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A reciprocal identity for sums of inverse tangents

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An identity is proved connecting two finite sums of inverse tangents. This identity is discretized version of Jacobi's imaginary transformation for the modular angle.

I. Introduction

Sums of inverse tangents have attracted a lot of attention. For example, the following sums of inverse tangents can be calculated in closed form:

$$\sum_{n\ge0}\arctan\frac{2}{(2n+1)^2}=\frac{\pi}{2}, \tag{1}$$

$$\sum_{n\ge1}(-1)^{n+1}\arctan\frac{1}{F_{2n}}=\arctan\frac{\sqrt{5}-1}{2}, \tag{2}$$

$$\sum_{n\ge1}\arctan\frac{\sinh x}{\cosh nx}=\frac{3\pi}{4}-\arctan e^x. \tag{3}$$

(1) is a classic sum evaluated first by Glaisher in [1]. The sum (2), where F_n is n-th Fibonacci number, was calculated by Hoggatt and Ruegels [3]. The sum (3) was noted in [5]. See [2] for further references and a brief summary of research in this direction.

All summations of the type (1) and (2) seem to be based on two methods: the telescopic principle, and the method of zeroes, as was noted in [4].

Even earlier, in his studies of elliptic functions, Jacobi proved identity of completely different kind of which he wrote in his treatise on elliptic functions "one is obliged to rank among the most elegant formulas" [6]:

$$\frac{1}{4}\arcsin k=\arctan q^{1/2}-\arctan q^{3/2}+\arctan q^{5/2}-\dots \tag{4}$$

Here $q=e^{-\pi K'/K}$, K is the complete elliptic integral of the first kind with modulus k

$$K=K(k)=\int_0^{\pi/2}\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}},$$

$K'=K(k')$ with $k'=\sqrt{1-k^2}$ being the complementary modulus. Together with the obvious relation

$$\arcsin k+\arcsin k'=\frac{\pi}{2},$$

this implies

$$\sum_{n\ge1}\chi(n)\arctan q^{n/2}+\sum_{n\ge1}\chi(n)\arctan {q'}^{n/2}=\frac{\pi}{8}, \tag{5}$$

where $q'=e^{-\pi K/K'}$, and $\chi(n)=\sin\frac{\pi n}{2}$ is Dirichlet character modulo 4.

The focus of this paper will be the following reciprocal identity for finite sums of inverse tangents

Theorem 1. *Let $n,m\in\mathbb{N}_0$. Then*

$$\sum_{|j|\le n}(-1)^{n+j}\arctan\left(\sqrt{1+\cos^2\tfrac{\pi j}{2n+1}}-\cos\tfrac{\pi j}{2n+1}\right)^{2m+1}$$
$$+\sum_{|k|\le m}(-1)^{m+k}\arctan\left(\sqrt{1+\cos^2\tfrac{\pi k}{2m+1}}-\cos\tfrac{\pi k}{2m+1}\right)^{2n+1}=\frac{\pi}{4}. \tag{6}$$

Note that when $n = m$ both sums in (6) are equal and we get a closed form summation:

Corollary 1.

$$\sum_{|j|\le n} (-1)^{n+j} \arctan\left(\sqrt{1+\cos^2\tfrac{\pi j}{2n+1}} - \cos\tfrac{\pi j}{2n+1}\right)^{2n+1} = \frac{\pi}{8}, \qquad n \in \mathbb{N}_0. \tag{7}$$

It is instructive to write (6) in another form by shifting the summation variable and simple rearrangement of terms

$$\sum_{j=1}^{2n} \chi_4(j) \arctan\left(\sqrt{1+\sin^2\tfrac{\pi j}{4n+2}} - \sin\tfrac{\pi j}{4n+2}\right)^{2m+1} + \sum_{k=1}^{2m} \chi_4(k) \arctan\left(\sqrt{1+\sin^2\tfrac{\pi j}{4m+2}} - \sin\tfrac{\pi j}{4m+2}\right)^{2n+1}$$
$$= \frac{\pi}{8} - \frac{1}{2}(-1)^n \arctan\left(\sqrt{2}-1\right)^{2m+1} - \frac{1}{2}(-1)^m \arctan\left(\sqrt{2}-1\right)^{2n+1}.$$

From this form of (6), it is evident that when $n, m \to \infty$ such that $m/n \to K'/K$, one recovers (5). Thus, (6) is discretized version of (5). Our proof is completely elementary and provides an elementary proof of the modular relation (5).

II. Proof of Theorem 1

We break the proof into a series of lemmas.

