ON A FUNCTION MODELING AN L-STEP SELF AVOIDING WALK

THEOPHILUS AGAMA

Abstract. We introduce and study the needle function

$$(\Gamma_{\vec{a}_1} \circ \mathbb{V}_m) \circ \cdots \circ (\Gamma_{\vec{a}_{\frac{l}{2}}} \circ \mathbb{V}_m) : \mathbb{R}^n \longrightarrow \mathbb{R}^n.$$

By exploiting the geometry of compression, we prove that this function is a function modeling an l-step self avoiding walk for $l \in \mathbb{N}$. We show that the total length of the l-step self-avoiding walk modeled by this function is of the order

$$\ll \frac{l}{2}\sqrt{n}\bigg(\max\{\sup(x_{j_k})\}_{\substack{1\leq j\leq \frac{l}{2}\\ 1\leq k\leq n}} + \max\{\sup(a_{j_k})\}_{\substack{1\leq j\leq \frac{l}{2}\\ 1\leq k\leq n}}\bigg)$$

and at least

$$\gg \frac{l}{2}\sqrt{n}\bigg(\min\{\mathrm{Inf}(x_{j_k})\}_{\substack{1\leq j\leq \frac{l}{2}\\ 1\leq k\leq n}} + \min\{\mathrm{Inf}(a_{j_k})\}_{\substack{1\leq j\leq \frac{l}{2}\\ 1\leq k\leq n}}\bigg).$$

1. Introduction

Self avoiding walk, roughly speaking, is a sequence of moves on the lattice that does not visit the same point more than once. It is somewhat akin to the graph theoretic notion of a path. It is a mathematical problem to determine a function that models self avoiding walks of any given number of steps. More formally, the problem states

Problem 1.1. Does there exist a function that models l-steps self-avoiding walks?

The problem had long been studied from mathematical perspective but unfortunately our understanding was not good enough. For instance the problem has recently been studied from the standpoint of network theory [2]. The problem also has great significance that extends beyond the shores of mathematics and its allied areas. For instance flurry of studies show that a good understanding of the underlying problem will certainly have its place in physics and chemistry about the long-term structural movement of substances such as polymers and certain proteins in the human anatomy[1],[3]. In this paper we find a function that models an n-step self avoiding walk. We leverage the method of compression and its accompanied estimates to study these things in much more detail. In particular we obtain the following result

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Theorem 1.1. *The map* $(\Gamma_{\vec{a}_1}\circ \mathbb{V}_m)\circ\cdots\circ(\Gamma_{\vec{a}_{\frac{l}{2}}}\circ \mathbb{V}_m):\mathbb{R}^n\longrightarrow \mathbb{R}^n$*, where*

$$(\Gamma_{\vec{a}_1}\circ \mathbb{V}_m)\circ\cdots\circ(\Gamma_{\vec{a}_k}\circ \mathbb{V}_m)$$

is the k*-fold needle function with mixed translation factors* $\vec{a}_1,\ldots,\vec{a}_k\in\mathbb{R}^n$*, is a function modeling* l*-step self avoiding walk.*

We also comment very roughly about the total length of the l-step self avoiding walk modeled by the needle function in the following result

Theorem 1.2. *The total length of the* l*-step self-avoiding walk modeled by the needle function* $(\Gamma_{\vec{a}_1}\circ \mathbb{V}_m)\circ\cdots\circ(\Gamma_{\vec{a}_{\frac{l}{2}}}\circ \mathbb{V}_m):\mathbb{R}^n\longrightarrow \mathbb{R}^n$ *for* $\vec{a_i}\in\mathbb{R}^n$ *with* $i=1,2\ldots,\frac{l}{2}$ *is of order*

$$\ll \frac{l}{2}\sqrt{n}\Bigg(\max\{\sup(x_{j_k})\}_{\substack{1\leq j\leq \frac{l}{2}\\ 1\leq k\leq n}}+\max\{\sup(a_{j_k})\}_{\substack{1\leq j\leq \frac{l}{2}\\ 1\leq k\leq n}}\Bigg)$$

and at least

$$\gg \frac{l}{2}\sqrt{n}\Bigg(\min\{\mathrm{Inf}(x_{j_k})\}_{\substack{1\leq j\leq \frac{l}{2}\\ 1\leq k\leq n}}+\min\{\mathrm{Inf}(a_{j_k})\}_{\substack{1\leq j\leq \frac{l}{2}\\ 1\leq k\leq n}}\Bigg).$$

2. **Preliminary results**

In this section we recall the notion of compression and its various statistics. We find this method very efficient and much more convenient in establishing the main result of this paper.

Definition 2.1. By the compression of scale $m\geq 1$ on $\mathbb{R}^n$ we mean the map $\mathbb{V}:\mathbb{R}^n\longrightarrow\mathbb{R}^n$ such that

$$\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]=\left(\frac{m}{x_1},\frac{m}{x_2},\ldots,\frac{m}{x_n}\right)$$

for $n\geq 2$ and with $x_i\neq 0$ for all $i=1,\ldots,n$.

It is important to notice a compression of scale $m\geq 1$ with $\mathbb{V}_m:\mathbb{R}^n\longrightarrow\mathbb{R}^n$ is a bijective map. In particular the compression $\mathbb{V}_m:\mathbb{R}^n\longrightarrow\mathbb{R}^n$ is a bijective map of order 2. To see why this is the case, let us suppose $\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]=\mathbb{V}_m[(y_1,y_2,\ldots,y_n)]$, then it follows that

$$\left(\frac{m}{x_1},\frac{m}{x_2},\ldots,\frac{m}{x_n}\right)=\left(\frac{m}{y_1},\frac{m}{y_2},\ldots,\frac{m}{y_n}\right).$$

It follows that $x_i=y_i$ for each $i=1,2,\ldots,n$. Surjectivity follows by definition of the map. Thus the map is bijective. The latter claim follows by noting that $\mathbb{V}_m^2[\vec{x}]=\vec{x}$.

Remark 2.2. The notion of compression is in some way the process of re scaling points in $\mathbb{R}^n$ for $n\geq 2$. Thus it is important to notice that a compression pushes points very close to the origin - with each coordinate smaller than a unit - away from the origin by certain scale and similarly draws points away from the origin - with each coordinate bigger than a unit - close to the origin.

2.1. **The mass and the gap of compression.**

Definition 2.3. By the mass of a compression of scale $m \geq 1$ we mean the map $\mathcal{M} : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$\mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) = \sum_{i=1}^{n} \frac{m}{x_i}.$$

with $x_i \neq 0$ for all $1 \leq i \leq n$.

