Riemann Hypothesis

Shekhar Suman

February 21, 2020

1 Abstract

The Riemann Zeta function is defined as the Analytic Continuation of the Dirichlet series,

$\zeta(s) = \sum_{n=1}^{\infty} 1/n^s, \ Re(s) > 1$

$The\ Riemann\ Zeta\ function\ is\ holomorphic\ in\ the\ complex\ plane\ except\ for\ a\ simple\ pole\ at\ s = 1$

$The\ non\ trivial\ zeroes(i.e\ those\ not\ at\ negative\ even\ integers)\ of\ the$

$Riemann\ Zeta\ function\ lie\ in\ the\ critical\ strip$

$0 < Re(s) < 1$

$Riemann's\ Xi\ function\ is\ defined\ as$ [4, $p.1$],

$\epsilon(s) \ = \ s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)/2$

$The\ zero\ of\ (s{-}1)\ cancels\ the\ pole\ of\ \zeta(s)\ ,\ and\ the\ real\ zeroes\ of\ s\ \zeta(s)\ are\ cancelled\ by\ the$

$simple\ poles\ of\ \Gamma(s/2)\ which\ never\ vanishes.$

$Thus,\ \epsilon(s)\ is\ an\ entire\ function\ whose\ zeroes\ are\ the\ non\ trivial\ zeroes\ of\ \zeta(s)$

$Further,\ \epsilon(s)\ satisfies\ the\ functional\ equation$

$\epsilon(1-s) \ = \ \epsilon(s)$

2 Statement of the Riemann Hypothesis

The Riemann Hypothesis states that all the non trivial zeroes of the Riemann Zeta function lie on the critical line Re(s)=1/2

3 Proof

The Riemann Xi function

defined as a Hadamard Product [2,p.37, Theorem 2.11] is, for all s∈$\mathbb{C}$ *we have*,

$$\epsilon(s) = \epsilon(0) \prod_{\rho}(1 - \frac{s}{\rho})$$

where if we combine the factors $(1-\frac{s}{\rho})$ *and* $(1-\frac{s}{(1-\rho)})$, *the product converges absolutely and uniformly on compact subsets of* $\mathbb{C}$

Also, $\epsilon(0) = 1/2$

Claim: Let, $\epsilon(s) \neq 0, for\ Im(s) \ \in \mathbb{R}^*$, (*where* $\mathbb{R}^*$ *denotes the set of all non zero real numbers*), *then* $Re(s) \neq 1/2$.

The functional equation of Riemann Xi function is

$\epsilon(1 - s) \ = \ \epsilon(s)$

Since, $\epsilon(s) \neq 0$

Thus,

$\epsilon(1 - s)/\epsilon(s) = 1.$

$\Rightarrow \mid \epsilon(1-s) \mid^2 / \mid \epsilon(s) \mid^2 = 1$

$\mid \epsilon(s) \mid^2 = \mid \epsilon(0) \prod_\rho (1-\frac{s}{\rho}) \mid^2$

$\mid \epsilon(1-s) \mid^2 = \mid \epsilon(0) \prod_\rho (1-\frac{(1-s)}{\rho}) \mid^2$

$\Rightarrow \mid \epsilon(1-s) \mid^2 / \mid \epsilon(s) \mid^2 = \prod_\rho \mid (1-\frac{1-s}{\rho}) \mid^2 / \prod_\rho \mid (1-\frac{s}{\rho}) \mid^2 = 1$

Let, s$= \sigma + it,\ 0 < Re(s) < 1,\ Im(s) \in \mathbb{R}^* (where\ \mathbb{R}^*\ denotes$

$the\ set\ of\ all\ non\ zero\ real\ numbers)$

$and\ \rho = a + ib,\ 0 < Re(\rho) < 1,\ Im(\rho) \in \mathbb{R}^* (where\ \mathbb{R}^*\ denotes$

