# Riemann Hypothesis

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## 1 Abstract

The Riemann Zeta function is defined as the Analytic Continuation
of the Dirichlet series

$\zeta(s) = \sum_{n=1}^{\infty} 1/n^s,\ Re(s) > 1$

*The Riemann Zeta function is holomorphic in the complex plane except for a simple pole at* $s = 1$

*The non trivial zeroes(i.e those not at negative even integers) of the*

*Riemann Zeta function lie in the critical strip*

$0 < Re(s) < 1$

*Riemann's Xi function is defined as*[4, p.1],

$\epsilon(s) \ = \ s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)/2$

*The zero of* $(s-1)$ *cancels the pole of* $\zeta(s)$ *, and the real zeroes of* $s\,\zeta(s)$ *are cancelled by the*

*simple poles of* $\Gamma(s/2)$ *which never vanishes.*

*Thus,* $\epsilon(s)$ *is an entire function whose zeroes are the non trivial zeroes of* $\zeta(s)$

*Further,* $\epsilon(s)$ *satisfies the functional equation*

$\epsilon(1-s) \ = \ \epsilon(s)$

## 2 Statement of the Riemann Hypothesis

The Riemann Hypothesis states that all the non trivial zeroes of the Riemann Zeta function lie on the critical line Re(s)=1/2

## 3 Proof

The Riemann Xi function

defined as a Hadamard Product [2,p.37, Theorem 2.11] is,

For all s$\in\mathbb{C}$ *we have*,

$\epsilon(s) = \epsilon(0)\prod_{\rho}(1-\frac{s}{\rho})$

*where if we combine the factors* $(1-\frac{s}{\rho})$ *and* $(1-\frac{s}{(1-\rho)})$, *the product converges absolutely and uniformly on compact subsets of* $\mathbb{C}$

*Also*, $\epsilon(0) = 1/2$

Claim: Let, $\epsilon(s) \neq 0, for\ Im(s)\ \in\mathbb{R}^*$ , (*where* $\mathbb{R}^*$ *denotes the set of all non zero real numbers*), *then* $Re(s) \neq 1/2$.

*The functional equation of Riemann Xi function is*

$\epsilon(1-s) = \epsilon(s)$

*Since*, $\epsilon(s) \neq 0$

*Thus*,

$\epsilon(1-s)/\epsilon(s) = 1.$

$\Rightarrow\mid \epsilon(1-s) \mid^2 / \mid \epsilon(s) \mid^2 = 1$

$\mid \epsilon(s) \mid^2 = \mid \epsilon(0) \prod_\rho (1-\frac{s}{\rho}) \mid^2$

$\mid \epsilon(1-s) \mid^2 = \mid \epsilon(0) \mid^2 \mid \prod_\rho (1-\frac{(1-s)}{\rho}) \mid^2$

$\Rightarrow\mid \epsilon(1-s) \mid^2 / \mid \epsilon(s) \mid^2 = \prod_\rho (1-\frac{1-s}{\rho}) \mid^2 / \prod_\rho (1-\frac{s}{\rho}) \mid^2 = 1$

Let, s$= \sigma + it,\ 0 < Re(s) < 1,\ Im(s) \in \mathbb{R}^*$(*where* $\mathbb{R}^*$ *denotes the set of all non zero real numbers*)

*and* $\rho = a + ib,\ 0 < Re(\rho) < 1,\ Im(\rho) \in \mathbb{R}^*$(*where* $\mathbb{R}^*$ *denotes the set of all non zero real numbers*)

$\mid \epsilon(1-s) \mid^2 / \mid \epsilon(s) \mid^2 =$

$\mid \epsilon(0) \mid^2 \prod_\rho \mid 1 - \frac{[1-(\sigma+it)]}{a+ib} \mid^2 / \mid \epsilon(0) \mid^2 \prod_\rho \mid 1 - \frac{(\sigma+it)}{a+ib} \mid^2 = 1$

