Riemann Hypothesis

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1 Abstract

The Riemann Zeta function is defined as the Analytic Continuation
of the Dirichlet series

$\zeta(s) = \sum_{n=1}^{\infty} 1/n^s, \ Re(s)>1$

$The\ Riemann\ Zeta\ function\ is\ holomorphic\ in\ the\ complex\ plane\ except\ for\ a\ simple\ pole\ at\ s = 1$

$The\ non\ trivial\ zeroes(i.e\ those\ not\ at\ negative\ even\ integers)\ of\ the$

$Riemann\ Zeta\ function\ lie\ in\ the\ critical\ strip$

$0 \leq Re(s) \leq 1$

$Riemann's\ Xi\ function\ is\ defined\ as[4,\ p.1],$

$\epsilon(s) \ = \ s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)/2$

$The\ zero\ of\ (s{-}1)\ cancels\ the\ pole\ of\ \zeta(s)\ ,and\ the\ real\ zeroes\ of\ s\,\zeta(s)\ are\ cancelled\ by\ the$

$simple\ poles\ of\ \Gamma(s/2)\ which\ never\ vanishes.$

$Thus,\ \epsilon(s)\ is\ an\ entire\ function\ whose\ zeroes\ are\ the\ non\ trivial\ zeroes\ of\ \zeta(s)(see[1, p.80])$

$Further,\ \epsilon(s)\ satisfies\ the\ functional\ equation$

$\epsilon(1-s) \ = \ \epsilon(s)$

2 Statement of the Riemann Hypothesis

The Riemann Hypothesis states that all the non trivial zeroes of the Riemann Zeta function lie on the critical line Re(s)=1/2

3 Proof

The Riemann Xi function [2, p.37, Theorem 2.11] is defined as

For all s$\in\mathbb{C}$ *we have*,

$\epsilon(s) = \epsilon(0)\prod_{\rho}(1-\frac{s}{\rho})$ ... (1)

where ρ *ranges over all the roots* ρ *of* $\epsilon(\rho)=0$ *and if we combine the factors* $(1-\frac{s}{\rho})$ *and* $(1-\frac{s}{(1-\rho)})$, *the product converges absolutely and uniformly on compact subsets of* $\mathbb{C}$

Also, $\epsilon(0)=1/2$

Let, $\epsilon(s)=0,\ 0\leq Re(s)\leq 1$ ... $(*)$

Since, $\epsilon(s)$ *satisfies the functional equation*

$\epsilon(1-s) = \epsilon(s)$

$\epsilon(1-s)=\epsilon(s)=0.$

From(1),

$\epsilon(1-s)=\epsilon(0)\prod_{\rho}(1-\frac{1-s}{\rho})=0$

$\epsilon(1-s)=\epsilon(0)\prod_{\rho}(\frac{\rho+s-1}{\rho})=0$

$\epsilon(0) = 1/2$ [2, $p.37$, $Theorem$ 2.11]

$\epsilon(1-s) = 1/2\prod_{\rho}(\frac{\rho+s-1}{\rho}) = 0$

$1/2\prod_{\rho}(\frac{\rho+s-1}{\rho}) = 0$

$\prod_{\rho}(\frac{\rho+s-1}{\rho}) = 0 \qquad \ldots \qquad (2)$

$Let,\ s = \sigma + it\ 0 \le \sigma \le 1$

$and\ let,\ \rho = a + ib$

$Since,\ \epsilon(\rho) = 0,$

$Thus,\ 0 \le Re(\rho) \le 1.(Since\ \epsilon(s)\ is\ zero\ free\ in\ Re(s) < 0$

$and\ Re(s) > 1.)$

$Thus,\ \rho = a + ib,\ 0\ , 0 \le a \le 1.$

$From\ (2),$

$\prod_{\rho}(\frac{\rho+s-1}{\rho}) = 0$

$Since,\ \epsilon(s) = 1/2\prod_{\rho}(1-\frac{s}{\rho})$

$\epsilon(1-\rho) = \epsilon(\rho) = 0.$

$Thus, \epsilon(s) = 1/2\prod_{\rho}(1-\frac{s}{\rho})(1-\frac{s}{1-\rho})$

$\mid \epsilon(s) \mid = \mid 1/2\prod_{\rho}(1-\frac{s}{\rho})(1-\frac{s}{1-\rho}) \mid$

$\mid \epsilon(s) \mid < \infty$ [2, $p.37$, $Theorem$ 2.11].

