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The risk ratio is logically inconsistent

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ABSTRACT

Many different measures of association are used by medical literature, the relative risk is one of these measures. However, to judge whether results of studies are reliable, it is essential to use among other measures of association which are logically consistent. In this paper, we will present how to deal with one of the most commonly used measures of association, the relative risk. The conclusion is inescapable that the relative risk is logically inconsistent and should not be used any longer.

Keywords: Statistical methods, logical consistency — — measures of relationships — relative risk

1. INTRODUCTION

The relation between data actually obtained (the sample) and hypotheses is studied by a mathematical and conceptual discipline called statistics. The data of a sample can be biased which can be a source of incorrect conclusions. However, in almost all scientific research, empirical data or facts are investigated by specific statistical methods in order to evaluate some hypotheses of a particular kind [1]. The statistical methods, in turn, need to be at least logically consistent. Central to the correctness of statistical methods is the problem of logical consistency, which concerns the justification of any statistical method. However, even if statistics provide us with various methods and means to evaluate hypotheses it is insightful to consider that statistics may harbour a large variety of errors and logical fallacies hidden sometimes behind highly abstract mathematical stuff. One of such commonly used statistical method is the risk ratio or relative risk (RR) which is designed to detect or to measure the relation between the exposure and the outcome.
Despite frequent use of RR, founded doubts regarding the correctness and logical consistency of RR are not automatically excluded. In any case, the issue is not how often is RR used, but whether RR is correct or not.

2. MATERIAL AND METHODS

From the beginning of statistics onward the same is interrelated with probability theory. However, what kinds of 'things'are probabilistic statements, or more generally under which circumstances are probability statements true or false?

2.1. *Material*

The subject of study in statistics is among other the relation between data and hypotheses. Summing up, it remains problematic to study anything with some definitions.

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[1] https://plato.stanford.edu/entries/statistics/

2.1.1. *Definitions*

Definition 2.1 (Independence).

The independence of two events A_t and B_t regarded from the standpoint of a certain observer was defined by de Moivre on page 7 as "... therefore, those two Events being independent, the Probability of their both happening will be 1/13 * 1/13 = 1/169 "[2] and Kolmogoroff [3] and other, as

$$p(B_t) \times p(A_t) = p(a_t) \tag{1}$$

where $p(A_t)$ denotes the probability of an event A_t at the Bernoulli trial t and $p(B_t)$ denotes the probability of another event B_t at the same Bernoulli trial t while $p(a_t)$ denotes the joint probability of $p(A_t \text{ AND } B_t)$ at the same Bernoulli trial t.

Definition 2.2 (Dependence).

The Dependence of two events A_t and B_t regarded from the standpoint of a certain observer is defined as

$$p(a_t) = (p(B_t) \times p(A_t))^{1/2} \tag{2}$$

where $p(A_t)$ denotes the probability of an event A_t at the Bernoulli trial t and $p(B_t)$ denotes the probability of another event B_t at the same Bernoulli trial t while $p(a_t)$ denotes the joint probability of $p(A_t \text{ AND } B_t)$ at the same Bernoulli trial t while the dependence of n events[4] follows as

$$p(a_{1,t}, a_{2,t}, ..., a_{n,t}) = (p(A_{1,t}) \times p(A_{2,t}) \times ... \times p(A_{n,t}))^{1/n} \tag{3}$$

Definition 2.3 (Basic relationships between probabilities).

In general, it is

$$p(A_t) = p(a_t) + p(b_t) \tag{4}$$

and

$$p(NotA_t) = 1 - p(A_t) = p(c_t) + p(d_t) \tag{5}$$

and

$$p(B_t) = p(a_t) + p(c_t) \tag{6}$$

and

$$p(NotB_t) = 1 - p(B_t) = p(b_t) + p(d_t) \tag{7}$$

where $p(a_t)$ denotes the joint probability of A_t and B_t. In general, it is

$$p(a_t) + p(b_t) + p(c_t) + p(d_t) = +1 \tag{8}$$

Definition 2.4 (Contingency table).

