Riemann Hypothesis

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1 Abstract

The Riemann Zeta function is defined as the Analytic Continuation of the Dirichlet series

$\zeta(s) = \sum_{n=1}^{\infty} 1/n^s, \; Re(s)>1$

$The\ Riemann\ Zeta\ function\ is\ analytic\ everywhere\ except\ for\ a\ simple\ pole\ at\ s = 1$

$The\ non\ trivial\ zeroes\ of\ the\ Riemann\ Zeta\ function\ lie\ in\ the\ critical\ strip$
$0 < Re(s) < 1$

$Riemann's\ Xi\ function\ is\ defined\ as,$

$\epsilon(s) \; = \; s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)/2$

$The\ zero\ of\ (s{-}1)\ cancels\ the\ pole\ of\ \zeta(s)\ , and\ the\ real\ zeroes\ of\ s\ \zeta(s)\ are\ cancelled\ by\ the$

$simple\ poles\ of\ \Gamma(s/2)\ which\ never\ vanishes.$

$Thus,\ \epsilon(s)\ is\ an\ entire\ function\ whose\ zeroes\ are\ the\ non\ trivial\ zeroes\ of\ \zeta(s)$

$Further,\ \epsilon(s)\ satisfies\ the\ functional\ equation$

$\epsilon(1-s) \; = \; \epsilon(s)$

2 Statement of the Riemann Hypothesis

The Riemann Hypothesis states that all the non trivial zeroes of the

Riemann Zeta function lie on the critical line Re(s)=1/2

3 Proof

The Riemann Xi function

defined as a Hadamard Product [2,p.37, Theorem 2.11] is,

For all s$\in\mathbb{C}$ $we\ have,$

$\epsilon(s) = \epsilon(0)\prod_{\rho}(1-\frac{s}{\rho})$

$where\ if\ we\ combine\ the\ factors\ (1-\frac{s}{\rho})\ and\ (1-\frac{s}{(1-\rho)}),\ the\ product$

$converges\ absolutely\ and\ uniformly\ on\ compact\ subsets\ of\ \mathbb{C}$

$Also,\ \epsilon(0) = 1/2$

Claim: If $\epsilon(s) \neq 0,\ Im(s)\ \in\mathbb{R}^*\ ,\ where\ \mathbb{R}^*\ denotes\ the\ non\ zero\ real\ numbers$

$,then\ Re(s) \neq 1/2.$

$The\ functional\ equation\ of\ Riemann\ Xi\ function\ is$

$\epsilon(1-s)\ =\ \epsilon(s)$

$Since, \epsilon(s) \neq 0$

$Thus,$

$\epsilon(1-s)/\epsilon(s) = 1.$

$\Rightarrow \mid \epsilon(1-s) \mid^2 / \mid \epsilon(s) \mid^2 = 1$

$\mid \epsilon(s) \mid^2 = \mid \epsilon(0) \prod_\rho (1-\frac{s}{\rho}) \mid^2$

$\mid \epsilon(1-s) \mid^2 = \mid \epsilon(0) \mid^2 \mid \prod_\rho (1-\frac{(1-s)}{\rho}) \mid^2$

$\Rightarrow \mid \epsilon(1-s) \mid^2 / \mid \epsilon(s) \mid^2 = \prod_\rho (1-\frac{1-s}{\rho}) \mid^2 / \prod_\rho (1-\frac{s}{\rho}) \mid^2 = 1$

Let, s= $\sigma + it,\ 0 < Re(s) < 1, Im(s) \in \mathbb{R}^*$

$and\ \rho = a + ib,\ 0 < Re(\rho) < 1, Im(\rho) \in \mathbb{R}^*$

$\mid \epsilon(1-s) \mid^2 / \mid \epsilon(s) \mid^2 =$

$\mid \epsilon(0) \mid^2 \prod_\rho \mid 1 - \frac{[1-(\sigma+it)]}{a+ib} \mid^2 / \mid \epsilon(0) \mid^2 \prod_\rho \mid 1 - \frac{(\sigma+it)}{a+ib} \mid^2 = 1$