$\mid \epsilon(1-s) \mid = \mid \epsilon(s) \mid < \infty$

$\epsilon(1-s)\ is\ absolutely\ convergent\ infinite\ product,\ thus\ it\ is\ a\ convergent\ infinite\ product.$

Since, $\epsilon(1-s)$ *is convergent infinite product*

The value of convergent infinite product is zero

if and only if atleast one of the factors is zero $.[5, p.287]$

$So, \epsilon(1-s) = 0 \Rightarrow \prod_\rho (\frac{\rho+s-1}{\rho}) = 0$

$(\frac{\rho_0+s-1}{\rho_0}) = 0,\ for\ some\ \rho_0 \in \mathbb{C}$

$\rho_0 + s - 1 = 0.$

$Putting,\ s = \sigma + it,\ 0 \leq \sigma \leq 1$

$and\ putting\ \ \rho_0 = a_0 + ib_0,\ 0 \leq a_0 \leq 1.$

$a_0 + ib_0 + \sigma + it - 1 = 0.$

$(a_0 + \sigma - 1) + i(b_0 + t) = 0$

$| (a_0 + \sigma - 1) + i(b_0 + t) |^2 = 0$

$(a_0 + \sigma - 1)^2 + (b_0 + t)^2 = 0$

$(a_0 + \sigma - 1)^2 = 0\ and\ (b_0 + t)^2 = 0.$

$(a_0 - \sigma + 2\sigma - 1)^2 = 0\ and\ b_0 = -t.$

$(a_0 - \sigma)^2 + (2\sigma - 1)^2 + 2(a_0 - \sigma)(2\sigma - 1) = 0$

$(a_0 - \sigma)^2 + (2\sigma - 1)(2\sigma - 1 + 2a_0 - 2\sigma) = 0$

$(a_0 - \sigma)^2 + (2\sigma - 1)(2a_0 - 1) = 0 \qquad ... \qquad (3)$

Since , the critical strip is $0 \leq Re(s) \leq 1$

$s = \sigma + it; 0 \leq \sigma \leq 1.$

$We\ discuss\ 2\ cases\ 0\ \leq \sigma \leq 1/2\ and\ 1/2 \leq \sigma \leq 1.$

$Case\ 1 : 0 \leq \sigma \leq 1/2$

$\rho = a + ib,\ 0 \leq a \leq 1$

$Claim : 0 \leq a \leq 1/2.$

$We\ prove\ the\ claim\ by\ contradiction.$

$Let, a \notin [0, 1/2]$

$Since\ 0 \leq a \leq 1 \Rightarrow 1/2 < a \leq 1.$

$From\ (1),$

$\epsilon(\sigma + it) = \epsilon(0) \prod_\rho (1 - \frac{\sigma+it}{a+ib})$

$\epsilon(\sigma + it) = \epsilon(0) \prod_\rho \frac{(a-\sigma)+i(b-t)}{a+ib})$

$Since, 1/2 < a \leq 1 \qquad ... \qquad (4)$

$Since, 0 \leq \sigma \leq 1/2$

$Thus, -1/2 \leq -\sigma \leq 0 \qquad ... \qquad (5)$

$Adding\ (4)\ and\ (5),\ we\ have$

$0 < a - \sigma \leq 1$

$\Rightarrow a - \sigma \neq 0\ \forall\ a \in (1/2, 1].$

$\Rightarrow (a - \sigma) + i(b - t) \neq 0\ \forall\ a \in (1/2, 1]\ and\ \ \forall\ b\ \in \mathbb{R}.$

$\Rightarrow \frac{(a-\sigma)+i(b-t)}{a+ib} \neq 0\ \forall\ a \in (1/2, 1]\ and\ \ \forall\ b\ \in \mathbb{R}.$

$Since\ \epsilon(s)\ is\ a\ convergent\ infinite\ product.$

$So,\ value\ of\ a\ convergent\ infinite\ product\ is\ zero$

$if\ and\ only\ if\ atleast\ one\ of\ the\ factors\ are\ zero.$

$Since\, all\, the\, factors\ \frac{(a-\sigma)+i(b-t)}{a+ib}\ are\, non\, zero\, \forall a \in (1/2, 1]\ and\ \forall\ b\ \in \mathbb{R}.$

