mathbf{p},t) &= \left[\hat{a}_{-\mathbf{p},\lambda}\bar{\tilde{\psi}}^{\mu C}_{\mathbf{p},\lambda} + \hat{a}^\dagger_{\mathbf{p},\lambda}\bar{\tilde{\psi}}^{\mu}_{\mathbf{p},\lambda}\right]\gamma^{(l}\partial^{m)}\left[\hat{a}_{\mathbf{p}+\mathbf{k},\lambda'}\tilde{\psi}_{\mu\mathbf{p}+\mathbf{k},\lambda'} + \hat{a}^\dagger_{-\mathbf{p}-\mathbf{k},\lambda'}\tilde{\psi}^C_{\mu\mathbf{p}+\mathbf{k},\lambda'}\right] \\ &\quad - \left[\hat{a}_{-\mathbf{p},\lambda}\bar{\tilde{\psi}}^{\mu C}_{\mathbf{p},\lambda} + \hat{a}^\dagger_{\mathbf{p},\lambda}\bar{\tilde{\psi}}^{\mu}_{\mathbf{p},\lambda}\right]\gamma^{((l}\partial_\mu\left[\hat{a}_{\mathbf{p}+\mathbf{k},\lambda'}\tilde{\psi}^{m)}_{\mathbf{p}+\mathbf{k},\lambda'} + \hat{a}^\dagger_{-\mathbf{p}-\mathbf{k},\lambda'}\tilde{\psi}^{m)C}_{\mathbf{p}+\mathbf{k},\lambda'}\right]. \end{aligned}$$

(3.15)

Notice that (3.4) implies that $\Lambda_{ij,lm}k_l = \Lambda_{ij,lm}k_m = 0$, which removes the linear k dependence from ∂_m in the first line of Eq. (3.15), similar to the case of scalars or spin-1/2 fermions [11]. However, in the second line of Eq. (3.15), ∂_μ leads to non-vanishing k_μ contracting with $\epsilon^\mu_{\mathbf{p},m}$, which is an important property of spin-3/2 gases.

The annihilation and creation operators lead to $2^4 = 16$ combinations among which only one contributes to non-trivial results:

$$\langle 0|\hat{a}_{-\mathbf{p},\lambda}\hat{a}_{\mathbf{k}+\mathbf{p},\kappa}\hat{a}^\dagger_{\mathbf{q},\lambda'}\hat{a}^\dagger_{\mathbf{k}'-\mathbf{q},\kappa'}|0\rangle = (2\pi)^6\delta^{(3)}(\mathbf{k}-\mathbf{k}')\{\delta^{(3)}(\mathbf{k}+\mathbf{p}-\mathbf{q})\delta_{\lambda,\kappa'}\delta_{\kappa,\lambda'} - \delta^{(3)}(\mathbf{p}+\mathbf{q})\delta_{\lambda,\lambda'}\delta_{\kappa,\kappa'}\}. \quad (3.16)$$

The two terms inside the brackets in (3.16) come from the Majorana nature, assumed for the spin-3/2 fields, and lead to the same results.

It is convenient to define:

$$\mathbf{p}' = \mathbf{p} + \mathbf{k}. \quad (3.17)$$

We now turn to the unequal-time correlator and write it in terms of 4-spinors:

$$\Pi^2(k,t,t') = 2\int \frac{d\mathbf{p}}{(2\pi)^3}\left[\bar{\mathbf{v}}^{(|\lambda|)}_{\mathbf{p},\frac{s}{2}}(t)\Delta^{\lambda s,\lambda' s'}_{ij}(t)\mathbf{u}^{(|\lambda'|)}_{\mathbf{p}',\frac{s'}{2}}(t)\right]\left[\bar{\mathbf{u}}^{(|\lambda'|)}_{\mathbf{p}',\frac{r'}{2}}(t')\Delta^{\lambda r,\lambda' r'}_{ij}(t')^*\mathbf{v}^{(|\lambda|)}_{\mathbf{p},\frac{r}{2}}(t')\right], \quad (3.18)$$