$\Rightarrow \mid \epsilon(1-s) \mid^2 / \mid \epsilon(s) \mid^2 =$

$\prod_\rho \mid 1 - \frac{[1-(\sigma+it)]}{a+ib} \mid^2 / \prod_\rho \mid 1 - \frac{(\sigma+it)}{a+ib} \mid^2 = 1$

$\Rightarrow \mid \epsilon(1-s) \mid^2 / \mid \epsilon(s) \mid^2 =$

$\prod_\rho \mid \frac{[(a+\sigma-1)+i(b+t)]}{a+ib} \mid^2 / \prod_\rho \mid \frac{(a-\sigma)+i(b-t)}{a+ib} \mid^2 = 1$

$\Rightarrow \mid \epsilon(1-s) \mid^2 / \mid \epsilon(s) \mid^2 =$

$\prod_\rho \frac{[(a+\sigma-1)^2+(b+t)^2]}{a^2+b^2} / \prod_\rho \frac{(a-\sigma)^2+(b-t)^2}{a^2+b^2} = 1$

$\prod_\rho \frac{[(a+\sigma-1)^2+(b+t)^2]}{a^2+b^2} / \prod_\rho \frac{(a-\sigma)^2+(b-t)^2}{a^2+b^2} = 1 \qquad .... \quad (*)$

Since,

$$0 < Re(s) < 1$$

$\Rightarrow\ a^2 + b^2 \neq 0\ \forall\ a \in (0,1)$.

$\Rightarrow \prod_\rho (a^2 + b^2) \neq 0$

$So,\ (*)\ gives,$

$\prod_\rho[(a+\sigma-1)^2+(b+t)^2]/\prod_\rho[(a-\sigma)^2+(b-t)^2]=1$

$\prod_\rho[(a-\sigma+2\sigma-1)^2+(b-t+2t)^2]/\prod_\rho[(a-\sigma)^2+(b-t)^2]=1$

$\prod_\rho[(a-\sigma)^2+(2\sigma-1)^2+2(a-\sigma)(2\sigma-1)+(b-t)^2+4t^2+4t(b-t)] = \prod_\rho[(a-\sigma)^2+(b-t)^2]$

$\prod_\rho[(a-\sigma)^2+(b-t)^2+(2\sigma-1)(2\sigma-1+2a-2\sigma)+4bt] = \prod_\rho[(a-\sigma)^2+(b-t)^2]$

$\prod_\rho[(a-\sigma)^2+(b-t)^2+(2\sigma-1)(2a-1)+4bt] = \prod_\rho[(a-\sigma)^2+(b-t)^2]$

$\prod_\rho[(a-\sigma)^2+(b-t)^2+(2\sigma-1)(2a-1)+4bt]/\prod_\rho[(a-\sigma)^2+(b-t)^2]=1$

$\prod_\rho[(a-\sigma)^2+(b-t)^2+(2\sigma-1)(2a-1)+4bt]/[(a-\sigma)^2+(b-t)^2]=1$

$\prod_\rho 1+\frac{(2\sigma-1)(2a-1)+4bt}{[(a-\sigma)^2+(b-t)^2]}=1 \quad \therefore \quad (1)$

$Let, t\in(-\infty,0)\cup(1/2,\infty)$

$Define\ a\ set$

H=$\{s=\sigma+it : t\in(-\infty,0)\cup(1/2,\infty)\}$

Since, $\epsilon(s)\neq 0\ \forall\ t\in\mathbb{R}^*$

$Therefore,\ \epsilon(s)\neq 0\ \forall\ s\in H.$

$Since,\ \epsilon(s)=\epsilon(0)\ \prod_\rho(1-\frac{s}{\rho})$

$\epsilon(\rho)=0 \quad \therefore \qquad (2)$

$Claim\ A : 0\le Im(\rho)\le 1/2\ or\ 0\le b\le 1/2.$

$We\ prove\ the\ claim\ by\ contradiction.$

$Let\ us\ assume,\ that\ 0 \leq b \leq\ 1/2\ is\ not\ true$

$\Rightarrow b \in (-\infty, 0) \cup (1/2, \infty)$

$\Rightarrow \rho = a + ib \in H.$

$Now\ since \epsilon(s) \neq 0\ \forall s \in H.$

$\Rightarrow \epsilon(\rho) \neq 0.$

$which\ is\ a\ contradiction\ since\ \epsilon(\rho) = 0\ (from\ (2)).$

$Thus, our\ assumption\ that\ b\ \in (-\infty, 0) \cup (1/2, \infty)\ is\ wrong.$

