Riemann Hypothesis

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1 Abstract

The Riemann Zeta function is defined as the Analytic Continuation of the Dirichlet series

$\zeta(s) = \sum_{n=1}^{\infty} 1/n^s, \ Re(s) > 1$

The Riemann Zeta function is holomorphic in the complex plane except for a simple pole at $s = 1$

The non trivial zeroes(i.e those not at negative even integers) of the

Riemann Zeta function lie in the critical strip

$0 \leq Re(s) \leq 1$

Riemann's Xi function is defined as[4, *p.*1],

$\epsilon(s) \ = \ s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)/2$

The zero of $(s-1)$ *cancels the pole of* $\zeta(s)$ *, and the real zeroes of* $s\,\zeta(s)$ *are cancelled by the*

simple poles of $\Gamma(s/2)$ *which never vanishes.*

Thus, $\epsilon(s)$ *is an entire function whose zeroes are the non trivial zeroes of* $\zeta(s)$*(see*[1, *p.*80])

Further, $\epsilon(s)$ *satisfies the functional equation*

$\epsilon(1-s) \ = \ \epsilon(s)$

2 Statement of the Riemann Hypothesis

The Riemann Hypothesis states that all the non trivial zeroes of the

Riemann Zeta function lie on the critical line Re(s)=1/2

3 Proof

The Riemann Xi function [2, p.37, Theorem 2.11] is defined as

For all s$\in\mathbb{C}$ *we have*,

$\epsilon(s) = \epsilon(0)\prod_\rho(1-\frac{s}{\rho})$ ... (1)

where ρ *ranges over all the roots* ρ *of* $\epsilon(\rho)=0$ *and if we combine the factors*

$(1-\frac{s}{\rho})$ *and* $(1-\frac{s}{(1-\rho)})$, *the product converges absolutely and uniformly on*

compact subsets of $\mathbb{C}$

Also, $\epsilon(0)=1/2$

Let, $\epsilon(s)=0,\ 0\leq Re(s)\leq 1$ ... $(*)$

Since, $\epsilon(s)$ *satisfies the functional equation*

$\epsilon(1-s) = \epsilon(s)$

$\epsilon(1-s)=\epsilon(s)=0.$

From(1),

$\epsilon(1-s)=\epsilon(0)\prod_\rho(1-\frac{1-s}{\rho})=0$

$\epsilon(1-s)=\epsilon(0)\prod_\rho(\frac{\rho+s-1}{\rho})=0$

$\epsilon(0) = 1/2$ [2, p.37 , Theorem 2.11]

$\epsilon(1-s) = 1/2\prod_{\rho}(\frac{\rho+s-1}{\rho}) = 0$

$1/2\prod_{\rho}(\frac{\rho+s-1}{\rho}) = 0$

$\prod_{\rho}(\frac{\rho+s-1}{\rho}) = 0 \qquad \ldots \qquad (2)$

Let, $s = \sigma + it \; 0 \le \sigma \le 1$

and let, $\rho = a + ib$

Since, $\epsilon(\rho) = 0$,

Thus, $0 \le Re(\rho) \le 1$.

(*Since* $\epsilon(s)$ *is zero free in* $Re(s) < 0$ *and* $Re(s) > 1$.)

Thus, $\rho = a + ib, \; 0 \;, 0 \le a \le 1$.

From (2),

$\prod_{\rho}(\frac{\rho+s-1}{\rho}) = 0$

Putting, $s = \sigma + it, \; 0 \le \sigma \le 1$ *and putting* $\rho = a + ib, \; 0 \le a \le 1$.

$\prod_{\rho}(\frac{a+ib+\sigma+it-1}{a+ib}) = 0$

$\prod_{\rho}(\frac{(a+\sigma-1)+i(b+t)}{a+ib}) = 0$

$\Rightarrow |\prod_{\rho}(\frac{(a+\sigma-1)+i(b+t)}{a+ib})|^2 = 0$

$\Rightarrow \prod_{\rho}|(\frac{(a+\sigma-1)+i(b+t)}{a+ib})|^2 = 0$

$\Rightarrow \prod_{\rho}(\frac{(a+\sigma-1)^2+(b+t)^2}{a^2+b^2}) = 0$

$\Rightarrow \prod_{\rho}(\frac{(a-\sigma+2\sigma-1)^2+(b+t)^2}{a^2+b^2}) = 0$

$\Rightarrow \prod_{\rho}(\frac{(a-\sigma)^2+(2\sigma-1)^2+2(a-\sigma)(2\sigma-1)+(b+t)^2}{a^2+b^2}) = 0$

$\Rightarrow \prod_\rho (\frac{(a-\sigma)^2+(2\sigma-1)(2\sigma-1+2a-2\sigma)+(b+t)^2}{a^2+b^2}) = 0$

$\Rightarrow \prod_\rho (\frac{(a-\sigma)^2+(2\sigma-1)(2a-1)+(b+t)^2}{a^2+b^2}) = 0$

$$\Rightarrow \prod_\rho (\frac{(a-\sigma)^2+(b+t)^2+(2\sigma-1)(2a-1)}{a^2+b^2}) = 0 \qquad ... \qquad (3)$$