Proposition 2.1. *Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ with $x_i \neq 0$ and $x_i \neq x_j$ for each $1 \leq i, j \leq n$, then the estimates holds*

$$m \log \left(1 - \frac{n-1}{\sup(x_j)}\right)^{-1} \ll \mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) \ll m \log \left(1 + \frac{n-1}{\mathrm{Inf}(x_j)}\right)$$

for $n \geq 2$.

Proof. Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ for $n \geq 2$ with $x_i \neq x_j$ for each $1 \leq i, j \leq n$. Then it follows that

$$\begin{aligned}\mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) &= m \sum_{j=1}^{n} \frac{1}{x_j} \\ &\leq m \sum_{k=0}^{n-1} \frac{1}{\mathrm{Inf}(x_j) + k}\end{aligned}$$

and the upper estimate follows by the estimate for this sum. The lower estimate also follows by noting the lower bound

$$\begin{aligned}\mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) &= m \sum_{j=1}^{n} \frac{1}{x_j} \\ &\geq m \sum_{k=0}^{n-1} \frac{1}{\sup(x_j) - k}.\end{aligned}$$

□

Definition 2.4. Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ with $x_i \neq 0$ for all $i = 1, 2 \ldots, n$. Then by the gap of compression, denoted $\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]$, we mean the expression

$$\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)] = \left|\left|\left(x_1 - \frac{m}{x_1}, x_2 - \frac{m}{x_2}, \ldots, x_n - \frac{m}{x_n}\right)\right|\right|$$

where $||\vec{x}||$ is the euclidean norm of the vector $\vec{x} = (x_1, x_2, \ldots, x_n)$ or the distance of a point $\vec{x} = (x_1, x_2, \ldots, x_n)$ relative to the origin in any euclidean space $\mathbb{R}^n$ for any $n \geq 2$.

Proposition 2.2. *Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ for $n \geq 2$ with $x_j \neq 0$ for $j = 1, \ldots, n$, then we have*

$$\begin{aligned}\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]^2 &= \mathcal{M} \circ \mathbb{V}_1\left[\left(\frac{1}{x_1^2}, \ldots, \frac{1}{x_n^2}\right)\right] + m^2 \mathcal{M} \circ \mathbb{V}_1[(x_1^2, \ldots, x_n^2)] - 2mn \\ &= \sum_{i=1}^{n} x_i^2 + m^2 \sum_{i=1}^{n} \frac{1}{x_i^2} - 2mn.\end{aligned}$$

In particular for all $(x_1,x_2,\ldots,x_n)\in \mathbb{R}^n$ *with* $x_i>1$ *for each* $1\leq i\leq n$*, we have the estimate*

$$\mathcal{G}\circ \mathbb{V}_m[(x_1,x_2,\ldots,x_n)]^2=\mathcal{M}\circ \mathbb{V}_1\left[\left(\frac{1}{x_1^2},\ldots,\frac{1}{x_n^2}\right)\right]-2mn + O\bigg(m^2\mathcal{M}\circ \mathbb{V}_1[(x_1^2,\ldots,x_n^2)]\bigg).$$

Proposition 2.2 offers us an extremely useful identity. It allows us to pass from the gap of compression on points to the relative distance to the origin. It tells us that points under compression with a large gap must be far away from the origin than points with a relatively smaller gap under compression. That is to say, the inequality holds

$$\mathcal{G}\circ \mathbb{V}_m[\vec{x}]\leq \mathcal{G}\circ \mathbb{V}_m[\vec{y}]$$

if and only if $||\vec{x}||\leq ||\vec{y}||$ for $\vec{x},\vec{y}\in \mathbb{R}^n$ with $x_i,y_j>1$ for each $1\leq i,j\leq n$. This important transference principle will be mostly put to use in obtaining our results.

Lemma 2.5 (Compression estimate)**.** *Let* $(x_1,x_2,\ldots,x_n)\in \mathbb{R}^n$ *for* $n\geq 2$ *with* $x_i>1$ *and* $x_i\neq x_j$ *for each* $1\leq i,j\leq n$*, then we have*

$$\mathcal{G}\circ \mathbb{V}_m[(x_1,x_2,\ldots,x_n)]^2\ll n\text{sup}(x_j^2)+m^2\log\left(1+\frac{n-1}{\text{Inf}(x_j)^2}\right)-2mn$$

and

$$\mathcal{G}\circ \mathbb{V}_m[(x_1,x_2,\ldots,x_n)]^2\gg n\text{Inf}(x_j^2)+m^2\log\left(1-\frac{n-1}{\text{sup}(x_j^2)}\right)^{-1}-2mn.$$

Proof. The estimates follows by leveraging the estimates in Proposition 2.1 and noting that

$$n\text{Inf}(x_j^2)\ll \mathcal{M}\circ \mathbb{V}_1\left[\left(\frac{1}{x_1^2},\ldots,\frac{1}{x_n^2}\right)\right]\ll n\text{sup}(x_j^2).$$

□

3. **Compression lines**

In this section we study the notion of lines induced under compression of a given scale and the associated geometry. We first launch the following language.

Definition 3.1. Let $\vec{x}=(x_1,x_2,\ldots,x_n)\in \mathbb{R}^n$ with $x_i\neq 0$ for each $1\leq i\leq n$. Then by the line $L_{\vec{x},\mathbb{V}_m[\vec{x}]}$ produced under compression $\mathbb{V}_m:\mathbb{R}^n\longrightarrow \mathbb{R}^n$ we mean the line joining the points $\vec{x}$ and $\mathbb{V}_m[\vec{x}]$ given by

$$\vec{r}=\vec{x}+\lambda(\vec{x}-\mathbb{V}_m[\vec{x}])$$

where $\lambda\in \mathbb{R}$.

Remark 3.2. In striving for the simplest possible notation and to save enough work space, we will choose instead to write the line produced under compression $\mathbb{V}_m:\mathbb{R}^n\longrightarrow \mathbb{R}^n$ by $L_{\mathbb{V}_m[\vec{x}]}$. Next we show that the lines produced under compression of two distinct points not on the same line of compression cannot intersect at the corresponding points and their images under compression.