$the\ set\ of\ all\ non\ zero\ real\ numbers)$

$\mid \epsilon(1-s) \mid^2 / \mid \epsilon(s) \mid^2 =$

$\mid \epsilon(0) \mid^2 \prod_\rho \mid 1 - \frac{[1-(\sigma+it)]}{a+ib} \mid^2 / \mid \epsilon(0) \mid^2 \prod_\rho \mid 1 - \frac{(\sigma+it)}{a+ib} \mid^2 = 1$

$\Rightarrow \mid \epsilon(1-s) \mid^2 / \mid \epsilon(s) \mid^2 =$

$\prod_\rho \mid 1 - \frac{[1-(\sigma+it)]}{a+ib} \mid^2 / \prod_\rho \mid 1 - \frac{(\sigma+it)}{a+ib} \mid^2 = 1$

$\Rightarrow \mid \epsilon(1-s) \mid^2 / \mid \epsilon(s) \mid^2 =$

$\prod_\rho \mid \frac{[(a+\sigma-1)+i(b+t)]}{a+ib} \mid^2 / \prod_\rho \mid \frac{(a-\sigma)+i(b-t)}{a+ib} \mid^2 = 1$

$\Rightarrow \mid \epsilon(1-s) \mid^2 / \mid \epsilon(s) \mid^2 =$

$\prod_\rho \frac{[(a+\sigma-1)^2+(b+t)^2]}{a^2+b^2} / \prod_\rho \frac{(a-\sigma)^2+(b-t)^2}{a^2+b^2} = 1 \qquad .... \quad (*)$

Since,

$$0 < Re(s) < 1$$

$\Rightarrow\ a^2+b^2 \neq 0\ \forall\ a \in (0,1)$.

$\Rightarrow \prod_\rho (a^2+b^2) \neq 0$

So, (∗) gives,

$\prod_\rho [(a+\sigma-1)^2+(b+t)^2]/\prod_\rho [(a-\sigma)^2+(b-t)^2] = 1$

$\prod_\rho [(a-\sigma+2\sigma-1)^2+(b-t+2t)^2]/\prod_\rho [(a-\sigma)^2+(b-t)^2] = 1$

$\prod_\rho [(a-\sigma)^2+(2\sigma-1)^2+2(a-\sigma)(2\sigma-1)+(b-t)^2+4t^2+4t(b-t)] = \prod_\rho [(a-\sigma)^2+(b-t)^2]$

$\prod_\rho [(a-\sigma)^2+(b-t)^2+(2\sigma-1)(2\sigma-1+2a-2\sigma)+4bt] = \prod_\rho [(a-\sigma)^2+(b-t)^2]$

$\prod_\rho [(a-\sigma)^2+(b-t)^2+(2\sigma-1)(2a-1)+4bt] = \prod_\rho [(a-\sigma)^2+(b-t)^2]$

$\prod_\rho [(a-\sigma)^2+(b-t)^2+(2\sigma-1)(2a-1)+4bt]/\prod_\rho [(a-\sigma)^2+(b-t)^2] = 1$

$\prod_\rho [(a-\sigma)^2+(b-t)^2+(2\sigma-1)(2a-1)+4bt]/[(a-\sigma)^2+(b-t)^2] = 1$

$\prod_\rho 1+\frac{(2\sigma-1)(2a-1)+4bt}{[(a-\sigma)^2+(b-t)^2]} = 1 \quad \therefore \quad (1)$

