Methods of the Global Optimization by Deformation of Functions (Part 1)

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Abstract

A new method of optimization by means of a redefinition of the function over a wider set and a deformation of the function on the initial and additional sets is proposed.

The method (a) reduces the initial complex problem of optimization to series of simplified problems, (b) finds the subsets containing the point of global minimum and finds the subsets containing better solutions that the given one, and (c) obtains a lower estimation of the global minimum.

Introduction

The classical approaches optimization problem is following:

Problem A. *Find a minimum of the given function.*

Together with problem A the following problems we are considered:

Problem B. *Find a smaller subset contains the all points of the global minimum.*

Problem C. *Find a subset of better solutions where the function is less that given value.*

Problem D. *Find a lower estimation of function.*

These non-classical approach B,C, and D require innovative methods, different from the well-known methods.
The author offers the new mathematical methods for the solution of these problems.

The new methods have turned out to be much more general, so that while solving one of the above problems, another may be solved in passing, which may help in the solution of the former. Thus, if a satisfactory lower estimate found, it can be compared with various engineering solutions and give rise to one very close to the optimum.

This method is applied to many mathematical problems of optimization. For example, functions of several variable, constrained optimization, linear and nonlinear programming, multivariable nonlinear problems described by regular differential equations and equations in partial derivatives, etc.

One can easy get from the given method to many well-known methods of optimization, for example, Lagrangian multiplier method, the penalty function method, the classical variational method, Pontragin's principle of maximum, dynamic programming and others.

At present, the most of researchers in optimization fields are using the traditional optimization problem – find a minimum of the given functional (Problem A). They look a single, local minimum. An engineer, however, is usually interested in a subset of quasi-optimal solutions. He must make sure that the optimum does not exceed a given value (Problem C). Also, a good estimation from below will indicate how far a given solution is from the optimum solution (Problem D). An addition an engineer usually has other considerations that cannot be introduced into a mathematical model or can lead to impractical complications. Approach C provides him with some choice.

Problem D is also of particular interest. If an estimate from bottom closes to the exact infimum of the function is found, the optimization can frequently be reduced to finding a quasi-optimal solution by trial and error.

Solution of the Problem B can significantly simplify the solution of any of the above problems, since it narrows the set containing optimal solution.

These non-classical Problems B, C and D require innovative methods, different from the well-known method of variational calculus, maximum principle and dynamic programming. This new method is general, so that while solving one of the above problems, another may be solved in passing, which may help in the solution of the former.

Thus, if a satisfactory estimate from below has been found, it can be compared with various engineering solutions and give rise to one very close to the optimum.

Our reasoning in this book is not complex. But we are using symbolic of set Theory, which many engineers forget. That way we are given these information in Appendix A of the book Bolonkin A.A., "Universal Optimization and its Application", LULU, 2017 [1].

In article, we are using the double numbering of formulae, theorems and drawings. The first figure in numbering formula or theorem notes the number of paragraph, the second figure is number formula or theorem in this paragraph. The first figure of drawings means the number of chapters, the second is the number of drawing.

Methods of β and γ functions

§1. Methods of β −functions

1. Statement of the Problem. Main theorems. Algorithm 1.

1^0. Statement of the Task. Assume that the state of the system is described by element *x*. A series of these elements form the set *X*={*x*}. The numerical function *I(x)* (functional) is defined and bounded by its lower estimate over *X*. The relationships and limitations imposed on the system yield a subset $X^* \subseteq X$.

Traditionally the problem of optimization has been set as follows:

A. Find a point of the minimum of the function *I(x)* over the set *X**.

We shall also consider the following problems:

B. Find a smaller subset $M \subset X^*$ that contains point *x** of global (absolute) minimum, $x^* \in M$.
C. Find a subset $N \subset X^*$ on which $I(x) \le c$, where $c \ge I(x)$.
D. Find the lower estimates of *I(x)* over *X**.

We will name the point (element) *x* the solution if *x* is resulting any presses, procedure, calculation or reasoning. It not means that *x* is point of optimum. We will tell the point x_1 is better solution than the point x_2, if $I(x_1) < I(x_2)$ and the point of the same solution, if $I(x_1) = I(x_2)$.

For simplicity we assume that the point of global minimum *x** exists in *X**, but this is not impotent limitation. The most results can be obtained without this assumption.

Let us introduce a set *Y={y}* and define a bounded numerical function (functional) *β(x,y)* over $X \times Y$. We shall call it ***β-functional***.

Then we set

$$J(x,y) = I(x) + \beta(x,y).$$

Call our initial problem of finding *x** and $I(x^*) = \inf I(x) = m, \quad x \in X^*$ **Problem 1**

and the problem of finding $\bar{x}$ and

$$\bar{J}(\bar{x}(y), y) = \inf[I(x) + \beta(x,y)], \quad x \in X \quad \textbf{Problem 2}$$

We assume that $\bar{x}(y)$ exist over $X \times Y$.

We deformed arbitrarily our functional *I(x)* by adding *β(x, y*). Moreover, we widened the domain of the deformed functional and arbitrarily defined it on the set *Y*. We should do so in such a way that problem 2 will be easier to solve.

It might seem that this makes no sense because we must find the points of minimum of our initial functional *I(x)*, i.e., solve Problem 1. But it appears that from the solution of the simpler Problem 2 we can obtain information about Problem 1. We can use freedom in choice of the functional *β(x,y)* and the set *Y* for such a deformation of functional *J(x,y)* and the set *Y* that we solve the initial Problem 1, but in an easier way.

2⁰. The Fundamental Theorem. The following main theorem establishes the relationship between Problem 1 and 2, as well as between Problems A, B, and C (The Principle 1 of Optimum).

Theorem 1.1. *Distinguishing between the sets containing: (1) The global minimum points, (2) only better solutions than the one given, (3) only worse solutions than one given.*

Assume $X^* \equiv X,\ \bar{x}(y)$ *are the points of global minimum in Problem 2. Then:*

(1) The points of global minimum in Problem 1 are contained in the set

$$M = \{x : \beta(x,y) \geq \beta(\bar{x}(y),y),\ \ y \in Y\};$$

(2) The set

$$N = \{x : J + I \leq \bar{J} + \bar{I},\quad y \in Y\}$$

contains the same or better solutions (that is over N, we have $I(x) \leq I(\bar{x})$ *);*

(3) The set

$$P = \{x : \beta(x,y) \leq \beta(\bar{x}(y),y),\ \ y \in Y\};$$

contains the same or worse solutions (that is over P $I(x) \geq I(\bar{x})$ *).*

Proof. 3. By subtracting the inequality $\beta(x,y) \leq \beta(\bar{x}(y),y)$ *from* $I(x) + \beta(x,y) \geq I(\bar{x}(y)) + \beta(\bar{x}(y),y)$ *we get* $I(x) \geq I(\bar{x})$

over *P*. Point 3 of the theorem is proved.

1. Point 1 of Theorem 1 is obvious because *X=M+P* and $I(x) \geq I(\bar{x})$ over *P*, we have $x^* \in M$. Point 1 of the theorem is proved.
2. By subtracting the inequality $J \geq \bar{J}$ *from* $J + I \leq \bar{J} + \bar{I}$ *we get* $I(x) \leq I(\bar{x})$ over *N*. Point 2 of the theorem is proved.
   Theorem 1 is proved.

If in sets *N* and *P* we write the strong inequality $\beta > \bar{\beta}$, then the set N will contain only better solutions and the set P will contain worse solutions that $I(\bar{x})$.

Theorem 1.1 is correct when *X*≠X*, but *M,N,P* contain elements from *X**.

Let us focus our attention on the fact that after solving the simpler Problem 2, we distinguished in our set *X* three subset: *M*, which contains a point of global minimum, subset *P*, containing the same or worse solutions, and subset *N*, which contains the same or better solutions.

Consequences:

1. Element $\bar{x}$ is the point of global minimum of the functional over the set *P⊆X*.

2. $\bar{x}$ is the element which gives the maximum of the functional *I(x)* over the set *N⊂X*.

3. If *X*⊆P*, then $\bar{x}$ is the point if global minimum Problem 1 over set *X**. In this case we have *M={x}.*

4. If $\beta=\beta(x)$, $x\in X$, *then*

$$M=\{x:\beta(x)\geq\beta(\bar{x})\},\qquad P=\{x:\beta(x)\leq\beta(\bar{x})\},\qquad N=\{x:J+I\geq\bar{J}+\bar{I}\quad .$$

Theorem 1 is correct when $X^*\neq X$, but M,N,P contain element from X^*.

5. Let $X^*\neq X$. If $X^*\cap M=\varnothing$, then $I(\bar{x})$ is the lower estimation $I(x)$ over the set X^* (because in this case we have $X^*\subseteq P$).

6. Let $X^*\neq X$. If $X^*\subset N$, then $I(\bar{x})$ is the top estimation $I(x)\leq I(\bar{x})$ over the set X^*.

If $\bar{x}\in X^*$, the sets M,N,P will always contain at least one element from the set X^*. This element is $\bar{x}$.

***Remarks*:**

1. $N\subseteq M$. The proof: Let us denote $\overset{o}{P}=P-\{\bar{x}\}$. Then $\overset{o}{P}\cap N=\varnothing$, because over $\overset{o}{P}$ we have $I(x)>I(\bar{x})$ and over N we have $I(x)\leq I(\bar{x})$. But $N\subset X$ and $M=X-\overset{o}{P}$. Hence $N\subseteq M$.
2. Assume the definitions of the sets N, P (see Theorem 1) contain strong inequalities. Then the set N will contain on; y better solutions and the set P – only worse solutions, compared to $\bar{x}$.
3. We can use the dependence of the sets M,N,P from y in order to change the "dimensions" of these sets.
4. β - functions exist and their number is infinite.
   The last statement is obvious because we can define β-functionals over the set $X\times Y$ in any possible way.