The 2 by 2 contingency table is able to provide a basic picture of the interrelation between two binomial distributed random variables and is of use to analyse the relationships between them. Karl Pearson was the first to use the term contingency table in his paper "On the Theory of Contingency and Its Relation to Association and Normal Correlation"[5].

In general, it is

$$p(A_t) = p(a_t) + p(b_t) \tag{9}$$

and

$$p(NotA_t) = 1 - p(A_t) = p(c_t) + p(d_t) \tag{10}$$

[2] https://doi.org/10.3931/e-rara-10420
[3] https://doi.org/10.1007/978-3-642-49888-6
[4] Ilija Barukčić, Die Kausalität, Hamburg: Wissenschaftsverlag, 1989, pp. 57-59.
[5] https://archive.org/details/cu31924003064833/page/n2/mode/2up

Relative risk		Outcome		Total
		YES	NO	
Exposed	YES	$p(a_t)$	$p(b_t)$	$p(A_t)$
	NO	$p(c_t)$	$p(d_t)$	$p(\underline{A}_t)$
	Total	$p(B_t)$	$p(\underline{B}_t)$	+1

and

$$p(B_t) = p(a_t) + p(c_t) \tag{11}$$

and

$$p(NotB_t) = 1 - p(B_t) = p(b_t) + p(d_t) \tag{12}$$

and

$$p(a_t) + p(b_t) + p(c_t) + p(d_t) = +1 \tag{13}$$

where $p(a_t)$ denotes the joint probability of A_t and B_t, $p(b_t)$ denotes the joint probability of A_t and Not B_t, $p(c_t)$ denotes the joint probability of not A_t and B_t and $p(d_t)$ denotes the joint probability of not A_t and Not B_t.

Definition 2.5 (Relative risk)**.**

The degree of association between the two binomial variables can be assessed by a number of very different coefficients, the relative risk [6] is one of them. Sir Ronald Aylmer Fisher (1890 - 1962) defined the relative risk in his publication "The Logic of Inductive Inference"[7] as:

$$RR(A_t, B_t) = \frac{\frac{p(a_t)}{p(A_t)}}{\frac{p(c_t)}{p(NotA_t)}} = \frac{p(a_t) \times p(NotA_t)}{p(A_t) \times p(c_t)} \tag{14}$$

That what scientist generally understand by relative risk is the ratio of a probability of an event occurring with an exposure versus the probability of an event occurring without an exposure. In other words, relative Risk = (Probability of event in exposed group) / (Probability of event in not exposed group). An $RR(A_t,B_t) = +1$ means that exposure does not affect the outcome or both are independent of each other while $RR(A_t,B_t)$ less than +1 means that the risk of the outcome is decreased by the exposure. In this context, an $RR(A_t,B_t)$ greater than +1 denotes that the risk of the outcome is increased by the exposure. Widely known problems with odds ratio [8] [9] and relative risk [10] are already documented [11] [12] in literature.

[6] https://www.ncbi.nlm.nih.gov/books/NBK430824/

[7] https://www.jstor.org/stable/pdf/2342435.pdf?seq=1

[8] https://www.ncbi.nlm.nih.gov/pubmed/9832001

[9] https://www.ncbi.nlm.nih.gov/pmc/articles/PMC6178613/

[10] https://www.ncbi.nlm.nih.gov/pmc/articles/PMC522855/

[11] https://www.crcpress.com/Principles-of-Biostatistics-Second-Edition/Pagano-Gauvreau/p/book/9781138593145

[12] https://www.biometricsociety.org/wp-content/uploads/2018/07/IBS-IBC2012-Final-Programme.compressed.pdf

Definition 2.6 (Exclusion relationship)**.**

The exclusion relationship is defined as

$$p(A_t \mid B_t) = p(b_t) + p(c_t) + p(d_t) = +1 \tag{15}$$

Definition 2.7 (Conditio sine qua non relationship)**.**

The conditio sine qua non relationship is defined as

$$p(A_t \longleftarrow B_t) = p(a_t) + p(b_t) + p(d_t) = +1 \tag{16}$$

Remark. Since thousands of years, human mankind is familiar with the concept of a necessary condition. For example, we all know that air or gaseous oxygen is a necessary for (human) life. Without gaseous oxygen, there is no (human) life. However, the first documented mathematiziation of the concept of a necessary condition (**conditio sine qua non**) was published by Barukčić 1989 [13]. Conditions may be necessary without being sufficient and vice versa. Sufficient conditions need not to be necessary. However, there may exist conditions which are both, necessary and sufficient.