where $\mathbf{v}^{(|\lambda|)}_{\mathbf{p},\frac{r}{2}} = i\gamma^0\gamma^2\bar{\mathbf{u}}^{|\lambda|T}_{\mathbf{p},\frac{r}{2}}$ and

$$\begin{aligned}\Delta^{\lambda s,\lambda' s'}_{ij}(t) &= \frac{1}{4}\Lambda_{ij,lm}\langle 1,\frac{1}{2},r,\frac{s}{2}|\frac{3}{2},\lambda\rangle\langle 1,\frac{1}{2},r',\frac{s'}{2}|\frac{3}{2},\lambda'\rangle \times \\ &\quad \{2\epsilon_{\mu\mathbf{p},r}\epsilon^\mu_{\mathbf{p}',r'}\, p^{(l}\gamma^{m)} - \epsilon_{\mu\mathbf{p},r}p'^\mu\epsilon^{(l}_{\mathbf{p}',r'}\gamma^{m)} - \epsilon_{\mu\mathbf{p}',r'}p^\mu\epsilon^{(l}_{\mathbf{p},r}\gamma^{m)}\}.\end{aligned} \quad (3.19)$$

We separate the calculation into two parts, $\lambda,\lambda' = \pm\frac{3}{2}$ and $\lambda,\lambda' = \pm\frac{1}{2}$, since the gravitational waves considered are produced mainly by relativistic states and the different helicity states in general are produced differently (see e.g. for gravitinos [17, 18]).

We first consider the case of $\lambda,\lambda' = \pm\frac{3}{2}$. Such a restriction can be thought as working in the massless limit for the spin-3/2 state. For a gravitino, this is the high energy limit before the spontaneous breaking of supersymmetry. The mode decomposition (3.7) reads:

$$\tilde{\psi}^\mu_{\mathbf{p},\pm\frac{3}{2}}(t) = \epsilon^\mu_{\mathbf{p},\pm 1}\,\mathbf{u}^{(3/2)}_{\mathbf{p},\pm\frac{1}{2}}(t). \quad (3.20)$$

We calculate the corresponding unequal-time correlator (3.18):

$$\Pi^2_{\frac{3}{2}}(k,t,t') = \frac{1}{32\pi^2}\int dp\,d\theta\; K^{(\frac{3}{2})}(p,k,\theta,m_{3/2})\;\; W^{(\frac{3}{2})}_{\mathbf{k},\mathbf{p}}(t)W^{(\frac{3}{2})*}_{\mathbf{k},\mathbf{p}}(t'), \quad (3.21)$$

where $\theta(\theta')$ is the angle between $\mathbf{k}$ and $\mathbf{p}(\mathbf{p}')$ and

$$\begin{aligned}K^{(\frac{3}{2})}(p,k,\theta,m_{3/2}) &= p^2k^2(5\sin^3\theta\sin^2\theta' + \sin^2(\theta-\theta')\sin\theta) + 4p^4\sin^4\theta\sin\theta'\} \\ &\quad +8\sin^2(\frac{\theta-\theta'}{2})\sin^2\theta\sin^2\theta',\end{aligned} \quad (3.22)$$

There is no final dependance on p' and θ' as these are expressed before integration as functions of p, k and θ:

$$p' = \sqrt{p^2+k^2+2kp\,\cos\theta}, \qquad \theta' = \arccos(\frac{p\,\cos\theta + k}{\sqrt{p^2+k^2+2kp\,\cos\theta}}). \quad (3.23)$$

We take the equation (3.20):

$$\tilde{\psi}^{\mu}_{\mathbf{p},\pm\frac{3}{2}}(t) = \epsilon^{\mu}_{\mathbf{p},\pm1}\,\mathbf{u}^{(3/2)}_{\mathbf{p},\pm\frac{1}{2}}(t).$$