The theorem 1 gives the Algorithm 1 (a β-functional method for finding the subsets that contains the points of global minimum or better solutions).

Algorithm 1. *Define $\beta_i(x,y)$ so that Problem 2 becomes easier to solve, and find sets M_i and N_i. Then $M=\cap M_i$ (that is not empty) is the set that contains the points of global minimum and $N=\cap N_i$ (if that is not empty) is subset contains min $\{I(\bar{x}_i)\}$ or better solutions.*

Note: The getting M is more "narrow" (contains less points x) subset then initial M. That means the finding x^* is easier. The decreasing of M is especially important in a "method of dynamic programming" because it is decreasing the number of computations.

Theorem 1.2. (The lower estimate) *Let us assume that $\beta(x,y)$ is a defined and bounded functional over $X\times Y$ then the lower estimate over X is*

$$I(x)\geq[I(\bar{x}(y))+\beta(\bar{x}(y),y)-\sup_X\beta(x,y)]\quad\text{for}\quad\forall y\in Y. \tag{1.1}$$

Proof: By adding the inequalities

$$I(x)+\beta(x,y)\geq I(\bar{x})+\beta(\bar{x}(y),y)\quad and\quad-\beta(x,y)\geq-\sup_X\beta(x,y)$$

over X, we get the estimate (1.2).

Remarks:

1. For case $\beta=\beta(x)$ the estimate (1.1) is

$$I(x)\geq\inf_X J(x)-\sup_X\beta(x)\ , \tag{1.1'}$$

2. When $X\neq X^*$ the estimate (1.1) is correct over X^*, because $X^*\subseteq X$. In this case we can use the better estimates:

$$I(x) \geq \inf_{X^*} J(x) - \sup_{X} \beta(x), \quad I(x) \geq \inf_{X} J(x) - \sup_{X^*} \beta(x), \quad I(x) \geq \inf_{X^*} J(x) - \sup_{X^*} \beta(x), \quad (1.1'')$$

When we found the set *M* for β_i the following estimate may be used

$$I(x) \geq \inf_{X^*} J(x) - \sup_{M} \beta(x), \qquad (1.1''')$$

The proof of (1.1'), (1.1''), (1.1''') is same the proof of theorem 1.2.

3. Dependence of the estimate (1.1) from *y* may be used for its improving

$$I(x) \geq \sup_{y}[\inf_{x^*} J(x) - \sup_{x} \beta(x)], \qquad (1.1^{IV})$$

When we use the estimates (1.1') - (1.1^{IV}) we decide the problem $\widehat{\beta} = \sup_{X} \beta$. It may be used for

Theorem 1.3. Assume *X=X**, $\bar{x}$ is point of a global minimum in the problem $\widehat{\beta} = \sup_{X} \beta$,

Then:

1) The points of global minimum in Problem 1 are contained in the set

$$M(y) = \{x : I + \beta \leq \widehat{I} + \widehat{\beta}, \quad y \in Y\}$$

Contains the same or better solutions.

2) The set

$$N(y) = \{x : \beta - I \geq \widehat{\beta} - \widehat{I}, \quad y \in Y\}$$

3) The set

$$P(y) = \{x : I + \beta \geq \widehat{I} + \widehat{\beta}, \quad y \in Y\}$$

Contains the same or worse solutions.

Here is $\widehat{I} = I(\widehat{x})$.

Proof of Theorem 1.3.

1, 3. By subtracting the inequality $\beta \leq \widehat{\beta}$ from $I + \beta \geq \widehat{I} + \widehat{\beta}$ we get $I \geq \widehat{I}$ over set *P*.
The statement 1, 2 follow from this.

2. By subtracting the inequality $\beta \geq \widehat{\beta}$ from $I - \beta \geq \widehat{I} - \widehat{\beta}$ and multiply this result by -1,
we get $I \leq \widehat{I}$ over N. The theorem 1,3 is proofed.

Remark:

For proof of the theorems 1.1-1.3 the existence of *x*, $\bar{x}$, $\widehat{x}$ is not important, but corresponding *inf* and *sup* must be existed.

Example 1.1.

Find minimum of functional

$$I = -e^{-x^4}\cos x^2 - \frac{0.1}{x^2 - 0.2x + 1}, \quad -\infty < x < \infty, \qquad (1.2)$$

Solution. Take

$$\beta(x) = \frac{0.1}{x^2 - 0.2x + 1}.$$

Then

$$J = I + \beta = e^{-x^4}\cos x^2.$$

The minimum of this *J* is obvious: $\bar{x} = 0$.

From theorem 1.1 we got the point of the global minimum is in set

M={x: $\beta(x) \geq \beta$(0)}

or

$$\frac{0,1}{x^2 - 0.2x + 1} \geq 0.1 \ ,$$

The solution of this inequality is

0≤ *x* ≤0.2 .

It's not difficult to find the point of global minimum in this small interval by any known method.

We get the lower estimate (theorem 1.2)

$$J(0) - \sup_x \beta = -1 - 0.101 = -1.101.$$

Value *I(0)* = -1.100 . We see *I(x)* for *x* = 0 is very close to global minimum.

Example 1.2

Find minimum

$$I = -\frac{0.1}{x^2 - 2x + 10} + \cos 4\pi x - 4\cos 2\pi x, \quad -\infty < x < \infty \qquad (1.3)$$

Solution: We take

$$\beta(x) = -\cos 4\pi x + 4\cos 2\pi x \ .$$

Then

$$J = I + \beta = \frac{0.1}{x^2 - 2x + 10}, \quad \bar{x} = 1.$$

This solution is global minimum of Problem1 over set

P = {x: β(x) = β(1)}

or

$$-\cos 4\pi x + 4\cos 2\pi x \leq 3.$$

We transform this inequality in

$$-8\sin^4\pi x \le 0 .$$

We see $P=\{x: |x|<\infty\}$. Therefore $P=X^*$. That means (see Consequence 1) $\bar{x}=1$ is point (and alone) of global minimum of the functional (1.3).

Example 1.3 .

More full, we are demonstrating the new method on following simple functional.

Find the absolute minimum of the functional

$$I=2x^4+x^2-2x+1 \text{ on the set } X^*=\{x: |x|<\infty\} . \qquad (1.4)$$

It is a simple example, which can be solved using well-known methods. For example, take the first derivative, make it equal to zero. Solve an algebraic 3-d order equation (it may not be a simple task) and then analyze the points so found with respect to maximum and minimum.

We shall try to solve this example by the above method as it follows from algorithm 1.

Let us introduce a series $\beta_i(x)$. As follows from Theorem 1.1 we have the sets M_i:

1) Take $\beta_1=2x$. Then $J=I+\beta_1=2x^4+x^2+1, \quad \bar{x}=0, \quad \text{from} \quad \beta\ge\overline{\beta} \quad \text{we} \quad \text{have} \quad M_1=\{x: x\ge 0\}$. As we see the domain which contain a global minimum have become less in two times.
2) Take $\beta_2=-x^2+2x$. Then $J=I+\beta_2=2x^4+1, \quad \bar{x}=0, \quad \text{from} \quad \beta\ge\overline{\beta} \quad \text{we} \quad \text{have} \quad M_1=\{x: 0\le x\le 2\}$.

   Our interval contained a global minimum is only $0\le x\le 2$.

   For given β_2 we can use an estimation of the functional which follows from Theorem 1.2.

$$I(x)\ge J(\bar{x})-\sup_X \beta_2(x)=1-\sup_X(-x^2+2x)=1-1=0,$$

where the point of supreme of β is $\hat{x}=1$.

From theorem 1.3 we have the additional set M:

$$M_3=\{x: J(x)\le J(\hat{x})\} \quad or \quad M_3=\{x: |x|\le 1\} .$$

As we see the set $M=M_2\cap M_3=\{x: 0\le x\le 1\}$, The global minimum of this problem is in the interval $0\le x\le 1$.

3) Take $\beta_3=2x^2+2x-0.5$. Then $J=I+\beta_3=2x^4-x^2-0.5$. From inf J we have $\bar{x}_{1,2}=\mp 0.5$.
4) Find for point x_1 set M:

$$\bar{x}_1=-0.5, \quad M_4=\{x: -0.5\le x\le 1.5\},$$

$$\bar{x}_2=0.5, \quad M_5=\{x: 0.5\le x\le 0.5\} .$$

The estimation gives $I(x)\ge 3/8-0=3/8$.

We see that the diameter of the set $M=\cap M_i$ decreases until reduces in the point $\bar{x}=0.5$. Therefore this point is one of the absolute minimum of the Problem 1 and $I(0.5)=3/8$.

5^0. The geometric illustration of Theorem 1.1 is given in fig, 1.1 for single variable. The curves *I(x), J(x), β(x), I(x)+0.5 β(x)* and point $\bar{x}$ are drawn. There are the sets *M, N, P. P* is set *x*, where

$\beta(x) \leq \beta(\bar{x})$, *M* is set *X\P* and *N* is set *x*, where $J(x)+0.5\beta(x) \leq J(\bar{x})+0.5\beta(\bar{x})$.