Since, $t \in \mathbb{R}^*$ *we discuss 2 cases :*

$t \in (-\infty,0) \ \cup\ (1/2,\infty)$ *and* $t \in\ (0,1/2]$

Case 1 : Let, $t \in (-\infty,0)\cup(1/2,\infty)$

Define a set,

H=$\{s=\sigma+it : Im(s) \in (-\infty,0)\cup(1/2,\infty)\}$

Since, $\epsilon(s) \neq 0\ \forall\ Im(s) \in \mathbb{R}^*$

$Therefore,\ \epsilon(s) \neq 0\ \forall\ s \in H.$

$Since,\ \epsilon(s) = \epsilon(0)\ \prod_{\rho}(1 - \frac{s}{\rho})$

$\epsilon(\rho) = 0 \quad \therefore.. \qquad (2)$

$Claim\ A : 0 \leq Im(\rho) \leq 1/2\ or\ 0 \leq b \leq 1/2.$

$We\ prove\ the\ claim\ by\ contradiction.$

$Let\ us\ assume,\ that\ Im(\rho) \notin [0, 1/2]$

$\Rightarrow Im(\rho) \in (-\infty, 0) \cup (1/2, \infty)$

$\Rightarrow \rho \in H.$

$Now\ since\ \epsilon(s) \neq 0\ \forall s \in H.$

$\Rightarrow \epsilon(\rho) \neq 0.$

$which\ is\ a\ contradiction\ since\ \epsilon(\rho) = 0\ (from\ (2)).$

$Thus, our\ assumption\ that\ Im(\rho)\ \in (-\infty, 0) \cup (1/2, \infty)\ is\ wrong.$

$Thus, 0 \leq Im(\rho) \leq 1/2. \quad \therefore.. \quad (3)$

$which\ proves\ Claim\ A$

$.But,\ Im(\rho) \in \mathbb{R}^*$

$\Rightarrow Im(\rho) \neq 0$

$Thus,\ 0 < Im(\rho) \leq 1/2\ or\ 0 < b \leq 1/2. \quad \therefore.. \quad (4)$

$Claim\ B : If\ \epsilon(s) \neq 0,\ Im(s) \in (-\infty, 0) \cup (1/2, \infty)\ then\ \sigma \neq 1/2.$

$We\ prove\ the\ claim\ by\ contradiction.$

$Let\ us\ assume,\ that\ \sigma = 1/2.$

$Then, by\ (1)$

$$\prod_\rho 1 + \frac{(2\sigma-1)(2a-1)+4bt}{[(a-\sigma)^2+(b-t)^2]} = 1 \quad \therefore \quad (5)$$

$Putting\ \sigma = 1/2\ in\ (5),$

$$\prod_\rho 1 + \frac{4bt}{[(a-1/2)^2+(b-t)^2]} = 1 \quad \therefore \quad (6)$$

$Now\ t \in (-\infty, 0)\ \cup\ (1/2, \infty),\ so\ we\ have\ two\ sub\ cases$

$, t \in (-\infty, 0)\ or\ t \in (1/2, \infty)$

$Case\ 1(a) : t \in (-\infty, 0)$

$Then, by\ (6)$

$$\prod_\rho 1 + \frac{4bt}{[(a-1/2)^2+(b-t)^2]} = 1$$

$$1 + \frac{4bt}{[(a-1/2)^2+(b-t)^2]} = \frac{(a-1/2)^2+(b-t)^2+4bt}{(a-1/2)^2+(b-t)^2]}$$

$$\Rightarrow 1 + \frac{4bt}{[(a-1/2)^2+(b-t)^2]} = \frac{(a-1/2)^2+(b+t)^2}{(a-1/2)^2+(b-t)^2]}$$

$$\Rightarrow 1 + \frac{4bt}{[(a-1/2)^2+(b-t)^2]} \geq 0. \quad \therefore \quad (7)$$

$Since,\ by\ (4)\ 0 < b \leq 1/2\ and\ t < 0$

$Thus,\ 4bt < 0.$

$$1 + \frac{4bt}{[(a-1/2)^2+(b-t)^2]} < 1 \quad \therefore \quad (8)$$

$From\ (7)\ and\ (8),$

$$0 \leq 1 + \frac{4bt}{[(a-1/2)^2+(b-t)^2]} < 1$$

$$Thus,\ 0 \leq \prod_\rho 1 + \frac{4bt}{[(a-\sigma)^2+(b-t)^2]} < 1$$

$which\ contradicts\ (6)\ since\ by\ (6),\ \prod_\rho 1 + \frac{4bt}{[(a-1/2)^2+(b-t)^2]} = 1$

$Case\ 1(b) : t \in (1/2, \infty)$

$t > 1/2\ and\ 0 < b \leq 1/2$

$\Rightarrow 4bt > 0.$

$\Rightarrow 1 + \frac{4bt}{[(a-1/2)^2+(b-t)^2]} > 1$

$\Rightarrow \prod_\rho 1 + \frac{4bt}{[(a-1/2)^2+(b-t)^2]} > 1$

$which\ contradicts\ (6)\ since\ by\ (6),\ \prod_\rho 1 + \frac{4bt}{[(a-1/2)^2+(b-t)^2]} = 1$