Definition 2.8 (Conditio per quam relationship)**.**

The conditio per quam relationship is defined [14] [15] [16] [17] [18] [19] [20] [21] as

$$p(A_t \longrightarrow B_t) = p(a_t) + p(c_t) + p(d_t) = +1 \tag{17}$$

2.1.2. *Axioms*

Axiom 1. Lex identitatis [22] [23] [24].

$$+1 = +1 \tag{18}$$

Axiom 2. Lex contradictionis[25] [26] [27].

$$+0 = +1 \tag{19}$$

2.2. *Methods*

2.2.1. *Proof methods*

Proof methods like a direct proof [28], proof by contradiction[29], modus ponens[30], modus inversus[31] [32] and other methods are of use to detect inconsistencies and inadequacies in scientific theories.

[13] Ilija Barukčić, Die Kausalität, Hamburg: Wissenschaftsverlag, 1989
[14] https://aip.scitation.org/doi/abs/10.1063/1.3567453
[15] https://aip.scitation.org/doi/abs/10.1063/1.4773147
[16] https://www.scirp.org/journal/paperinformation.aspx?paperid=69478
[17] https://www.scirp.org/journal/paperinformation.aspx?paperid=67272
[18] http://www.ijapm.org/show-64-515-1.html
[19] https://www.sciencedirect.com/science/article/pii/S1875389211006626
[20] https://view.publitas.com/amph/rjr_2018_4_art-02/page/1
[21] http://jddtonline.info/index.php/jddt/article/view/3385
[22] https://www.scirp.org/journal/paperinformation.aspx?paperid=69478
[23] https://www.ncbi.nlm.nih.gov/nlmcatalog/101656626
[24] https://doi.org/10.22270/jddt.v9i2.2389
[25] https://www.ncbi.nlm.nih.gov/nlmcatalog/101656626
[26] https://doi.org/10.22270/jddt.v9i2.2389
[27] https://doi.org/10.22270/jddt.v10i1-s.3856
[28] http://www.ijmttjournal.org/Volume-65/Issue-7/IJMTT-V65I7P524.pdf
[29] https://aip.scitation.org/doi/abs/10.1063/1.3567453
[30] http://www.ijmttjournal.org/Volume-65/Issue-7/IJMTT-V65I7P524.pdf
[31] http://www.ijmttjournal.org/Volume-65/Issue-7/IJMTT-V65I7P524.pdf
[32] https://vixra.org/pdf/1911.0410v1.pdf

3. RESULTS

3.1. *Independence of A_t and B_t*

Theorem 1 (Independence of $\mathbf{A_t}$ and $\mathbf{B_t}$).

Claim.
In general, under circumstances of independence of of A_t and B_t, it is

$$p(B_t) = \frac{p(a_t)}{p(A_t)} \tag{20}$$

Proof By Modus Ponens.
The premise of modus ponens [33] in the case of independence according to de Moivre [34] and Kolmogoroff [35] and other, is that

$$p(B_t) \times p(A_t) = p(a_t) \tag{21}$$

Dividing by $p(A_t)$, we obtain

$$\frac{p(B_t) \times p(A_t)}{p(A_t)} = \frac{p(a_t)}{p(A_t)} \tag{22}$$

At the end, the conclusion

$$p(B_t) = \frac{p(a_t)}{p(A_t)} \tag{23}$$

is true.
Quod erat demonstrandum.

3.2. *Independence of Not A_t and B_t*

Theorem 2 (Independence of not $\mathbf{A_t}$ and $\mathbf{B_t}$).