We can see that $N \subset M$.

In fig.1.2 we see sets *M, N, P* for the case when $I(x_1,x_2)$ is function of two variables x_1 and x_2.

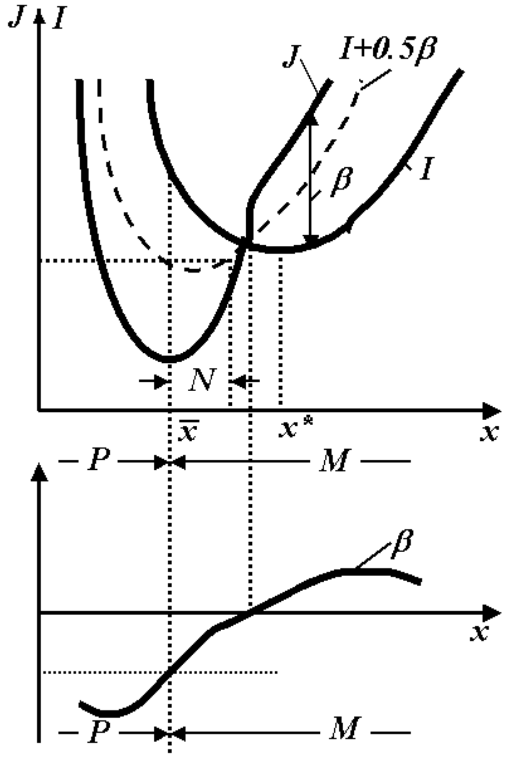


Fig. 1.

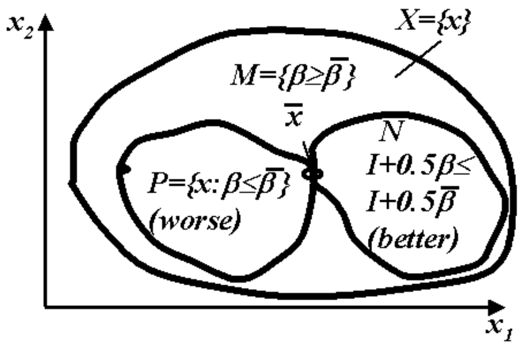


Fig. 2.

Fig.1.1. Geometric illustration of Theorem 1.1 for case of single variable.

Fig.1.2. Sets *M, N, P* for case of two variable.

2. About Convergence of Algorithm 1.

Consider condition of convergence $\inf_{x\in X} J(x), \quad x \in X \quad to \quad \inf_{x\in X^8} J(x), \quad x \in X^* \quad and \quad \bar{x} \quad to \quad x^*$ for Algorithm 1. when we have the succession $\beta_i(x)$, I = 1,2,… This succession gives the succession of the sets M_i, N_i and values of functionals $J(\bar{x}_i)$.

The succession $\{\inf J(\bar{x}_i)\}$

For $i \to \infty$ is monotonous decreasing and bounded of bottom, that's way it has a limit. If this limit equals one of lower estimates, that $J(\bar{x}) = I(x^*)$.

Let us to consider now convergence of diameter *d(M), d(N)* of sets $M=\cap M_i$, $N=\cap N_i$ for $i\to\infty$.

This convergence is also monotonous decreasing and bounded of bottom: $d \geq 0$. Therefore, it has a limit.

We have got the following simply criterion of convergence

Theorem 1.4. *Assume, the point of the absolute minimum of functional I(x) over set X=X* is single.*

If $d(M)\to 0$, than $x=\lim M(i)=x^*$, $i\to\infty$.

In this case the set contained of point of global minimum $M=\cap M_i$ decrease in point. Therefore, this point is the point of the absolute minimum of Problem 1.

Let us take succession of function $W_s(x)$, s =1,2,... . Take $\beta_i(x)$ as

$$\beta_i = \sum_{s=1}^{i} c_s W_s(x) \tag{1.5}$$

where c_s is constants.

We will take these constants c_s from condition

$$\Delta_i = \min_c [I(\bar{x}_i) - \inf_X J_i(x) + \sup_X \beta_i(x)].$$

The value ΔI is difference functional from its lower estimate. Other words value Δ show how much value $I(\bar{x}_i)$ differs from optimum. We name this number Δ-estimate (*delta-estimate*). It is obvious that succession $\{\Delta_i\}$ is monotonous decreasing because every next sum (1.5) contains previous sum. It is also limited of bottom ($\Delta_i \geq 0$). Therefore the succession $\{\Delta_i\}$ converge.

From destination Δ_i we get the following

Theorem 1.5. *If* $\Delta_i \to 0$, *then* $\inf_X J(x) \to \inf_{X^*} I(x)$.

Theorem 1.6. *Assume X=X*, $\beta_i = c_i\beta(x)$, I(x), β(x) is continuous and β(x) is limited on X.*

Then, if $c_i \to 0$ we have $J(x) \to m = \inf I(x)$ over X.*

Statement of Theorem 1.6 follows from continuous *J(x).*

This theorem may be useful for finding of the local minimum of *I(x)* by way of methods of successive approximations. Assume c_1=1 and problem inf *J(x)* can decided simply. Because functional *J(x)* is continuous, we can wait, that small change of *c* gives small changing (moving) $\bar{x}$.

Therefore $\bar{x}$ is good the initial approximation for $c_2 < c_1$. It is known, that a good initial approximation is very important for speed of convergence. We come to *x** by decreasing *c* to 0.

These criterions of convergence may be used for solutions Problem *A, B, C*, D (see §1,A).

3. Modification of the Theorem 1.1

Over we have considered the case, when we are looking for the additional function $\beta(x,y)$ such us the problem 2 became simpler for solution.

But sometimes it's more comfortable to take such function *J*(*x,y*) that the problem $\inf_X J(x,y)$ became easy for solution.

In this case Theorem 1.1. better to write as following

Theorem 1.1'. *Assume* $X^{\bullet} \equiv X, \quad \bar{x}(y)$ *is the point of global minimum in Problem 2 .*

$$\bar{J} = \inf_{X} J(x.y)$$

Then

1) *The points of global minimum in Problem 1 are contained in the set*

$$M(y) = \{x: \; J - I \geq \bar{J} - \bar{I}, \quad y \in Y\}.$$

2) *The set*

$$N(y) = \{x: \; J + I \leq \bar{J} + \bar{I}, \quad y \in Y\}$$

Contains the better or same solutions.

3) *The set*

$$P(y) = \{x: \; J - I \leq \bar{J} - \bar{I}, \quad y \in Y\}$$

Contains worse or same solutions.

This Theorem is correct if $J = kJ_1$, where k = const > 0.

4. Method of big steps in set of better solutions. Algorithm2.

From the Theorem 1.1 we can get the following

Algorithm 2 (Method of big steps in set of better solutions)

Take any point x_1 from X^* and such function $J_1(x)$ that point x_1 is its minimum. Find the set N_1 of better solutions. Take from this set a point x_2 and such function $J_2(x)$ that x_2 is its minimum. Find the set N_2 and so on.

It is obvious that $N_1 \supseteq N_2 \supseteq N_3 \supseteq \ldots$. Let us suppose that result of this process is following - set N_i become point x_N .

Theorem 1.7. *Assume X^* is open set, $I(x)$, $J_i(x)$ are continuously and differential (of Freshe) on X^*.*

Then point x_N is a stationary point of the function $I(x)$ over X^*.

Proof in Appendix 4°.

Theorem 1.8. *If in point x_N we have*

$$\beta(x_N) - I(x_N) = \sup_{X^*}[\beta(x) - I(x)],$$

Then x_N is point of global minimum of Problem 1.

Proof is in Appendix 5°.

If conditions of Theorem 1.8 is executed only in small sphere around point x_N then x_N is point of local minimum of Problem 1.

The example for illustration of this method (for tests of constrained minimum) will be given in § 4 (remark 4.3).

We can get the direction in the set *N*, if we calculi a gradient of function in *N.*

The advantages this method with comparison of gradient method is big steps. When you are in set *N*, you have not a danger of to get worthier solution than given one. This can substantially decrease amount of calculation.

5. Method of β - function for Problems with constrains

A) Assume *I*(*x*) is function by its lower estimate over set *X*. The subset $X^* \neq \varnothing$ is separated from *X* by functions

$$F_i(x) = 0 \quad i = 1,2,\ldots,k, \quad \Phi_j(x) \le 0, \quad j = 1,2,\ldots,q, \tag{1.6}$$

where *x* - is *n* - dimensional vector of numerical values.

Take β-function as following (we have a sum for lower index *i,j*)

$$\beta(x,y) = \lambda_i(x,y)F_i(x) + \omega_j(x,y)\Phi_j(x),$$

where $\lambda_i(x,y)$, $\Phi_j(x,y)$ are functions of *x,y,* $y \in Y$, $\omega_j(x,y) \ge 0$.

Write *J*-function

$$J(x,y) = I(x) + \lambda_i(x,y)F_i(x) + \omega_j(x,y)\Phi_j(x). \tag{1.8}$$

Theorem 1.9. *Assume exist* $x^* \in X^*$*, y is fixed.*

In other $\bar{x}$ *to be a point of global minimum of function I(x) over X* necessary and enough to exist of function* $\beta(x,y)$ *such as*

$$1)\ J(\bar{x},y) = \inf_{x\in X} J(x,y), \quad 2)\ \bar{x} \in X^*, \quad 3)\ \omega_j(x,y) \ge 0 \quad over \quad X, \quad 4)\ \beta(\bar{x},y) = 0, \tag{1.9}$$

The **proof** in Appendix 6°.