Lemma 3.3. *Let $\vec{a} = (a_1, a_2, \dots, a_n), \vec{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ with $\vec{a} \neq \vec{x}$ and $a_i, x_j > 1$ for all $1 \leq i, j \leq n$. If the point $\vec{a}$ lies on the corresponding line $L_{\mathbb{V}_m[\vec{x}]}$, then $\mathbb{V}_m[\vec{a}]$ also lies on the same line.*

Proof. Pick arbitrarily a point $\vec{a} = (a_1, a_2, \dots, a_n)$ with $a_i > 1$ for each $1 \leq i \leq n$ and close to the point $\vec{x} = (x_1, x_2, \dots, x_n)$ on the line $L_{\mathbb{V}_m[\vec{x}]}$ produced under compression induced on $\vec{x} \in \mathbb{R}^n$ with $x_i > 1$ for each $1 \leq i \leq n$. Suppose on the contrary that $\mathbb{V}_m[\vec{a}]$ cannot live on the same line with $\vec{a}$. Then $\mathbb{V}_m[\vec{a}]$ must be away from the line $L_{\mathbb{V}_m[\vec{x}]}$. Produce the compression line $L_{\mathbb{V}_m[\vec{a}]}$ by joining the point $\vec{a}$ to the point $\mathbb{V}_m[\vec{a}]$ by a straight line. Since the point $\vec{a}$ is closer to the origin that the point $\vec{x}$, it follows from Proposition 2.2

$$\mathcal{G} \circ \mathbb{V}_m[\vec{x}] > \mathcal{G} \circ \mathbb{V}_m[\vec{a}].$$

Again pick a point $\vec{c} = (c_1, c_2, \dots, c_n)$ with $c_i > 1$ for all $1 \leq i \leq n$ and close to the point $\vec{a}$ on the line $L_{\mathbb{V}_m[\vec{a}]}$. Then under the assumption it follows that the point $\mathbb{V}_m[\vec{c}]$ must be away from the line. Produce the compression line $L_{\mathbb{V}_m[\vec{c}]}$ by joining the points $\vec{c}$ to $\mathbb{V}_m[\vec{c}]$. Since the point $\vec{c}$ is closer to the origin that the point $\vec{a}$, it follows from Proposition 2.2 the following decreasing sequence of compression gaps - lengths of distinct lines

$$\mathcal{G} \circ \mathbb{V}_m[\vec{x}] > \mathcal{G} \circ \mathbb{V}_m[\vec{a}] > \mathcal{G} \circ \mathbb{V}_m[\vec{c}].$$

By repeating this argument under the underlying contrary assumption, we obtain an infinite descending sequence of compression gaps - lengths of distinct lines

$$\mathcal{G} \circ \mathbb{V}_m[\vec{x}] > \mathcal{G} \circ \mathbb{V}_m[\vec{a_1}] > \cdots > \mathcal{G} \circ \mathbb{V}_m[\vec{a_n}] > \cdots .$$

This proves the Lemma. $\square$

It is important to point out that Lemma 3.3 is the ultimate tool we need to show that certain function is indeed a function modeling l-step self avoiding walk. We first launch such a function as an outgrowth of the notion of compression. Before that we launch our second Lemma. One could think of this result as an extension of Lemma 3.3.

Lemma 3.4. *Let $\vec{a} = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$ and $\vec{b} = (b_1, b_2, \dots, b_n) \in \mathbb{R}^n$ be points with identical configurations with $\vec{a} \neq \vec{b}$ and $a_i, b_j > 0$ for $1 \leq i, j \leq n$. If the corresponding lines $L_{\mathbb{V}_m[\vec{a}]} : r_1 = \vec{a} + \lambda(\vec{a} - \mathbb{V}_m[\vec{a}])$ and $L_{\mathbb{V}_m[\vec{b}]} : r_2 = \vec{b} + \mu(\vec{b} - \mathbb{V}_m[\vec{b}])$ for $\mu, \lambda \in \mathbb{R}$ intersect, then*

$$\vec{a} - \mathbb{V}_m[\vec{a}] \parallel \vec{b} - \mathbb{V}_m[\vec{b}].$$

Proof. First consider the points $\vec{a} = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$ and $\vec{b} = (b_1, b_2, \dots, b_n) \in \mathbb{R}^n$ with $\vec{a} \neq \vec{b}$ and $a_i, b_j > 0$ for $1 \leq i, j \leq n$ with corresponding lines $L_{\mathbb{V}_m[\vec{a}]} : r_1 = \vec{a} + \lambda(\vec{a} - \mathbb{V}_m[\vec{a}])$ and $L_{\mathbb{V}_m[\vec{b}]} : r_2 = \vec{b} + \mu(\vec{b} - \mathbb{V}_m[\vec{b}])$ for $\mu, \lambda \in \mathbb{R}$. Suppose they intersect at the point $\vec{s}$, then it follows that the point $\mathbb{V}_m[\vec{s}]$ lies on the lines $L_{\mathbb{V}_m[\vec{a}]} : r_1 = \vec{a} + \lambda(\vec{a} - \mathbb{V}_m[\vec{a}])$ and $L_{\mathbb{V}_m[\vec{b}]} : r_2 = \vec{b} + \mu(\vec{b} - \mathbb{V}_m[\vec{b}])$ and the result follows immediately. $\square$

Lemma 3.3 combined with Lemma 3.4 tells us that the line produced by compression on points with certain configuration away from other lines of compression are not intersecting. We leverage this principle to show that a certain function indeed models a self-avoiding walk.

Remark 3.5. Next we show that the lines produced under compression and their corresponding lines under translation are non-intersecting.

Proposition 3.1. *Let $L_{\mathbb{V}_m[\vec{x}]}$ and $L_{\mathbb{V}_m[\vec{y}]}$ be two distinct lines under compression. Then the corresponding lines $L_{\Gamma_{\vec{a}}\circ\mathbb{V}_m[\vec{x}]}$ and $L_{\Gamma_{\vec{a}}\circ\mathbb{V}_m[\vec{y}]}$ for a fixed $\vec{a}\in\mathbb{R}^n$ are distinct and non-intersecting.*

Proof. Suppose the lines $L_{\Gamma_{\vec{a}}\circ\mathbb{V}_m[\vec{x}]}$ and $L_{\Gamma_{\vec{a}}\circ\mathbb{V}_m[\vec{y}]}$ for a fixed $\vec{a}\in\mathbb{R}^n$ intersect and let $\vec{s}$ be their point of intersection. Then it follows that there exist some $1\geq k_1,k_2>0$ such that $\Gamma_{k_1\vec{a}}\circ\mathbb{V}_m[\vec{x}]=\vec{s}$ and $\Gamma_{k_2\vec{a}}\circ\mathbb{V}_m[\vec{y}]=\vec{s}$. Then we can write

$$\begin{aligned}\mathbb{V}_m[\vec{x}]&=\Gamma^{-1}_{k_1\vec{a}}\circ\Gamma_{k_2\vec{a}}\circ\mathbb{V}_m[\vec{y}]\\&=\Gamma_{(k_2-k_1)\vec{a}}\circ\mathbb{V}_m[\vec{y}].\end{aligned}$$

