Lagrangian Field Approach to Einstein-Maxwell Equation in Curved Spacetime along with Relativistic dust.

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ABSTRACT

This paper aims at deriving the Einstein-Maxwell field equation along with relativistic dust from lagrangian field theory or the variational principle by taking the variations of action for gravitation, electromagnetism and for relativistic dust where in general the fields are functions of the metric.

Keywords:- ***variational principle, metric, Einstein-Hilbert action, Einstein-Maxwell field equation, Relativistic dust.***

1. INTRODUCTION

Before writting down an action for General Relativity, we need to refer to another key assumption. General Relativity assumes that no fields other than metric mediate the Gravitational interaction. Any field other than metric is considered to be matter and should be included in the matter action such as the stress energy momentum tensor. Therefore the general structure for the action should include a Lagrangian for Gravity which depends only on the metric and a Lagrangian for the matter which depends on the matter fields. The Lagrangian should be a generally covariant scalar if it is to lead to covariant equations. The simply generally covariant scalar that one can construct is the Ricci scalar, so it becomes obvious to include Ricci scalar in the action for Gravitation.

2. ACTION FOR GRAVITATION, ELECTROMAGNETISM AND RELATIVISTIC DUST

Action for Gravitation also known as Einstein-Hilbert action is as follow,

$$S_{EH} = \frac{c^4}{16\pi G}\int R\sqrt{-g}\, d^4x$$

Where $R = R_{\mu\nu}g^{\mu\nu}$ is the contraction of Ricci tensor known as Ricci scalar or the scalar curvature, $\sqrt{-g}$ is the determinant of metric tensor, c is the speed of light and G is the universal Gravitation constant in its standard units.

Action for Electromagnetism is as follow,

$$S_{EM} = \frac{-1}{4\mu_0}\int F^{\alpha\beta}F_{\alpha\beta}\sqrt{-g}\, d^4x$$

Where $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ is the Electromagnetic field tensor, $F^{\alpha\beta} = F_{\lambda\varrho}g^{\alpha\lambda}g^{\beta\varrho}$ and μ_0 is the permeability of free space in its standard units.

Action for relativistic dust is as follow,

$$S_{RD} = -c\int \rho\sqrt{v^\mu v_\mu}\sqrt{-g}\, d^4x$$

Where ρ is the mass density, v^μ is the four-velocity and c is the speed of light in its standard units.

3. VARIATION OF ACTION AND EINSTEIN-MAXWELL FIELD EQUATIONsss

Variational principle states that,

$$\delta S = 0$$

Where,

$$\delta S = \delta S_{EH} + \delta S_{EM} + \delta S_{RD}$$

Therefore we need to take variations of the actions and equate them to zero after adding them.

Variation of Einstein-Hilbert action after setting $R = R_{\mu\nu} g^{\mu\nu}$ is as follows,

$$\delta S_{EH} = \frac{c^4}{16\pi G} \delta \int R_{\mu\nu} g^{\mu\nu} \sqrt{-g}\, d^4x$$

$$\delta S_{EH} = \frac{c^4}{16\pi G} \int (\delta R_{\mu\nu} g^{\mu\nu} \sqrt{-g} + R_{\mu\nu} \delta g^{\mu\nu} \sqrt{-g} + R\delta\sqrt{-g}\,) d^4x$$

$$\delta\sqrt{-g} = \frac{-1}{2}\sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}$$

$$\delta S_{EH} = \frac{c^4}{16\pi G} \int (\delta R_{\mu\nu} g^{\mu\nu} \sqrt{-g}\,) d^4x + \frac{c^4}{16\pi G} \int (R_{\mu\nu} \delta g^{\mu\nu} \sqrt{-g} - \frac{1}{2} R \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}\,) d^4$$

$$\delta S_{EH} = \frac{c^4}{16\pi G} \int (\delta R_{\mu\nu} g^{\mu\nu} \sqrt{-g}\,) d^4x + \frac{c^4}{16\pi G} \int (R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}) \delta g^{\mu\nu} \sqrt{-g} d^4x$$