Theorem 1.10. (The lower estimation)

Assume y is fixed, $\bar{x}$ *is point of minimum (1.8) for conditions* $\omega_j(x,y) \ge 0$.

Then $J(\bar{x},y)$ *is lower estimation of function I(x) on X*.*

Proof: On set *X** we have $\lambda_i F_i \equiv 0, \quad \omega_j \Phi_j \le 0$ (that is $\beta(\bar{x},y) \le 0$). Since over *X** we have $J(\bar{x},y) \le I(x)$. Theorem is proved.

Likely a common case for β - function we can get the sets

$$M = \{x : \beta \ge \overline{\beta}\}, \quad N = \{x : J + I \le \dot{J} + \bar{I}\}, \quad P = \{x : \beta \le \overline{\beta}\}$$

and in this case.

Freedom in choice of *y* we can use for improvement of estimation and decrease sizes of sets *M, N*. Remark only that $\bar{x} = \bar{x}(y)$ and for every *y* corresponding $\bar{x}$ you must find inf *J*(*x,y*), $x \in X$.

Remark: We can take β-function (1.7) in form

$$\beta(x)=\frac{1}{2}a\sum_{i=1}^{k}F_i^2(x)+\sum_{j=1}^{q}a^{\Phi_j(x)}.$$

It is possible to show for some conditions: [$I(x)$, $\Phi_j(x)$, $F_i(x)$ are continuous, x is compact set, x^* is close set and don't contain separated points; $x^*\in X^*$ and exist], when $a\to\infty$, we have $\bar{J}\to m,\quad \bar{x}=x^*$.

B) Assume $F_i(x)=0$ in (1.6) absent, i.e. the Problem is

$$I(x)=\min,\quad \Phi_j(x)\le 0,\quad j=1,2,\ldots,q$$

For solution of this problem we can use following **algorithm**:

1. Take any functions $\omega(x,y)$ (it's may be less zero) and find the point $\bar{x}(y)$ of global minimum (one may be implicit form $\xi(\bar{x},y)=0$) of general numerical function

$$J=I(x)+\sum\omega_j(x,y)\Phi_j(x)\quad on\quad X. \tag{1.12}$$

2. Solve equations

$$\xi(\bar{x},y)=0,\quad \omega_j(\bar{x},y)\Phi_j(\bar{x})=0,\quad j=1,2,\ldots,q \tag{1.13}$$

3. Select from these solutions such which satisfy inequalities

$$\omega_j(\bar{x},\bar{y})\ge 0,\quad j=1,2,\ldots,q. \tag{1.14}$$

These are points of global minimum of Problem (1.11) because all request the theorem 1.4 is satisfy.

We can solve (1.13) by different ways. For example, find $\bar{x}$ from equation $\xi(\bar{x},y)=0$ and substitute in the last equations (1.13)

$$\omega_j(\bar{x}(y),y)\Phi_j(\bar{x}(y))=0,\quad j=1,2,\ldots,q \tag{1.15}$$

Find y from this system of equations. Select from these solutions such which satisfy inequalities

$$\omega_j(\bar{x}(y),y)\ge 0,\quad j=1,2,\ldots,q, \tag{1.16}$$

or we can find y from $\xi(\bar{x},y)=0$ and substitute in the last equations (1.13) and find $\bar{x}$.

6. Application the method of β- functions to linear programming.

The Problem of Linear Programming is

$$I=\sum_{i=1}^{n}c_ix_i=\min,\quad \sum_{j=1}^{n}a_{kj}x_j-b_k\le 0,\quad k=1,2,\ldots,m \tag{1.17}$$

Here $c_i,\ a_{kj},\ b_k$ are constant.

Take $\omega_j=y_i$. Then equation (1.13) are

$$y_k(\sum_{j=1}^{n}a_{kj}x_j-b_k)=0,\quad k=1,2,\ldots,m \tag{1.18}$$

$$c_i+\sum_{j=1}^{m}a_{ij}y_j=0,\quad i=1,2,\ldots,n \tag{1.19}$$

Selective from (1.18) *l* equations $(l \le n,\ l \le m,\ l = \max)$ and *l* variables x_j such that determinant $|a_{kj}| \ne 0$. Find $\widetilde{x}_j$ from these *l* linear equations (1.18) (corresponded $y_k \ne 0$).

If this solution don't satisfy inequalities (1.17), we take *l* other equations and repeat this procedure (process) while we find $\widetilde{x}_j$ which satisfy (1.17). If these equations absent, we take *l* -1 equations (1.18) and repeat process, than *l* - 2 equations and so on, while we get *l* = 0.

If solution, which satisfy (1.17), absent that inequality (1.17) is conflicting (incompatible) and cannot be solved.

Assume that by using this procedure we find the solution $\widetilde{x}_j$, that satisfy (1.17). Take in (1.19) all y_j, which don't belong the taken questions (1.18), equal zero and find *y* from equation (1.19). If all $\widetilde{y}_j \ge 0$ then $\widetilde{x}_j$ is point of minimum of problem (1.17). If part of $\widetilde{y}_j < 0$, then we change corresponded equations (1.18) by other and repeat this process while get all $\bar{y}_j \ge 0$.

We can suppose that this process makes all $\widetilde{y}_j \ge 0$. Inequality $\widetilde{y}_j \ge 0$ means that anti-gradient has direction into internal of the corresponding constraints. Because our problem and constrains are linear, anti-gradient, which has direction into constrains, will has this direction in any point of corresponding hyper plate (1.17). It means that this procedure will increase the amount of $y_j \ge 0$.

Example 1.4.

Find minimum of Problem

$$I = x_1 + x_2, \quad -x_1 \le 0, \quad -x_2 \le 0, \quad x_1 - 1 \le 0, \quad x_2 - 1 \le 0. \tag{1.20}$$

The equations (1.18),(1.19) are

$$\begin{aligned} &-y_1 x_1 = 0, \quad y_3(x_1 - 1) = 0, \quad 1 - y_1 + y_3 - 0, \\ &-y_2 x_2 = 0, \quad y_4(x_2 - 1) = 0, \quad 1 - y_2 + y_4 = 0. \end{aligned} \tag{1.21}$$

Chose equations $x_1 - 1 = 0, \quad x_2 - 1 = 0$. From solution of them we have $\widetilde{x}_1 = 1, \quad \widetilde{x}_2 = 1$. They satisfy (1.20). From the first column of (1.21) we get $y_1 - y_2 = 0$, and from the last column (1.21) we find $y_3 = y_4 = -1$. Inequality $y_i \ge 0$ is not satisfied. Change equalities by others $\widetilde{x}_1 = 0, \quad \widetilde{x}_2 = 0$. We get $\widetilde{y}_1 = \widetilde{y}_2 = 1 > 0$. Hence $\widetilde{x}_1 = \widetilde{x}_2 = 0$ is point of the global minimum.

Example 1.5.

Find point of global minimum in Problem

$$I = -x_1 - x_2, \quad x_1 + x_2 \le 0.$$

Solution. Write equations (1.18),(1.19)

$$y(x_1 + x_2) = 0, \quad -1 + y,$$

From $x_1 + x_2 = 0$ we get $\widetilde{x}_1 = -\widetilde{x}_2$. From -1+*y* = 0 we get *y* = 1 > 0. Sense any $\widetilde{x}_1 = -\widetilde{x}_2$ is optimal.

7. Application of method β-function to quadratic programming.

This problem is following:

$$I = \sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij} x_i x_j, \quad \sum a_{kj} x_j - b_k \le 0, \quad k = 1,2,\ldots,m. \tag{1.22}$$

Assume that quadratic form in function (1.22) is positive. If don't consider constraints in (1.22), it is obvious the point of minimum in this problem is $x_j^* = 0.$ If this point satisfies inequalities in (1.22), the process of solution is finished. In particular, we have this case when all $b_k \geq 0$. We consider not triviality case. Take $\omega_j = y_j$. Equations (1.13) and (1.14) are:

$$y_k(\sum_{j=1}^{n} a_{kj}x_j - b_k) = 0 \quad i,k = 1,2,\ldots,n; \quad \sum_{j=1}^{n} c_{ij}x_j + \sum_{j=1}^{m} y_l a_{jk} = 0, \quad y_k = 0. \qquad (1.23)$$

Later procedure is analogous of the Linear Programming.

Example 1.6.

Problem are:

$$I = 0.5x_1^2 + 0.5x_2^2, \quad -x_1 - x_2 + 1 \leq 0, \quad x_1 - 1 \leq 0, \quad x_2 - 1 \leq 0. \qquad (1.24)$$

The equations (1.23)

$$\begin{aligned} &y_1(-x_2 - x_1 + 1) = 0, \quad y_2(x_1 - 1) = 0, \quad y_2(x_2 - 1) = 0 \\ &x_1 - y_1 + y_2 = 0, \quad x_2 - y_1 + y_2 = 0 \end{aligned} \qquad (1.25)$$

Take the 2-nd and 3-rd equations. We get $\widetilde{x}_1 = \widetilde{x}_2 = 1$. The inequalities (1.24) are satisfied, but from two the last equations (1.25) for $y_1 = 0$ we have $\widetilde{y}_2 = \widetilde{y}_3 = -1$. It is contrary the request $\widetilde{y}_i \geq 0$.