$Since\ ,\ the\ critical\ strip\ is\ 0 \leq Re(s) \leq 1; s = \sigma + it.$

$Thus, 0 \leq \sigma \leq 1.$

$We\ discuss\ 2\ cases\ 0\ \leq \sigma \leq 1/2\ and\ 1/2 \leq \sigma \leq 1.$

$Case\ 1 : 0 \leq \sigma \leq 1/2$

$\rho = a + ib,\ 0 \leq a \leq 1$

$Claim : 0 \leq a \leq 1/2.$

$We\ prove\ the\ claim\ by\ contradiction.$

$Let, a \notin [0, 1/2]$

$Since 0 \leq a \leq 1 (\rho = a + ib\ is\ a\ non\ trivial\ zero\ of\ \epsilon(s)[3, p.3]$

$Thus, a \notin [0, 1/2]\ and\ 0 \leq a \leq 1 \Rightarrow 1/2 < a \leq 1.$

$\epsilon(s) = \epsilon(0) \prod_\rho (1 - \frac{s}{\rho})$

$s = \sigma + it$

$\rho = a + ib$

$\epsilon(\sigma + it) = \epsilon(0) \prod_\rho (1 - \frac{\sigma+it}{a+ib})$

$\epsilon(\sigma + it) = \epsilon(0) \prod_\rho \frac{(a-\sigma)+i(b-t)}{a+ib})$

$Since, 1/2 < a \leq 1 \qquad ... \qquad (4)$

$Since, 0 \leq \sigma \leq 1/2$

$Thus, -1/2 \leq -\sigma \leq 0 \qquad ... \qquad (5)$

$Adding\ (4)\ and\ (5),\ we\ have$

$0 < a - \sigma \leq 1$

$\Rightarrow a - \sigma \neq 0.$

$\Rightarrow (a-\sigma) + i(b-t) \neq 0.$

$\Rightarrow \frac{(a-\sigma)+i(b-t)}{a+ib} \neq 0.$

$Since,\ \epsilon(0) = 1/2\ \Rightarrow \epsilon(0)\frac{(a-\sigma)+i(b-t)}{a+ib} \neq 0.$

$\epsilon(s) \neq 0.$

$But\ in\ (*)\ ,\ we\ have\ assumed\ that\ \epsilon(s) = 0.\ So\ we\ get\ a\ contradiction.$

$So,\ our\ assumption\ that\ a \notin [0, 1/2]\ is\ wrong.$

$Thus, a \in [0, 1/2]$

$0 \leq a \leq 1/2$

$From\ (3),$

$\prod_\rho (\frac{(a-\sigma)^2+(b+t)^2+(2\sigma-1)(2a-1)}{a^2+b^2}) = 0$

$\prod_\rho (\frac{(a-\sigma)^2+(b+t)^2+(1-2\sigma)(1-2a)}{a^2+b^2}) = 0$

Since, $0 \leq \sigma \leq 1/2 \Rightarrow 1 - 2\sigma \geq 0 \qquad ... \qquad (6)$

$Since, 0 \leq a \leq 1/2 \Rightarrow 1 - 2a \geq 0 \qquad ... \qquad (7)$

$From\ (6)\ and\ (7)\ , (1-2\sigma)(1-2a) \geq 0. \qquad ... \qquad (8)$

$From\ (3),\ \prod_\rho(\frac{(a-\sigma)^2+(b+t)^2+(2\sigma-1)(2a-1)}{a^2+b^2}) = 0.$

$$(\frac{(a_0-\sigma)^2+(b_0+t)^2+(2\sigma-1)(2a_0-1)}{a_0^2+b_0^2}) = 0,\ for\ some\ a_0+ib_0 \in\mathbb{C}. \quad ... \quad (9)$$

$Since,\ 0 \leq a \leq 1/2\ Thus,\ 1-2a \geq 0 \Rightarrow 1-2a_0 \geq 0$

$From\ (9),$

$(\frac{(a_0-\sigma)^2+(b_0+t)^2+(2\sigma-1)(2a_0-1)}{a_0^2+b_0^2}) = 0\ for\ some\ a_0+ib_0 \in\mathbb{C}.$

$From\ (8),$

$(1-2\sigma)(1-2a) \geq 0.\ Also, (a_0-\sigma)^2+(b_0+t)^2 \geq 0.$

$Thus, (\frac{(a_0-\sigma)^2+(b_0+t)^2+(2\sigma-1)(2a_0-1)}{a_0^2+b_0^2}) = 0$

$\Rightarrow (a_0-\sigma)^2 = 0\ and\ (b_0+t)^2 = 0\ and\ (2\sigma-1)(2a_0-1) = 0.$

$\Rightarrow a_0 = \sigma\ and\ b_0 = -t\ and\ (2\sigma-1)(2\sigma-1) = 0 (Since,\ a_0 = sigma)$