For determing the variation of Ricci tensor we derive palatini identity,

$$\delta R_{\mu\nu} = (\delta\Gamma^{\lambda}_{\mu\nu})_{;\lambda} - (\delta\Gamma^{\lambda}_{\mu\lambda})_{;\nu}$$

By varying the Ricci tensor:

$$R_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu,\lambda} - \Gamma^{\lambda}_{\mu\lambda,\nu} + \Gamma^{\alpha}_{\mu\nu}\Gamma^{\lambda}_{\alpha\lambda} - \Gamma^{\lambda}_{\alpha\nu}\Gamma^{\alpha}_{\mu\lambda}$$

We find

$$\delta R_{\mu\nu} = \delta\Gamma^{\lambda}_{\mu\nu,\lambda} - \delta\Gamma^{\lambda}_{\mu\lambda,\nu} + \delta\Gamma^{\alpha}_{\mu\nu}\Gamma^{\lambda}_{\alpha\lambda} + \delta\Gamma^{\lambda}_{\alpha\lambda}\Gamma^{\alpha}_{\mu\nu} - \delta\Gamma^{\lambda}_{\alpha\nu}\Gamma^{\alpha}_{\mu\lambda} - \delta\Gamma^{\alpha}_{\mu\lambda}\Gamma^{\lambda}_{\alpha\nu}$$

To evaluate this expression we need to compute $\delta\Gamma^{\lambda}_{\mu\nu}$. For this purpose, we will use the property

$$\delta g^{\lambda\delta} = -g^{\varrho\lambda} g^{\sigma\delta} \delta g_{\varrho\sigma}$$

If we define

$$\Gamma_{\mu\nu,\delta} = g_{\delta\lambda}\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}(g_{\mu\delta,\nu} + g_{\nu\delta,\mu} - g_{\mu\nu,\delta})$$

We can write the variation of christoffel's symbols as follows

$$\delta\Gamma^{\lambda}_{\mu\nu} = \delta\left[g^{\lambda\delta}\Gamma_{\mu\nu,\delta}\right] = \delta g^{\lambda\delta}\Gamma_{\mu\nu,\delta} + g^{\lambda\delta}\delta\Gamma_{\mu\nu,\delta}$$

$$= -g^{\varrho\lambda} g^{\sigma\delta} \delta g_{\varrho\sigma} \Gamma_{\mu\nu,\delta} + g^{\lambda\varrho}\delta\Gamma_{\mu\nu,\varrho}$$

$$= -g^{\lambda\varrho}\delta g_{\varrho\sigma}\Gamma^{\sigma}_{\mu\nu} + g^{\lambda\varrho}\frac{1}{2}\left[\delta g_{\mu\varrho,\nu} + \delta g_{\nu\varrho,\mu} - \delta g_{\mu\nu,\varrho}\right]$$

$$= g^{\lambda\varrho}\frac{1}{2}\left[\delta g_{\mu\varrho,\nu} + \delta g_{\nu\varrho,\mu} - \delta g_{\mu\nu,\varrho} - 2\Gamma^{\sigma}_{\mu\nu}\delta g_{\varrho\sigma}\right]$$

$$= \frac{1}{2} g^{\lambda\varrho} \begin{bmatrix} \left(\delta g_{\mu\varrho,\nu} - \Gamma^{\alpha}_{\mu\nu}\delta g_{\alpha\varrho} - \Gamma^{\alpha}_{\mu\varrho}\delta g_{\alpha\mu}\right) + \left(\delta g_{\nu\varrho,\mu} - \Gamma^{\alpha}_{\nu\mu}\delta g_{\alpha\varrho} - \Gamma^{\alpha}_{\varrho\mu}\delta g_{\alpha\nu}\right) \\ -\left(\delta g_{\mu\nu,\varrho} - \Gamma^{\alpha}_{\mu\varrho}\delta g_{\alpha\nu} - \Gamma^{\alpha}_{\nu\varrho}\delta g_{\alpha\mu}\right) \end{bmatrix}$$

$$= \frac{1}{2} g^{\lambda\varrho}\left[\delta g_{\mu\varrho,\nu} + \delta g_{\nu\varrho,\mu} - \delta g_{\mu\nu,\varrho}\right]$$