Take the 1-st equation in (1.25). We have $\widetilde{x}_2 = 1 - \widetilde{x}_1$. Solve it together with equations $\widetilde{x}_1 - \widetilde{y}_1 = 0, \quad \widetilde{x}_2 - \widetilde{y}_1 = 0$ we get $\widetilde{x}_1 = \widetilde{x}_2 = 1/2, \quad \widetilde{y}_1 = \widetilde{y}_2 = 1/2 > 0$. Hence $x_1 = x_2 = 1/2$ is point of global minimum.

Appendix to #1. Proof of Theorems.

1°. Proof of Theorem 1.1. Proof of:

Statement 3. By subtracting the inequality $\beta(x,y) \leq \beta(\bar{x}(y),y)$ from $I(x) + \beta(x,y) \geq I(\bar{x}(y)) + \beta(\bar{x}(y),y)$ we get $I(x) \geq I(\bar{x}) \quad \text{over} \quad P$. Statement 3 of the Theorem 1.1 is proved.

Statement 1 of the Theorem 1.1 is obvious because *X=M+P* and $I(x) \geq I(\bar{x}) \quad \text{over} \quad P$, we have $x^* \in M$. Statement 1 of Theorem 1.1 is proved.

Statement 2. By subtracting the inequality $J \geq \bar{J} \quad \text{from} \quad J + I \leq \bar{J} + \bar{I}$ we get $I(x) \leq I(\bar{x}) \quad \text{over} \quad N$.

Theorem 1.1 is proved.

2°. Proof of Theorem 1.2. By adding the inequality

$I(x) + \beta(x,y) \geq I(\bar{x}(y)) + \beta(\bar{x}(y),y)$ and $-\beta(x,y) \geq -\sup_X \beta(x,y)$ over *X,* we get the estimate (1.2).

3°. Proof of Theorem 1.3. *Statements 1, 3*. By subtracting the inequality $\beta \leq \hat{\beta}$ from $I + \beta \geq \hat{I} + \hat{\beta}$ we get $I \geq \hat{I}$ over set *P*. Statement 1 follow from this.

Statement 2. By subtracting the inequality $\beta \geq \hat{\beta}$ from $\beta - I \geq \hat{\beta} - \hat{I}$ and multiply this result by -1, we get $I \leq \hat{I}$ over N, The theorem 1.3 is proved.

4°. Proof of Theorem 1.7. Assume x_N is point of the minimum of the objective function *J*(*x*).Therefore $J'(x_N) = 0$ because *J*(*x*) is continuously and differential, x_N is single point N_i on set *X** since this is (see Theorem 1.1')

$$I(x)+J(x)\ge I(x_N)+J(x_N)\,.$$

This means that $J_i(x_N)=\inf_X[I(x)+J(x)]$. The function *I*(*x*), *J*(*x*) are continuously and differential, hence $I'(x_N)+J'(x_N)=0$. But $J'(x_N)=0$, therefore $I'(x_N)=0$. Theorem 1.7 is proved.

5°. Proof of Theorem 1.8. By subtracting the inequality $\beta\ge\beta_N$ from $\beta-I\le\beta_N-I_N$ we get $I\ge I_N$ over set *X**. The Theorem 1.8 is proved.

6°. Proof of Theorem 1.9.

Sufficiency. From "1)" of (1.9) we have

$$I+\lambda_i F_i+\omega_j\Phi_j\ge\bar I+\bar\lambda_i\bar F_i+\bar\omega_j\bar\Phi_j\,.$$

From this and "4)" (1.9) we get $I+\lambda_i F_i+\omega_j\Phi_j\ge\bar I$. Look it inequality over X*. On X* we have $\lambda_i F_i=0,\quad \omega_j\Phi_j\le 0$ hence $I(x)\ge I(\bar x)$. Because $\bar x\in X^*$ hence $\bar x$ is the point of global minimum of *I*(*x*) on *X**.

Necessity. (Method of designing). Assume that $x^*\in X$ exists. Design β(*x,y*) following way. Take $\lambda_i\equiv 0\quad \text{on}\quad X^*$ and take functions $\lambda_i,\ \omega_j\ge 0$ such us $J(x)>m\quad on\quad set\quad X\setminus X^*$. Then we have as the result of our design

$$J(x^*)=\inf_{x\in X^*}J(x),\quad x^*\in X^*,\quad \omega_j\ge 0,\quad \bar\beta=0\,.$$

The theorem 1.9 is proved.

§2. Method of combining of the extremes.

Let us to have the problems:

Problem 1 $\quad I(x^*)=\inf I(x),\quad x\in X^*$;

Problem 2 $\quad J(\bar x)=\inf[I(x)+\beta(x)],\quad x\in X$;

Problem 3 $\quad \beta(\hat x)=\sup\beta(x),\quad x\in X.$

Assume that all points $x^*,\bar x,\hat x$ are exist.

Theorem 2.1. *Let X=X*, then for every couple* $(\bar x_i,\hat x_i)$ *which satisfy the condition* $\bar x_i=\hat x_i$ *we have*

$$\bar x_i=\hat x_i=x_i^*\,.$$

Proof. Let $\bar x_i=\hat x_i$ Then

$$\inf J(x)-\sup\beta(x)=J(\bar x_i)-\beta(\bar x_i)=I(\bar x_i)-\beta(\bar x_i)-\beta(\bar x_i)=I(\bar x_i)\,.$$

But with other side from Theorem 1.2 we have $\inf J(x)-\sup\beta(x)\le\inf I$. That is $I(\bar x_i)\le I(x^*)$. As x^* is point of global minimum and *X=X** hence must be only $I(\bar x_i)=I(x_i^*)$. As far as $\bar x_i$ and x_i^* exist we can find the point of minimum x_i^* such that $\bar x_i=x_i^*$. Theorem 2.1 is proved.

Theorem 2.2. *Let X=X*. If exist at least one of the couple* $(\bar x_i,\hat x_i)$ *such that* $\bar x_1=\hat x_1$, *then in every point* x_i^* *we have*

1) $x_i^*=\hat x_i,\quad 2)\ x_i^*=\bar x_i$.

Proof. 1. Assume the contrast: $\bar{x}_i \neq x_i^*$. Than summarize $I(\bar{x}_i) = I(x_i^*) \quad and \quad \beta(x_i^*) < \beta(\bar{x}_i) = \beta(\hat{x}_i)$ we get $J(x_i^*) < J(\bar{x}_i)$. This contrast $J(\bar{x}_i) = \inf J(x)$.

2. Add $J(\bar{x}_i) = J(x_i^*) \quad and \quad \beta(x_i^*) = \beta(\bar{x}_i) = \beta(\hat{x}_i)$ we get $J(x_i^*) = J(\bar{x}_i)$, hence $x_i^* = \bar{x}_i$. Theorem 2.2 is proved.

From Theorems 2.1, 2.2 we have

Consequence:

If we want to find all points of minimum of Problem 1 it necessary and sufficiently to find all corresponding couple $(\bar{x}_i, \hat{x}_i)$.

We shall call the Problems 1 and 2 equivalents if all correspondent points of minimum of these Problems are coincided.

From Theorem 2.2 we have:

1. For equivalence of Problems 1, 2 is sufficient to exist one couple such that $\bar{x}_i = \hat{x}_i$.

2. Let exist β-functional and although one of couple $(\bar{x}_i, \hat{x}_i)$ such that $\bar{x}_i = \hat{x}_i$.

Then any points of minimum of Problem 2 and point of maximum of Problem 3 is point of minimum of Problem 1, and back, any point of minimum of Problem 1 is point of minimum of Problem 2 and point of minimum of Problem 3.

Remarks:

1. If $\beta(\bar{x}) = 0, \quad \text{then} \quad \inf J(x) = \inf I(x)$.

2. If $\bar{x} = \hat{x}$, then the lower estimate (1.1) in §1 coincide with infimum of the functional *I*(*x*).

From consequence 1 §2 we have the following

Algorithm 3. (Method of combining the extremes)

Let us take some bounded functional β (x,y) where y is an element of the set Y. We solve this problem

$$\inf[I(x) + \beta(x, y)], \quad x \in X^*$$

and find the point of minimum

$$\bar{x}_1 = \bar{x}_1(y).$$

From

$$\sup \beta(x, y)$$

we find

$$\bar{x}_2 = \bar{x}_2(y).$$

After this we equate

$$\bar{x}_1(y) = \bar{x}_2(y) \tag{2.1}$$

and from this equation of the combination of extreme we find the roots y_i.

These roots are the points of minimum for Problem 1:

$$\bar{x} = \bar{x}_1(y_i) = \bar{x}_2(y_i)$$

Since the Problem of finding of minimum is reduced to Problem of finding at least one root of equation of the combination of extremes (2.1).

The exist and difficulty of finding of roots depend from choose of β-functional, from freedom of its deformation, which give the "*y*" relation.

Note that is differ from the regular method of finding of minimum. In the usual method we take partial derivatives, equal its zero, get the set equation and from them we find only the stationary (extreme) points. They may be points local minimum, maximum, or inflection. By this method we find points of global minimum.

Thus, we find the connect two various (different) problems.

The existence of solution in equation of the combination of extremes is sufficient condition for the existence of absolute minimum of functional in Problem 1.

The mathematic has good achievements in the field of existence of solution of equations. And equation (2.1) give connection between these problems and give some opportunity in solving of optimal problems.

Note also that equation (2.1) not requests that functional was continuous and differential function, hence it has wider domain for application.