If $\epsilon^{\mu}_{\mathrm{p},\pm1}\ \boldsymbol{u}^{(3/2)}_{\mathrm{p},\pm\frac{1}{2}}(\mathrm{t})$;

$$\epsilon^{\mu}_{\mathrm{p},\pm1} = 1{,}08643 \times 10^{-28};$$

$$\boldsymbol{u}^{(3/2)}_{\mathrm{p},\pm\frac{1}{2}}(\mathrm{t}) = (6{,}582 \times 10^{-16})^{3/2} = 1{,}6886390 \times 10^{-23}$$

(6,582×(10^(-16)))^1,5 = 1,68863901E-23

$(1{,}6886390 \times 10^{-23}) + (1{,}08643 \times 10^{-28}) = 1{,}68864986 \times 10^{-23}$

Thence:

$$\tilde{\psi}^{\mu}_{\mathrm{p},\pm\frac{3}{2}}(t) = \epsilon^{\mu}_{\mathrm{p},\pm1}\ \boldsymbol{u}^{(3/2)}_{\mathrm{p},\pm\frac{1}{2}}(t) = 1{,}68864986 \times 10^{-23}$$

We note that

$$\epsilon^{\mu}_{\mathrm{p},\pm1} = 1{,}08643 \times 10^{-28} = \left\{\frac{\pi^{1/4}}{\Gamma\left(\frac{3}{4}\right)} = 1 + \int_0^{\infty} \frac{e^{-t^2/2}}{\sqrt{2\pi}} \left[\frac{4e^{\pi}\left(e^{2\pi} - \cos\left(\sqrt{2\pi}t\right)\right)}{e^{4\pi} - 2e^{2\pi}\cos\left(\sqrt{2\pi}t\right) + 1}\right]\right\} \times 10^{-2}$$

that is the Ramanujan's integral.

We have that, from (3.5):

$$\frac{d\rho_{GW}}{d\log k}(k,t) \simeq \frac{Gk^3}{\pi^3 a^4(t)} \int dp\, d\theta\, K(p,k,\theta,m_{3/2})\ \{|I_c(k,p,\theta,t)|^2 + |I_s(k,p,\theta,t)|^2\}, \tag{3.32}$$

where

$$I_c(k,p,\theta,t) = \int_{t_i}^{t} \frac{dt'}{a(t')}\cos(kt')\,\tilde{W}^{(\frac{1}{2})}_{\mathbf{k},\mathbf{p}}(t'), \qquad I_s(k,p,\theta,t) = \int_{t_i}^{t} \frac{dt'}{a(t')}\sin(kt')\,\tilde{W}^{(\frac{1}{2})}_{\mathbf{k},\mathbf{p}}(t') \tag{3.33}$$

parameterize the spectrum of helicity-1/2 component. Then, (3.32) is the master equation for gravitational waves produced from non-adiabatic spin-3/2 gases.

Note 1

Analytic continuation (sometimes called simply "continuation") provides a way of extending the domain over which a complex function is defined. The most common application is to a complex analytic function determined near a point z_0 by a power series

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k. \tag{1}$$

Such a power series expansion is in general valid only within its radius of convergence. However, under fortunate circumstances (that are very fortunately also rather common!), the function f will have a power series expansion that is valid within a larger-than-expected radius of convergence, and this power series can be used to define the function outside its original domain of definition. This allows, for example, the natural extension of the definition trigonometric, exponential, logarithmic, power, and hyperbolic functions from the real line $\mathbb{R}$ to the entire complex plane $\mathbb{C}$. Similarly, analytic continuation can be used to extend the values of an analytic function across a branch cut in the complex plane.

Let f_1 and f_2 be analytic functions on domains Ω_1 and Ω_2, respectively, and suppose that the intersection $\Omega_1 \cap \Omega_2$ is not empty and that $f_1 = f_2$ on $\Omega_1 \cap \Omega_2$. Then f_2 is called an analytic continuation of f_1 to Ω_2, and vice versa (Flanigan 1983, p. 234). Moreover, if it exists, the analytic continuation of f_1 to Ω_2 is unique.

This uniqueness of analytic continuation is a rather amazing and extremely powerful statement. It says in effect that knowing the value of a complex function in some finite complex domain uniquely determines the value of the function at *every* other point.