$\Rightarrow a_0 = \sigma\ and\ b_0 = -t\ and\ (2\sigma-1)^2 = 0$

$\Rightarrow (2\sigma-1)^2 = 0$

$\Rightarrow \sigma = 1/2$

Case 2: $1/2 \leq \sigma \leq 1$

$\rho = a+ib,\ 0 \leq a \leq 1$

$Claim : 1/2 \leq a \leq 1.$

$We\ prove\ the\ claim\ by\ contradiction.$

$Let, a \notin [1/2, 1]$

$Since,\ 0 \leq a \leq 1(\ \rho = a+ib\ is\ a\ non\ trivial\ zero\ of\ \epsilon(s)\ [3\ , p.3]$

$Thus,\ a \notin [1/2, 1]\ and\ 0 \le a \le 1 \Rightarrow 0 \le a < 1/2.$

$\epsilon(s) = \epsilon(0) \prod_\rho (1 - \frac{s}{\rho})$

$s = \sigma + it$

$\rho = a + ib$

$\epsilon(\sigma + it) = \epsilon(0) \prod_\rho (1 - \frac{\sigma+it}{a+ib})$

$\epsilon(\sigma + it) = \epsilon(0) \prod_\rho \frac{(a-\sigma)+i(b-t)}{a+ib})$

$Since, 0 \le a < 1/2 \qquad ... \qquad (10)$

$Since, 1/2 \le \sigma \le 1$

$Thus, -1 \le -\sigma \le -1/2 \qquad ... \qquad (11)$

$Adding\ (10)\ and\ (11)\ ,\ we\ have$

$-1 \le a - \sigma < 0$

$\Rightarrow a - \sigma \neq 0.$

$\Rightarrow (a - \sigma) + i(b - t) \neq 0.$

$\Rightarrow \frac{(a-\sigma)+i(b-t)}{a+ib} \neq 0.$

$Since,\ \epsilon(0) = 1/2 \Rightarrow \epsilon(0)\frac{(a-\sigma)+i(b-t)}{a+ib} \neq 0.$

$\epsilon(s) \neq 0.$

$But\ ,\ we\ have\ assumed\ that\ \epsilon(s) = 0.\ So\ we\ get\ a\ contradiction.$

$So,\ our\ assumption\ that\ a \notin [1/2, 1]\ is\ wrong.$

$Thus, a \in [1/2, 1]$

$From(3),$

$\prod_\rho(\frac{(a-\sigma)^2+(b+t)^2+(2\sigma-1)(2a-1)}{a^2+b^2}) = 0$

$Since, 1/2 \leq \sigma \leq 1 \Rightarrow 2\sigma - 1 \geq 0 \quad ... \quad (12)$

$Since, 1/2 \leq a \leq 1 \Rightarrow 2a - 1 \geq 0 \quad ... \quad (13)$

$From\ (12)\ and\ (13),\ (2\sigma-1)(2a-1) \geq 0. \quad ... \quad (14)$

$From(3),\ \prod_\rho(\frac{(a-\sigma)^2+(b+t)^2+(2\sigma-1)(2a-1)}{a^2+b^2}) = 0.$

$(\frac{(a_1-\sigma)^2+(b_1+t)^2+(2\sigma-1)(2a_1-1)}{a_1^2+b_1^2}) = 0\ for\ some\ a_1+ib_1\ \in\mathbb{C}. \quad ... \quad (15)$

$Since,\ 1/2 \leq a \leq 1.\ Thus, 2a-1 \geq 0 \Rightarrow 2a_1 - 1 \geq 0$

$From(15),$

$(\frac{(a_-\sigma)^2+(b_1+t)^2+(2\sigma-1)(2a_1-1)}{a_1^2+b_1^2}) = 0\ for\ some\ a_1 + ib_1 \in\mathbb{C}.$

$From(14),$

$(1-2\sigma)(1-2a) \geq 0.\ Also, (a_1-\sigma)^2 + (b_1+t)^2 \geq 0.$

$Thus, (\frac{(a_1-\sigma)^2+(b_1+t)^2+(2\sigma-1)(2a_0-1)}{a_0^2+b_0^2}) = 0$

$\Rightarrow (a_1-\sigma)^2 = 0\ and\ (b_1+t)^2 = 0\ and\ (2\sigma-1)(2a_1-1) = 0.$

$\Rightarrow a_1 = \sigma\ and\ b_1 = -t\ and\ (2\sigma-1)(2\sigma-1) = 0 (Since, a_1 = \sigma)$

$\Rightarrow a_1 = \sigma\ and\ b_1 = -t\ and\ (2\sigma-1)^2 = 0$

$\Rightarrow (2\sigma-1)^2 = 0$

$\Rightarrow \sigma = 1/2$

$So,\ in\ both\ the\ cases\ \sigma = 1/2.$

$\Rightarrow Re(s) = 1/2.\ This\ proves\ the\ Riemann\ Hypothesis.$

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