If point of minimum cannot be getting in explicit form than we can write this equation in form

$$\varphi_1(x, y) = 0, \quad \varphi_2(x, y) = 0, \tag{2.1'}$$

where function φ_1, φ_2 are got from

$$\inf_X J(x, y), \quad \sup_X \beta(x, y).$$

Example 2.1. Find a point of minimum of functional

$$I = 2x^4 + x^2 - 2x + 1, \quad -\infty < x < \infty$$

<u>*Solution*</u>: Use algorithm 3. Take

$$\beta = -yx^2 + 2x.$$

Than

$$J = I + \beta = 2x^4 + (1 - y)x^2 + 1.$$

Denote x^2=w and substitute in J:

$$J = 2w^2 + (1 - y)w + 1.$$

Find point of minimum this functional

$$J'_w = 4w + (1 - y) = 0, \quad \overline{w} = \bar{x}_1^2 = \frac{1}{4}(y - 1)$$

and point of maximum functional β:

$$\beta(x) = -yx^2 + 2x, \quad \beta'_x = -2yx + 2 = 0, \quad \bar{x}_2 = 1/y.$$

Equate $\bar{x}_1$ to $\bar{x}_2$

$$\bar{x}_1^2 = \bar{x}_2^2 \quad \frac{1}{4}(y-1) = \frac{1}{y^2}, \quad y^3 - y^2 - 4 = (y-2)(y^2+y+1)$$

This equation has only alone root $\bar{y} = 2$. Since $\bar{x} = \frac{1}{y} = \frac{1}{2}$.

§3. Remark about γ-functional

A) Let us take

$$\beta(x) = [\gamma(x) - 1]I(x) \tag{3.1}$$

then

$$J(x) = I(x)\gamma(x).$$

This form of common functional is sometimes more comfortable because we can chouse the multiplier to *I*(*x*) which make *J*(*x*) simpler.

Using our results about β-functional for this case we get following:

If *X*=*X** and we finding the point of global minimum Problem 2:

$$\inf_X J(x) = \inf_X [I(x)\gamma(x)] \tag{3.2}$$

than

1) Set

$$M = \{x : J - I \geq \bar{J} - \bar{I}, \quad x \in X\}$$

contains the point of global minimum of Problem 1;

2) Set

$$N = \{x : \ I\gamma + I \leq \bar{I}\bar{\gamma} + \bar{I}, \quad x \in X\}$$

contains the better or same solutions than $\bar{x}$ (that is over *N*, we have $I(x) \leq I(\bar{x})$);

3) Set

$$P = \{x : J - I \leq \bar{J} - \bar{I}, \quad x \in X\}$$

contains the worse or same solutions than $\bar{x}$ (that is over *P*, we have $I(x) \geq I(\bar{x})$).

All these statements follow from (3.1) and Theorem 1.1.

Lower estimate (from Theorem 1.3 and (3.1) look as

$$I(x) \geq \inf_X J - \sup_X (J - I). \tag{3.3}$$

Condition of equivalence of Problem 1 and 2 (theorem 2.1) in this case (*X=X**) is:

$\bar{x}$ and $\hat{x}$, which are founded from problems

$$\inf_X J(x) \quad and \quad \sup_{X^s}[J(x) = I(x)],$$

must equal respectively.

Algorithm 3 (Method of combining the extremes) is used for this case without change.

B) However, for this case we get some new results.

Let define functional $\gamma(x,y) \neq 0$ over set $X \times Y$. We call it as γ-functional. Take functional

$$J(x,y) = I(x)\gamma(x,y)$$

Theorem 3.1.

Assume X=X, $\bar{x}$ is point of global minimum of Problem 2:*

$$\inf J(x), \quad x \in X, \quad where \quad J = I(x)\gamma(x),$$

Then:

1) *Set*

$$P = \{x: \ 0 < \gamma \le \bar{\gamma}\}$$

contains worth or same solutions of Problem 1 (that is $I(x) \ge I(\bar{x})$ over P);

2) *Set*

$$N = \{x: \ 0 > \bar{\gamma} \ge \gamma\}$$

contains better or same solution of Problem 1 (that is $I(x) \le I(\bar{x})$ over N);

3) *The point of global minimum is in set $M = X \setminus \beta, \quad where \quad \overset{o}{P} = \{x: \ 0 < \gamma < \bar{\gamma}\}$.*

Proof: 1. From inequalities $I\gamma \ge \bar{I}\bar{\gamma}, \quad 0 < \gamma \le \bar{\gamma}$ we have $I \ge \bar{I}\bar{\gamma}/\gamma, \quad \bar{\gamma}/\gamma \ge 1$. That is $I \ge \bar{I}$.

2. From inequalities $I\gamma \ge \bar{I}\bar{\gamma}, \quad 0 > \gamma \ge \bar{\gamma}$ we get $I \le \bar{I}\bar{\gamma}/\gamma, \quad \bar{\gamma}/\gamma \le 1$. That is $I \le \bar{I}$.

3. Because *X=M+P* and $M \cap \overset{o}{P} \neq 0$, we have $M = X - \overset{o}{P}$. Theorem is proved.

Theorem 3.2. *Assume $\sup_X \gamma > 0$. Then we have the lower estimation*

$$I(x) = \frac{\bar{J}}{\sup \beta} \quad on \quad X. \tag{3.4}$$

If $\sup_X \gamma(x,y) > 0 \quad for \quad \forall y \in Y$, we have the lower estimate

$$I(x) \geq \sup_{Y}\left(\frac{\bar{J}}{\sup\limits_{X} X}\right). \qquad (3.4)'$$

Proof: 1) For written conditions from $I\gamma \geq \bar{I}\bar{\gamma}$ we got $I \geq \bar{J}/\gamma \quad and \quad I \geq \bar{J}/\sup\limits_{X}\gamma$.

2) Take this estimate by *y*, we get expression (3.4)'.

Example 3.1. Find the lower estimate for functional

$$I = (x^2 - \cos x + 1)e^{(x-1)^2} \quad -\infty < x < \infty.$$

Take

$$\gamma = e^{-(x-1)^2}.$$

Then

$$J = x^2 - \cos x + 1.$$

If it is obvious the point of minimum this functional

$$\bar{x} = 0, \quad \bar{\gamma} = 1 > 0, \quad \sup_{X}\gamma = 1.$$

Use the estimate (3.4) we get $I(x) \geq 0$. But for *x* = 0 we have *I*(0) = 0. That way *x* = 0 is point of global minimum.

§4. Application β- function to the multi-variable nonlinear problems of constrained optimization and to problems described by regular differential equations.

A) The first problem is following. Find minimum of functional

$$I=f_0(x), \qquad (4.1)$$

Where *x*-*n*-dimensional vector, which satisfy independent equations

$$f_i(x) = 0, \quad i = 1,2,\ldots,m \leq n. \qquad (4.2)$$

Functions *f*(*x*) is defended in the open domain *n*-dimensional vector of space *X*. The admissible set *X** separate from *X* by equations (4.2).

Let us take some functional β(*x*), such that to find

$$\inf[f_0(x) = \beta(x)] \quad \text{on} \quad X^*.$$

It is easier to solve.

Then from solution of Problem 2 in accordance with theorems of §1 we get the following information about Problem 1:

1) The point of global minimum is in set $M = \{x:\ \beta(x) \geq \beta(\bar{x})\}$;
2) The set $N = \{x:\ 2f_0 + \beta \leq 2\bar{f}_0 + \bar{\beta}\}$ contains better and same solutions (that is $f_0(x) \leq f_0(\bar{x}) \quad \text{on} \quad N$);
3) The set $P = \{x:\ \beta(x) \leq \beta(\bar{x})\}$ contains worth and same solutions (that is $f_0(x) \geq f_0(\bar{x})$ on P;
4) If $X=X^*\subseteq P$, that $\bar{x}$ is point of global minimum of problem 1 (consequence 3 of §1).

Let us assume we widen the set *X** for simplification of solution. For example, we reject the part of constrains (4.2). Then we have

5) If $X^* \cap M = \varnothing$, then $J(\bar{x})$ is lower estimation $f_0(x)$ on *X** (consequence 5, §1).
It is more comfortable some times to take the suitable J(x) at first and find the point minimum of problem inf *J*(*x*) on *X**.

Then the corresponding sets will be (from theorem 1.1')

$$M = \{x:\ J - I \ge \bar{J} - \bar{I}\},\ \ N = \{x:\ J + I \le \bar{J} + \bar{I}\},\ \ P = \{x:\ J - I \le \bar{J} - \bar{I}\}.$$

If we solve the problem $\beta(\hat{x}) = \sup \beta(x) \quad on \quad X \supseteq X^*$ we get the additional lower estimate

$$f_0(x) \ge f_0(\bar{x}) + \beta(\bar{x}) - \beta(\hat{x}),$$

(theorem 1.3) and set

$$M = \{x:\ f_0 + \beta \le \hat{f}_0 + \hat{\beta}\},\ \ N = \{x:\ \beta - f_0 \ge \hat{\beta} - \hat{f}_0\},\ \ P = \{x:\ f_0 + \beta \ge \hat{f}_0 + \hat{\beta}\}.$$

(theorem 1.4).

Take series β_i we can get the solution of one from Problems of §1 or to facilitate the solution of Problem 1.

The example for case *X**=*X* was over (see Examples 1.1-1.3). Explain by simple examples (how you can apply the method β-functional for case, when *X**≠*X* that is problem with constrains.