By means of analytic continuation, starting from a representation of a function by any one power series, any number of other power series can be found which together define the value of the function at all points of the domain. Furthermore, any point can be reached from a point without passing through a singularity of the function, and the aggregate of all the power series thus obtained constitutes the analytic expression of the function (Whittaker and Watson 1990, p. 97)

From

https://dilucia.wordpress.com/tag/prolungamento-analitico-di-una-funzione/

Note in Italian

VIII.55.- Prolungamento analitico di una funzione.-

Supponiamo che :

una funzione f(x) sia ***indefinitamente derivabile in un intervallo]a,b[*** *e* ***sviluppabile in questo intervallo]a,b[in serie di Taylor***.-

$$f(x) = \sum_{k=0}^{\infty} c_k \cdot (x - x_o)^k \quad ove\ c_k = \frac{f^k(x_o)}{k!}$$

Consideriamo questa serie nel ***dominio complesso*** *sostituendo l'argomento reale* ***x*** *con* ***l'argomento complesso z = x + i·y***

$$f(z)=\sum_{k=0}^{\infty} c_k\cdot(z-x_o)^k \quad ove\ c_k=\frac{f^k(x_o)}{k!}$$

Il dominio di convergenza di questa serie *é* ***un cerchio di raggio R e centro nel punto*** x_o

Il numero R é comunque tale che tutto l'intervallo]a,b[, nel quale la serie

$$f(x)=\sum_{k=0}^{\infty} c_k\cdot(x-x_o)^k \quad ove\ c_k=\frac{f^k(x_o)}{k!}$$

é ***convergente****, é contenuto nel* ***cerchio*** $|z-x_o|<R$

poichè, come é noto, in tutti i ***punti esterni al cerchio di convergenza*** $|z-x_o|<R$ *una serie di potenze é* ***divergente****.-*

Nei punti z ∈]a,b[,

la somma della serie $f(z)=\sum_{k=0}^{\infty} c_k\cdot(z-x_o)^k \quad ove\ c_k=\frac{f^k(x_o)}{k!}$ *é uguale, per ipotesi, a* ***f(x)****.*

Nei punti z ∉]a,b[,

La somma della serie $f(z)=\sum_{k=0}^{\infty} c_k\cdot(z-x_o)^k \quad ove\ c_k=\frac{f^k(x_o)}{k!}$ *negli altri punti del cerchio di convergenza é* ***una funzione di z****; la chiameremo* ***Prolungamento Analitico*** *della funzione f(x) al* ***dominio complesso*** *e la indicheremo con*

$$f: z\in C\rightarrow y=f(z)=\sum_{k=0}^{\infty} c_k\cdot(z-x_o)^k \in C$$

In particolare,

se la serie $f(x)=\sum_{k=0}^{\infty} c_k\cdot(x-x_o)^k \quad ove\ c_k=\frac{f^k(x_o)}{k!}$ *é convergente per tutti gli x reali,*

la serie $f(z)=\sum_{k=0}^{\infty} c_k\cdot(z-x_o)^k \quad ove\ c_k=\frac{f^k(x_o)}{k!}$ *é* ***convergente*** *per tutti gli* ***z complessi*** *e* ***il prolungamento analitico della funzione f(x)*** *risulta così definita in tutto* ***il piano complesso****.–*

Let's now take the wave function of the Universe of Hawking's paper:

$$\psi_0(a_0)=2\cos\left[\frac{(H^2a_0^2-1)^{3/2}}{3H^2}-\frac{\pi}{4}\right]$$

For the values we have given previously, it becomes:

$$2\cos\left(\frac{0{,}671919\cdot 24\pi^2}{0{,}5}-\frac{\pi}{4}\right)$$

2cos[((0,671919*24*9.86960)/0.5)-(3,141592653/4)]

integrate 2 cos[((0.671919*24*9.86960)/0.5))-((3.141592653/4))] [1.08643]

$$\frac{\pi^{1/4}}{\Gamma\left(\frac{3}{4}\right)}=1{,}08643$$

$$\int 2\cos\left(\left(\frac{0.671919\times 24\times 9.86960}{0.5}-\frac{3.141592653}{4}\right)^{\circ}\right)1.08643\,dx =$$
$$1.60277\,x+\text{constant}$$