Since $\delta g_{\mu\nu}$is a tensor, $\delta\Gamma^{\lambda}_{\mu\nu}$ is also a tensor. Therefore the expression,

$$\delta R_{\mu\nu} = (\delta\Gamma^{\lambda}_{\mu\nu})_{;\lambda} - (\delta\Gamma^{\lambda}_{\mu\lambda})_{;\nu}$$

Can be evaluated with the usual rules of covariant differentiation of tensors. Thus we find,

$$(\delta\Gamma^{\lambda}_{\mu\nu})_{;\lambda} - (\delta\Gamma^{\lambda}_{\mu\lambda})_{;\nu}$$

$$= \delta\Gamma^{\lambda}_{\mu\nu,\lambda} - \delta\Gamma^{\lambda}_{\mu\lambda,\nu} + \delta\Gamma^{\alpha}_{\mu\nu}\Gamma^{\lambda}_{\alpha\lambda} - \delta\Gamma^{\lambda}_{\alpha\nu}\Gamma^{\alpha}_{\mu\lambda} + \Gamma^{\alpha}_{\mu\nu}\delta\Gamma^{\lambda}_{\alpha\lambda} - \Gamma^{\lambda}_{\alpha\nu}\delta\Gamma^{\alpha}_{\mu\lambda}.$$

Comparison with the equation

$$\delta R_{\mu\nu} = \delta\Gamma^{\lambda}_{\mu\nu,\lambda} - \delta\Gamma^{\lambda}_{\mu\lambda,\nu} + \delta\Gamma^{\alpha}_{\mu\nu}\Gamma^{\lambda}_{\alpha\lambda} + \delta\Gamma^{\lambda}_{\alpha\lambda}\Gamma^{\alpha}_{\mu\nu} - \delta\Gamma^{\lambda}_{\alpha\nu}\Gamma^{\alpha}_{\mu\lambda} - \delta\Gamma^{\alpha}_{\mu\lambda}\Gamma^{\lambda}_{\alpha\nu}$$

Shows that

$$\delta R_{\mu\nu} = (\delta\Gamma^{\lambda}_{\mu\nu})_{;\lambda} - (\delta\Gamma^{\lambda}_{\mu\lambda})_{;\nu}$$

$$g^{\mu\nu}\delta R_{\mu\nu} = g^{\mu\nu}\left[(\delta\Gamma^{\lambda}_{\mu\nu})_{;\lambda} - (\delta\Gamma^{\lambda}_{\mu\lambda})_{;\nu}\right] = (g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\nu})_{;\lambda} - (g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\lambda})_{;\nu}$$

$$(g^{\mu\nu}\delta\Gamma^{\alpha}_{\mu\nu} - g^{\mu\alpha}\delta\Gamma^{\lambda}_{\mu\lambda})_{;\alpha}$$

Which is divergence of a vector, therefore by gauss theorem such term vanishes when integrated over the 4 dimensional volume. Therefore,

$$\delta R_{\mu\nu} g^{\mu\nu} = 0$$

$$\delta S_{EH} = \frac{c^4}{16\pi G}\int (R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu})\delta g^{\mu\nu}\sqrt{-g}\,d^4x$$