Example 4.1. Find minimum of functional

$$I = x \quad on \quad x^2 + y^2 - 1 = 0\ .$$

Take any admissible point, for example $\bar{x}_0 = 1, \quad \bar{y}_0 = 0 \quad and \quad J(x)$ functional as

$$J_1 = (x - x_0)^2.$$

The point of minimum of this functional is obvious $\bar{x} = x_0$. The set *M*, containing the point of global minimum, is

$$J_1 - I \ge \bar{J}_1 - \bar{I}, \quad that \quad is \quad (x-1)^2 - x \ge -1 \quad or \quad |x - 3/2| \ge 3/2$$

The boundaries of this inequality together with admissible subset (circle) draw on fig.1.3a. We see the point of absolute minimum is in left half of circle.

Take now the admissible point $\bar{x}_0 = -1, \quad y = 0$ and *J* -functional in more common case as

$$J_2 = c(x - x_0)^2, \quad c > 0.$$

Then *M* set is

$$cx^2 + 2cx + c - x \ge 1.$$

Take *c* = 0.5. Then we get $|x| \ge 1$ (fig. 1.3b).

Set *M* contain only two admissible point: x_1=1 and x_2= -1. But point x_1=1 from the J_1 cannot be the point of absolute minimum. Since the point of global minimum is $\bar{x} = -1, \quad \bar{y} = 0$.

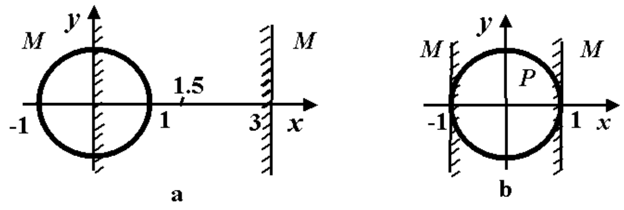


Fig. 1.3

<u>Example 4.2</u>. Find the point of global minimum of functional with constrain

$$I = x^2 - x + y^2 - 2y + 1, \quad y - \ln\left(x \pm \sqrt{x-1}\right) = 0 .$$

Take *J* functional

$$J = (x - x_0)^2 + (y - y_0)^2 .$$

The set *M* is separated by inequality

$$J - I \geq \bar{J} - \bar{I}, \quad or \quad 2y(1 - y_0) \geq (2x_0 - 1)x + a ,$$

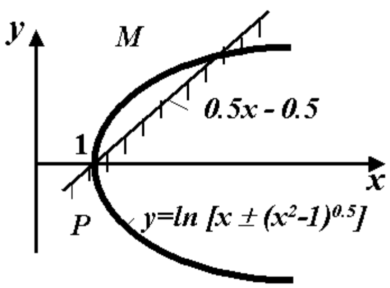
where

$$a - x_0 - 2x_0^2 + 2y_0 - 2y_0^2 .$$

Take the admissible point $x_0 = -1, \quad y_0 = 0$. Then

$$M = \left\{x, y : y \geq \frac{1}{2}x - \frac{1}{2}\right\} \qquad \text{(Fig.1.4)}.$$

From drawing we see *M* is small domain and find the point of global minimum no difficult.



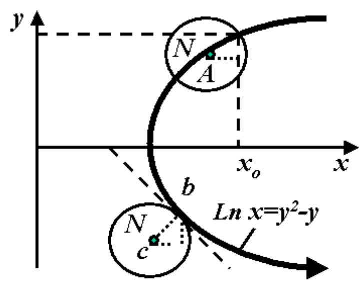


Fig. 1.4, Fig.1.5.

<u>Example 4.3</u>. Given functional and constrains is

$$I = 2x + 2y, \quad \ln x = y^2 = y$$

Take

$$J = (x - x_0)^2 - (y - y_0)^2 ,$$

where couple x_0, y_0 is admissible point.

The set *N* is separated with according Theorem 1.1 by inequality $J + I \le \bar{J} + \bar{I}$, that is

$$\left[x-(x_0-1)^2\right]+\left[y+(y_0-1)\right]^2 \le 2.$$

This is interior of the circle (fig.1.5).

Assume that a center of this circle is located in the point *A*. The set *N* intersect with admissible curve ln $x = y^2 - y$. If we take a point x_0, y_0 from this intersection, we will descent along this curve whole the set *N* become by point. This take place in point *B*, where the tangent to permissible curve has the angle -45^0 (because the center of the circle is located from point x_0,y_0 from -1, -1, that is the angle +45^0, (fig. 1.5). Any moving from this point will return us to it.

May be shown that the point *B* is the point of global minimum.

Take into consideration when we have used the methods of β-functional we have not used in continuously and differ of functional (4.1) and constancies (4.2) unlike from known methods (for example, theory of extreme functions).

B) Consider how we can apply the methods given in §1 to optimization problems are described by regular differential equations. Below we write the statement of problem, which we widely use in future.

Assume that the moving of object is described by set of independent differential equations

$$\dot{x}_i = f_i(t,x,u), \quad i = 1,2,\ldots,n, \quad t \in T = [t_1 t_2], \tag{4.3}$$

where $x(t)$ is *n* - dimensional continually piece-differential vector-function of the phase coordinates, $x \in G(t)$; $u(t)$ is *n* - dimensional function which continuous on *T* except the limited number of point where it can have discontinuities of the 1-st form, $u \in U$ is an independed variable. Boundary values t_1, t_2 is given, $x(t_1) \in G(t_1)$, $x(t_2) \in G(t_2)$.

The aim function is

$$I = F(x_1,x_2) + \int_{t_1}^{t_2} f_0(t,x,u)dt, \quad x_1 = x(t_1), \quad x_2 = x(t_2). \tag{4.4}$$

Functions $F(x_1,x_2)$, $f_i(t,x,u)$, $i = 0,1,\ldots,n$ are continuous over $T \times G \times U$. Set of continuous, almost

everywhere differentiable functions $x(t) \in G(t)$ we denote *D*. Set of pies-continuous functions $x(t) \in U$, we denote *V*. Set of couple $x(t)$, $u(t)$ which satisfy these requirements and almost everywhere comply with equations (4.3) we shall call **admissible** and denote *Q*, $Q \subset D \times V$.

Consider the problems:

a) Find the couple $u^*(t)$, $x^*(t) \in D$, which give the minimum of function (4.4) (Traditional statement).
b) Find sup-set $N \subset G \times U \times T$ such that any admissible curve from *N* we have $I(x) \le c$, where *c* is constant.
c) Find the lower estimate of $I(x)$ over *Q*.

Take the function $\int_{t_1}^{t_2} \beta(t,x,u)dt$, where $\beta(t,x,u)$ is a definite and continuous function on $T \times G \times U$.

Theorem 4.1. *Let us assume that $F \equiv 0$ and Problem 2 is solved. That means*

$$J(\bar{x},\bar{u}) = \inf J(x,u) \quad \text{on} \quad Q,$$

where

$$J = \int_{t_1}^{t_2} [f_0(t,x,u) + \beta(t,x,u)]dt .$$

Then:

1) Set

$$N = \{t,x,u:\ 2f_0 + \beta \le 2\bar{f}_0 + \overline{\beta},\quad t \in T\}$$

contains the same or better solutions of Problem 1.

3) Set

$$P = \{t,x,u:\ \beta \le \overline{\beta},\quad t \in T\}$$

contains the same or worse solutions of Problem 1.

Proof: 1. On set Q from N we have

$$\int_{t_1}^{t_2} (2f_0 + \beta)dt \le \int_{t_1}^{t_2} (2\bar{f}_0 + \overline{\beta})dt .$$

Subtract from this inequality following

$$\int_{t_1}^{t_2} (f_0 + \beta)dt \ge \int_{t_1}^{t_2} (\bar{f}_0 + \overline{\beta})dt , \tag{4.5}$$

we get over *Q* from *N*

$$\int_{t_1}^{t_2} f_0 dt \le \int_{t_1}^{t_2} \bar{f}_0 dt .$$

2. By analogy with above, subtract from inequality

$$\int_{t_1}^{t_2} \beta dt \le \int_{t_1}^{t_2} \overline{\beta} dt$$

the inequality (4.5) we get over Q from *P*

$$\int_{t_1}^{t_2} f_0 dt \ge \int_{t_1}^{t_2} \bar{f}_0 dt .$$

The Theorem 4.1 is proved.

Sets *N*, *P* not empty. They contain at least one curve from Q. This curve is $\bar{x}(t), \bar{u}(t) \in Q$.

If we solve the additional problem

$$\sup_Q \int_{t_1}^{t_2} \beta dt ,$$

we get additional information about sets *N*, *P* and lower estimate. It is following

Theorem 4.2. *Let us assume F ≡ 0 and solved the Problem*

$$\sup \int_{t_1}^{t_2} \beta(t,x,u)dt \quad \text{on} \quad Q .$$

Then

1) *Set*

$$N = \{t, x, u: \ \beta - f_0 \ge \hat{\beta} - \hat{f}_0, \quad t \in T\}$$

contains the same or better solutions:

2) *Set*

$$P = \{t, x, u: \ f_0 + \beta \le \hat{f} + \hat{\beta}, \quad t \in T\}$$

contains the same or worse solutions.