Indefinite

$$\int 2\cos\left(\left(\frac{0.671919\times 24\times 9.86960}{0.5}-\frac{3.141592653}{4}\right)^{\circ}\right)dx = 1.47526\,x+\text{constant}$$

So:

$$1{,}08643\int 2\cos\left[\frac{\left(\frac{9\pi}{16}-1\right)^{3/2}}{\frac{3l^2}{72\pi^2}}-\frac{\pi}{4}\right]=1{,}08643\int 2\cos\left(\frac{0{,}671919\cdot 24\pi^2}{0{,}5}-\frac{\pi}{4}\right)=$$

$$=1{,}60277x$$

Generically:

$$1{,}08643\int 2\cos\left[\frac{(H^2a_0^2-1)^{3/2}}{3H^2}-\frac{\pi}{4}\right]=1{,}60277x$$

$$\frac{\pi^{1/4}}{\Gamma\left(\frac{3}{4}\right)}=1+\int_0^{\infty}\frac{e^{-t^2/2}}{\sqrt{2\pi}}\left[\frac{4e^{\pi}\left(e^{2\pi}-\cos\left(\sqrt{2\pi}t\right)\right)}{e^{4\pi}-2e^{2\pi}\cos\left(\sqrt{2\pi}t\right)+1}\right]=1{,}08643481$$

The value 1.602773 is practically equal to the electric charge of a positron (1.602176). Recall that the universe, according to the "brane-world" scenario, originates from the collision of two branes. From what has been deduced from the mathematical point of view, the hypothesis that the two branes are nothing more than a positron and an electron that collide appears increasingly plausible, annihilating and thus originating a pair of mediating bosons. One of them, due to the infinite possibilities of the quantum vacuum and its energy, expands in an inflationary way giving rise to a universe, while the other remains in quantum form, identifying with the 6 extra compact dimensions in a toroidal space (dimensions of energy, identifiable with real plans of existence and / or degrees of freedom).

The positron, as well as the electron, are fermionic strings. The action of them is given by the following equations:

From:

SLAC-PUB-3088
March 1983 - T
SUPERSPACE GEOMETRY OF FERMIONIC STRINGS
EMIL MARTINEC

$$S_{eff} = \left[B + \left(\frac{6-d}{4}\right) \ell n\epsilon\right]\chi(M) + +C \int_M d^2z\, \hat{e}^{-1} e^{-2\psi}$$
$$+ \frac{\beta d}{4\sqrt{\pi\epsilon}} \int_{\partial M} d\hat{s}\, d^2\theta\, \delta(\theta) e^{-\psi} + \left(E + \frac{\beta d}{4\pi}\right) \int_{\partial M} ds\, d^2\theta\, k$$
$$+ \left(\frac{10-d}{2\pi}\right)\left[\int_M d^2z\, \hat{e}^{-1}\left[\frac{1}{2}\hat{e}^{\alpha}\,\psi\, \hat{e}_{\alpha}\,\psi + \hat{R}\,\psi\right] + \int_{\partial M} d\hat{s}\, d^2\theta\, \hat{k}\,\psi\right]$$
$$+ (\text{Ker terms}) + (\text{indep of } \psi)$$

Or for supersymmetric boundary conditions:

$$S_{eff}^{matter+ghost} = \left(\frac{10-d}{2\pi}\right)\left[\int_M d^2z \ \hat{e}^{-1}\left[\frac{1}{2}(\hat{e}^\alpha \psi)(\hat{e}_\alpha \psi) + \frac{1}{2}\hat{R}\psi\right] + \int_{\partial M} d\hat{s} \ d^2\theta \ \hat{k}\psi\right] .$$