It follows that either the point $\mathbb{V}_m[\vec{x}]$ lies on the line $L_{\Gamma_{\vec{a}}\circ\mathbb{V}_m[\vec{y}]}$ or the point $\mathbb{V}_m[\vec{y}]$ lies on the line $L_{\Gamma_{\vec{a}}\circ\mathbb{V}_m[\vec{x}]}$. Without loss of generality, we let the point $\mathbb{V}_m[\vec{x}]$ lie on the line $L_{\Gamma_{\vec{a}}\circ\mathbb{V}_m[\vec{y}]}$. Under the underlying assumption, the following equations hold

$$\Gamma_{k_1\vec{a}}\circ\mathbb{V}_m[\vec{x}]=\mathbb{V}_m[\vec{x}]\quad\text{and}\quad\Gamma_{k_2\vec{a}}\circ\mathbb{V}_m[\vec{y}]=\mathbb{V}_m[\vec{x}].$$

This is absurd since the lines $L_{\mathbb{V}_m[\vec{x}]}$ and $L_{\mathbb{V}_m[\vec{y}]}$ are distinct. □

4. The ball induced by compression

In this section we introduce the notion of the ball induced by a point $\vec{x}=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$ under compression of a given scale. We study the geometry of the ball induced under compression. We launch more formally the following language.

Definition 4.1. Let $(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$ with $x_i\neq 0$ for all $1\leq i\leq n$. Then by the ball induced by $(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$ under compression of scale m, denoted $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]}[(x_1,x_2,\ldots,x_n)]$ we mean the inequality

$$\left|\left|\vec{y}-\frac{1}{2}\left(x_1+\frac{m}{x_1},x_2+\frac{m}{x_2},\ldots,x_n+\frac{m}{x_n}\right)\right|\right|\leq\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)].$$

A point $\vec{z}=(z_1,z_2,\ldots,z_n)\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]}[(x_1,x_2,\ldots,x_n)]$ if it satisfies the inequality.

Remark 4.2. Next we prove that smaller balls induced by points should essentially be covered by the bigger balls in which they are embedded. We state and prove this statement in the following result.

For simplicity we will on occasion choose to write the ball induced by the point $\vec{x}=(x_1,x_2,\ldots,x_n)$ under compression as

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}].$$

We adopt this notation to save enough work space in many circumstances.

Theorem 4.3. *Let $\vec{y}=(y_1,y_2,\ldots,y_n),\vec{z}=(z_1,z_2,\ldots,z_n)\in\mathbb{R}^n$ with $z_i>1$ and $y_i>1$ for all $1\leq i\leq n$. Then $\vec{z}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ if and only if*

$$\mathcal{G}\circ\mathbb{V}_m[\vec{z}]\leq\mathcal{G}\circ\mathbb{V}_m[\vec{y}].$$

Proof. Let $\vec{z} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ for $\vec{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{R}^n$ with $z_i > 1$ for all $1 \leq i \leq n$, then it follows that $||\vec{y}|| > ||\vec{z}||$. Suppose on the contrary that

$$\mathcal{G} \circ \mathbb{V}_m[\vec{z}] > \mathcal{G} \circ \mathbb{V}_m[\vec{y}],$$

then it follows from Proposition 2.2 that $||\vec{y}|| < ||\vec{z}||$, which is a contradiction. Conversely, suppose

$$\mathcal{G} \circ \mathbb{V}_m[\vec{z}] \leq \mathcal{G} \circ \mathbb{V}_m[\vec{y}]$$

then it follows from Proposition 2.2 that $||\vec{z}|| \leq ||\vec{y}||$. It follows that

$$\begin{aligned}\left|\left|\vec{z} - \frac{1}{2}\left(y_1 + \frac{m}{y_1}, \ldots, y_n + \frac{m}{y_n}\right)\right|\right| &\leq \left|\left|\vec{y} - \frac{1}{2}\left(y_1 + \frac{m}{y_1}, \ldots, y_n + \frac{m}{y_n}\right)\right|\right| \\ &= \frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[\vec{y}].\end{aligned}$$

This certainly implies $\vec{z} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ and the proof of the theorem is complete. □

4.1. **Remark.** As an alternative to the requirement that $\vec{x} \in \mathbb{R}^n$ with $x_i \neq 0$ for all $1 \leq i \leq n$, we will work under the more convenient regime that each coordinate of the vector $x_i > 1$ for each $1 \leq i \leq n$. This requirement has no restriction on the geometry since for a vector $\vec{x} = (x_1, x_2, \ldots, x_n)$ with $x_i = 1$ for each $1 \leq i \leq n$ the coordinates of the corresponding compression vector is still $\mathbb{V}_1[\vec{x}] = (1, 1, \ldots, 1)$. It is also easy to see that for those points $\vec{x} = (x_1, x_2, \ldots, x_n)$ with $0 < x_i < 1$ the corresponding compression vector will be a point $\vec{y} = (y_1, y_2, \ldots, y_n)$ with $y_i > 1$ for each $1 \leq i \leq n$ and vice-versa. Thus in the sequel we will only use points $\vec{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ with $x_i > 1$ for all $1 \leq i \leq n$ for our constructions, since the remaining points with unlike properties will automatically be obtained as compression images. It has to be said that this requirement will turn to be natural in our studies in the sequel.

Theorem 4.4. *Let $\vec{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ with $x_i > 1$ for all $1 \leq i \leq n$. If $\vec{y} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ then*

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}] \subseteq \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}].$$

Proof. First let $\vec{y} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ and suppose for the sake of contradiction that

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}] \not\subseteq \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}].$$

Then there must exist some $\vec{z} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ such that $\vec{z} \notin \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$. It follows from Theorem 4.3 that

$$\mathcal{G} \circ \mathbb{V}_m[\vec{z}] > \mathcal{G} \circ \mathbb{V}_m[\vec{x}].$$

It follows that

$$\begin{aligned}\mathcal{G} \circ \mathbb{V}_m[\vec{y}] &\geq \mathcal{G} \circ \mathbb{V}_m[\vec{z}] \\ &> \mathcal{G} \circ \mathbb{V}_m[\vec{x}]\end{aligned}$$

which is absurd, thereby ending the proof. □

Remark 4.5. Theorem 4.4 tells us that points confined in certain balls induced under compression should by necessity have their induced ball under compression covered by these balls in which they are contained.

4.2. **Interior points and the limit points of balls induced under compression.** In this section we launch the notion of an interior and the limit point of balls induced under compression. We study this notion in depth and explore some connections.