Claim.
In general, under circumstances of independence, it is

$$p(B_t) = \frac{p(c_t)}{p(NotA_t)} \tag{24}$$

Proof By Modus Ponens.
The premise of modus ponens in the case of independence according to de Moivre [36] and Kolmogoroff [37] and other, is that

$$p(B_t) \times p(NotA_t) = p(c_t) \tag{25}$$

Dividing by $p(\text{Not } A_t)$, we obtain

$$\frac{p(B_t) \times p(NotA_t)}{p(NotA_t)} = \frac{p(c_t)}{p(NotA_t)} \tag{26}$$

At the end, the conclusion

$$p(B_t) = \frac{p(c_t)}{p(NotA_t)} \tag{27}$$

is true.
Quod erat demonstrandum.

[33] http://www.ijmttjournal.org/archive/ijmtt-v65i7p524
[34] https://doi.org/10.3931/e-rara-10420
[35] https://doi.org/10.1007/978-3-642-49888-6
[36] https://doi.org/10.3931/e-rara-10420
[37] https://doi.org/10.1007/978-3-642-49888-6

3.3. *Case $p(a_t) = 0$: The relative risk RR is defined*

Theorem 3 (**Case $\mathbf{p(a_t) = 0}$: The relative risk RR is defined**).

Claim.
In general, under circumstances $p(a_t) = 0$, the relative risk RR is determined as

$$RR(A_t, B_t) = \frac{\frac{p(a_t)}{p(A_t)}}{\frac{p(c_t)}{p(notA_t)}} = \frac{p(a_t) \times p(notA_t)}{p(A_t) \times p(c_t)} = 0. \tag{28}$$

Proof By Modus Ponens.
The premise of modus ponens is that the relative risk RR is true. Thus far, it is

$$RR(A_t, B_t) = \frac{p(a_t) \times p(notA_t)}{p(A_t) \times p(c_t)} \tag{29}$$

Under conditions where $p(a_t) = 0$, it is

$$RR(A_t, B_t) = \frac{0 \times p(notA_t)}{p(A_t) \times p(c_t)} \tag{30}$$

Under these circumstances the conclusion

$$RR(A_t, B_t) = 0 \tag{31}$$

is true.
Quod erat demonstrandum.

Remark. Theoretically, the relative risk has the potential to detect an exclusion relationship, but only if RR = 0. The following figure may illustrate this relationship again.

Relative risk		Outcome: YES	Outcome: NO	Total
Exposed	YES	$p(a_t)$	$p(b_t)$	$p(A_t)$
	NO	$p(c_t)$	$p(d_t)$	$p(\underline{A}_t)$
	Total	$p(B_t)$	$p(\underline{B}_t)$	+1

3.4. *Case $p(b_t) = 0$: The relative risk RR is defined*

Theorem 4 (Case $\mathbf{p(b_t) = 0}$: The relative risk RR is defined).

Claim.
In general, under circumstances $p(b_t) = 0$, the relative risk RR is determined as

$$RR(A_t, B_t) = \frac{p(notA_t)}{p(c_t)} \tag{32}$$

Proof By Modus Ponens.
The premise of modus ponens is that the relative risk RR is true. Thus far, it is

$$RR(A_t, B_t) = \frac{p(a_t) \times p(notA_t)}{p(A_t) \times p(c_t)} \tag{33}$$

which is equivalent with

$$RR(A_t, B_t) = \frac{p(a_t) \times p(notA_t)}{(p(a_t) + p(b_t)) \times p(c_t)} \tag{34}$$

Under conditions where $p(b_t) = 0$, the equation before changes to

$$RR(A_t, B_t) = \frac{p(a_t) \times p(notA_t)}{(p(a_t) + 0) \times p(c_t)} \tag{35}$$

or to

$$RR(A_t, B_t) = \frac{p(a_t) \times p(notA_t)}{p(a_t) \times p(c_t)} \tag{36}$$

Under circumstances where $p(b_t) = 0$ the conclusion

$$RR(A_t, B_t) = \frac{p(notA_t)}{p(c_t)} \tag{37}$$

is true.
Quod erat demonstrandum.

Remark. Theoretically, the relative risk RR has the potential to detect a **conditio per quam** relationship, but only if RR > 0. However, a significant and positive relative risk does not provide evidence of a conditio per quam relationship. Furthermore and depending on study design, an exsiting conditio per quam relationship must not be detected by the relative risk. The following figure may illustrate the relationship again.