$Thus, 0 \leq b \leq 1/2. \quad \therefore.. \quad (3)$

$which\ proves\ Claim\ A\ .But,\ b \in \mathbb{R}^*$

$\Rightarrow b \neq 0$

$Thus,\ 0 < b \leq 1/2. \quad \therefore.. \quad (4)$

$Claim\ B : If\ \epsilon(s) \neq 0\ then\ \sigma \neq 1/2.$

$We\ prove\ the\ claim\ by\ contradiction.$

$Let\ us\ assume,\ that\ \sigma = 1/2.$

$Then, by\ (1)$

$\prod_\rho 1 + \frac{(2\sigma-1)(2a-1)+4bt}{[(a-\sigma)^2+(b-t)^2]} = 1 \quad \therefore.. \quad (5)$

$Putting\ \sigma = 1/2\ in\ (5),$

$\prod_\rho 1 + \frac{4bt}{[(a-1/2)^2+(b-t)^2]} = 1 \quad \therefore.. \quad (6)$

$Now\, t \in (-\infty, 0) \cup (1/2, \infty),\ so\, we\, have\, two\, cases\, , t \in (-\infty, 0)\, and$

$t \in (1/2, \infty)$

$Case 1 : t \in (-\infty, 0)$

$Then, by\ (6)$

$\prod_{\rho} 1 + \frac{4bt}{[(a-1/2)^2+(b-t)^2]} = 1$

$1 + \frac{4bt}{[(a-1/2)^2+(b-t)^2]} = \frac{(a-1/2)^2+(b-t)^2+4bt}{(a-1/2)^2+(b-t)^2]}$

$\Rightarrow 1 + \frac{4bt}{[(a-1/2)^2+(b-t)^2]} = \frac{(a-1/2)^2+(b+t)^2}{(a-1/2)^2+(b-t)^2]}$

$$\Rightarrow 1 + \frac{4bt}{[(a-1/2)^2+(b-t)^2]} \geq 0. \quad \therefore \quad (7)$$

$Since,\ by\ (4)\ 0 < b \leq 1/2\ and\ t < 0$

$Thus,\ 4bt < 0.$

$1 + \frac{4bt}{[(a-1/2)^2+(b-t)^2]} < 1 \quad \therefore \quad (8)$

$Frome (7) and (8),$

$$0 \leq 1 + \frac{4bt}{[(a-1/2)^2+(b-t)^2]} < 1$$

$Thus,\ 0 \leq \prod_{\rho} 1 + \frac{4bt}{[(a-\sigma)^2+(b-t)^2]} < 1$

$which\ contradicts\ (6)\ since\ by\ (6),\ \prod_{\rho} 1 + \frac{4bt}{[(a-1/2)^2+(b-t)^2]} = 1$

$Case 2 : t \in (1/2, \infty)$

$t > 1/2\ and\ 0 < b \leq 1/2$

$\Rightarrow 4bt > 0.$

$\Rightarrow 1 + \frac{4bt}{[(a-1/2)^2+(b-t)^2]} > 1$

$\Rightarrow \prod_\rho 1 + \frac{4bt}{[(a-1/2)^2+(b-t)^2]} > 1$

$which\ contradicts$ (6) $since\ by$ (6), $\prod_\rho 1 + \frac{4bt}{[(a-1/2)^2+(b-t)^2]} = 1$

$So,\ in\ both\ the\ cases\ we\ get\ a\ contradiction\ .Hence\ ,our\ assumption\ that\ \sigma = 1/2\ is\ wrong$

$Thus\ ,\sigma \neq 1/2.$

$We\ proved\ above\ that\ if\ \epsilon(s) \neq 0\ ,then\ Re(s) \neq 1/2\ Hence,\ Claim\ B\ is\ proved.$

$But,\ by\ Riemann\ Hypothesis\ we\ assumed\ that\ \epsilon(s) = 0.$

$Thus,\ if\ \epsilon(s) = 0\ then\ Re(s) = 1/2.$

4 References:-

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