Definition 4.6. Let $\vec{y}=(y_1,y_2,\ldots,y_n)\in\mathbb{R}^n$ with $y_i\neq 0$ for all $1\leq i\leq n$. Then a point $\vec{z}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ is an interior point of the ball in which it is contained if

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{z}]}[\vec{z}]\subseteq\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

for most $\vec{x}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$. An interior point $\vec{z}$ of the ball is then said to be a limit point of the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ if and only if

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{z}]}[\vec{z}]\subseteq\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

for all $\vec{x}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$

Remark 4.7. Next we prove that there must exist an interior and limit point in any ball induced by points under compression of any scale in any dimension.

Theorem 4.8. *Let $\vec{x}=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$ with $x_i>1$ for all $1\leq i\leq n$. Then the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ contains an interior point and a limit point.*

Proof. Let $\vec{x}=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$ with $x_i>1$ for all $1\leq i\leq n$ and suppose on the contrary that $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ contains no limit point. Then pick

$$\vec{z}_1\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

for $\vec{z}_1\neq\vec{x}$. Then by Theorem 4.4 and Theorem 4.3, it follows that

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{z}_1]}[\vec{z}_1]\subset\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

with $\mathcal{G}\circ\mathbb{V}_m[\vec{z}_1]<\mathcal{G}\circ\mathbb{V}_m[\vec{x}]$. Again pick $\vec{z}_2\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{z}_1]}[\vec{z}_1]$ for $\vec{z}_2\neq\vec{z}_1$. Then by employing Theorem 4.4 and Theorem 4.3, we have

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{z}_2]}[\vec{z}_2]\subset\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{z}_1]}[\vec{z}_1]$$

with $\mathcal{G}\circ\mathbb{V}_m[\vec{z}_2]<\mathcal{G}\circ\mathbb{V}_m[\vec{z}_1]$. By continuing the argument in this manner, we obtain the infinite descending sequence of the gap of compression

$$\mathcal{G}\circ\mathbb{V}_m[\vec{x}]>\mathcal{G}\circ\mathbb{V}_m[\vec{z}_1]>\mathcal{G}\circ\mathbb{V}_m[\vec{z}_2]>\cdots>\mathcal{G}\circ\mathbb{V}_m[\vec{z}_n]>\cdots$$

thereby ending the proof of the theorem. ◻

Now we state and prove a result that in some way makes our earlier imposition in Remark 4.1 a natural one and gives more meaning to our work in further sequel.

Proposition 4.1. *The point $\vec{x}=(x_1,x_2,\ldots,x_n)$ with $x_i=1$ for each $1\leq i\leq n$ is the limit point of the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{y}]}[\vec{y}]$ for any $\vec{y}=(y_1,y_2,\ldots,y_n)\in\mathbb{R}^n$ with $y_i>1$ for each $1\leq i\leq n$.*

Proof. Applying the compression $\mathbb{V}_1:\mathbb{R}^n\longrightarrow\mathbb{R}^n$ on the point $\vec{x}=(x_1,x_2,\ldots,x_n)$ with $x_i=1$ for each $1\leq i\leq n$, we obtain $\mathbb{V}_1[\vec{x}]=(1,1,\ldots,1)$ so that $\mathcal{G}\circ\mathbb{V}_1[\vec{x}]=0$ and the corresponding ball induced under compression $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{x}]}[\vec{x}]$ contains only the point $\vec{x}$. It follows by Definition 4.8 the point $\vec{x}$ must be the limit point of the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{x}]}[\vec{x}]$. It follows that

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{x}]}[\vec{x}]\subseteq\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{y}]}[\vec{y}]$$

for any $\vec{y}=(y_1,y_2,\ldots,y_n)\in\mathbb{R}^n$ with $y_i>1$ for all $1\leq i\leq n$. For if the contrary

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{x}]}[\vec{x}]\not\subseteq\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{y}]}[\vec{y}]$$

holds for some $\vec{y}=(y_1,y_2,\ldots,y_n)\in\mathbb{R}^n$ with $y_i>1$ for each $1\leq i\leq n$, then there must exists some point $\vec{z}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{x}]}[\vec{x}]$ such that $\vec{z}\notin\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{y}]}[\vec{y}]$. Since $\vec{x}$ is the only point in the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{x}]}[\vec{x}]$, it follows that

$$\vec{x}\notin\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{y}]}[\vec{y}].$$

Appealing to Theorem 4.3, we have the corresponding inequality of compression gaps

$$\mathcal{G}\circ\mathbb{V}_1[\vec{x}]>\mathcal{G}\circ\mathbb{V}_1[\vec{y}]$$

so that by appealing to Proposition 2.2 and the ensuing remarks, we have the inequality of their corresponding distance relative to the origin

$$||\vec{x}||>||\vec{y}||.$$

This is a contradiction, since by our earlier assumption $\vec{y}=(y_1,y_2,\ldots,y_n)\in\mathbb{R}^n$ with $y_i>1$ for each $1\leq i\leq n$. Thus the point $\vec{x}=(x_1,x_2,\ldots,x_n)$ with $x_i=1$ for each $1\leq i\leq n$ must be the limit point of any ball of the form

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{y}]}[\vec{y}]$$

for any $\vec{y}=(y_1,y_2,\ldots,y_n)\in\mathbb{R}^n$ with $y_i>1$ for each $1\leq i\leq n$. □

4.3. **Admissible points of balls induced under compression.** We launch the notion of admissible points of balls induced by points under compression. We study this notion in depth and explore some possible connections.

Definition 4.9. Let $\vec{y}=(y_1,y_2,\ldots,y_n)\in\mathbb{R}^n$ with $y_i\neq 0$ for all $1\leq i\leq n$. Then $\vec{y}$ is said to be an admissible point of the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ if

$$\left|\left|\vec{y}-\frac{1}{2}\left(x_1+\frac{m}{x_1},\ldots,x_n+\frac{m}{x_n}\right)\right|\right|=\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}].$$

Remark 4.10. It is important to notice that the notion of admissible points of balls induced by points under compression encompasses points on the ball. These points in geometrical terms basically sit on the outer of the induced ball. Next we show that all balls can in principle be generated by their admissible points.

Theorem 4.11. *The point $\vec{y}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ is admissible if and only if*

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]=\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

and $\mathcal{G}\circ\mathbb{V}_m[\vec{y}]=\mathcal{G}\circ\mathbb{V}_m[\vec{x}]$.