$So,\ in\ both\ the\ cases\ we\ get\ a\ contradiction\ .Hence\ ,our\ assumption\ that$

$\sigma = 1/2\ is\ wrong$

$Thus\ ,\ \sigma \neq 1/2.$

$We\ proved\ above\ that\ if\ \epsilon(s) \neq 0$

$and\ if\ Im(s) \in (-\infty, 0) \cup (1/2, \infty)\ ,then$

$Re(s) \neq 1/2\ Hence,\ Claim\ B\ is\ proved.$

Case 2:

$$0 < Im(s) \leq 1/2\ or\ 0 < t \leq 1/2.$$

Define a set

$L =$ {s= $\sigma + it : Im(s) \in (0, 1/2]$}

Since, $\epsilon(s) \neq 0\ \forall\ Im(s) \in \mathbb{R}^*$

$Therefore,\ \epsilon(s) \neq 0\ \forall\ s \in L.$

$Since,\ \epsilon(s) = \epsilon(0)\ \prod_{\rho}(1 - \frac{s}{\rho})$

$\epsilon(\rho) = 0 \quad \therefore \qquad (9)$

$Claim\ C : Im(\rho) \in (-\infty, 0] \cup (1/2, \infty).$

$We\ prove\ the\ claim\ by\ contradiction.$

$Let\ us\ assume,\ that\ Im(\rho) \notin (-\infty, 0] \cup (1/2, \infty)$

$\Rightarrow 0 < Im(\rho) \leq 1/2$

$\Rightarrow \rho \in L.$

$Now\ since\ \epsilon(s) \neq 0\ \forall s \in L.$

$\Rightarrow \epsilon(\rho) \neq 0.$

$which\ is\ a\ contradiction\ since\ \epsilon(\rho) = 0\ (from\ (9)).$

$Thus, our\ assumption\ that\ Im(\rho) \notin (-\infty, 0] \cup (1/2, \infty)\ is\ wrong.$

$Thus, Im(\rho) \in (-\infty, 0] \cup (1/2, \infty)$

$But,\ we\ had\ Im(\rho) \in \mathbb{R}^*$

$Thus, Im(\rho) \in (-\infty, 0) \cup (1/2, \infty) \quad \therefore \quad (10)$

$which\ proves\ Claim\ C\ .$

$Claim\ D : If\ \epsilon(s) \neq 0, Im(s) \in (0, 1/2]\ then\ \sigma \neq 1/2.$

$We\ prove\ the\ claim\ by\ contradiction.$

$Let\ us\ assume,\ that\ \sigma = 1/2.$

$Then, by$ (1) ,

$$\prod_\rho 1+\frac{(2\sigma-1)(2a-1)+4bt}{[(a-\sigma)^2+(b-t)^2]}=1 \quad \therefore \quad (11)$$

$Putting\ \sigma = 1/2\ in$ (11),

$$\prod_\rho 1+\frac{4bt}{[(a-1/2)^2+(b-t)^2]}=1 \quad \therefore \quad (12)$$

$Since\ , by$ (10) $Im(\rho) =\ b \in (-\infty, 0)\cup(1/2, \infty)$ $so\,we\,have$ 2 $subcases$

$b \in (-\infty, 0)$ and $b \in (1/2, \infty)$. $Also$ $, 0 < t \leq 1/2$

$Case$ $2(a) : b \in (-\infty, 0) 0 < t \leq 1/2$

$Then, by$ (12)

$$\prod_\rho 1+\frac{4bt}{[(a-1/2)^2+(b-t)^2]}=1$$

$$1+\frac{4bt}{[(a-1/2)^2+(b-t)^2]} = \frac{(a-1/2)^2+(b-t)^2+4bt}{(a-1/2)^2+(b-t)^2]}$$

$$\Rightarrow 1+\frac{4bt}{[(a-1/2)^2+(b-t)^2]} = \frac{(a-1/2)^2+(b+t)^2}{(a-1/2)^2+(b-t)^2]}$$

$$\Rightarrow 1+\frac{4bt}{[(a-1/2)^2+(b-t)^2]} \geq 0. \quad \therefore \quad (13)$$