$\Rightarrow\mid \epsilon(1-s) \mid^2 / \mid \epsilon(s) \mid^2 =$

$\prod_\rho \mid 1 - \frac{[1-(\sigma+it)]}{a+ib} \mid^2 / \prod_\rho \mid 1 - \frac{(\sigma+it)}{a+ib} \mid^2 = 1$

$\Rightarrow\mid \epsilon(1-s) \mid^2 / \mid \epsilon(s) \mid^2 =$

$\prod_\rho \mid \frac{[(a+\sigma-1)+i(b+t)]}{a+ib} \mid^2 / \prod_\rho \mid \frac{(a-\sigma)+i(b-t)}{a+ib} \mid^2 = 1$

$\Rightarrow\mid \epsilon(1-s) \mid^2 / \mid \epsilon(s) \mid^2 =$

$\prod_\rho \frac{[(a+\sigma-1)^2+(b+t)^2]}{a^2+b^2} / \prod_\rho \frac{(a-\sigma)^2+(b-t)^2}{a^2+b^2} = 1 \quad .... \quad (*)$

Since,

$$0 < Re(s) < 1$$

$\Rightarrow\ a^2 + b^2 \neq 0\ \forall\ a \in (0,1)$ .

$\Rightarrow \prod_\rho (a^2+b^2) \neq 0$

$So,\ (*)\ gives,$

$\prod_\rho [(a+\sigma-1)^2+(b+t)^2]/\prod_\rho [(a-\sigma)^2+(b-t)^2]= 1$

$\prod_\rho [(a-\sigma+2\sigma-1)^2+(b-t+2t)^2]/\prod_\rho [(a-\sigma)^2+(b-t)^2]= 1$

$\prod_\rho [(a-\sigma)^2+(2\sigma-1)^2+2(a-\sigma)(2\sigma-1)+(b-t)^2+4t^2+4t(b-t)] = \prod_\rho [(a-\sigma)^2+(b-t)^2]$

$\prod_\rho [(a-\sigma)^2+(b-t)^2+(2\sigma-1)(2\sigma-1+2a-2\sigma)+4bt] = \prod_\rho [(a-\sigma)^2+(b-t)^2]$

$\prod_\rho [(a-\sigma)^2+(b-t)^2+(2\sigma-1)(2a-1)+4bt] = \prod_\rho [(a-\sigma)^2+(b-t)^2]$

$\prod_\rho [(a-\sigma)^2+(b-t)^2+(2\sigma-1)(2a-1)+4bt]/\prod_\rho [(a-\sigma)^2+(b-t)^2]= 1$

$\prod_\rho [(a-\sigma)^2+(b-t)^2+(2\sigma-1)(2a-1)+4bt]/[(a-\sigma)^2+(b-t)^2]= 1$

$\prod_\rho 1+\frac{(2\sigma-1)(2a-1)+4bt}{[(a-\sigma)^2+(b-t)^2]} = 1 \quad \therefore \quad (1)$