Proof. First let $\vec{y}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ be admissible and suppose on the contrary that

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]\neq\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}].$$

Then there exist some $\vec{z}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ such that

$$\vec{z}\notin\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}].$$

Applying Theorem 4.3, we obtain the inequality

$$\mathcal{G}\circ\mathbb{V}_m[\vec{y}]<\mathcal{G}\circ\mathbb{V}_m[\vec{z}]\leq\mathcal{G}\circ\mathbb{V}_m[\vec{x}].$$

By leveraging Proposition 2.2, it follows that $||\vec{x}|| < ||\vec{y}||$ or $||\vec{y}|| < ||\vec{x}||$. By joining this points to the origin by a straight line, this contradicts the fact that the point $\vec{y} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ is an admissible point. Now we notice that $\vec{y} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ certainly implies $\mathcal{G} \circ \mathbb{V}_m[\vec{y}] \leq \mathcal{G} \circ \mathbb{V}_m[\vec{x}]$. Conversely we notice as well that $\vec{x} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$, which certainly implies $\mathcal{G} \circ \mathbb{V}_m[\vec{x}] \leq \mathcal{G} \circ \mathbb{V}_m[\vec{y}]$ by Theorem 4.3. Thus the conclusion follows. Conversely, suppose

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}] = \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

and $\mathcal{G} \circ \mathbb{V}_m[\vec{y}] = \mathcal{G} \circ \mathbb{V}_m[\vec{x}]$. Then it follows that the point $\vec{y}$ must satisfy the inequality

$$\begin{aligned}\left|\left|\vec{z} - \frac{1}{2}\left(y_1 + \frac{m}{y_1}, \ldots, y_n + \frac{m}{y_n}\right)\right|\right| &= \left|\left|\vec{z} - \frac{1}{2}\left(x_1 + \frac{m}{x_1}, \ldots, x_n + \frac{m}{x_n}\right)\right|\right| \\ &\leq \frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[\vec{x}].\end{aligned}$$

It follows that

$$\begin{aligned}\frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[\vec{x}] &= \left|\left|\vec{y} - \frac{1}{2}\left(x_1 + \frac{m}{x_1}, \ldots, x_n + \frac{m}{x_n}\right)\right|\right| \\ &\leq \frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[\vec{x}]\end{aligned}$$

and $\vec{y}$ is indeed admissible, thereby ending the proof. □

Proposition 4.2. *Compression balls are non-overlapping.*

Proof. Pick arbitrarily points $\vec{x}, \vec{y} \in \mathbb{R}^n$ with $x_i, y_i \neq 0$ for all $i = 1, \ldots, n$ such that $\vec{x} \neq \vec{y}$ with $||\vec{x}|| \neq ||\vec{y}||$. Then it follows that $||\vec{x}|| < ||\vec{y}||$ or $||\vec{x}|| > ||\vec{y}||$. Without loss of generality, let us assume that $||\vec{x}|| < ||\vec{y}||$, then it follows from Proposition 2.2

$$\mathcal{G} \circ \mathbb{V}_m[\vec{x}] < \mathcal{G} \circ \mathbb{V}_m[\vec{y}]$$

or

$$\mathcal{G} \circ \mathbb{V}_m[\vec{y}] < \mathcal{G} \circ \mathbb{V}_m[\vec{x}].$$

By appealing to Theorem 4.3 and Theorem 4.4, it follows that

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}] \subset \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$$

or

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}] \subset \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}].$$

This completes the proof since the points $\vec{x}$ and $\vec{y}$ with $||\vec{x}|| \neq ||\vec{y}||$ were chosen arbitrarily. □

Next we show that there must exists some point in a bigger ball whose induced ball under compression has admissible points way off a certain line in the underlying ball. We find the following Lemma useful.

Lemma 4.12. *The point $\frac{\vec{y}_i + \mathbb{V}_m[\vec{y}_i]}{2}$ with $y_i \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ for all $i \in \mathbb{N}$ is on the line $L_{\mathbb{V}_m}[\vec{x}]$ if and only if the limits point $\vec{z}$ of the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ is on the line $L_{\mathbb{V}_m[\vec{x}]}$.*

Proof. Let $\frac{\vec{y}_i+\mathbb{V}_m[\vec{y}_i]}{2}$ for $i \in \mathbb{N}$ with $y_i \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ be on the line $L_{\mathbb{V}_m[\vec{x}]}$ with $\vec{x} \neq \vec{y}_i$ for all $i \in \mathbb{N}$. Then by Lemma 3.3, It follows that $\mathbb{V}_m[\vec{y}_i] \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ with $\mathcal{G}\circ\mathbb{V}_m[\vec{y}_i] < \mathcal{G}\circ\mathbb{V}_m[\vec{x}]$. Let us now construct the ball induced by compression on this point given by $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}_i]}[\vec{y}_i]$ and by Proposition 4.4

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}_i]}[\vec{y}_i] \subset \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

and $\frac{\vec{y}_i+\mathbb{V}_m[\vec{y}_i]}{2} = \frac{1}{2}\bigg(y_{i_1} + \frac{m}{y_{i_1}}, \ldots, y_{i_n} + \frac{m}{y_{i_n}}\bigg)$ is on the line $L_{\mathbb{V}_m[\vec{x}]}$. By repeating the argument by choosing a point in this manner in the much smaller ball the first part of the result follows. Conversely, suppose the limit point lies on the line $L_{\mathbb{V}_m[\vec{x}]}$ and there exist a ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}_j]}[\vec{y}_j] \subset \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ such that the point $\frac{\vec{y}_j+\mathbb{V}_m[\vec{y}_j]}{2}$ does not live on the line $L_{\mathbb{V}_m[\vec{x}]}$. It follows that the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}_j]}[\vec{y}_j]$ and the line $L_{\mathbb{V}_m[\vec{x}]}$ are overlapping, since the limit point is on the line $L_{\mathbb{V}_m[\vec{x}]}$. By Lemma 3.3 the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}_j]}[\vec{y}_j]$ overlaps some ball induced by some point under compression on the line $L_{\mathbb{V}_m[\vec{x}]}$. This is absurd since compression balls are non-overlapping. ◻

Theorem 4.13. *There exist some $\vec{z} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ such that admissible points of the induced ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{z}]}[\vec{z}]$ are not on the line $L_{\mathbb{V}_m[\vec{x}]}$.*

Proof. Consider the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ and suppose on the contrary that for any point $\vec{z} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ the corresponding induced ball under compression $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{z}]}[\vec{z}]$ intersects the compression line $L_{\mathbb{V}_m[\vec{x}]}$. Then by Lemma 4.12 the limit point of the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ is on the line $L_{\mathbb{V}_m[\vec{x}]}$. It follows from Lemma 4.12 the point $\frac{\vec{z}+\mathbb{V}_m[\vec{z}]}{2}$ must lie on the line $L_{\mathbb{V}_m[\vec{x}]}$. This is a contradiction since the point $\vec{z}$ is an arbitrary point in the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ and so is the point $\frac{\vec{z}+\mathbb{V}_m[\vec{z}]}{2}$. This completes the proof of the theorem. ◻

5. **The needle function**

In this section we introduce and study the needle function. We combine the geometry of lines under compression and the geometry of balls under compression to prove that this function is a function modeling an l- step self avoiding walk.