$\Rightarrow \epsilon(0) \prod_{\rho} \frac{(a-\sigma)+i(b-t)}{a+ib} \neq 0.$

$\epsilon(s) \neq 0.$

$But\ in\ (*)\ ,\ we\, have\, assumed\, that\ \epsilon(s) = 0.\ So\, we\, get\, a\, contradiction.$

$So,\ our\ assumption\ that\ a \notin [0, 1/2]\ is\ wrong.$

$Thus, a \in [0, 1/2]$

$0 \leq a \leq 1/2$

$From\ (3),$

$(a_0 - \sigma)^2 + (2\sigma - 1)(2a_0 - 1) = 0$

$Since, 0 \leq \sigma \leq 1/2 \Rightarrow 1 - 2\sigma \geq 0 \qquad ... \qquad (6)$

$Since, 0 \leq a \leq 1/2 \Rightarrow 1 - 2a \geq 0 \Rightarrow (1 - 2a_0) \geq 0) \qquad ... \qquad (7)$

$From\ (6)\ and\ (7)\ , (1 - 2\sigma)(1 - 2a_0) \geq 0$

$\Rightarrow (2\sigma - 1)(2a_0 - 1) \geq 0. \qquad ... \qquad (8)$

$Using\ (8)\ in\ (a_0 - \sigma)^2 + (2\sigma - 1)(2a_0 - 1) = 0$

$(a_0 - \sigma)^2 = 0\ and\ (2\sigma - 1)(2a_0 - 1) = 0$

$a_0 = \sigma\ and\ (2\sigma - 1)(2a_0 - 1) = 0$

$Putting\ a_0 = \sigma\ in\ (2\sigma - 1)(2a_0 - 1) = 0$

$(2\sigma - 1)(2\sigma - 1) = 0$

$(2\sigma - 1)^2 = 0$

$\Rightarrow \sigma = 1/2.$

Case 2: $1/2 \leq \sigma \leq 1$

$\rho = a + ib,\ 0 \leq a \leq 1$

$Claim : 1/2 \leq a \leq 1.$

$We\ prove\ the\ claim\ by\ contradiction.$

$Let, a \notin [1/2, 1]$

$Since,\ 0 \leq a \leq 1 \Rightarrow 0 \leq a < 1/2.$

$From\ (1),$

$\epsilon(\sigma + it) = \epsilon(0) \prod_\rho (1 - \frac{\sigma+it}{a+ib})$

$\epsilon(\sigma + it) = \epsilon(0) \prod_\rho \frac{(a-\sigma)+i(b-t)}{a+ib})$

$Since, 0 \leq a < 1/2 \qquad ... \qquad (9)$

$Since, 1/2 \leq \sigma \leq 1$

$Thus, -1 \leq -\sigma \leq -1/2 \qquad ... \qquad (10)$

$Adding\ (9)\ and\ (10)\ ,\ we\ have$

$-1 \leq a - \sigma < 0$

$\Rightarrow a - \sigma \neq 0\ \forall\ a \in [0, 1/2).$

$\Rightarrow (a-\sigma)+i(b-t) \neq 0 \ \forall \ a \in \ [0,1/2) \ and \ \forall \ b \in \mathbb{R}.$

$\Rightarrow \frac{(a-\sigma)+i(b-t)}{a+ib} \neq 0 \ \forall \ a \in \ [0,1/2) \ and \ \forall \ b \in \mathbb{R}.$

$Since \ \epsilon(s) \ is \ a \ convergent \ infinite \ product.$

$So, \ value \ of \ a \ convergent \ infinite \ product \ is \ zero$

$if \ and \ only \ if \ atleast \ one \ of \ the \ factors \ are \ zero.$

$Since \ all \ the \ factors \ \frac{(a-\sigma)+i(b-t)}{a+ib} \ are \ non \ zero \ \forall \ a \in [0,1/2) \ and \ \forall \ b \ \in \mathbb{R}.$