Here $\hat{f}_0 = f_0(t, \hat{x}, \hat{u}), \quad \hat{x}(t), \quad \hat{u}(t)$ is curve of extreme

$$\sup \int_{t_1}^{t_2} \beta(t) \quad \text{on} \quad Q .$$

Proof: 1. Over *Q* from *N* we have

$$\int_{t_1}^{t_2} (\beta - f_0) dt \ge \int_{t_1}^{t_2} (\hat{\beta} - \hat{f}_0) dt$$

Subtract from this inequality the following

$$\int_{t_1}^{t_2} \beta dt \le \int_{t_1}^{t_2} \hat{\beta} dt ,$$

we get

$$\int_{t_1}^{t_2} f_0 dt \le \int_{t_1}^{t_2} \hat{\hat{f}}_0 dt .$$

2. By analogy, subtract $\int_{t_1}^{t_2} \beta dt \le \int_{t_1}^{t_2} \hat{\hat{\beta}} dt$ from

$$\int_{t_1}^{t_2} (f_0 + \beta) dt \ge \int_{t_1}^{t_2} (\hat{\hat{f}}_0 + \hat{\hat{\beta}}) dt$$

we get

$$\int_{t_1}^{t_2} f_0 dt \ge \int_{t_1}^{t_2} \hat{\hat{f}}_0 dt .$$

The Theorem 4.2 is proved.

Theorem 4.3. *(Lower estimation).*

Assume F ≡ 0, the ends of x(t) are fixed, β (t,x,u) is defined and bounded on G×U×T.

Then there is lower estimate of Problem 1:

$$I(x, u) \ge \int_T [f_0(t, \bar{x}, \bar{u}) + \beta(t, \bar{x}, \bar{u}) - \beta(t, \hat{x}, \hat{u})] dt \tag{4.6}$$

Proof: Subtract $\int_T \beta dt \le \int_T \sup \beta dt$ from inequality

$$\int_T (f_0 + \beta) dt \ge \int_T (\bar{f}_0 + \overline{\beta}) dt$$

we get (4.6). The theorem 4.3 is proved.

Consequence 1: Couple $\bar{x}, \bar{u}$ is curve of absolute minimum of Problem 1 over set *N*.

Consequence 2: If set P⊒T×G×U (or accessible) than $\bar{x}, \bar{u}$ (or $\hat{x}, \hat{u}$) is curve of global minimum of problem 1 over *Q*.

Similar results we can get for case, when $F \neq 0$ and ends of *x(t)* can move.

Example 4.4. Assume the problem is described by conditions:

$$I = \int_0^1 (x^2 + e^u)dt, \quad \dot{x} = u, \quad |u| \leq 1, \quad x(0) = 1, \quad x(1) = 0.$$

Use the theorem 4.1. Take $\beta = -e^{+u}$. We get the problem

$$I = \int_0^1 x^2 dt, \quad \dot{x} = u, \quad |u| \leq 1, \quad x(0) = 1, \quad x(1) = 0.$$

Its solution is $\bar{x} = -t, \quad \bar{u} = -1, \quad 0 \leq t \leq 1.$

Find set *P*: $\beta \leq \overline{\beta}. \quad That \quad is \quad e^u \geq e^{-1}, \quad u \geq 1$.

But value *u* < -1 is not acceptable. Since *P* is cover all admissible set points *t,x,u*. That way $\bar{x} = -t$.

Is the curve of global minimum (see Consequence 2).

Example 4.2. Find of minimum in problem

$$I = \int_0^2 (|x| + 0.5x^2)dt, \quad \dot{x} = u, \quad |u| \leq 1, \quad x(0) = 1, \quad x(2) = 0.$$

We have here undifferentiated function in integral. Known methods us variational calculation or principle of maximum are not been used.

Change this problem following "good" (easy) problem:

$$I = \int_0^2 0.5x^2 dt, \quad \dot{x} = u, \quad |u| \leq 1, \quad x(0) = 1, \quad x(2) = 0$$

and find

$$\sup_{x(t)} L.$$

The solution is shown in Fig. 1.6.

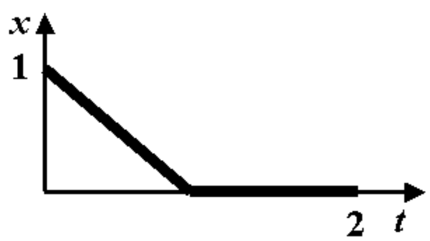


Fig. 1.6. For Example, 4.5.

By according the theorem 4.2

$$P = \{x :\ |x| \geq |\bar{x}|\},$$

that means set *P* cover all accessible domain. Since obtained, solution is curve of global minimum of Problem 1.

5. Method of β - function in minimizing sequences

A) The sequence $\{x_s\}$ such that $I(x_s) \underset{s\to\infty}{\to} \inf I(x)$ on the set *X** is named as a minimizing sequence (for Problem 1).

We must design these sequences in the successive approximation methods and in case, when extreme is absent in an allowable (admissible) subset.

Theorem 5.1. *Assume $\beta(x) \leq 0$ on X* and there exists sequence $\{x_s\} \in X^*$ such, that*

$$J(x_s) \to \inf J \quad \text{for} \quad s\to\infty \quad \text{on} \quad X \tag{5.1}$$

Then: 1) $I(x_s) \underset{s\to\infty}{\to} m = \inf I(x) \quad on \quad X^*$*;*

2) Any sequence $\{x_s\} \in X$*, which satisfy* (5.1) *or* $I(x_s) \to \inf_X J$*, minimize I(x) on X*, minimize*

and J(x) on X.

<u>Proof: 1</u>. Because $\beta(x) \leq 0$ on *X**, we have $\inf_X J \leq I(x)$. That is $\inf_X J \leq \inf_{X^*} J$. From $\{x_s\} \in X^*$ and (5.1) we have that

$$\inf_X J = \inf_{X^*} I\ . \tag{5.2}$$

That is $I(x_s) \to m$.

2. From (5.1) and (5.2) we have the statement 2 of the theorem.

3. From $I(x_s) \underset{s\to\infty}{\to} m$ and (5.2) we have $J(x_s) \to \inf J \quad \text{for} \quad s\to\infty$ on X. Theorem is proved.

Remark. The requirement $\beta(x) \leq 0$ on *X** of the theorem 5.1 we can change by the requirement $\sup_{X^*} \beta \leq 0 \quad \text{on} \quad X^*$ because from sup $\beta \leq 0$ on *X** we have $\beta(x) \leq 0$ on *X**.

Theorem 5.2. *Assume there exist the sequence $\{x_s\} \in X^*$ such that*

$$J(x_s) \underset{s\to\infty}{\to} \inf J(x) \ \text{ on } X \text{ (or } X^*) \quad \text{and} \quad \beta(x_s) \to \sup \beta \quad \text{on } X \text{ (or } X^*) \tag{5.3}$$

Then this sequence is minimized.

<u>Proof</u>*:* From $I(x_s) + \beta(x_s) \to \inf J$ and $\beta(x_s) \to \sup \beta$ we get that $I(x_s) \to \inf J - \sup \beta$. Because $I(x_s) \geq \inf J - \sup \beta$ *and there exist* $\{x_s\} \in X^*$ *we have* $I(x_s) \to m = \inf J - \sup \beta$. Q.E.D.

Remark: From (1.1) and (1.1') we see that *X* and X* in (5.3) we can take in any combinations.

B) Let us consider a case now, when we have both a sequence of elements $\{x_s\}$ and a sequence of functions $\{\beta_i(x)\}$.

Theorem 5.3. *In order that a sequence* $\{x_s\} \in X^*$ *minimize function I(x) on set X*. It is sufficient that there exist a sequence of functions {*β_i*(x)} such that*

(1) $\beta_i(x) \le 0$ over *X* for all i*;

(2) *There exist numbers* $q_i = \inf_X J_i$, $q = \lim_{i\to\infty} q_i$;

(3) $J(x_s) \to q$ *or* $I(x_s) \to q$ *if* $s \to \infty$.

This theorem may be proved easy, because *q* = inf *I* over set *X**.

From theorems 2.1, 2.3 we have next *statement*:

If there exist one sequence which satisfy theorem 2.3 than any other sequence which belong to set *X*, $\{x_s\} \in X$ and satisfy the condition $I(x_s) \to q$ or $J(x_s) \to q$ is minimize for Problem 1.

Appendix.

1. **Operations with signs** inf **and** sup**.**

Below there shown the characteristics of signs **inf** and **sup,** which can be useful for solution of problems. The proof is simply and no given. We assume that are shown constrains have place in domain of definition of function.

$$1.\ \inf[-f(x)] = -\sup f(x), \quad \sup[-f(x)] = -\inf f(x).$$
$$2.\ \inf cf(x) = c\inf f(x) \quad \text{if} \quad c = const > 0;$$
$$\inf cf(x) = -c\inf f(x) \quad \text{if} \quad c = const < 0.$$
$$3.\ \inf[c + f(x)] = c + \inf f(x),$$
$$4.\ \inf\frac{1}{f(x)} = \frac{1}{\sup(x)} \quad \text{if} \quad f(x) \ne 0.$$

5. If $\bar{x}(t)$ can have breaks and $f(t,\bar{x}(t))$ has integrality then

$$\inf_{x(t)} \int_{t_1}^{t_2} f[t,x(t)]dt = \int_{t_1}^{t_2} \inf_x f(t,x)dt\,.$$

6. Assume *f(φ)* is monotone function, $\partial f/\partial\varphi$ is continuous. Then

$$\inf f[\varphi(x)] = f[\inf_X \varphi(x)] \quad \textit{if} \quad \partial f/\partial\varphi > 0,$$

$$\inf f[\varphi(x)] = f[\sup_X \varphi(x)] \quad \textit{if} \quad \partial f/\partial\varphi < 0\,.$$

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