Definition 5.1. By the needle function of scale m and translation factor $\vec{a}$, we mean the composite map

$$\Gamma_{\vec{a}} \circ \mathbb{V}_m : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

such that for any $\vec{x} \in \mathbb{R}^n$

$$\Gamma_{\vec{a}} \circ \mathbb{V}_m[\vec{x}] = \vec{y}$$

where $\vec{x} = (x_1, x_2, \ldots, x_n)$ with $x_i \neq 0$ for $1 \leq i \leq n$ and $\Gamma_{\vec{a}}[\vec{x}] = (x_1 + a_1, x_2 + a_2, \ldots, x_n + a_n)$.

Proposition 5.1. *The needle function $\Gamma_{\vec{a}} \circ \mathbb{V}_m : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a bijective map of order 2.*

Proof. We remark that the translation with translation factor $\vec{a}$ for a fixed $\vec{a}$ given by $\Gamma_{\vec{a}} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a bijective map. The result follows since the composite of bijective maps is still bijective. ◻

Theorem 5.2. *The map* $(\Gamma_{\vec{a}_1}\circ \mathbb{V}_m)\circ \cdots \circ (\Gamma_{\vec{a}_{\frac{l}{2}}}\circ \mathbb{V}_m):\mathbb{R}^n\longrightarrow \mathbb{R}^n$*, where*

$$(\Gamma_{\vec{a}_1}\circ \mathbb{V}_m)\circ \cdots \circ (\Gamma_{\vec{a}_k}\circ \mathbb{V}_m)$$

is the k-fold needle function with mixed translation factors $\vec{a}_1,\ldots,\vec{a}_k\in \mathbb{R}^n$, is a function modeling an l-step self avoiding walk.

Proof. Pick arbitrarily a point $\vec{x}=(x_1,x_2,\ldots,x_n)\in \mathbb{R}^n$ for $n\geq 2$ with $x_i>1$ for each $1\leq i\leq n$ and apply the compression $\mathbb{V}_m[\vec{x}]$ and construct the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ \mathbb{V}_m[\vec{x}]}[\vec{x}]$. Now choose a point $\vec{u}\in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ \mathbb{V}_m[\vec{x}]}[\vec{x}]$ and construct the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ \mathbb{V}_m[\vec{u}]}[\vec{u}]$ so that admissible points do not sit on the compression line $L_{\mathbb{V}_m[\vec{x}]}$. Let us now join the point $\mathbb{V}_m[\vec{x}]$ to the closest admissible point $\vec{t}$ of $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ \mathbb{V}_m[\vec{u}]}[\vec{u}]$ on the line $L_{\mathbb{V}_m[u]}$ under a suitable translation vector $\vec{a}_1\neq \vec{O}$. Let us now traverse the line produced under compression to the line produced by translation of the point $\Gamma_{\vec{a}_1}(\mathbb{V}_m[\vec{x}])$ with the starting point $\vec{x}$ to $\mathbb{V}_m[\vec{x}]$ and from $\mathbb{V}_m[\vec{x}]$ to $\Gamma_{\vec{a}_1}(\mathbb{V}_m[\vec{x}])=\vec{t}$ and finally from $\vec{t}$ to $\mathbb{V}_m[\vec{t}]$. The upshot is a 3-step self avoiding walk. Again we choose a point $\vec{s}\in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ \mathbb{V}_m[\vec{u}]}[\vec{u}]$ so that the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ \mathbb{V}_m[\vec{s}]}[\vec{s}]$ satisfies the relation

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ \mathbb{V}_m[\vec{s}]}[\vec{s}]\subset \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ \mathbb{V}_m[\vec{u}]}[\vec{u}]$$

and with the property that admissible points of the inner ball are not allowed to sit on the compression line $L_{\mathbb{V}_m[\vec{u}]}$. We then join the point $\mathbb{V}_m[\vec{t}]$ to the closest admissible point of the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ \mathbb{V}_m[\vec{s}]}[\vec{s}]$ on the line $L_{\mathbb{V}_m[\vec{s}]}$ by the translation $\Gamma_{\vec{a}_2}$ under a suitable translation factor $\vec{a}_2\neq \vec{O}$. By traversing all these lines starting from the point $\vec{x}$ to $\mathbb{V}_m[\vec{x}]$, $\vec{z}=\mathbb{V}_m[\vec{x}]$ to $\Gamma_{\vec{a}_1}[\vec{z}]$, $\Gamma_{\vec{a}_1}[\vec{z}]$ to $\mathbb{V}_m\circ \Gamma_{\vec{a}_1}[\vec{z}]$ and finally from $\mathbb{V}_m\circ \Gamma_{\vec{a}_1}[\vec{z}]$ to $\Gamma_{\vec{a}_2}\circ \mathbb{V}_m\circ \Gamma_{\vec{a}_2}[\vec{z}]$, we obtain a 4-step self avoiding walk. By continuing this argument $\frac{l}{2}$ number of times, we produce an l-step self avoiding walk. This completes the proof. ☐

We remark that we can certainly do more than this by estimating the total length of the self-avoiding walk modeled by this function in the following result.

Theorem 5.3. *The total length of the l-step self-avoiding walk modeled by the needle function* $(\Gamma_{\vec{a}_1}\circ \mathbb{V}_m)\circ \cdots \circ (\Gamma_{\vec{a}_{\frac{l}{2}}}\circ \mathbb{V}_m):\mathbb{R}^n\longrightarrow \mathbb{R}^n$ *for* $\vec{a_i}\in \mathbb{R}^n$ *with* $i=1,2\ldots,\frac{l}{2}$ *is of order*

$$\ll \frac{l}{2}\sqrt{n}\bigg(\max\{\sup(x_{j_k})\}_{\substack{1\leq j\leq \frac{l}{2}\\ 1\leq k\leq n}}+\max\{\sup(a_{j_k})\}_{\substack{1\leq j\leq \frac{l}{2}\\ 1\leq k\leq n}}\bigg)$$

and at least

$$\gg \frac{l}{2}\sqrt{n}\bigg(\min\{\mathrm{Inf}(x_{j_k})\}_{\substack{1\leq j\leq \frac{l}{2}\\ 1\leq k\leq n}}+\min\{\mathrm{Inf}(a_{j_k})\}_{\substack{1\leq j\leq \frac{l}{2}\\ 1\leq k\leq n}}\bigg).$$