$Since,\ b \in (-\infty, 0),\ 0 < t \leq 1/2$

$Thus,\ 4bt < 0.$

$$1+\frac{4bt}{[(a-1/2)^2+(b-t)^2]} < 1 \quad \therefore \quad (14)$$

$From$ (13) and (14),

$$0 \leq 1+\frac{4bt}{[(a-1/2)^2+(b-t)^2]} < 1$$

$Thus,\ 0 \leq \prod_\rho 1+\frac{4bt}{[(a-\sigma)^2+(b-t)^2]} < 1$

$which\ contradicts\ (12)\ since\ by\ (12),\ \prod_\rho 1+\frac{4bt}{[(a-1/2)^2+(b-t)^2]}=1$

$Case\ 2(b) : b \in (1/2,\infty),\ 0<t\leq 1/2$

$\Rightarrow 4bt>0.$

$\Rightarrow 1+\frac{4bt}{[(a-1/2)^2+(b-t)^2]}>1$

$\Rightarrow \prod_\rho 1+\frac{4bt}{[(a-1/2)^2+(b-t)^2]}>1$

$which\ contradicts\ (12)\ since\ by\ (12),\ \prod_\rho 1+\frac{4bt}{[(a-1/2)^2+(b-t)^2]}=1$

$So,\ in\ both\ the\ cases\ we\ get\ a\ contradiction\ .Hence\ ,our\ assumption\ that$

$\sigma=1/2\ is\ wrong$

$Thus\ ,\ \sigma\neq 1/2.$

$We\ proved\ above\ that\ if\ \epsilon(s)\neq 0\ and\ if\ Im(s)\in(0,1/2]\ ,then$

$Re(s)\neq 1/2\ Hence,\ Claim\ D\ is\ proved.$

$Combining\ Claim\ B\ and\ Claim\ D\ we\ see\ that\ \epsilon(s)\neq 0,$

$Im(s)\in(-\infty,0)\cup(1/2,\infty)\ implies\ Re(s)\neq 1/2$

$and\ \epsilon(s)\neq 0, Im(s)\in(0,1/2]\ implies\ Re(s)\neq 1/2$

$Thus\ ,\epsilon(s)\neq 0, Im(s)\in\mathbb{R}^*$

$But,\ by\ Riemann\ Hypothesis\ we\ assumed\ that$

$\epsilon(s)=0\ for\ 0<Re(s)<1\ and\ \epsilon(s)\neq 0\ \Rightarrow\ Re(s)\neq 1/2.$

$thus,\ \epsilon(s)=0,\ must\ imply\ Re(s)=1/2.\ So,\ Riemann\ Hypothesis\ is\ true\ \forall\ Im(s)\in\mathbb{R}^*$

4 References:-

1. E. C. Titchmarsh, D. R. Heath-Brown - The theory of the Riemann Zeta function [2nd ed] Clarendon Press; Oxford University Press (1986).

2. Kevin Broughan - Equivalents of the Riemann Hypothesis : Arithmetic Equivalents Cambridge University Press (2017) .

3. Kevin Broughan - Equivalents of the Riemann Hypothesis : Analytic Equivalents Cambridge University Press (2017).

4. A Monotonicity of Riemann's Xi function and a reformulation of the Riemann Hypothesis, Periodica Mathematica Hungarica - May 2010.

5. Tom M. Apostol - Introduction to Analytical Number Theory (1976).

6. https://www.claymath.org/millennium-problems/riemann-hypothesis.

7. A note on S(t) and the zeros of the Riemann zeta-function - DA Goldston, SM Gonek.

8. 4. Lars Ahlfors - Complex analysis [3 ed.] McGraw -Hill (1979).

Contact:

Shekhar Suman.

I.I.T. Delhi M.Sc. (2015-17).

Email: shekharsuman068@gmail.com