Relative risk		Outcome		Total
		YES	NO	
Exposed	YES	$p(a_t)$	$p(b_t)$	$p(A_t)$
	NO	$p(c_t)$	$p(d_t)$	$p(\underline{A}_t)$
	Total	$p(B_t)$	$p(\underline{B}_t)$	+1

3.5. *Case $p(c_t) = 0$: The relative risk RR is not defined*

Theorem 5 (**Case $\mathbf{p(c_t) = 0}$: The relative risk RR is not defined**).

Claim.
In general, under circumstances $p(c_t) = 0$, the relative risk RR is not defined due to

$$RR(A_t, B_t) = \frac{p(a_t) \times p(d_t)}{0} \tag{38}$$

Proof.
The premise of modus ponens is that the relative risk RR is true. Thus far, again it is

$$RR(A_t, B_t) = \frac{p(a_t) \times p(notA_t)}{p(A_t) \times p(c_t)} \tag{39}$$

which is equivalent with

$$RR(A_t, B_t) = \frac{p(a_t) \times (p(c_t) + p(d_t))}{(p(a_t) + p(b_t)) \times p(c_t)} \tag{40}$$

Under conditions where $p(c_t) = 0$, the equation before changes to

$$RR(A_t, B_t) = \frac{p(a_t) \times (0 + p(d_t))}{(p(a_t) + p(b_t)) \times 0} \tag{41}$$

or to

$$RR(A_t, B_t) = \frac{p(a_t) \times (0 + p(d_t))}{(p(a_t) + p(b_t)) \times 0} \tag{42}$$

or to

$$RR(A_t, B_t) = \frac{p(a_t) \times p(d_t)}{0} \tag{43}$$

However, today, the division by zero is not accepted. Therefore, the conclusion that

$$RR(A_t, B_t) = \frac{p(a_t) \times p(d_t)}{0} \tag{44}$$

the relative risk RR is not defined under circumstances where $p(c_t) = 0$ is true.
Quod erat demonstrandum.

Remark. Theoretically, a conditio sine qua non relationship is determined by the fact that $p(c_t) = 0$. However, under these circumstances the relative risk RR collapses into logical absurdity and cannot detect a necessary condition, a **conditio sine qua non** at all. The following figure may illustrate the relationship again.

Relative risk		Outcome		Total
		YES	NO	
Exposed	YES	$p(a_t)$	$p(b_t)$	$p(A_t)$
	NO	$p(c_t)$	$p(d_t)$	$p(\underline{A}_t)$
	Total	$p(B_t)$	$p(\underline{B}_t)$	+1

Under conditions of **a necessary and sufficient condition** is determined by $\mathbf{p(c_t) = 0}$ **AND** $\mathbf{p(b_t) = 0}$. However, even under these circumstances, the relative risk breaks together too, because

$$RR(A_t, B_t) = \frac{p(a_t) \times (0 + p(d_t))}{(p(a_t) + 0) \times 0} \tag{45}$$

4. DISCUSSION

The relative risk is a measure of association used in the statistical analysis of the data of different studies. Unfortunately, this publication has recognised the fundamental problems as associated with the relative risk. The relative risk depends to much on study design and can lead to contradictory and highly misleading results. The relative risk cannot recognise the conditio sine qua non relationship (theorem 5) and fails in principle on the conditio per quam relationship. The relative risk [38] is logically inconsistent, unreliable and highly dangerous, and will not be helpful either for decision makers, who will be unable to rely on the results achieved by the relative risk and to translate the same into effective interventions or action, or scientists, who will be unable to relate the relationship between two events to a causal mechanism.

5. CONCLUSION

There are many studies in clinical research published which rely on the relative risk. In this publication, we have investigated the interior logic of the relative risk. We cannot rely on the relative risk. The relative risk is logically inconsistent and completely useless, the relative risk is refuted. The hope that this will help clinicians an others when reading medical literature.

[38] https://www.ncbi.nlm.nih.gov/pmc/articles/PMC5841621/