$Since,\ t \in \mathbb{R}^*\ we\ \ discuss\ \ 2\ cases:$

$t \in (-\infty, 0)\ \cup\ (1/2, \infty)\ and\ t \in\ (0, 1/2]$

$Case\ 1 : Let, t \in (-\infty, 0) \cup (1/2, \infty)$

$Define\ a\ set$

H=$\{s = \sigma + it : Im(s) \in (-\infty, 0) \cup (1/2, \infty)\}$

Since, $\epsilon(s) \neq 0\ \forall\ Im(s) \in \mathbb{R}^*$

$Therefore,\ \epsilon(s) \neq 0\ \forall\ s \in H.$

$Since,\ \epsilon(s) = \epsilon(0)\ \prod_\rho (1 - \frac{s}{\rho})$

$\epsilon(\rho) = 0 \quad \therefore \quad (2)$

$Claim\ A : 0 \leq Im(\rho) \leq 1/2\ or\ 0 \leq b \leq 1/2.$

$We\ prove\ the\ claim\ by\ contradiction.$

$Let\ us\ assume,\ that\ Im(\rho) \notin [0, 1/2]$

$\Rightarrow Im(\rho) \in (-\infty, 0) \cup (1/2, \infty)$

$\Rightarrow \rho \in H.$

$Now\ since\ \epsilon(s) \neq 0\ \forall s \in H.$

$\Rightarrow \epsilon(\rho) \neq 0.$

$which\ is\ a\ contradiction\ since\ \epsilon(\rho) = 0\ (from\ (2)).$

$Thus, our\ assumption\ that\ Im(\rho)\ \in (-\infty, 0) \cup (1/2, \infty)\ is\ wrong.$

$Thus, 0 \leq Im(\rho) \leq 1/2. \quad \therefore \quad (3)$

$which\ proves\ Claim\ A$

$.But,\ Im(\rho) \in \mathbb{R}^*$

$\Rightarrow Im(\rho) \neq 0$

$Thus,\ 0 < Im(\rho) \leq 1/2\ or\ 0 < b \leq 1/2. \quad \therefore \quad (4)$

$Claim\ B : If\ \epsilon(s) \neq 0, Im(s) \in (-\infty, 0) \cup (1/2, \infty)\ then\ \sigma \neq 1/2.$

$We\ prove\ the\ claim\ by\ contradiction.$

$Let\ us\ assume,\ that\ \sigma = 1/2.$

$Then, by$ (1)

$$\prod_\rho 1+\frac{(2\sigma-1)(2a-1)+4bt}{[(a-\sigma)^2+(b-t)^2]}=1 \quad \therefore \quad (5)$$

$Putting\ \sigma=1/2\ in$ (5),

$$\prod_\rho 1+\frac{4bt}{[(a-1/2)^2+(b-t)^2]}=1 \quad \therefore \quad (6)$$

$Now\ t\in(-\infty,0)\ \cup\ (1/2,\infty),\ so\ we\ have\ two\ sub\ cases$

$, t\in(-\infty,0)\ or t\in(1/2,\infty)$

$Case\ 1(a): t\in(-\infty,0)$

$Then, by$ (6)

$$\prod_\rho 1+\frac{4bt}{[(a-1/2)^2+(b-t)^2]}=1$$
$$1+\frac{4bt}{[(a-1/2)^2+(b-t)^2]}=\frac{(a-1/2)^2+(b-t)^2+4bt}{(a-1/2)^2+(b-t)^2]}$$
$$\Rightarrow 1+\frac{4bt}{[(a-1/2)^2+(b-t)^2]}=\frac{(a-1/2)^2+(b+t)^2}{(a-1/2)^2+(b-t)^2]}$$
$$\Rightarrow 1+\frac{4bt}{[(a-1/2)^2+(b-t)^2]}\geq 0. \quad \therefore \quad (7)$$

$Since,\ by$ (4) $0<b\leq 1/2\ and\ t<0$

$Thus,\ 4bt<0.$

$$1+\frac{4bt}{[(a-1/2)^2+(b-t)^2]}<1 \quad \therefore \quad (8)$$

$From$ (7) $and$ (8),

$$0\leq 1+\frac{4bt}{[(a-1/2)^2+(b-t)^2]}<1$$

$Thus,\ 0\leq\prod_\rho 1+\frac{4bt}{[(a-\sigma)^2+(b-t)^2]}<1$

$which\ contradicts$ (6) $since\ by$ (6), $\prod_\rho 1+\frac{4bt}{[(a-1/2)^2+(b-t)^2]}=1$