Variation of Electromagnetic field tensor after setting $F^{\alpha\beta} = F_{\lambda\varrho}g^{\alpha\lambda}g^{\beta\varrho}$ is as follows,

$$\delta S_{EM} = \frac{-1}{4\mu_0}\delta\int F_{\alpha\beta}F_{\lambda\varrho}g^{\alpha\lambda}g^{\beta\varrho}\sqrt{-g}\,d^4x$$

$$\delta S_{EM} = \frac{-1}{4\mu_0}\int ((\delta g^{\alpha\lambda}g^{\beta\varrho})F_{\alpha\beta}F_{\lambda\varrho}\sqrt{-g} + F^{\alpha\beta}F_{\alpha\beta}\delta\sqrt{-g})d^4x$$

$$\delta S_{EM} = \frac{-1}{4\mu_0}\int ((2\delta g^{\alpha\lambda})F_{\alpha\beta}F^{\beta}_{\lambda}\sqrt{-g} - F^{\alpha\beta}F_{\alpha\beta}\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu})d^4x$$

$$\delta S_{EM} = \frac{-1}{2\mu_0}\int (F_{\mu\beta}F^{\beta}_{\nu} - F^{\alpha\beta}F_{\alpha\beta}\frac{1}{4}g_{\mu\nu})\delta g^{\mu\nu}\sqrt{-g}\,d^4x$$

Variation of the action for Relativistic dust is as follows,

$$\delta S_{RD} = -c\delta\int \rho\sqrt{v^{\mu}v_{\mu}}\sqrt{-g}\,d^4x$$

$$\delta S_{RD} = -c\delta\int \sqrt{p^{\mu}p_{\mu}}\,d^4x$$

Where$\sqrt{p^{\mu}p_{\mu}} = \rho\sqrt{v^{\mu}v_{\mu}}\sqrt{-g}$ is the four momentum density and ρ is the mass density

$$\delta S_{RD} = -c\int \frac{\delta g^{\mu\nu}p_{\mu}p_{\nu}}{2\sqrt{p^{\alpha}p_{\alpha}}}d^4x$$

$$\delta S_{RD} = -c\int \frac{p_{\mu}p_{\nu}}{2\sqrt{p^{\alpha}p_{\alpha}}}\delta g^{\mu\nu}d^4x$$

$$\delta S_{RD} = -c\int \frac{\rho v_{\mu}\rho v_{\nu}\sqrt{-g}^{2}}{2\rho c\sqrt{-g}}\delta g^{\mu\nu}d^4x$$

$$\delta S_{RD} = -\frac{1}{2}\int \rho v_\mu v_\nu \sqrt{-g}\,\delta g^{\mu\nu} d^4x$$

Now as we have derived all the variations of the action we can now define stress energy momentum tensor as follows,

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta(S_{EH} + S_{EM} + S_{RD})}{\delta g^{\mu\nu}}$$

Therefore

$$\frac{1}{\sqrt{-g}}\frac{\delta(S)}{\delta g^{\mu\nu}} = \frac{1}{\sqrt{-g}}\frac{\delta(S_{EH})}{\delta g^{\mu\nu}} + \frac{1}{\sqrt{-g}}\frac{\delta(S_{EM})}{\delta g^{\mu\nu}} + \frac{1}{\sqrt{-g}}\frac{\delta(S_{RD})}{\delta g^{\mu\nu}} = 0$$

$$\frac{c^4}{16\pi G}(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}) - \frac{1}{2\mu_0}\left(F_{\mu\beta}F_\nu^\beta - F^{\alpha\beta}F_{\alpha\beta}\frac{1}{4}g_{\mu\nu}\right) - \frac{1}{2}\rho v_\mu v_\nu = 0$$

Therefore the desired equation is,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}\frac{1}{\mu_0}\left(F_{\mu\beta}F_\nu^\beta - F^{\alpha\beta}F_{\alpha\beta}\frac{1}{4}g_{\mu\nu}\right) + \frac{8\pi G}{c^4}\rho v_\mu v_\nu$$

4. CONCLUSION

When worked independently with the variation of action we can derive the equations that led to Schwarzchild soloution and Reissner-Nordstrom soloution for spherically symmetric and charged blackhole.

5. REFRENCES

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