$\Rightarrow \epsilon(0) \prod_{\rho} \frac{(a-\sigma)+i(b-t)}{a+ib} \neq 0.$

$\epsilon(s) \neq 0.$

But , we have assumed that $\epsilon(s) = 0. \ So \ we \ get \ a \ contradiction.$

$So, \ our \ assumption \ that \ a \notin [1/2,1] \ is \ wrong.$

$Thus, a \in [1/2,1]$

From (2),

Since, $\epsilon(s) = 1/2 \prod_{\rho}(1-\frac{s}{\rho})$

$\epsilon(1-\rho) = \epsilon(\rho) = 0.$

$Thus, \epsilon(s) = 1/2 \prod_{\rho}(1-\frac{s}{\rho})(1-\frac{s}{1-\rho})$

$\mid \epsilon(s) \mid = \mid 1/2 \prod_{\rho}(1-\frac{s}{\rho})(1-\frac{s}{1-\rho}) \mid$

$\mid \epsilon(s) \mid < \infty$ [2, $p.37$, $Theorem$ 2.11].

$\mid \epsilon(1-s) \mid = \mid \epsilon(s) \mid < \infty$

$\epsilon(1-s) \ is \ absolutely \ convergent \ infinite \ product, \ thus \ it \ is \ a \ convergent \ infinite \ product.$

Since, $\epsilon(1-s)$ *is convergent infinite product*

The value of convergent infinite product is zero

if and only if atleast one of the factors is zero .[5, *p*.287]

$So, \epsilon(1-s) = 0 \Rightarrow \prod_{\rho}(\frac{\rho+s-1}{\rho}) = 0$

$(\frac{\rho_1+s-1}{\rho_1}) = 0,\ for\ some\ \rho_1 \in \mathbb{C}$

$\rho_1 + s - 1 = 0.$

$Putting,\ s = \sigma + it,\ 0 \leq \sigma \leq 1$

$and\ putting\ \ \rho_1 = a_1 + ib_1,\ 0 \leq a_1 \leq 1.$

$a_1 + ib_1 + \sigma + it - 1 = 0.$

$(a_1 + \sigma - 1) + i(b_1 + t) = 0$

$\mid (a_1 + \sigma - 1) + i(b_1 + t) \mid^2 = 0$

$(a_1 + \sigma - 1)^2 + (b_1 + t)^2 = 0$

$(a_1 + \sigma - 1)^2 = 0\ and\ (b_1 + t)^2 = 0.$

$(a_1 - \sigma + 2\sigma - 1)^2 = 0\ and\ b_1 = -t.$

$(a_1 - \sigma)^2 + (2\sigma - 1)^2 + 2(a_1 - \sigma)(2\sigma - 1) = 0$

$(a_1 - \sigma)^2 + (2\sigma - 1)(2\sigma - 1 + 2a_1 - 2\sigma) = 0$

$(a_1 - \sigma)^2 + (2\sigma - 1)(2a_1 - 1) = 0$

$(a_1 - \sigma)^2 + (2\sigma - 1)(2a_1 - 1) = 0$

$Since, 1/2 \leq \sigma \leq 1 \Rightarrow 2\sigma - 1 \geq 0 \qquad ... \qquad (11)$

$Since, 1/2 \leq a \leq 1 \Rightarrow 2a - 1 \geq 0 \Rightarrow 2a_1 - 1 \geq 0) \qquad ... \qquad (12)$

$From\ (11)\ and\ (12)\ , (2\sigma - 1)(2a_1 - 1) \geq 0. \qquad ... \qquad (13)$

$Using\ (13)\ in\ (a_1 - \sigma)^2 + (2\sigma - 1)(2a_1 - 1) = 0$

$(a_1 - \sigma)^2 = 0\ and\ (2\sigma - 1)(2a_1 - 1) = 0$

$a_1 = \sigma\ and\ (2\sigma - 1)(2a_1 - 1) = 0$

$Putting\ a_1 = \sigma\ in\ (2\sigma - 1)(2a_1 - 1) = 0$

$(2\sigma - 1)(2\sigma - 1) = 0$

$(2\sigma - 1)^2 = 0$

$\Rightarrow \sigma = 1/2.$

So, in both the cases $\sigma = 1/2$.

$\Rightarrow Re(s) = 1/2.$ *This proves the Riemann Hypothesis.*

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