$Case\ 1(b): t\in(1/2,\infty)$

$t>1/2\ and\ 0<b\leq 1/2$

$\Rightarrow 4bt > 0.$

$\Rightarrow 1 + \frac{4bt}{[(a-1/2)^2+(b-t)^2]} > 1$

$\Rightarrow \prod_\rho 1 + \frac{4bt}{[(a-1/2)^2+(b-t)^2]} > 1$

$which\ contradicts$ (6) $since\ by$ (6), $\prod_\rho 1 + \frac{4bt}{[(a-1/2)^2+(b-t)^2]} = 1$

$So,\ in\ both\ the\ cases\ we\ get\ a\ contradiction\ .Hence\ ,our\ assumption\ that$

$\sigma = 1/2\ is\ wrong$

$Thus\ ,\ \sigma \neq 1/2.$

$We\ proved\ above\ that\ if\ \epsilon(s) \neq 0\ and\ if\ Im(s) \in (-\infty, 0) \cup (1/2, \infty)\ ,then$

$Re(s) \neq 1/2\ Hence,\ Claim\ B\ is\ proved.$

Case 2: 0¡Im(s)≤ $1/2 or 0 < t \leq 1/2.$

Define a set

$L =$ {s= $\sigma + it : Im(s) \in (0, 1/2]$}

Since, $\epsilon(s) \neq 0\ \forall\ Im(s) \in \mathbb{R}^*$

$Therefore,\ \epsilon(s) \neq 0\ \forall\ s \in H.$

$Since,\ \epsilon(s) = \epsilon(0)\ \prod_\rho (1 - \frac{s}{\rho})$

$\epsilon(\rho) = 0 \quad \therefore \quad$ (9)

$Claim\ C : Im(\rho) \in (-\infty, 0] \cup (1/2, \infty).$

$We\ prove\ the\ claim\ by\ contradiction.$

$Let\ us\ assume,\ that\ Im(\rho) \notin (-\infty, 0] \cup (1/2, \infty)$

$\Rightarrow 0 < Im(\rho) \leq 1/2$

$\Rightarrow \rho \in L.$

$Now\ since\ \epsilon(s) \neq 0\ \forall s \in L.$

$\Rightarrow \epsilon(\rho) \neq 0.$

$which\ is\ a\ contradiction\ since\ \epsilon(\rho) = 0\ (from\ (9)).$

$Thus, our\ assumption\ that\ Im(\rho) \notin (-\infty, 0] \cup (1/2, \infty)\ is\ wrong.$

$Thus, Im(\rho) \in (-\infty, 0] \cup (1/2, \infty)$

$But\ we\ had\ Im(\rho) \in \mathbb{R}^*$

$Thus, Im(\rho) \in (-\infty, 0) \cup (1/2, \infty) \quad \therefore \quad (10)$

$which\ proves\ Claim\ C\ .$

Claim D: If $\epsilon(s) \neq 0, Im(s) \in (0, 1/2]\ then\ \sigma \neq 1/2.$

$We\ prove\ the\ claim\ by\ contradiction.$

$Let\ us\ assume,\ that\ \sigma = 1/2.$

Then , by (1)

$$\prod_\rho 1 + \frac{(2\sigma-1)(2a-1)+4bt}{[(a-\sigma)^2+(b-t)^2]} = 1 \quad \therefore \quad (12)$$

$Putting\ \sigma = 1/2\ in\ (12),$

$\prod_\rho 1+\frac{4bt}{[(a-1/2)^2+(b-t)^2]}=1 \quad \therefore.. \quad (13)$

$Since\,, by\ (10)\ Im(\rho) = \ b \in (-\infty,0) \cup (1/2,\infty)\ so\, we\, have\ 2\ subcases\ b \in (-\infty,0) and b \in (1/2,\infty) Also, 0<t\leq 1/2$

$Case\ 2(a): b\in(-\infty,0) 0<t\leq 1/2$

$Then, by\ (6)$

$\prod_\rho 1+\frac{4bt}{[(a-1/2)^2+(b-t)^2]}=1$

$1+\frac{4bt}{[(a-1/2)^2+(b-t)^2]}=\frac{(a-1/2)^2+(b-t)^2+4bt}{(a-1/2)^2+(b-t)^2]}$