Here $d = 26$

For the second equation , we have: $\frac{10-}{2\pi} = -\frac{16}{2\pi} = -\frac{8}{\pi} = -2{,}546479 ...$

We note that $\pi^{1/e} = 1.523\ 671\ 054\ 858\ 931\ 718\ 386\ 285$ e che

$\pi^{-1/e} = 0.656\ 309\ 639\ 020\ 204\ 707\ 493\ 834$

we observe that 2,546479 * 0,6563096 = -1,671278613… value very near to the proton mass $1{,}6726231 \times 10^{-27}$ kg

thence:

$$S_{eff}^{matter+ghost} = \left(\frac{10-d}{2\pi}\right)\left[\int_M d^2z \ \hat{e}^{-1}\left[\frac{1}{2}(\hat{e}^\alpha \psi)(\hat{e}_\alpha \psi) + \frac{1}{2}\hat{R}\psi\right] + \int_{\partial M} d\hat{s} \ d^2\theta \ \hat{k}\psi\right] .$$

$$\left(\frac{10-d}{2\pi}\right) = S_e^{m+g} \cdot \frac{1}{\left[\int_M d^2z\hat{e}^{-1}\left[\frac{1}{2}(\hat{e}^\alpha\psi)(\hat{e}_\alpha\psi) + \frac{1}{2}\widehat{R\psi}\right] + \int_{\partial M} d\hat{s}d^2\theta\hat{k}\psi\right]}$$

$$-2{,}546479 = S_e^{m+g} \cdot \frac{1}{\left[\int_M d^2z\hat{e}^{-1}\left[\frac{1}{2}(\hat{e}^\alpha\psi)(\hat{e}_\alpha\psi) + \frac{1}{2}\widehat{R\psi}\right] + \int_{\partial M} d\hat{s}d^2\theta\hat{k}\psi\right]}$$

and:

$$S_e^{m+g} = -2{,}546479\left[\int_M d^2z\hat{e}^{-1}\left[\frac{1}{2}(\hat{e}^{\alpha}\psi)(\hat{e}_{\alpha}\psi) + \frac{1}{2}\widehat{R\psi}\right] + \int_{\partial M} d\hat{s}d^2\theta\hat{k}\psi\right]$$

$$S_e^{m+g} = -2{,}546479 \times 0{,}6563096 = -1{,}671278613$$

If instead from 1,04719755 multiply by 0,61803398, we obtain: 0,64720

from which $-2{,}546479 \times 0{,}64720 = -1{,}6480812088$.

Thence, we have for:

$$\int_0^{\infty} \frac{1+\left(\frac{x}{b+1}\right)^2}{1+\left(\frac{x}{a}\right)^2} \cdot \frac{1+\left(\frac{x}{b+2}\right)^2}{1+\left(\frac{x}{a+1}\right)^2} \ldots dx = \frac{1}{2}\pi^{1/2}\frac{\Gamma\left(a+\frac{1}{2}\right)\Gamma(b+1)\Gamma\left(b-a+\frac{1}{2}\right)}{\Gamma(a)\Gamma\left(b+\frac{1}{2}\right)\Gamma(b-a+1)} =$$

$$= 1{,}047197551196598\ldots$$

$$0{,}61803398\left[\int_0^{\infty} \frac{1+\left(\frac{x}{b+1}\right)^2}{1+\left(\frac{x}{a}\right)^2} \cdot \frac{1+\left(\frac{x}{b+2}\right)^2}{1+\left(\frac{x}{a+1}\right)^2}\right] \ldots dx$$

$$= 0{,}61803398\left[\frac{1}{2}\pi^{1/2}\frac{\Gamma\left(a+\frac{1}{2}\right)\Gamma(b+1)\Gamma\left(b-a+\frac{1}{2}\right)}{\Gamma(a)\Gamma\left(b+\frac{1}{2}\right)\Gamma(b-a+1)}\right] =$$

$$= 0{,}64720\ldots$$

Thus:

$$\left[\int_M d^2z\hat{e}^{-1}\left[\frac{1}{2}(\hat{e}^{\alpha}\psi)(\hat{e}_{\alpha}\psi) + \frac{1}{2}\widehat{R\psi}\right] + \int_{\partial M} d\hat{s}d^2\theta\hat{k}\psi\right] =$$

$$= 0{,}61803398\left[\int_0^{\infty} \frac{1+\left(\frac{x}{b+1}\right)^2}{1+\left(\frac{x}{a}\right)^2} \cdot \frac{1+\left(\frac{x}{b+2}\right)^2}{1+\left(\frac{x}{a+1}\right)^2}\right] \ldots dx$$

$$= 0{,}61803398\left[\frac{1}{2}\pi^{1/2}\frac{\Gamma\left(a+\frac{1}{2}\right)\Gamma(b+1)\Gamma\left(b-a+\frac{1}{2}\right)}{\Gamma(a)\Gamma\left(b+\frac{1}{2}\right)\Gamma(b-a+1)}\right] = 0{,}64720$$

From Wikipedia:

Ramanujan approximation formula to π

We have the following Ramanujan formula

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801}\sum_{k=0}^{\infty}\frac{(4k)!(1103+26390k)}{(k!)^4 396^{4k}}$$

For k = 2, and inverting the formula, we obtain:

1/(((2sqrt2/(9801) * sum ((((4n)! (1103+26390n))))/((((n!)^4*396^(4n)))), n = 0 to 2)))

Input interpretation:

$$\frac{1}{2\times\frac{\sqrt{2}}{9801}\sum_{n=0}^{2}\frac{(4n)!\,(1103+26\,390\,n)}{(n!)^4\times 396^{4n}}}$$

n! is the factorial function

Result:

$$\frac{2\,286\,635\,172\,367\,940\,241\,408\,\sqrt{2}}{1\,029\,347\,477\,390\,786\,609\,545} \approx$$
3.141592653589793238462649065702758898156677480462334781168399595644739794558841580205059234965983146

3.14159265… = π

Possible closed forms:

$\pi \approx 3.14159265358979323846264338327$

$\log(\mathcal{G}_{Gc}) \approx 3.14159265358979323846264338327$

$\sqrt{6\zeta(2)} \approx 3.14159265358979323846264$33832

$\frac{1}{2\mathcal{P}_A} \approx 3.14159265358979323846264$33832

$\frac{1}{2C_{PTH}} \approx 3.14159265358979323846264$33832

$\frac{128}{45\bar{s}_{ld}} \approx 3.14159265358979323846264$33832

$\frac{3(-50-81e+299e^2)}{79-427e+397e^2} \approx 3.14159265358979323846$03988

$\log\left(\frac{1}{63}\left(23\left(\sqrt{2}-6\right)+114e+303e^2-210\pi-33\pi^2\right)\right) \approx$
3.14159265358979323842471 4

root of $108x^4+1717x^3-6952x^2+258x+4045$ near $x = 3.14159$ $\approx$
3.1415926535897932384 52342

root of $15134x^3-53597x^2+28993x-31352$ near $x = 3.14159$ $\approx$
3.1415926535897932384 28911

$\frac{1}{\text{root of } 4045x^4+258x^3-6952x^2+1717x+108 \text{ near } x = 0.31831} \approx$
3.1415926535897932384 52342

$\frac{1}{\text{root of } 31352x^3-28993x^2+53597x-15134 \text{ near } x = 0.31831} \approx$
3.1415926535897932384 28911

root of $305x^5-1062x^4+316x^3-159x^2+97x+1579$ near $x = 3.14159$ $\approx$
3.1415926535897932384 93361

$\frac{6167950454}{1963319607} \approx 3.14159265358979323838637$

from which, multiplying by 0.16394033883189229775 = ((((-979 + 398 sqrt(π) - 210 π + 86 π^(3/2) + 51 π^2)/(95 π)))), we obtain:

((((-979 + 398 sqrt(π) - 210 π + 86 π^(3/2) + 51 π^2)/(95 π))))*(((1/(((2sqrt2/(9801) * sum ((((4n)! (1103+26390n))))/((((n!)^4*396^(4n)))), n = 0 to 2))))))^2