Lemma 1. *The following identity holds for $n, m \in \mathbb{N}_0$ and $j \in \mathbb{Z}$*

$$2\arctan\left(\sqrt{1+\cos^2\tfrac{\pi j}{2n+1}} - \cos\tfrac{\pi j}{2n+1}\right)^{2m+1} = \frac{\pi}{2} - \arctan\left(\sinh(2m+1)\alpha_j\right)$$

where α_j is the positive solution of $\sinh\alpha_j = \cos\frac{\pi j}{2n+1}$.

Proof. Denote $s = 2m+1$:

$$\begin{aligned}
2\arctan\left(\sqrt{1+\cos^2\tfrac{\pi j}{2n+1}} - \cos\tfrac{\pi j}{2n+1}\right)^{s} &= 2\arctan\left(\cosh\alpha_j - \sinh\alpha_j\right)^s \\
&= 2\arctan e^{-s\alpha_j} \\
&= \frac{\pi}{2} - \left(\arctan e^{s\alpha_j} - \arctan e^{-s\alpha_j}\right) \\
&= \frac{\pi}{2} - \arctan\frac{e^{s\alpha_j} - e^{-s\alpha_j}}{2}.
\end{aligned}$$

Since $\dfrac{e^x - e^{-x}}{2} = \sinh x$ the proof is complete. □

Lemma 2. *For $n, m \in \mathbb{N}_0$, $j \in \mathbb{Z}$, and α_j as was defined in the previous lemma, one has*

$$\frac{\pi}{2} - \arctan\left(\sinh(2m+1)\alpha_j\right) = (-1)^m \sum_{|k|\le m} \arctan\frac{\cos\frac{2\pi k}{2m+1}}{\cos\frac{\pi j}{2n+1}},$$

Proof. Using properties of complex numbers we write

$$\begin{aligned}
\frac{\pi}{2} - \arctan\left(\sinh(2m+1)\alpha_j\right) &= \arg(i) + \arg\left(1 - i\sinh(2m+1)\alpha_j\right) \\
&= \arg\left(\sinh(2m+1)\alpha_j + i\right)
\end{aligned}$$

$$= (-1)^m \arg\left(\sinh(2m+1)\alpha_j + \sinh \tfrac{\pi i(2m+1)}{2}\right).$$

This expression can be factorised according to the formula

$$\sinh(2m+1)a + \sinh(2m+1)b = 2^{2m} \prod_{|k|\le m}\left(\sinh a + \sinh\left(b - \frac{2\pi i k}{2m+1}\right)\right),$$

its validity is easy to check by standard methods: both sides are polynomials in with leading coefficient and zeroes

thus

$$\begin{aligned}
\frac{\pi}{2} - \arctan\left(\sinh(2m+1)\alpha_j\right) &= (-1)^m \arg\left(2^{2m}\prod_{|k|\le m}\left(\sinh\alpha_j + \sinh\left(\frac{\pi i}{2} - \frac{2\pi i k}{2m+1}\right)\right)\right)\\
&= (-1)^m \arg\left(\prod_{|k|\le m}\left(\cos\frac{\pi j}{2n+1} + i\cos\frac{2\pi k}{2m+1}\right)\right)\\
&= (-1)^m \sum_{|k|\le m} \arctan\frac{\cos\frac{2\pi k}{2m+1}}{\cos\frac{\pi j}{2n+1}},
\end{aligned}$$

as required. □

Lemma 3. *For $n, m \in \mathbb{N}_0$, $j \in \mathbb{Z}$, one has*

$$\sum_{|k|\le m} \arctan\frac{\cos\frac{2\pi k}{2m+1}}{\cos\frac{\pi j}{2n+1}} = \sum_{|k|\le m} (-1)^k \arctan\frac{\cos\frac{\pi k}{2m+1}}{\cos\frac{\pi j}{2n+1}}.$$

Proof. Let f be an odd function. Then

$$\begin{aligned}
\sum_{|k|\le m}(-1)^k f\left(\cos\frac{\pi k}{2m+1}\right) &= \sum_{|k|\le m} f\left(\cos\left(\frac{\pi k}{2m+1} - \pi k\right)\right)\\
&= \sum_{|k|\le m} f\left(\cos\frac{2\pi k m}{2m+1}\right)\\
&= \sum_{|k|\le m} f\left(\cos\frac{2\pi k}{2m+1}\right).
\end{aligned}$$

The last equality is explained as follows. First, note that cos has period 2π. The sum $\sum_{|k|\le m}$ is over residue class mod $2m+1$. When $m > 0$, the numbers m and $2m+1$ are coprime. Hence, when k runs over residue class mod $2m+1$, the set of numbers km runs over residue class mod $2m+1$.