$\Rightarrow 1+\frac{4bt}{[(a-1/2)^2+(b-t)^2]}=\frac{(a-1/2)^2+(b+t)^2}{(a-1/2)^2+(b-t)^2]}$

$$\Rightarrow 1+\frac{4bt}{[(a-1/2)^2+(b-t)^2]}\geq 0. \quad \therefore.. \quad (13)$$

$Since,\ b\in(-\infty,0) 0<t\leq 1/2$

$Thus,\ 4bt<0.$

$1+\frac{4bt}{[(a-1/2)^2+(b-t)^2]}<1 \quad \therefore.. \quad (14)$

$From\ (13)\ and\ (14),$

$$0\leq 1+\frac{4bt}{[(a-1/2)^2+(b-t)^2]}<1$$

$Thus,\ 0\leq \prod_\rho 1+\frac{4bt}{[(a-\sigma)^2+(b-t)^2]}<1$

$which\ contradicts\ (6)\ since\ by\ (6),\ \prod_\rho 1+\frac{4bt}{[(a-1/2)^2+(b-t)^2]}=1$

$Case\ 2(b): b\in(1/2,\infty) 0<t\leq 1/2$

$\Rightarrow 4bt>0.$

$\Rightarrow 1+\frac{4bt}{[(a-1/2)^2+(b-t)^2]}>1$

$\Rightarrow \prod_\rho 1+\frac{4bt}{[(a-1/2)^2+(b-t)^2]}>1$

*which contradicts* (6) *since by* (6), $\prod_\rho 1+\frac{4bt}{[(a-1/2)^2+(b-t)^2]}=1$

*So, in both the cases we get a contradiction .Hence , our assumption that*

$\sigma = 1/2$ *is wrong*

*Thus* , $\sigma \neq 1/2$.

*We proved above that if* $\epsilon(s) \neq 0 0 < Im(s) \leq 1/2$ *and if* $Im(s) \in (0, 1/2]$ *, then*

$Re(s) \neq 1/2$ *Hence, Claim D is proved.*

*Combining Claim B and Claim D we see that* $\epsilon(s) \neq 0, Im(s) \in (-\infty, 0) \cup (1/2, \infty)$ *implies* $Re(s) \neq 1/2 \epsilon(s) \neq 0, Im(s) \in (0, 1/2]$

*implies* $Re(s) \neq 1/2$

*Thus* , $\epsilon(s) \neq 0, Im(s) \in \mathbb{R}^*$

*But, by Riemann Hypothesis we assumed that*

$\epsilon(s) = 0$ *and* $\epsilon(s) \neq 0\ \forall\ Re(s) \neq 1/2$.

*thus we must have* $Re(s) = 1/2$.

## 4 References:-

1. E. C. Titchmarsh, D. R. Heath-Brown - The theory of the Riemann Zeta function [2nd ed] Clarendon Press; Oxford University Press (1986).

2. Kevin Broughan - Equivalents of the Riemann Hypothesis : Arithmetic Equivalents Cambridge University Press (2017) .

3. Kevin Broughan - Equivalents of the Riemann Hypothesis : Analytic Equivalents Cambridge University Press (2017).

4. A Monotonicity of Riemann's Xi function and a reformulation of the Riemann Hypothesis, Periodica Mathematica Hungarica - May 2010.

5. Tom M. Apostol - Introduction to Analytical Number Theory (1976).

6. https://www.claymath.org/millennium-problems/riemann-hypothesis.

7. A note on S(t) and the zeros of the Riemann zeta-function - DA Goldston, SM Gonek.

8. 4. Lars Ahlfors - Complex analysis [3 ed.] McGraw -Hill (1979).

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