Proof. We note that the total length of the l-step self avoiding walk modeled by the needle function is given by the expression

$$\sum_{i=1}^{\frac{l}{2}}\mathcal{G}\circ \mathbb{V}_m[\vec{x_i}]+\sum_{i=1}^{\frac{l}{2}}||\vec{a}_i||$$

and the result follows by applying the estimates in Lemma 2.5. ☐

6. **A combinatorial interpretation**

In this section we provide a combinatorial twist of the main result in this paper. We reformulate Theorem 5.2 in the language of graphs. We launch the following language:

Definition 6.1 (Compression graphs)**.** By a compression graph G of order $k>1$ induced by $\vec{x}_1=(u_1,u_2,\ldots,u_n)\in\mathbb{R}^n$ with $u_i\neq 0$ for all $1\leq i\leq n$, we mean the pair (V,E) where V is the vertex set

$$V:=\{\vec{x}_1,\mathbb{V}_m[\vec{x}_1],\Gamma_{\vec{a}_1}(\mathbb{V}_m[\vec{x}_1])=\vec{x}_2,\mathbb{V}_m[\vec{x}_2],\ldots,\Gamma_{\vec{a}_s}(\mathbb{V}_m[\vec{x}_{k-1}])=\vec{x}_k,\mathbb{V}_m[\vec{x}_k]\}$$

and E the set of edges

$$E:=\left\{\mathbb{L}_{\vec{x}_1,\mathbb{V}_m[\vec{x}_1]},\mathbb{L}_{\mathbb{V}_m[\vec{x}_1],\vec{x}_2},\ldots,\mathbb{L}_{\vec{x}_k,\mathbb{V}_m[\vec{x}_k]}\right\}.$$

We now state a graph-theoretic version of Theorem 5.2.

Theorem 6.2 (A combinatorial version)**.** *There exists a compression graph of order $l+1$ with $l\in\mathbb{N}$ and whose edges are paths.*

Proof. Pick $\vec{x}_1=(u_1,u_2,\ldots,u_n)\in\mathbb{R}^n$ very far away from the origin with the property that $u_i>1$ and for all $1\leq i\leq n$. Next we apply the compression $\mathbb{V}_1$ on $\vec{x}_1$ and obtain a point $\mathbb{V}_1[\vec{x}]\in\mathbb{R}^n$. Let us join the point $\vec{x}_1$ to the point $\mathbb{V}_1[\vec{x}_1]$ by a straight line so that by traversing this line starting from $\vec{x}_1$ to $\mathbb{V}_1[\vec{x}_1]$, we have the **path** $\mathbb{L}_{\vec{x}_1,\mathbb{V}_1[\vec{x}_1]}$. Next we pick a point $\vec{x}_2=(v_1,v_2,\ldots,v_n)\in\mathbb{R}^n\bigcap\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{x}_1]}[\vec{x}_1]$ with $v_i>1$ for all $1\leq i\leq n$ such that no admissible point of the induced ball of the compression on $\vec{x}_2$ namely

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{x}_2]}[\vec{x}_2]$$

sit on the compression line $\mathbb{L}_{\vec{x}_1,\mathbb{V}_1[\vec{x}_1]}$. Next apply the compression $\mathbb{V}_1$ on $\vec{x}_2$ and we obtain the point $\mathbb{V}_1[\vec{x}_2]$. Next let us join the point $\vec{x}_2$ to the point $\mathbb{V}_1[\vec{x}_2]$ by a straight line. Let us now apply the translation $\Gamma_{\vec{a}_1}$ under some suitable translation vector to join the point $\mathbb{V}_1[\vec{x}_1]$ to exactly one the points $\vec{x}_2,\mathbb{V}_1[\vec{x}_2]$ whose distance is minimal. Without loss of generality let us assume that the point $\vec{x}_2$ is closer to $\mathbb{V}_1[\vec{x}_1]$ than $\mathbb{V}_1[\vec{x}_2]$ to $\mathbb{V}_1[\vec{x}_1]$ and apply the translation

$$\Gamma_{\vec{a}_1}(\mathbb{V}_1[\vec{x}_1])=\vec{x}_2.$$

Next we join the point $\mathbb{V}_1[\vec{x}_1]$ to the point $\vec{x}_2$ by a straight line $\mathbb{L}_{\mathbb{V}_1[\vec{x}_1],\vec{x}_2}$ and the point $\vec{x}_2$ to the point $\mathbb{V}_1[\vec{x}_2]$ by the straight line $\mathbb{L}_{\vec{x}_2,\mathbb{V}_1[\vec{x}_2]}$. By traversing the line $\mathbb{L}_{\vec{x}_1,\mathbb{V}_1[\vec{x}_1]}$ starting from $\vec{x}_1$ to $\mathbb{V}_1[\vec{x}_1]$ and the line $\mathbb{L}_{\mathbb{V}_1[\vec{x}_1],\vec{x}_2}$ continuing from $\mathbb{V}_1[\vec{x}_1]$ to $\vec{x}_2$ and the line $\mathbb{L}_{\vec{x}_2,\mathbb{V}_1[\vec{x}_2]}$ continuing from $\vec{x}_2$ to $\mathbb{V}_1[\vec{x}_2]$, we obtain a **path** induced by **four** vertices. By repeating this argument and in the sense of proof of Theorem 5.2, we obtain a compression graph $G=(V,E)$ with vertex and edge set

$$V:=\{\vec{x}_1,\mathbb{V}_m[\vec{x}_1],\Gamma_{\vec{a}_1}(\mathbb{V}_m[\vec{x}_1])=\vec{x}_2,\mathbb{V}_m[\vec{x}_2],\ldots,\Gamma_{\vec{a}_s}(\mathbb{V}_m[\vec{x}_l])=\vec{x}_{l+1},\mathbb{V}_m[\vec{x}_{l+1}]\}$$

and E the set of edges

$$E:=\left\{\mathbb{L}_{\vec{x}_1,\mathbb{V}_m[\vec{x}_1]},\mathbb{L}_{\mathbb{V}_m[\vec{x}_1],\vec{x}_2},\ldots,\mathbb{L}_{\vec{x}_{l+1},\mathbb{V}_m[\vec{x}_{l+1}]}\right\}$$

where the points in set V are the vertices and each line in E are the edges of the graph G, with the edges being a path. □

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Department of Mathematics, African Institute for Mathematical science, Ghana
E-mail address: theophilus@aims.edu.gh/emperordagama@yahoo.com

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