Input interpretation:

$$\frac{-979+398\sqrt{\pi}-210\pi+86\pi^{3/2}+51\pi^2}{95\pi}\left(\frac{1}{2\times\frac{\sqrt{2}}{9801}\sum_{n=0}^{2}\frac{(4n)!(1103+26390n)}{(n!)^4\times 396^{4n}}}\right)^2$$

$n!$ is the factorial function

Result:

$$(10457400823020319557061574451198754635644928(-979+398\sqrt{\pi}-210\pi+86\pi^{3/2}+51\pi^2))/(100657841775023715279808521462319247085167375\pi)\approx 1.61803$$

1.61803

Alternate forms:

$$-((10457400823020319557061574451198754635644928(979-398\sqrt{\pi}+210\pi-86\pi^{3/2}-51\pi^2))/(100657841775023715279808521462319247085167375\pi))$$

$$-\frac{62744404938121917342369446707192527813869568}{2875938336429249007994529184637692773861925}-\frac{10237795405736892846363281387723580788296384512}{100657841775023715279808521462319247085167375\pi}+\frac{4162045527562087183710506631577104344986681344}{100657841775023715279808521462319247085167375\sqrt{\pi}}+\frac{899336470779747481907295402803092898665463808\sqrt{\pi}}{100657841775023715279808521462319247085167375}+\frac{533327441974036297410140297011136486417891328\pi}{100657841775023715279808521462319247085167375}$$

We note that:

((((-979 + 398 sqrt(π) - 210 π + 86 π^(3/2) + 51 π^2)/(95 π))))

Input:

$$\frac{-979+398\sqrt{\pi}-210\pi+86\pi^{3/2}+51\pi^2}{95\pi}$$

Decimal approximation:

0.1639403388318922976747203068673421170221584916608790344 43...

0.16394033883…

Property:

$\frac{-979+398\sqrt{\pi}-210\pi+86\pi^{3/2}+51\pi^2}{95\pi}$ is a transcendental number

Alternate forms:

$$-\frac{979-398\sqrt{\pi}+210\pi-86\pi^{3/2}-51\pi^2}{95\pi}$$

$$-\frac{42}{19}-\frac{979}{95\pi}+\frac{398}{95\sqrt{\pi}}+\frac{86\sqrt{\pi}}{95}+\frac{51\pi}{95}$$

Series representations:

$$\frac{-979+398\sqrt{\pi}-210\pi+86\pi^{3/2}+51\pi^2}{95\pi}=$$
$$-\frac{42}{19}-\frac{979}{95\pi}+\frac{86\sqrt{\pi}}{95}+\frac{51\pi}{95}+\frac{398\sqrt{-1+\pi}\sum_{k=0}^{\infty}(-1+\pi)^{-k}\binom{\frac{1}{2}}{k}}{95\pi}$$

$$\frac{-979+398\sqrt{\pi}-210\pi+86\pi^{3/2}+51\pi^2}{95\pi}=$$
$$-\frac{42}{19}-\frac{979}{95\pi}+\frac{86\sqrt{\pi}}{95}+\frac{51\pi}{95}+\frac{398\sqrt{-1+\pi}\sum_{k=0}^{\infty}\frac{(-1)^k(-1+\pi)^{-k}\left(-\frac{1}{2}\right)_k}{k!}}{95\pi}$$

$$\frac{-979+398\sqrt{\pi}-210\pi+86\pi^{3/2}+51\pi^2}{95\pi}=$$
$$-\frac{42}{19}-\frac{979}{95\pi}+\frac{86\sqrt{\pi}}{95}+\frac{51\pi}{95}+\frac{398\sqrt{z_0}\sum_{k=0}^{\infty}\frac{(-1)^k\left(-\frac{1}{2}\right)_k(\pi-z_0)^k z_0^{-k}}{k!}}{95\pi}$$

for (not ($z_0\in\mathbb{R}$ and $-\infty<z_0\leq 0$))

References

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