To complete the proof of the lemma set $f(x) = \arctan\dfrac{x}{\cos\frac{\pi j}{2n+1}}$. □

Lemma 4.

$$\sum_{|j|\le n}(-1)^j = (-1)^n, \qquad n \in \mathbb{N}_0.$$

Proof. The sum is trivial when $n = 0$. Let's assume that $n > 0$. Then

$$\sum_{|j|\le n}(-1)^j = (-1)^n\frac{1-(-1)^{2n+1}}{1-(-1)} = (-1)^n. \quad \square$$

Now, we are in a position to prove Theorem 1. According to lemmas 1-4 we have that the LHS of the equation (6) equals

$$\sum_{|j|\le n}(-1)^{n+j}\frac{1}{2}\sum_{|k|\le m}(-1)^{m+k}\arctan\frac{\cos\frac{\pi k}{2m+1}}{\cos\frac{\pi j}{2n+1}}+\sum_{|k|\le m}(-1)^{m+k}\frac{1}{2}\sum_{|j|\le n}(-1)^{n+j}\arctan\frac{\cos\frac{\pi j}{2n+1}}{\cos\frac{\pi k}{2m+1}}$$
$$=\frac{1}{2}(-1)^{n+m}\sum_{|j|\le n}\sum_{|k|\le m}(-1)^{j+k}\frac{\pi}{2}\,\mathrm{sign}\left(\cos\frac{\pi j}{2n+1}\cos\frac{\pi k}{2m+1}\right)$$
$$=\frac{\pi}{4}(-1)^{n+m}\sum_{|j|\le n}(-1)^{j}\sum_{|k|\le m}(-1)^{k}=\frac{\pi}{4}.$$

III. Other reciprocal relations?

In our previous paper[8], we have found many relations of the form $P(n,m)=P(m,n)$, for finite products of trigonometric functions. However, the identity in Theorem 1 is of the type $S(n,m)+S(m,n)=C$, where C is independent of n and m. There is simple method to find other relations of this type. It is based on the solution of Dirichlet problem on a finite rectangular grid. For example

$$m\sum_{j=1}^{n}(-1)^{j}\cot\frac{\pi j}{2n}\,\frac{\sinh y\alpha_j}{\sinh m\alpha_j}\sin\frac{\pi jx}{n}+n\sum_{k=1}^{m}(-1)^{k}\cot\frac{\pi k}{2m}\,\frac{\sinh x\beta_k}{\sinh n\beta_k}\sin\frac{\pi ky}{m}=-xy, \tag{8}$$

where $1\le x\le n,\ 1\le y\le m$ are integers and $\cos\frac{\pi j}{n}+\cosh\alpha_j=2,\ \cos\frac{\pi j}{m}+\cosh\beta_j=2.$

Laplace operator on a finite rectangular grid is defined as

$$\Delta f(x,y,k)=f(x-1,y)+f(x+1,y)+f(x,y-1)+f(x,y+1)-4f(x,y).$$

One can see that the RHS of (8) satisfies the discrete Laplace equation

$$\Delta f(x,y)=0,\quad (0<x<n,0<y<m) \tag{9}$$

on a rectangular grid of size $n\times m$. Also $-xy=f_1(x,y)+f_2(x,y)$, where $f_1(x,y)$ and $f_2(x,y)$ are solutions of the Laplace equation with boundary conditions

$$\begin{cases} f_1(0,y)=f_1(n,y)=0,\ 0\le y\le m,\\ f_1(x,0)=0, f_1(x,m)=xm,\ 0\le x\le n,\end{cases} \tag{10}$$

$$\begin{cases} f_2(x,0)=f_2(x,m)=0,\ 0\le x\le n,\\ f_2(0,y)=0, f_2(n,y)=ny,\ 0\le y\le m.\end{cases} \tag{11}$$

Partial solutions of Laplace equation corresponding to boundary conditions (10) and (11) are given by, respectively

$$u_j^{(1)}(x,y)=\sin\frac{\pi jx}{n}\sinh y\alpha_j,\quad (1\le j\le n).$$
$$u_k^{(2)}(x,y)=\sin\frac{\pi ky}{m}\sinh x\beta_k,\quad (1\le k\le m).$$

In fact this method is quite well known and there are many examples in electrodynamics and heat conduction problems in physics (e.g., [7]).

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